Search and endogenous growth: when Romer meets Lagos and Wright

Chu, Angus C. and Lai, Ching-Chong and Liao, Chih-Hsing

Durham University, Academia Sinica, National Chengchi University

February 2012

Search and Endogenous Growth: When Romer Meets Lagos and Wright

Angus C. Chu, Durham University Ching-Chong Lai, Academic Sinica
Chih-Hsing Liao, National Chengchi University

February 2012

Abstract

In this note, we develop a search-based monetary growth model to analyze the growth and welfare effects of inflation. We introduce endogenous growth via capital externality into a two-sector search model and compare the effects of inflation to those from a standard cash-in-advance (CIA) growth model. We find two important differences between the two approaches. First, while the growth effect of inflation operates solely through endogenous labor supply in the CIA model, the growth effect of inflation operates through an additional consumption effect in the decentralized market in the search model. Second, we quantitatively evaluate the welfare cost of inflation and find that the search model exhibits a larger (smaller) welfare gain than the CIA model when we decrease the growth rate of money supply to achieve the Friedman rule (zero inflation). These contrasting results are due to a non-linearity in welfare as a function of inflation in the search model.

JEL classification: E41, O41, O42.
Keywords: economic growth, inflation, monetary policy.

Chu: angusccc@gmail.com. Durham Business School, Durham University, Durham, United Kingdom.
Lai: cclai@econ.sinica.edu.tw. Institute of Economics, Academia Sinica, Taipei, Taiwan.
Liao: 93258507@nccu.edu.tw. Department of Economics, National Chengchi University, Taipei, Taiwan.

1 Introduction

In this note, we analyze the effects of inflation and monetary policy on economic growth and social welfare. Although this important issue in monetary economics has received much attention and careful analysis in previous studies, our analysis provides some novel elements and results. To highlight the novelty of this study, it is helpful to first discuss two related branches of literature in monetary economics. First, this study relates to the search-based literature on money and capital formation; see for example, Shi (1999), Menner (2006), Williamson and Wright (2010), Aruoba *et al.* (2011) and Waller (2011). This branch of literature analyzes the relationship between money and capital formation in a search-theoretic framework without considering economic growth as an endogenous process. Second, this study also relates to the branch of literature on inflation and economic growth; see for example, Wang and Yip (1992), Gomme (1993), Dotsey and Ireland (1996), Mino (1997) and more recently, Itaya and Mino (2003, 2007). This branch of literature analyzes the growth and welfare effects of inflation by modeling money demand based on the classical approach, such as a cash-in-advance (CIA) constraint, money in utility and transaction costs, without considering search. In this study, we attempt to provide a bridge between these two branches of literature by analyzing the growth and welfare effects of inflation in a search-based monetary growth model. In summary, we introduce endogenous growth via capital externality as in Romer (1986) into a two-sector search model based on Lagos and Wright (2005), Aruoba *et al.* (2011) and Waller (2011). We also compare the growth and welfare effects of inflation from the search model to those from a standard CIA endogenous-growth model and find the following important differences between these two approaches.

Qualitatively, while the growth effect of inflation operates solely through endogenous labor supply in the CIA model, the growth effect of inflation operates through an additional consumption effect in the decentralized market in the search model. Intuitively, a higher inflation increases the cost of holding money and reduces consumption in the decentralized market that requires the use of money for transactions. As a result of lower consumption in the decentralized market, capital demand decreases causing a reduction in capital accumulation and economic growth regardless of whether or not labor supply is endogenous. However, when labor supply is exogenous in the CIA model, inflation has no effect on economic growth.

Quantitatively, we evaluate the welfare cost of inflation and find that the search model exhibits a *larger* welfare gain than the CIA model when we decrease the inflation rate from its empirical level in the US towards the Friedman rule. This finding is consistent with previous studies, such as Lagos and Wright (2005) and Aruoba *et al.* (2011); however, we also discover a novel finding that the search model exhibits a *smaller* welfare gain than the CIA model when we decrease the inflation rate to zero. These contrasting results arise because social welfare is a non-linear function of money growth in the search model. Due to this non-linearity, when the change in inflation is small (large), the search model exhibits a smaller (larger) welfare effect than the CIA model. Therefore, it is not always the case that the search model exhibits a larger welfare effect of inflation than the CIA model, as often claimed in the literature.

The rest of this note is organized as follows. In Section 2, we set up the two-sector search model. In Section 3, we analyze the growth and welfare effects of inflation in the search model. In Section 4, we present a canonical CIA model and analyze the growth and welfare effects of inflation. In Section 5, we calibrate the two models to provide a quantitative comparison of the two approaches. The final section concludes.

2 A search-based monetary growth model

The two-sector search model is due to Lagos and Wright (2005). Aruoba *et al.* (2011) extend the Lagos-Wright model by introducing capital accumulation, whereas Waller (2011) further extends the model in Aruoba *et al.* (2011) by allowing for exogenous technological progress. Our model is based on Waller (2011), but we introduce capital externality into his model to generate endogenous growth. In what follows, we describe the basic features of the search-based monetary growth model.

2.1 Households

There is a unit measure of identical and infinitely-lived households in discrete time. In each period, households engage in economic activities first in the decentralized market (hereafter DM) and then in the centralized market (hereafter CM). The DM and the CM are distinguished as follows. In period t, households first enter the DM where they consume or produce special goods q_t. In this market, each meeting is random and anonymous so that money becomes essential.[1] Once the round of DM trade is completed, households proceed to the CM where they consume and produce general goods as in standard growth models. Following the common approach in the literature, we assume that there is no discounting between the DM and the CM within each period, and the discount factor between any two consecutive periods is $\beta \in (0, 1)$. In what follows, we first discuss households' optimization in the CM.

2.1.1 Households' optimization in the CM

In the CM, households have an instantaneous utility function $u_t = \ln x_t - Ah_t$, which is increasing in the consumption of general goods x_t and decreasing in the supply of labor h_t. The parameter $A > 0$ determines the disutility of labor supply. Let $W(m_t, k_t)$ and $V(m_t, k_t)$ denote the period-t value functions for households in the CM and the DM respectively. m_t is the nominal money balance and k_t is the capital stock owned by households in period t. The maximization problem of households in the CM can be expressed as

$$W(m_t, k_t) = \max_{x_t, h_t, m_{t+1}, k_{t+1}} [\ln x_t - Ah_t + \beta V(m_{t+1}, k_{t+1})] \tag{1}$$

[1] Following the standard approach in the literature, we assume that capital cannot serve as a medium of exchange; see Williamson and Wright (2010) and Aruoba *et al.* (2011) for a useful discussion. Lagos and Rocheteau (2008) show that even when capital serves as a competing medium of exchange, fiat money can still be valued and used as a medium of exchange.

subject to a sequence of budget constraints given by

$$k_{t+1} + \frac{m_{t+1}}{p_t} = w_t h_t + (1 + r_t - \delta)k_t + \frac{m_t}{p_t} + \tau_t - x_t. \tag{2}$$

p_t is the price of general goods. w_t is the real wage rate (denominated in the price of general goods). r_t is the rental price of capital. The parameter $\delta \geq 0$ is the depreciation rate of capital. τ_t is a real lump-sum transfer from the government.

From standard optimization, the optimality condition for consumption in the CM is

$$\frac{1}{x_t} = \frac{A}{w_t}. \tag{3}$$

Equation (3) implies that all households consume the same amount of general goods x_t in the CM regardless of their holdings of capital and money. This useful property results from the quasi-linear utility function, which is a standard simplifying assumption in this branch of model to eliminate any dispersion in money holdings that arises from trades in the DM.[2] The standard intertemporal optimality conditions for the accumulation of capital and money are respectively

$$\frac{1}{x_t} = \beta V_k(m_{t+1}, k_{t+1}), \tag{4}$$

$$\frac{1}{p_t x_t} = \beta V_m(m_{t+1}, k_{t+1}). \tag{5}$$

Equations (3) to (5) imply that all households enter the DM in the next period with the same holdings of capital and money. In addition, the familiar envelope conditions are

$$W_k(m_t, k_t) = \frac{1 + r_t - \delta}{x_t}, \tag{6}$$

$$W_m(m_t, k_t) = \frac{1}{p_t x_t}. \tag{7}$$

2.1.2 Households' optimization in the DM

In the DM, a household either becomes (a) a buyer, (b) a seller or (c) a nontrader. The probability of becoming a buyer is $\sigma \in (0, 0.5)$, and the probability of becoming a seller is also $\sigma \in (0, 0.5)$. The probability of becoming a nontrader is $1 - 2\sigma > 0$. As $\sigma \to 0$, monetary policy would have no effects on economic growth and social welfare. This taste-and-technology-shock specification shows a matching technology that buyers meet with sellers and is a standard feature of the Lagos-Wright model. As a result of this taste-and-technology shock, the value of entering the DM is

$$V(m_t, k_t) = \sigma V^b(m_t, k_t) + \sigma V^s(m_t, k_t) + (1 - 2\sigma)W(m_t, k_t), \tag{8}$$

where $V^b(.)$ and $V^s(.)$ are the values of being a buyer and a seller respectively.

[2]See for example, Rocheteau and Wright (2005) and Aruoba *et al.* (2011) for a useful discussion.

To analyze $V^b(.)$ and $V^s(.)$, we consider the following functional forms for the buyers' preference and the sellers' production technology. In the DM, each buyer's utility $\ln q_t^b$ is increasing and concave in the consumption of special goods. Each seller produces special goods q_t^s by combining her capital k_t and effort e_t subject to the following Cobb-Douglas production function.

$$q_t^s = z_t^{1-\alpha} k_t^{\alpha} e_t^{\eta}, \tag{9}$$

where z_t denotes aggregate technology. To achieve endogenous growth, we will follow Romer (1986) to assume that capital has a positive externality effect on aggregate technology such that $z_t = \overline{k}_t$, where $\overline{k}_t$ is the aggregate holding of capital in the economy.[3] The parameter $\alpha \in (0,1)$ determines capital share. To ensure constant returns to scale, we will impose $\eta = 1-\alpha$ for labor share; however, it would be useful for us to first present the analysis with η in order to isolate the effects of capital and labor shares.

Rewriting equation (9), we can express the utility cost of production as

$$e\left(\frac{q_t^s}{z_t}, \frac{k_t}{z_t}\right) = \left(\frac{q_t^s}{z_t}\right)^{1/\eta} \left(\frac{k_t}{z_t}\right)^{-\alpha/\eta}. \tag{10}$$

Buyers purchase special goods q_t^b by spending money d_t^b, whereas sellers earn money d_t^s by producing special goods q_t^s.[4] Given these terms of trade, the values of being a buyer and a seller are respectively[5]

$$V^b(m_t, k_t) = \ln q_t^b + W(m_t - d_t^b, k_t), \tag{11}$$

$$V^s(m_t, k_t) = -e\left(\frac{q_t^s}{z_t}, \frac{k_t}{z_t}\right) + W(m_t + d_t^s, k_t). \tag{12}$$

Differentiating (11) and (12) and substituting them into (8), we can obtain the following envelope condition for m_t.

$$\begin{aligned} V_m(m_t, k_t) &= (1-2\sigma) W_m(m_t, k_t) + \sigma \left[\frac{1}{q_t^b} \frac{\partial q_t^b}{\partial m_t} + W_m(m_t - d_t^b, k_t) \left(1 - \frac{\partial d_t^b}{\partial m_t}\right) \right] \\ &\quad + \sigma \left[-e_1 \left(\frac{q_t^s}{z_t}, \frac{k_t}{z_t}\right) \frac{1}{z_t} \frac{\partial q_t^s}{\partial m_t} + W_m(m_t + d_t^s, k_t) \left(1 + \frac{\partial d_t^s}{\partial m_t}\right) \right], \end{aligned} \tag{13}$$

where $W_m(m_t, k_t) = W_m(m_t - d_t^b, k_t) = W_m(m_t + d_t^s, k_t) = 1/(p_t x_t)$ from (7). Similarly, we can obtain the following envelope condition for k_t.

$$\begin{aligned} V_k(m_t, k_t) &= (1-2\sigma) W_k(m_t, k_t) + \sigma \left[\frac{1}{q_t^b} \frac{\partial q_t^b}{\partial k_t} - W_m(m_t - d_t^b, k_t) \frac{\partial d_t^b}{\partial k_t} + W_k(m_t - d_t^b, k_t) \right] \\ &\quad + \sigma \left[-e_1 \left(\frac{q_t^s}{z_t}, \frac{k_t}{z_t}\right) \frac{1}{z_t} \frac{\partial q_t^s}{\partial k_t} - e_2 \left(\frac{q_t^s}{z_t}, \frac{k_t}{z_t}\right) \frac{1}{z_t} + W_m(m_t + d_t^s, k_t) \frac{\partial d_t^s}{\partial k_t} + W_k(m_t + d_t^s, k_t) \right], \end{aligned} \tag{14}$$

[3] It is useful to note that $k_t = \overline{k}_t$ in equilibrium.

[4] As a result of these different money holdings at the end of the DM, households supply different amounts of labor in the CM that eliminate any dispersion in money holdings.

[5] Adding a disutility parameter to the supply of effort in the DM would not change our qualitative and quantitative results. Therefore, we follow Aruoba *et al.* (2011) and Waller (2011) to normalize this parameter to unity.

where $W_k(m_t, k_t) = W_k(m_t - d_t^b, k_t) = W_k(m_t + d_t^s, k_t) = (1 + r_t - \delta)/x_t$ from (6).

To solve the marginal value of holding money (13) and capital (14), we consider a competitive equilibrium with price taking as in Aruoba *et al.* (2011) and Waller (2011).[6] Under price taking, once buyers and sellers are matched, they both act as price takers. Given the price $\widetilde{p}_t$ of special goods, buyers choose q_t^b to maximize

$$V^b(m_t, k_t) = \max_{q_t^b}[\ln q_t^b + W(m_t - \widetilde{p}_t q_t^b, k_t)] \tag{15}$$

subject to the budget constraint

$$d_t^b = \widetilde{p}_t q_t^b \le m_t. \tag{16}$$

In the DM, buyers spend all their money,[7] so that the money constraint implies that

$$q_t^b = m_t/\widetilde{p}_t. \tag{17}$$

As for sellers' maximization problem in the DM, it is given by

$$V^s(m_t, k_t) = \max_{q_t^s}\left[-e\left(\frac{q_t^s}{z_t}, \frac{k_t}{z_t}\right) + W(m_t + \widetilde{p}_t q_t^s, k_t)\right]. \tag{18}$$

Sellers' optimal supplies of special goods can be obtained from the following condition.

$$e_1\left(\frac{q_t^s}{z_t}, \frac{k_t}{z_t}\right)\frac{1}{z_t} = \widetilde{p}_t W_m(m_t + \widetilde{p}_t q_t^s, k_t) \Leftrightarrow \frac{1}{\eta}e\left(\frac{q_t^s}{z_t}, \frac{k_t}{z_t}\right) = \frac{\widetilde{p}_t q_t^s}{p_t x_t}, \tag{19}$$

where the second equality of (19) makes use of (7) and (10).

Using (17) and (19), we can obtain $\partial q_t^b/\partial m_t = 1/\widetilde{p}_t$, $\partial d_t^b/\partial m_t = 1$, and $\partial d_t^s/\partial k_t = \widetilde{p}_t\,(\partial q_t^s/\partial k_t)$, whereas the other partial derivatives, $\partial q_t^b/\partial k_t$, $\partial d_t^b/\partial k_t$, $\partial q_t^s/\partial m_t$ and $\partial d_t^s/\partial m_t$, in (13) and (14) are zero. Substituting these conditions, $q_t^b = q_t^s = q_t$ and (19) into (13) and (14), we can derive the following conditions.

$$V_m(m_t, k_t) = \frac{1-\sigma}{p_t x_t} + \frac{\sigma}{\widetilde{p}_t q_t}, \tag{20}$$

$$V_k(m_t, k_t) = \frac{1 + r_t - \delta}{x_t} - \frac{\sigma}{z_t}e_2\left(\frac{q_t}{z_t}, \frac{k_t}{z_t}\right). \tag{21}$$

[6] In addition to the competitive equilibrium with price taking, Aruoba *et al.* (2011) and Waller (2011) also consider bargaining between buyers and sellers to determine the terms of trade, which is also a common approach in the literature. In the present study, we only consider the competitive equilibrium with price taking because of economic growth. In the case of generalized Nash bargaining as in Aruoba *et al.* (2011) or proportional bargaining as in Waller (2011), the bargaining condition is incompatible with balanced growth because the buyers' utility, which determines their surplus, is increasing overtime due to economic growth whereas the sellers' disutility of effort is stationary on a balanced growth path. In Appendix A, we demonstrate this problem under proportional bargaining and show that only a special case in which buyers gain all surplus is consistent with balanced growth. The same result can also be shown for the case of generalized Nash bargaining.

[7] See Appendix B for a proof. Intuitively, due to the opportunity cost of holding money and the possibility of not being a buyer in the DM, households do not carry a sufficient amount of money to the DM. Therefore, if a household turns out to be a buyer in the DM, it would be optimal to spend all the money on q_t^b.

Intuitively, (20) states that the marginal benefit of holding money is the sum of the marginal utility from being able to consume special goods with probability σ (i.e., the household becomes a buyer in the DM) and the marginal utility from spending the money, which is also a valuable asset in the CM, on general goods with probability $1-\sigma$ (i.e., the household does not become a buyer in the DM). Equation (21) states that the marginal benefit of holding capital is the sum of the marginal utility from spending the capital return $1+r_t-\delta$ on general goods in the CM and the expected marginal utility from having to exert less effort (recall that $e_2 < 0$) in producing special goods in the DM with probability σ (i.e., the household becomes a seller in the DM).[8]

2.2 Firms in the CM

General goods are produced by using capital k_t and production labor h_t with the following Cobb-Douglas production function.

$$y_{x,t} = z_t^{1-\alpha} k_t^{\alpha} h_t^{\eta}, \tag{22}$$

where aggregate technology is $z_t = \overline{k}_t$ as before. The producers act competitively by taking output and input prices as given. The conditional demand functions for capital and production labor are respectively

$$r_t = \alpha z_t^{1-\alpha} k_t^{\alpha-1} h_t^{\eta}, \tag{23}$$

$$w_t = \eta z_t^{1-\alpha} k_t^{\alpha} h_t^{\eta-1}. \tag{24}$$

2.3 Monetary authority

Let $\mu_t = (m_{t+1} - m_t)/m_t$ denote the growth rate of money supply that is exogenously set by the monetary authority. Given the definition of real money balance m_t/p_t (denominated in the price of general goods), its evolution can then be expressed as

$$\frac{m_{t+1}}{p_{t+1}} = \left(\frac{1+\mu_t}{1+\pi_t}\right)\frac{m_t}{p_t}, \tag{25}$$

where π_t is the inflation rate that is endogenous and determines the cost of holding money. In each period, the monetary authority issues money to finance a lump-sum transfer that has a real value of $\tau_t = (m_{t+1} - m_t)/p_t = \mu_t m_t/p_t$.

[8] Following Aruoba *et al.* (2011) and Waller (2011), we assume that the stock of capital does not depreciate within a period even upon usage in the DM. Capital depreciation only occurs at the end of a period after usage in the CM.

2.4 Competitive equilibrium

The competitive equilibrium is a sequence of allocations $\{h_t, x_t, y_{x,t}, q_t, d_t, m_{t+1}, k_{t+1}\}_{t=0}^{\infty}$, a sequence of prices $\{w_t, r_t, p_t, \widetilde{p}_t, \pi_t\}_{t=0}^{\infty}$ and a sequence of policies $\{\mu_t, \tau_t\}_{t=0}^{\infty}$. Also, in each period, the following conditions hold.

- In the CM, households choose $\{h_t, x_t, m_{t+1}, k_{t+1}\}$ to maximize (1) subject to (2) taking $\{w_t, r_t, p_t, \tau_t\}$ as given;
- In the DM, buyers and sellers choose $\{q_t, d_t\}$ to maximize their value functions taking $\{\widetilde{p}_t\}$ as given;
- Competitive firms in the CM produce $\{y_{x,t}\}$ to maximize profit taking $\{w_t, r_t\}$ as given;
- The real value of aggregate consumption includes consumption in the CM and the DM such that $c_t \equiv (p_t x_t + \sigma \widetilde{p}_t q_t)/p_t$;
- The real value of aggregate output includes output in the CM and the DM such that $y_t \equiv (p_t y_{x,t} + \sigma \widetilde{p}_t q_t)/p_t$;
- The capital stock accumulates through investment from general goods such that $k_{t+1} = y_{x,t} - x_t + (1-\delta)k_t$;
- The monetary authority balances its budget such that $\tau_t = \mu_t m_t / p_t$.

2.5 Balanced growth path

In this subsection, we consider the dynamic properties of the model. Given that the monetary authority sets a stationary growth rate of money supply (i.e., $\mu_t = \mu$ for all t), Proposition 1 shows that the economy jumps to a unique and locally stable balanced growth path. The proof is relegated to Appendix C. Given this balanced growth behavior of the model, we analyze the effects of monetary policy on the balanced growth path in the next section.

Proposition 1 *Given a stationary sequence of monetary policy (i.e., $\mu_t = \mu$ for all t), the economy jumps to a unique and stable balanced growth path.*

3 Growth and welfare effects of monetary policy

In this section, we analyze the effects of monetary policy on the balanced growth path along which the supply of labor is stationary. Given the equilibrium condition $\overline{k}_t = k_t$, variables, such as output, consumption, capital and real money balance, exhibit a common growth rate g. Using (4), (10) and (21), we obtain

$$g \equiv \frac{x_{t+1}}{x_t} - 1 = \beta \left(1 + r - \delta + \sigma \frac{\alpha f_c f_d^{1/\eta}}{\eta} \right) - 1, \tag{26}$$

where $r = \alpha h^\eta$ from (23). The variables $f_c \equiv x/k$ and $f_d \equiv q/k$ denote the steady-state consumption-capital ratios in the CM and the DM respectively.

We first make use of (5), (10), (19), (20) and (25) to derive the steady-state consumption-capital ratio in the DM. We obtain

$$f_d = \left[\frac{\sigma\eta}{(1+\mu)/\beta - (1-\sigma)} \right]^\eta. \tag{27}$$

f_d must be positive because $\mu > \beta - 1 > \beta(1-\sigma) - 1$.[9] Equation (27) shows that the consumption-capital ratio in the DM is decreasing in the growth rate of money supply, and this result can be shown as follows.

$$\frac{\partial f_d}{\partial \mu} = -\frac{\eta}{\beta} \frac{(\sigma\eta)^\eta}{[(1+\mu)/\beta - (1-\sigma)]^{1+\eta}} < 0. \tag{28}$$

Intuitively, a higher money growth rate increases inflation, which in turn increases the cost of consumption in the DM, where money is needed for transactions.

As for f_c, we make use of (23), (24), (26) and the capital-accumulation equation $k_{t+1} = y_{x,t} - x_t + (1-\delta)k_t$ to derive

$$f_c = \frac{(1-\alpha\beta)h^\eta + (1-\beta)(1-\delta)}{1 + \sigma\alpha\beta f_d^{1/\eta}/\eta}, \tag{29}$$

where aggregate labor h is still an endogenous variable and can be determined with the following condition.

$$A h^{1-\eta} f_c = \eta, \tag{30}$$

which uses (3) and (24). We use (30) to derive

$$\frac{\partial f_c}{\partial \mu} = -\frac{\eta(1-\eta)}{A h^{2-\eta}} \frac{\partial h}{\partial \mu}. \tag{31}$$

As for the derivative of h, we substitute (27) and (30) into (29) and then take the differentials of h with respect to μ to obtain

$$\frac{dh}{d\mu} = -\frac{\alpha(f_d^{1/\eta}/\eta)^2}{A(1-\alpha\beta) + (1-\eta)\left(1 + \sigma\alpha\beta f_d^{1/\eta}/\eta\right)/h} < 0. \tag{32}$$

[9] It can be shown that as $\mu \to \beta - 1$, the nominal interest rate approaches the lower bound of zero. Here the nominal interest rate refers to the nominal rate of return on a conventional interest-bearing bond that pays interests in the CM (but not in the DM) of each period.

Substituting (32) into (31) shows that $\partial f_c/\partial\mu > 0$. In summary, a higher money growth rate induces households to increase leisure and shift consumption from the DM to the CM.

Substituting (29) into (26), we obtain

$$g = \alpha\beta h^{\eta} + \frac{(1-\alpha\beta)h^{\eta} + (1-\beta)(1-\delta)}{1 + 1/(\sigma\alpha\beta f_d^{1/\eta}/\eta)} + \beta(1-\delta) - 1. \tag{33}$$

From (33), it is easy to see that the growth rate g is decreasing in μ because $\partial h/\partial\mu < 0$ and $\partial f_d/\partial\mu < 0$. It is useful to note that there are two channels through which μ causes a negative effect on economic growth. The first channel is endogenous labor supply, which is standard in monetary growth models. Intuitively, a decrease in labor supply reduces the marginal product of capital thereby reducing capital accumulation.

The second channel is through the consumption-capital ratio f_d in the DM. The intuition for the presence of this second channel can be explained as follows. A higher inflation increases the cost of holding money, thereby reducing the real money balance held by households and the value of goods traded in the DM. As a result, capital demand is depressed reducing the growth rate. To separate the consumption and labor-supply effects, we briefly consider the limiting case $\eta \to 0$.

$$\lim_{\eta\to 0} g = \alpha\beta + \frac{1-\alpha\beta + (1-\beta)(1-\delta)}{1 + [(1+\mu)/\beta - (1-\sigma)]/(\sigma^2\alpha\beta)} + \beta(1-\delta) - 1, \tag{34}$$

where we have used $f_d^{1/\eta}/\eta = \sigma/[(1+\mu)/\beta - (1-\sigma)]$ from (27). Therefore, even when the search-based monetary growth model approaches the case without endogenous labor supply, inflation continues to have a detrimental effect on economic growth. This result stands in stark contrast to the CIA growth model as we will show in the next section.

Proposition 2 *A higher money growth rate μ reduces economic growth through two channels: (a) endogenous labor supply h, and (b) the consumption-capital ratio f_d in the DM.*

Next, we examine the welfare effects of monetary policy. In this two-sector search model, households engage in two types of economic activities in the DM and the CM every period. On the balanced growth path, the lifetime utility U of households that includes the utility from the CM and the expected utility from the DM can be expressed as

$$(1-\beta)U = \underbrace{\sigma \ln q_0 - \sigma\left(\frac{q_0}{k_0}\right)^{1/\eta}}_{DM} + \underbrace{\ln x_0 - Ah}_{CM} + \underbrace{\frac{\beta(1+\sigma)}{1-\beta}\ln(1+g)}_{growth}. \tag{35}$$

Substituting $q_0 = f_d k_0$ and $x_0 = f_c k_0$ into (35) and then normalizing initial k_0 to unity, (35) simplifies to

$$(1-\beta)U = \sigma \ln f_d - \sigma f_d^{1/\eta} + \ln f_c - Ah + \frac{\beta(1+\sigma)}{1-\beta}\ln(1+g). \tag{36}$$

Differentiating (36) with respect to μ yields

$$(1-\beta)\frac{\partial U}{\partial \mu} = \frac{\sigma}{f_d}\left(1-\frac{f_d^{1/\eta}}{\eta}\right)\underset{-}{\frac{\partial f_d}{\partial \mu}} + \frac{1}{f_c}\underset{+}{\frac{\partial f_c}{\partial \mu}} - A\underset{-}{\frac{\partial h}{\partial \mu}} + \frac{\beta(1+\sigma)}{(1-\beta)(1+g)}\underset{-}{\frac{\partial g}{\partial \mu}}, \tag{37}$$

where $f_d^{1/\eta}/\eta < 1$ from (27) because $\mu > \beta - 1$. A higher money growth rate (a) decreases the consumption-capital ratio f_d in the DM, (b) increases the consumption-capital ratio f_c in the CM, (c) decreases labor supply h in the CM, and (d) decreases economic growth g. Effects (a) and (d) hurt welfare, whereas effects (b) and (c) improve welfare. Although it appears that the overall effect of money growth on welfare is ambiguous, we show below that higher money growth is in fact detrimental to social welfare.

Comparing the equilibrium allocations and the first-best allocations, we find that (a) $f_d < f_d^*$, (b) $f_c > f_c^*$, (c) $h < h^*$, and (d) $g < g^*$, where the variables with superscript * denote first-best allocations.[10] In other words, there is too little consumption in the DM due to the cost of holding money. In the CM, there is too much consumption and too little labor supply due to capital externality. Finally, the equilibrium growth rate is also suboptimally low. Therefore, increasing the money growth rate that forces the equilibrium allocations to deviate further from the first-best allocations is detrimental to welfare. On the other hand, decreasing the money growth improves welfare, and the Friedman rule (given by $\mu \to \beta - 1$) is optimal in this model. However, although the Friedman rule is optimal, it does not achieve the first-best allocations due to the presence of capital externality.[11]

Proposition 3 *A higher money growth rate μ reduces social welfare, and the Friedman rule is optimal but does not achieve the first-best allocations due to capital externality.*

4 A cash-in-advance model

In this section, we develop a monetary growth model with a CIA constraint to compare the effects of monetary policy in this model with the effects in the search model. We make two changes to the search model. First, we eliminate the DM so that $c_t = x_t$ in the CM is the only consumption goods now. Second, we introduce the following cash-in-advance constraint on consumption.

$$\xi c_t \leq \frac{m_t}{p_t}, \tag{38}$$

where $\xi \in (0, 1]$. Equation (38) implies that households hold the real money balance to facilitate the purchase of consumption goods c_t. Following Dotsey and Ireland (1996), we assume that only a fraction of consumption expenditures is subject the CIA constraint, and

[10] In Appendix D, we derive the first-best allocations of the search model and prove these inequalities.

[11] It is useful to note that the Friedman rule is not always optimal under price taking in the search model. For example, Rocheteau and Wright (2005) show that the Friedman rule is not optimal when there exist search externalities.

this specification allows us to perform a more realistic quantitative analysis on the welfare cost of inflation. As $\xi \to 0$, monetary policy would have no effects on growth and welfare.

On the balanced growth path, the growth rate of consumption is given by

$$g = \beta (1 + r - \delta) - 1 = \beta (1 + \alpha h^{\eta} - \delta) - 1. \tag{39}$$

In contrast to the search model, (39) shows an important implication that monetary policy affects economic growth solely through endogenous labor supply h in the CIA model. In other words,

$$\frac{\partial g}{\partial \mu} = \frac{\beta \alpha \eta}{h^{1-\eta}} \frac{\partial h}{\partial \mu}. \tag{40}$$

In the case of exogenous labor supply (i.e., h is constant), $\partial g / \partial \mu = 0$. We relegate the derivation of h to Appendix E. In summary, the equilibrium h is implicitly determined by the following condition.

$$\frac{\eta}{(1-\alpha\beta)h + (1-\beta)(1-\delta)h^{1-\eta}} = \frac{A}{\beta} \left[\xi(1+\mu) + \beta(1-\xi)\right], \tag{41}$$

which shows that h is decreasing in μ. Taking the total differentials of h with respect to μ in (41) and then substituting $dh/d\mu$ into (40) yield

$$\frac{\partial g}{\partial \mu} = -\frac{\alpha A \xi [(1-\alpha\beta)h + (1-\beta)(1-\delta)h^{1-\eta}]^2}{h^{1-\eta}\left[(1-\alpha\beta) + (1-\beta)(1-\delta)(1-\eta)/h^{\eta}\right]} < 0. \tag{42}$$

Intuitively, a monetary expansion induces households to reduce their money holdings and consumption via the CIA constraint. As a result, households consume more leisure instead. The reduction in labor supply decreases the marginal product of capital, thereby reducing economic growth.

As for social welfare, we can derive households' lifetime utility U as follows.

$$(1-\beta)U = \ln c_0 - Ah + \frac{\beta}{1-\beta} \ln(1+g) = \ln f - Ah + \frac{\beta}{1-\beta} \ln(1+g). \tag{43}$$

where $f \equiv c_0/k_0$ and initial k_0 is normalized to unity. Furthermore, from the capital-accumulation equation, one can show that

$$f = h^{\eta} - g - \delta = (1-\alpha\beta)h^{\eta} + (1-\beta)(1-\delta), \tag{44}$$

where the second equality uses (39). Equation (44) implies that $\partial f/\partial \mu$ and $\partial h/\partial \mu$ have the same sign. Finally, differentiating (43) with respect to μ yields

$$(1-\beta)\frac{\partial U}{\partial \mu} = \frac{1}{f}\underbrace{\frac{\partial f}{\partial \mu}}_{-} - A\underbrace{\frac{\partial h}{\partial \mu}}_{-} + \frac{\beta}{(1-\beta)(1+g)}\underbrace{\frac{\partial g}{\partial \mu}}_{-}. \tag{45}$$

Although $\partial U/\partial \mu$ appears to be ambiguous in (45), the Friedman rule also holds in the CIA model.[12] Intuitively, labor supply h in the equilibrium is suboptimally low, so that any increase in μ that further reduces h is harmful to social welfare. On the other hand, decreasing the money growth rate improves welfare, and the Friedman rule is optimal. However, as in the search model, although the Friedman rule is optimal in the CIA model, it does not achieve the first-best allocations due to the presence of capital externality.

[12] See Appendix E for derivations.

5 Quantitative analysis

In this section, we calibrate the two models in order to perform a numerical investigation to evaluate the effects of monetary policy on economic growth and social welfare. We consider two policy objectives (a) price stability (or equivalently, zero inflation) and (b) the Friedman rule. Both of these policy objectives are commonly analyzed in the literature; see for example, Dotsey and Ireland (1996), Lucas (2000), Lagos and Wright (2005) and Aruoba *et al.* (2011).

To calibrate the structural parameters, we will match the models' implied moments to data in the US. One empirical moment that we consider is the ratio of money holding to consumption expenditures. The larger this ratio is, the larger the welfare effects of monetary policy would be. To ensure robustness, we consider two monetary aggregates, currency and M1, as alternative measures of money held by households for the purpose of facilitating transactions. On the one hand, currency holding by households is a subset of the money holding that is subject to the cost of inflation. On the other hand, M1 includes interest-bearing assets, such as demand deposits, which are partly immune to the depreciation effect of inflation. Therefore, we consider the welfare cost of inflation computed based on currency as a lower bound and the welfare cost computed based on M1 as an upper bound.

5.1 Calibration

We begin by characterizing a benchmark economy, in which each structural parameter is either set to a conventional value or matched to an empirical moment in the US. The discount factor β is set to 0.952 to match an annual discount rate of 5%. The capital-share parameter α is set to 0.3, which implies a labor share $\eta = 1 - \alpha$ of 0.7. We consider an initial money growth rate of 5.8%, so that the annual inflation rate is 3% (i.e., the average inflation rate in the US from 1990 to 2008) when the economy grows at an annual growth rate of 2.7% (i.e., the average output growth rate in the US from 1990 to 2008). We choose a value for the depreciation rate δ to match the investment-capital ratio of 0.07 (i.e., the average investment-capital ratio in the US from 1990 to 2008), and this value of δ is 0.043.

When we match the average ratio of *currency* to households' consumption expenditures in the US, we set the probability σ to 0.009 in the search model and the cash-in-advance parameter ξ to 0.075 in the CIA model. In this case, the money-consumption ratio is 0.075. When we match the ratio of *M1* to households' consumption expenditures, we set σ to 0.025 and ξ to 0.185. In this case, the money-consumption ratio is 0.185. Finally, when money is measured by currency, we set the leisure parameter A to 3.087, so that the long-run growth rate in the search model is 2.7%. As for the CIA model, we set A to 3.059, so that the long-run growth rate in the CIA model is also 2.7%. When money is measure by M1, we set A to 3.104 in the search model and 3.022 in the CIA model respectively to match the long-run growth rate of 2.7%. We summarize these parameter values in Table 1a. As for the equilibrium values of the key variables, we report them in Table 1b. The total consumption-capital ratio of 0.335 and the capital-output ratio of 2.472 from the model are in line with empirical moments ($c/k = 0.308$ and $k/y = 2.223$) in the US.

Table 1a: Benchmark parameter values

Parameter	α	η	β	μ	σ	ξ	δ	A
Currency specification								
The search model	0.300	0.700	0.952	0.058	0.009	-	0.043	3.087
The CIA model	0.300	0.700	0.952	0.058	-	0.075	0.043	3.059
M1 specification								
The search model	0.300	0.700	0.952	0.058	0.025	-	0.043	3.104
The CIA model	0.300	0.700	0.952	0.058	-	0.185	0.043	3.022

Table 1b: Benchmark equilibrium values

Variable	g	π	h	f_d	f_c	c/k	i/k	k/y
Currency specification								
The search model	0.027	0.030	0.274	0.127	0.334	0.335	0.070	2.472
The CIA model	0.027	0.030	0.274	-	-	0.335	0.070	2.472
M1 specification								
The search model	0.027	0.030	0.273	0.240	0.333	0.335	0.070	2.472
The CIA model	0.027	0.030	0.274	-	-	0.335	0.070	2.472

5.2 Numerical results

Given the above set of parameter values, we consider the following policy experiments. First, we lower μ from 0.058 to a value that achieves zero inflation, and this value is 0.027. In this case, the inflation rate decreases from 3% to 0% in both models under both money specifications. As for the Friedman rule, we lower μ from 0.058 to -0.048, so the nominal interest rate decreases and approaches zero. In Table 2, we report the results, which are expressed in percent changes, except for g and U. The changes in g are expressed in percentage point, and the changes in U are expressed in the usual equivalent variations in annual consumption.

Table 2: Growth and welfare effects of a lower μ

	$\Delta(f_d)\,\%$	$\Delta(f_c)\,\%$	$\Delta(c/k)\,\%$	$\Delta(h)\,\%$	$\Delta(g)\,\%$	$\Delta(U)\,\%$
Zero inflation						
Currency specification						
The search model	24.649	-0.002	0.023	0.008	0.003	0.229
The CIA model	-	-	0.160	0.265	0.021	0.355
M1 specification						
The search model	20.789	-0.014	0.131	0.046	0.018	0.659
The CIA model	-	-	0.387	0.641	0.052	0.867
The Friedman rule						
Currency specification						
The search model	512.706	-0.079	0.751	0.262	0.100	2.732
The CIA model	-	-	0.554	0.918	0.074	1.233
M1 specification						
The search model	224.754	-0.194	1.850	0.649	0.248	5.747
The CIA model	-	-	1.365	2.265	0.183	3.076

In Table 2, we see that reducing the money growth rate has the following effects. First, it raises the supply of labor in both models; however, this labor-supply effect is much larger in the CIA model than in the search model. This expansion in labor supply serves to increase economic growth in both models. Second, it raises the consumption-capital ratio f_d in the DM, and this increase in f_d also serves to increase economic growth in the search model. Comparing $\Delta(f_d)\%$ under zero inflation and the Friedman rule, we see that the increase in f_d is disproportionately larger under the Friedman rule than under zero inflation. From (27), we see that f_d is a decreasing and convex function in μ, and this property has the following implications on the growth and welfare effects of inflation.

We find that the equilibrium growth rate g increases in both models under both money specifications. However, it is interesting to note that although the search model exhibits a *larger* growth effect than the CIA model in the case of the *Friedman rule*, the search model exhibits a *smaller* growth effect than the CIA model in the case of *zero inflation*. For example, under the M1 specification, the positive growth effect from decreasing inflation from 3% to the Friedman rule in the search model is 0.25% compared to 0.18% in the CIA model. However, also under the M1 specification, the growth effect from decreasing inflation from 3% to zero inflation in the search model is 0.018% compared to 0.052% in the CIA model. These contrasting results are also robust to the currency specification.

Examining the effects on social welfare, we also see a similar pattern. In other words, the search model exhibits a larger (smaller) welfare effect than the CIA model in the case of the Friedman rule (zero inflation). For example, under the M1 specification, the welfare gain from decreasing inflation from 3% to the Friedman rule in the search model is 5.7% compared to 3.1% in the CIA model. However, also under the M1 specification, the welfare gain from decreasing inflation from 3% to zero inflation in the search model is 0.66% compared to 0.87% in the CIA model. These contrasting results are also robust to the currency specification. In Figures 1 and 2, we plot welfare changes against the money growth rate under the M1 and currency specifications respectively and find that the welfare effect of money growth in the CIA model is approximately linear whereas the welfare effect of money growth in the search model is convex. As a result, the welfare effect in the search model dominates (is dominated by) the CIA model under large (small) changes in μ.

[Insert Figures 1 and 2 here]

We have conducted a number of robustness checks for the above results. We find that the only cases in which the welfare effect in one model always dominates the effect in the other model are the followings. When we decrease the value of ξ in the AK model while holding all other parameters constant, the welfare effect of inflation in the CIA model decreases and eventually becomes always dominated by the search model. Similarly, when we decrease the value of σ in the search model while holding all other parameters constant, the welfare effect of inflation in the search model decreases and eventually becomes always dominated by the CIA model. Also, when we increase the value of A in the search model while keeping the value of A in the CIA model constant, the welfare effect of money growth in the search model decreases and eventually becomes always dominated by the effect in the CIA model.[13]

[13] It is useful to note that due to the different values of A and the resulting different growth rates of output, the rate of inflation that corresponds to each money growth rate would be different across the two models.

Similarly, when we increase the value of A in the CIA model, the welfare effect of money growth in the CIA model decreases and eventually becomes always dominated by the search model. However, all these changes would give rise to different calibrated values for the implied moments, such as the money-consumption ratio and the growth rate of output, from the two models making them improper comparisons.

Therefore, we conclude that when the two models are calibrated to match the same set of empirical moments, the welfare gain from decreasing inflation would be larger (smaller) in a canonical search-based monetary growth model with price taking than in a canonical CIA growth model if the change in the growth rate of money supply is large (small). It is possible that this result may change when additional features, such as bargaining and search externalities, are added to the search model, but then the different result would be driven by an interaction between search and these additional features rather than driven by search per se.[14]

6 Conclusion

In this note, we have compared the growth and welfare effects of inflation between a search-based monetary growth model and a canonical CIA growth model. We find that the qualitative effects of inflation on economic growth and social welfare are similar across the two models. In other words, a monetary expansion is detrimental to economic growth and social welfare in both models. Nevertheless, we find some important differences between the two approaches. First, the growth effect of inflation in the search model operates through *both* endogenous labor supply and the consumption-capital ratio in the DM as compared to only the labor-supply channel in the CIA model. Second, the two approaches provide different quantitative implications. Specifically, it is not always the case that the search model exhibits a larger welfare effect of inflation than the CIA model, as often claimed in the literature. Given these interesting differences and the relative tractability of recent vintages of search models, it would be a fruitful direction for future research to further revisit the interesting implications of monetary policy on economic growth and social welfare using variants of the search-based monetary growth model.

[14] See Craig and Rocheteau (2008) for an interesting analysis on the welfare cost of inflation in a search model under different pricing mechanims with search externalities.

References

[1] Aruoba, S., Waller, C., and Wright, R., 2011. Money and capital. *Journal of Monetary Economics*, 58, 98-116.

[2] Craig, B., and Rocheteau, G., 2008. Inflation and welfare: A search approach. *Journal of Money, Credit and Banking*, 40, 89-119.

[3] Dotsey, M., and Ireland, P., 1996. The welfare cost of inflation in general equilibrium. *Journal of Monetary Economics*, 37, 29-47.

[4] Gomme, P., 1993. Money and growth revisited: Measuring the costs of inflation in an endogenous growth model. *Journal of Monetary Economics*, 32, 51-77.

[5] Itaya, J., and Mino, K., 2003. Inflation, transaction costs and indeterminacy in monetary economies with endogenous growth. *Economica*, 70, 451-470.

[6] Itaya, J., and Mino, K., 2007. Technology, preference structure, and the growth effect of money supply. *Macroeconomic Dynamics*, 11, 589-612.

[7] Lagos, R., and Rocheteau, G., 2008. Money and capital as competing media of exchange. *Journal of Economic Theory*, 142, 247-258.

[8] Lagos, R., and Wright, R., 2005. A unified framework for monetary theory and policy evaluation. *Journal of Political Economy*, 113, 463-484.

[9] Lucas, R., 2000. Inflation and welfare. *Econometrica*, 68, 247-274.

[10] Menner, M., 2006. A search-theoretic monetary business cycle with capital formation. *Contributions to Macroeconomics*, 6, Article 11.

[11] Mino, K., 1997. Long-run effects of monetary expansion in a two-sector model of endogenous growth. *Journal of Macroeconomics*, 19, 635-655.

[12] Rocheteau, G., and Wright, R., 2005. Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium. *Econometrica*, 73, 175-202.

[13] Romer, P., 1986. Increasing returns and long-run growth. *Journal of Political Economy*, 94, 1002-1037.

[14] Shi, S., 1999. Search, inflation and capital accumulation. *Journal of Monetary Economics*, 44, 81-104.

[15] Waller, C., 2011. Random matching and money in the neoclassical growth model: Some analytical results. *Macroeconomic Dynamics*, 15, 293-312.

[16] Wang, P., and Yip, C., 1992. Alternative approaches to money and growth. *Journal of Money, Credit and Banking*, 24, 553-62.

[17] Williamson, S., and Wright, R., 2010. New monetarist economics: Models. In: B. Friedman and M. Woodford (ed.). *Handbook of Monetary Economics*, 3, 25-96. Elsevier.

Appendix A: Bargaining in the search model

In this appendix, we show that proportional bargaining is incompatible with balanced growth unless we focus on a special case in which buyers obtain all surplus.[15] If a buyer with state $\left(m_t^b, k_t^b\right)$ is matched with a seller with state (m_t^s, k_t^s), then a proportional bargaining problem, in which the buyer's gains from trade are a fixed share θ of the trade surplus can be expressed as

$$\underset{q_t}{Max}\, \underbrace{\ln q_t + W(m_t^b - d_t, k_t^b)}_{buyers'\ payoff} - \underbrace{W(m_t^b, k_t^b)}_{buyers'\ threat\ point} = \theta \underbrace{\left[\ln q_t - e\left(\frac{q_t}{z_t}, \frac{k_t^s}{z_t}\right)\right]}_{trade\ surplus}, \tag{A1}$$

subject to the budget constraint $d_t \leq m_t^b$. Using (7) and substituting $d_t = m_t^b$ into (A1), the bargaining condition for special goods is

$$(1-\theta)\ln q_t = \frac{Am_t^b}{p_t w_t} - \theta e\left(\frac{q_t}{z_t}, \frac{k_t^s}{z_t}\right). \tag{A2}$$

In (A2), q_t is increasing overtime due to economic growth whereas $m_t^b / \left(p_t w_t\right)$, q_t/z_t and k_t^s/z_t are stationary on a balanced growth path. As a result, the bargaining condition is incompatible with balanced growth unless the buyer obtains all surplus (i.e., $\theta = 1$).

Appendix B: Consumption in the DM

In this appendix, we show that it is optimal for the buyers to spend all their money to consume special goods in the DM. Differentiating (15) with respect to q_t^b yields

$$\frac{\partial V^b(m_t, k_t)}{\partial q_t^b} = \frac{1}{q_t^b} - \widetilde{p}_t W_m(m_t - \widetilde{p}_t q_t^b, k_t) = \frac{1}{q_t^b} - \frac{\widetilde{p}_t}{p_t x_t}, \tag{B1}$$

where the second equality follows from (7). The second-order condition shows that $V^b(.)$ is globally concave in q_t^b and reaches a maximum at $q_t^b = p_t x_t / \widetilde{p}_t$. In what follows, we show that $q_t^b = m_t/\widetilde{p}_t < p_t x_t/\widetilde{p}_t$ implying that the money constraint must be binding because $q_t^b < p_t x_t/\widetilde{p}_t \Leftrightarrow \partial V^b(.)/\partial q_t^b > 0$. Setting $q_t^b = q_t^s = q_t$ and $z_t = \overline{k}_t = k_t$ in (10) and (19), we have

$$\frac{\widetilde{p}_t q_t}{p_t x_t} = \frac{1}{\eta}\left(\frac{q_t}{z_t}\right)^{1/\eta}, \tag{B2}$$

where $q_t/z_t = f_d$. From (27), we know that $f_d^{1/\eta}/\eta < 1$ because $\mu > \beta - 1$. Therefore, $\widetilde{p}_t q_t = m_t < p_t x_t$.

[15] The same result can be shown for the case of generalized Nash bargaining.

Appendix C: Dynamic properties of the search model

In this appendix, we show that the economy in the search model always jumps to a unique and locally stable balanced growth path given a stationary sequence of monetary policy (i.e., $\mu_t = \mu$ for all t). Combining (17) and (19) and using (10), we can obtain

$$\frac{m_t}{p_t x_t} = \frac{1}{\eta}\left(\frac{q_t}{k_t}\right)^{1/\eta}. \tag{C1}$$

Here we define $f_{d,t} \equiv q_t/k_t$ as the ratio between consumption and capital in the DM and make use of (5), (17), (20), (25) and (C1) to derive

$$(1+\mu)\left(\frac{f_{d,t}}{f_{d,t+1}}\right)^{1/\eta} = \beta\left[1+\sigma+\frac{\sigma\eta}{f_{d,t+1}^{1/\eta}}\right]. \tag{C2}$$

Combining (3), (5), (21), (23) and (24) and using (10) to yield

$$\frac{k_{t+1}}{k_t}\left(\frac{h_{t+1}}{h_t}\right)^{\eta-1} = \beta\left[1+\alpha h_{t+1}^{\eta}-\delta+\left(\frac{\alpha\sigma}{A}\right)h_{t+1}^{\eta-1}f_{d,t+1}^{1/\eta}\right]. \tag{C3}$$

In addition, the capital-accumulation equation is $k_{t+1}/k_t = y_{x,t}/k_t - x_t/k_t + 1 - \delta = h_t^{\eta} - \eta h_t^{\eta-1}/A + 1 - \delta$. Applying this equation to (C3) yields

$$\left(h_t^{\eta}-\frac{\eta h_t^{\eta-1}}{A}+1-\delta\right)\left(\frac{h_{t+1}}{h_t}\right)^{\eta-1} = \beta\left[1+\alpha h_{t+1}^{\eta}-\delta+\left(\frac{\alpha\sigma}{A}\right)h_{t+1}^{\eta-1}f_{d,t+1}^{1/\eta}\right]. \tag{C4}$$

Log-linearizing (C2) and (C4) around the steady-state equilibrium yields the following deterministic system:

$$\begin{bmatrix}\log\left(h_{t+1}/h\right)\\ \log\left(f_{d,t+1}/f_d\right)\end{bmatrix} = \underbrace{\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}}_{Jacobian\ matrix}\begin{bmatrix}\log\left(h_t/h\right)\\ \log\left(f_{d,t}/f_d\right)\end{bmatrix}, \tag{C5}$$

where

$$a_{11} = \frac{\eta h+\beta\left(1-\eta\right)\left(1-\delta+\alpha h^{\eta}+\alpha\sigma h^{\eta-1}f_d^{1/\eta}/A\right)+\eta\left(1-\eta\right)h^{\eta-1}/A}{\left(\alpha\beta\eta\right)h+\beta\left(1-\eta\right)\left(1-\delta+\alpha h^{\eta}\right)} > 1,$$

$$a_{12} = -\frac{\left(\alpha\beta\sigma\right)h^{\eta-1}f_d^{1/\eta}/A\eta}{\left(\alpha\beta\eta\right)h+\beta\left(1-\eta\right)\left(1-\delta+\alpha h^{\eta}\right)}\left\{\frac{\sigma}{\sigma-\left(\alpha\beta\sigma\right)\left(1+\mu\right)\left[\left(1+\mu\right)/\beta-\left(1-\sigma\right)\right]}\right\} < 0,$$

$$a_{21} = 0,$$

$$a_{22} = \frac{\sigma}{\sigma-\left(\alpha\beta\sigma\right)\left(1+\mu\right)\left[\left(1+\mu\right)/\beta-\left(1-\sigma\right)\right]} > 1.$$

Let s_1 and s_2 be the two characteristic roots of the dynamic system. The trace and determinant of Jacobian are given by

$$\mathrm{Tr} = s_1 + s_2 = a_{11} + a_{22} > 0, \tag{C6}$$

$$\mathrm{Det} = s_1 s_2 = a_{11} a_{22} > 0. \tag{C7}$$

As indicated in (C6) and (C7), the dynamic system exists two unstable characteristic roots ($s_1 = a_{11} > 1$ and $s_2 = a_{22} > 1$). Given h and f_d are jump variables, two unstable characteristic roots imply that the economy jumps to a unique and locally stable balanced growth path.

Appendix D: First-best allocations of the search model

In this appendix, we derive the first-best allocations of the search model and compare them with the equilibrium allocations. The planner chooses all quantities directly, taking all relevant information into account. Here money is not essential. The planner's problem is

$$J(k_t) = \max_{q_t, x_t, h_t, k_{t+1}} \left\{ \sigma \ln q_t - \sigma \left(\frac{q_t}{k_t} \right)^{1/\eta} + \ln x_t - A h_t + \beta J(k_{t+1}) \right\}, \tag{D1}$$

subject to the capital-accumulation equation

$$k_{t+1} = k_t h_t^{\eta} - x_t + (1 - \delta) k_t. \tag{D2}$$

From standard dynamic optimization, the optimality conditions for q_t and h_t are respectively

$$\frac{q_t}{k_t} = \eta^{\eta}, \tag{D3}$$

$$A h_t^{1-\eta} \left(\frac{x_t}{k_t} \right) = \eta. \tag{D4}$$

The intertemporal optimality condition for capital accumulation is

$$\frac{1}{x_t} = \beta J_k(k_{t+1}), \tag{D5}$$

and the envelope condition is

$$J_k(k_t) = \frac{\sigma}{\eta k_t} \left(\frac{q_t}{k_t} \right)^{1/\eta} + \frac{1}{x_t} (1 - \delta + h_t^{\eta}). \tag{D6}$$

Combining (D5) and (D6), we can derive the first-best balanced growth rate g^* given by

$$g^* \equiv \frac{x_{t+1}}{x_t} - 1 = \beta \left[1 + (h^*)^{\eta} - \delta + \sigma \frac{f_c^* \, (f_d^*)^{1/\eta}}{\eta} \right] - 1, \tag{D7}$$

where $f_c^* \equiv (x/k)^*$ and $f_d^* \equiv (q/k)^*$ denote the first-best consumption-capital ratios for general goods and special goods respectively. From (D3), we can obtain

$$\frac{(f_d^*)^{1/\eta}}{\eta} = 1. \tag{D8}$$

As for f_c^*, combining (D7) and (D2) yields

$$f_c^* = \frac{(1-\beta)\,(h^*)^{\eta} + (1-\beta)(1-\delta)}{1 + \sigma\beta\,(f_d^*)^{1/\eta}/\eta}. \tag{D9}$$

Rewriting (D4) yields

$$A(h^*)^{1-\eta} f_c^* = \eta. \tag{D10}$$

Equations (D7), (D8), (D9) and (D10) determine the first-best allocations $\{g^*, f_d^*, f_c^*, h^*\}$.

Comparing (D8) and (27) shows that $f_d < f_d^*$ because $\mu > \beta - 1$. Substituting (D10) into (D9) yields

$$f_c^* = \frac{(1-\beta)[\eta/(Af_c^*)]^{\eta/(1-\eta)} + (1-\beta)(1-\delta)}{1 + \sigma\beta\,(f_d^*)^{1/\eta}/\eta}, \tag{D11}$$

where $(f_d^*)^{1/\eta}/\eta$ is determined by (D8). Substituting (30) into (29) yield

$$f_c = \frac{(1-\alpha\beta)[\eta/(Af_c)]^{\eta/(1-\eta)} + (1-\beta)(1-\delta)}{1 + \sigma\alpha\beta f_d^{1/\eta}/\eta}, \tag{D12}$$

where $f_d^{1/\eta}/\eta$ is determined by (27). Comparing (D11) and (D12) shows that $f_c > f_c^*$ because $f_d < f_d^*$ and $\alpha < 1$. Given $f_c > f_c^*$, (D10) and (30) imply that $h < h^*$. Rewriting (D2) yields

$$g = \frac{k_{t+1}}{k_t} - 1 = h^{\eta} - f_c - \delta. \tag{D13}$$

Given that $h < h^*$ and $f_c > f_c^*$, it must be the case that $g < g^*$.

Appendix E: The CIA model

In this appendix, we present the detailed derivations of the CIA model. The household chooses a sequence of allocations $\{c_t,\, h_t,\, m_{t+1},\, k_{t+1}\}_{t=0}^{\infty}$ to maximize (1) subject to (2) and (38), taking as given w_t, r_t and τ_t. The optimality condition for labor supply is

$$\frac{1}{c_t} = \frac{A}{w_t} + \lambda_t \xi, \tag{E1}$$

where λ_t denotes the Lagrange multiplier for the cash-in-advance constraint in (38). The standard intertemporal optimality conditions for the accumulation of capital and money are respectively

$$\frac{A}{w_t} = \beta W_k(m_{t+1}, k_{t+1}), \tag{E2}$$

$$\frac{A}{p_t w_t} = \beta W_m(m_{t+1}, k_{t+1}). \tag{E3}$$

In addition, the familiar envelope conditions are

$$W_k(m_t, k_t) = \frac{A(1 + r_t - \delta)}{w_t}, \tag{E4}$$

$$W_m(m_t, k_t) = \frac{A + \lambda_t w_t}{p_t w_t}. \tag{E5}$$

Based on (E1), (E3) and (E5), we can derive

$$\frac{1}{p_{t+1}c_{t+1}} = A\left[\frac{\xi}{\beta p_t w_t} + \frac{(1-\xi)}{p_{t+1}w_{t+1}}\right]. \tag{E6}$$

Combining (E2) and (E4), we obtain $(1 + r_{t+1} - \delta)/w_{t+1} = 1/(\beta w_t)$. Substituting this condition into (E6), we obtain

$$\frac{1}{p_{t+1}c_{t+1}} = A\left[\frac{\xi(1 + r_{t+1} - \delta)}{p_t w_{t+1}} + \frac{(1-\xi)}{p_{t+1}w_{t+1}}\right]. \tag{E7}$$

On the balanced growth path, variables, k_t, y_t, c_t, w_t and m_t/p_t, grow at the same rate. Substituting (23), (24) and (25) into (E7), the steady-state equilibrium h is determined by the following condition.

$$\frac{\eta}{h^{1-\eta}(c/k)} = \frac{A}{\beta}\left[\xi(1+\mu) + \beta(1-\xi)\right], \tag{E8}$$

where $c/k = h^\eta - g - \delta = (1 - \alpha\beta)h^\eta + (1 - \beta)(1 - \delta)$. Therefore, (E8) is identical to (41) in the main text. The Fisher equation is $(1 + R) = (1 + r - \delta)(1 + \pi)$, where R denotes the nominal interest rate. From (25) and (39), we have

$$R = \frac{1+\mu}{\beta} - 1. \tag{E9}$$

Substituting (E9) into (E8), we can derive

$$\frac{\eta}{fh^{1-\eta}} = A(1 + \xi R). \tag{E10}$$

From (40) and (44), equation (45) can be arranged as

$$(1-\beta)\frac{\partial U}{\partial \mu} = \left[\frac{\eta(1-\alpha\beta)}{fh^{1-\eta}} + \frac{\alpha\eta\beta^2}{(1-\beta)(1+g)h^{1-\eta}} - A\right]\frac{\partial h}{\partial \mu}. \tag{E11}$$

Substituting (39) and (E10) into (E11) and using condition $f = (1 - \alpha\beta)h^\eta + (1 - \beta)(1 - \delta)$, we can obtain

$$(1-\beta)\frac{\partial U}{\partial \mu} = \left\{(1-\alpha\beta)A\xi R + \frac{\alpha\beta A\left[\xi R f + (1-\alpha)\beta h^\eta\right]}{(1-\beta)(1+g)}\right\}\frac{\partial h}{\partial \mu} < 0. \tag{E12}$$

Based on (E9), it is easy to see that

$$\frac{\partial R}{\partial \mu} = \frac{1}{\beta} > 0. \tag{E13}$$

Equation (E12) shows that a rise in μ decreases social welfare whereas (E13) shows that the nominal interest rate rises in μ. Therefore, social welfare is maximized as the nominal interest rate approaches zero; in other words, the Friedman rule holds in the CIA model.

Appendix F: Figures

Figure 1: The welfare cost of inflation under the M1 specification

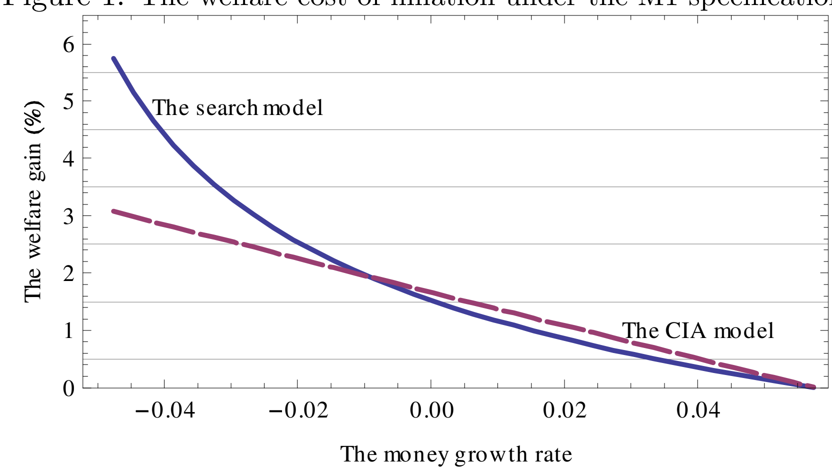

Figure 2: The welfare cost of inflation under the currency specification