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Abstract

In this paper the existence and stability of equilibriums in an evolutionary game theory model of the labour market is studied by using the Lyapunov method. The model displays multiple equilibriums and it is shown that the Nash Equilibriums of the static game are evolutionary stable equilibrium in the game theory evolutionary set up. In this vein a complete characterization of the dynamics of an evolutionary model of the labour market is provided.

Keywords: Evolutionary game theory approach, labour market, informal economy, Lyapunov function.

JEL Classification: E26, J62, C73.

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1. Introduction

In the present paper the dynamics of the labour market is studied by using an evolutionary game theory approach. The starting point is the model developed by Araujo and Souza (2010) who departing from a microeconomic point of view of agents' choice making and going through a macroeconomic assessment of formal and informal sectors behaviours delineate optimal policies that foresee the trade-off between tax collecting and incentive creation to workers and firms to operate in the formal sector.

In fact there are a number of papers acknowledging that there is a correspondence between the labour market and the stage of economic development [see Acemoglu (1998, 2002)]. Greenwood and Yorukoglu (1997), for instance, maintain that the adoption of technical change requires equally specific human capital in addition to physical capital, and an increase in labour skills facilitates the adoption of new technologies. Hendricks (2000) models growth through technology adoption focusing on the complementariness between technologies and skills. Workers' skills and technological profile of firms are therefore complementary: the level of the former limits the profile of technologies that firms can use, while this latter determines the rate of learning. Benhabib and Spiegel (1994), focusing on the role of human capital in economic development suggest that the role of the former is to facilitate the adoption of technology from abroad and at the same time, to create a domestic technology.

Hence, there exists a consensus that the presence of skilled workers implies a better environment for skill-complementary technologies, and it encourages further upgrading of productivity of skilled workers. On one hand, firms operating in a labour market thickly populated by high skilled workers may choose a better technological profile to match those skills. On the other hand, workers in an environment in which firms demand high skilled workers, find incentives to improve their skills. This view is supported by a number of authors. Snower (1994), for instance, shows how a country can fall into a "low-skill, bad-job trap," characterized by a vicious cycle of low productivity, deficient training, and low-skilled jobs, preventing the economy from competing effectively in the markets for skill-intensive products. Redding (1996) also points to the existence of a low growth trap in which a large proportion of the workforce is unskilled, firms have little incentive to provide good jobs (requiring high skills and providing high wages), and if few good jobs are available, workers have little incentive to acquire skills.

Following this rationale, Lavezzi (2006) have emphasized the role of skill resources as a crucial constraint on the selection of the technological profile to be implemented in developing economies. This author focuses on the dynamics of human capital accumulation – framed by a Markov chain – where human capital accumulation and technology adoption are interrelated processes. For workers the crucial issue is the type of firms they interact with, and likewise for firms, it is the type of workers they hire. In high-skill equilibrium, for example, workers expect firms to invest in technology and then invest in human capital. Thus, firms find it optimal to invest, and therefore expectations are fulfilled in equilibrium.

The connection between skills and formality, which is one of assumptions of the present paper, was addressed by Rausch (1991) in a model in which agents with highest ability become formal managers. Managers with more ability would naturally run larger firms and employ more capital; for this reason they choose to join the formal sector, where they face a lower cost of capital and do not face limits on capital deployment. Hence in this model limited access to capital goods is not the only constraint that firms and workers face when they decide for the informal sector.

In this paper we intend to provide a characterization of the dynamics of the labour market by studying the stability of an evolutionary game theory model of the labour market presented by Araujo and Souza (2010) by using the Lyapunov method. Following this approach our study consider that workers and firms' decision to engage in the formal or informal sector[1] as the outcome of rational decisions based not only on the expected pay-offs in each of the sectors but also on the interaction with other agents. In this vein our framework is similar to the search and matching models but with the advantage of endogenizing the probabilities of matching between firms and workers.

In this vein the model presented here accommodates a varied growth experience of both developed and less developed economies, in which both technological adoption and labour skills play a crucial role in the determination of the stage of the labour market in an evolutionary dynamic framework. We conclude that when profits in the formal and informal sectors are positive the final outcome of the interplay between skills and technologies is dependent upon the economy's initial conditions, akin to path dependence. This paper is structured as follows: in the next section we present the model with its main properties. In section 2 we present the model and in section 3 we study the local stability. Lyapunov stability is studied in section 4 and section 5 concludes.

2. The Evolutionary Model

The model departs from Araujo and Souza (2010) and corresponds to an asymmetric evolutionary game where there are two populations of interacting agents [See Gintis (2000)]: workers and firms. It is assumed that each identical worker has two

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[1] It is important to bear in mind that in this paper we do not view informality as the result of exclusion but rather as the outcome of rational decisions by firms and workers [see Hirschman (1970)].

possible strategies that is, supply labour in either formal or informal market at each period of time. Let N be the number of workers, N_f the number of workers that choose to supply labour in the formal sector – the formal strategy – and N_i be the number of workers that choose the informal sector – the informal strategy. Let n_f and n_i be the proportions of workers that chooses the formal and informal strategies respectively, with $n_i + n_f = 1$. By choosing a strategy does not mean that the worker will be employed since to be hired it depends on matching a firm that has chosen the same strategy. Otherwise the worker will be unemployed. If she chooses the formal strategy then there exists a probability σ, $0 \leq \sigma \leq 1$, of finding a job in a formal firm. In this vein her instantaneous expected utility, U_f^e, is assumed to be given by:

$$U_f^e = \sigma u[(1-\tau)w_f] + (1-\sigma)u(0) \tag{1}$$

Where $u(.)$ is a concave utility function, and w_f is the real wage discounted by the income tax τ, $0 < \tau < 1$. Expression (1) shows that if the worker chooses the formal strategy there is no probability of punishment but she faces uncertainty related to finding or not a firm that also chooses the formal strategy to hire her, what happens with probability σ. By assuming that $u(0) = 0$ expression (1) reduces to:

$$U_f^e = \sigma u[(1-\tau)w_f] \tag{1'}$$

However, if the worker decides to act in the informal sector his expected utility, U_i^e, is given by[2]:

[2] An important difference between this approach and the one developed by Fortin et al (1997) is that in our model we model explicitly the possibility of being caught due to the operation in the informal sector while they consider that the firm in the informal sector faces a cost in order to avoid to be caught. The insight is that the higher the production of the firm the higher the cost in order to conceal its production.

$$U_i^e = \phi u[(1-\rho)w_i + \rho(w_i - m)] + (1-\phi)u(0) \qquad (2)$$

Where ϕ, $0 \leq \phi \leq 1$, is the probability of finding a job in the informal sector and w_i is the wage paid in the informal sector. The probability of being caught due to the operation in the informal sector is given by ρ, $0 \leq \rho \leq 1$. In this case the worker pays a fine, denoted by *m*, due to the choice of acting in the informal sector. These variables are assumed to be exogenous. Expression (2) shows that the worker who chooses the informal strategy faces two kinds of uncertainty: the first is related to the possibility of not finding a firm that chooses the informal strategy and the second is related to the possibility of being caught if hired by an informal firm. This expression may be rewritten as:

$$U_i^e = \phi u[w_i - \rho m] \qquad (2)'$$

In order to model the demand side of the labour market, let us assume following the literature of search and matching – see e.g. Pissarides (2000) – that the number of firms, denoted by L, is equal to the number of workers[3], that is $L = N$. Let L_f be the number of firms that chooses the formal strategy and L_i the number of firms that chooses the informal strategy. Analogous to the case of labour supply, each firm can demand labour in only one of the markets in each period of time. Let η_i be the proportion of firms that chooses the informal strategy and η_f, the proportion of firms that chooses the formal strategy, with $\eta_i + \eta_f = 1$.

[3] This is a usual assumption in the search and matching models and here it is adopted for tractability only. For a treatment of the labor market dynamics by using an evolutionary model in which the processes of vacancy setting is modeled through a process of searching and matching see Fagiolo et al (2004).

Following Pissarides (2000) assume that each firm hires only one worker who produces a fixed amount of product at a time. The price of the product is normalized to 1 and the amount of production in the formal sector is exogenously given by y_f. Being θ, $0 \le \theta \le 1$ the probability of a firm that chooses the formal sector to find a worker that decides to supply labour in this sector, the profit of the firm if it decides to operate in the formal sector is given by:

$$\Pi_f^e = \theta[(1-\gamma)y_f - w_f] \tag{3}$$

Where γ, $0 < \gamma < 1$, stands for the costs for being in the formal sector. Expression (3) shows that each firm has to pay γy_f as taxes. Both y_f and γ are assumed to be exogenous. If there is no matching between the formal worker and the formal firm then the profit of the firm is equal to zero, what occurs with a probability $1 - \theta$. In the informal sector the firm is also assumed to hire only one worker, but now it produces a smaller amount of product than in the formal operation due to limited access to public goods, capital goods etc. Let y_i be the amount of product in informal operation, with $y_i < y_f$. In this vein the profit of the firm in the informal sector is given by:

$$\Pi_i^e = \lambda\{(1-\psi)[y_i - w_i] + \psi[y_i - w_i - e]\} \tag{4}$$

Where λ, $0 \le \lambda \le 1$ is the probability of matching a worker in the informal sector, and ψ, $0 \le \psi \le 1$, is the probability[4] that the firm faces of paying a fine, expressed by e, due to the operation in the informal labour market. After some algebraic manipulations expression (4) yields:

$$\Pi_i^e = \lambda(y_i - w_i - \psi e) \tag{5}$$

[4] We assume that this probability is the same of finding a worker in the informal sector. This assumption is made for the sake of convenience only but it expresses the fact that once a worker in the informal sector is detected then the corresponding firm is also found.

Since it is assumed that each firm hires only one worker the ratio of labour demanded in the formal sector, η_f , and the ratio of labour demanded in the informal sector, η_i , is proportional to the amount of firms in each sector. It is important to recall that if a worker who chooses the formal strategy does not match a firm with this strategy – an informal firm – then the pay-off of both worker and firm will be equal to zero. In this case the firm is assumed to produce zero output and the worker does not earn wage. This situation can be identified as unemployment from the viewpoint of the worker. We could assume alternatively that if a worker that chooses the formal sector matches a firm in the informal sector then both will obtain positive pay-offs but smaller than the pay-offs if both worker and firm choose the formal sector or informal sector simultaneously. It is easy to see that this game has two pure Nash equilibrium namely {*f*,*f*} and {*i*,*i*} together with a mixed strategy equilibrium, in which both workers and firms randomly choose between being formal or informal.

In order to evaluate the dynamics of entrance and withdrawal of workers in the formal market we use a version of the dynamic replicator as proposed by Hofbauer and Sigmund (2003) adapted to the study of the labour market according to Araujo and Souza (2010). The dynamic movement of workers between the two strategies, namely formal and informal is given by the following expressions:

$$\dot{N}_f = N_f\left[U^e_f - \overline{U}_{f,i}\right] \tag{6}$$

$$\dot{N}_i = N_i\left[U^e_i - \overline{U}_{f,i}\right] \tag{7}$$

Where $\overline{U}_{f,i}$ is the average pay-off given by: $\overline{U}_{f,i} = n_f U^e_f + n_i U^e_i$. By inserting expressions (1) and (2) into (6) and (7) it is possible to show after some algebraic manipulation[5] that it yields the following equations for the dynamic behaviour of the ratios of workers in the formal and informal sectors.

$$\dot{n}_f = n_f n_i \{\sigma u[(1-\tau)w_f] - \phi u[w_i - \rho m]\} \tag{8}$$

$$\dot{n}_i = n_f n_i \{\phi u[w_i - \rho m] - \sigma u[(1-\tau)w_f]\} \tag{9}$$

These expressions show that the central planner can affect the supply of the labour in each sector by choosing the taxation, τ, the probability of caught the worker in the informal sector, ρ , and the fine to be paid in the informal sector, *m*. Until this point of the analysis the values of σ and ϕ are exogenously considered but a further inquire on this probabilities by using a Bayesian inference may show that $\sigma = \eta_f$ and $\phi = 1 - \eta_f$. Remember that firms have only two strategies, namely formal and informal. Even in the case where there is no matching between a firm choosing the formal strategy and a worker choosing the informal strategy their strategies are 'formal' and 'informal' despite the fact that the worker will be unemployed and the firm will produce nothing in that period of time. Hence all firms can be grouped into one of these categories: 'formal' or 'informal'. The probability that a worker faces of finding a 'formal' firm is given by $\sigma = \frac{L_f}{L} = \eta_f$ and the probability of finding a 'informal' firm is given by $\phi = \frac{L_i}{L} = \eta_i$. Hence expression (8) may be rewritten as:

$$\dot{n}_f = n_f n_i \{\eta_f u[(1-\tau)w_f] - (1-\eta_f)u[w_i - \rho m]\} \tag{8'}$$

Following the same approach for the labour demand, the dynamic replicators for the firms are given by:

$$\dot{L}_f = L_f \left(\Pi_f^e - \overline{\Pi}_{f,i}\right) \tag{10}$$

[5] See Araujo and Souza (2010) for the derivation of expressions (8) and (9) from (3) and (4).

$$\dot{L}_i = L_i\left(\Pi_i^e - \overline{\Pi}_{f,i}\right) \tag{11}$$

Where Π_f^e stands for the expected profit of the formal extrategy and Π_i^e stands for the expected profit of the informal extrategy and $\overline{\prod}_{f,i}$ represents the average expected profit in the economy which is the average payoff for firms, given by: $\overline{\prod}_{f,i} = \eta_f \Pi_f^e + \eta_i \Pi_i^e$. By replacing expressions (4) and (5) into expressions above and considering that $\eta_i + \eta_f = 1$ we obtain the following dynamic replicator in the simplex form:

$$\dot{\eta}_f = \eta_f \eta_i \{\theta[(1-\gamma)y_f - w_f] - \lambda[y_i - w_i - \rho m]\} \tag{12}$$

$$\dot{\eta}_i = \eta_i \eta_f \{\lambda[y_i - w_i - \rho m] - \theta[(1-\gamma)y_f - w_f]\} \tag{13}$$

By following the same rationale adopted for the labour supply it is possible to conclude that $\theta = n_f$ and $\lambda = 1 - n_f$. Expression (12) may then be rewritten as:

$$\dot{\eta}_f = \eta_f \eta_i \{n_f[(1-\gamma)y_f - w_f] - (1-n_f)[y_i - w_i - \rho m]\} \tag{12'}$$

In the next section we analyze the steady state equilibrium from the system formed by expressions (8), (9), (12) and (13). Firstly an assessment of the local stability is made and then propositions concerning the Lyapunov stability are presented.

3. Local Stability

According to Vega-Redondo (1996, p. 50), a singular point x^* of a dynamic system is an asymptotically stable equilibrium of it if:

I) There exists some neighbourhood V of x^* such that all trajectories starting in V satisfy $x(t) \to x^*$ as $t \to \infty$.

II) It is Lyapunov stable, i.e. given any neighbourhood U_1 of x^* there exists another neighbourhood U_2 of x^* such that all trajectories with $x(0) \in U_2$ satisfy $x(t) \in U_1$, $\forall t > 0$.

In this section we analyse the first requirement which is in fact a test on the local stability of the singular points of the dynamical system while in the next section we consider the second test which is test on the Lyapunov stability. In order to classify the equilibrium points of system formed by expressions (8)-(9) and (12)-(13) it is useful to remember that $\eta_f + \eta_i = 1$ and $n_f + n_i = 1$ and, hence it is not necessary to consider explicitly expressions (9) and (13). Let us rewrite the system as:

$$\dot{n}_f = n_f n_i f(\eta_f) \tag{8)''}$$

$$\dot{\eta}_f = \eta_f \eta_i g(n_f) \tag{12)''}$$

Where:

$$f(\eta_f) = \eta_f u[(1-\tau)w_f] - (1-\eta_f)u[w_i - \rho m] \tag{14}$$

$$g(n_f) = n_f[(1-\gamma)y_f - w_f] - (1-n_f)[y_i - w_i - \rho m] \tag{15}$$

The equilibrium or steady state solution of the model is obtained by considering that: $\dot{n}_f = \dot{\eta}_f = 0$. From expression (8)'' we have three possibilities, namely: $n_f = 0$, $n_f = 1$ or $f(\eta_f) = 0$. From expression (12)' we also have three possibilities, namely: $\eta_f = 0$, $\eta_f = 1$ or $g(n_f) = 0$. Hence, from the combination of these possibilities we have the following possible solutions: (i) $n_f = 0$, $\eta_f = 0$; (ii) $n_f = 0$, $\eta_f = 1$; (iii) $n_f = 1$, $\eta_f = 0$; (iv) $n_f = 1$, $\eta_f = 1$; (v) $f(\eta_f) = 0$, $g(n_f) = 0$; (vi) $n_f = 0$, $g(n_f) = 0$; (vii) $n_f = 1$, $g(n_f) = 0$; (viii) $\eta_f = 0$, $f(\eta_f) = 0$ and (ix) $\eta_f = 1$, $f(\eta_f) = 0$. Let us exclude those equilibrium in which the profits or utility function has to be equal to 0 to hold. Consider for instance case (vi): if $n_f = 0$ and $g(0) = 0$ then the profit of the firm in the informal sector is given by: $y_i - w_i - \rho m = 0$. Since the variables in this expression are exogenously given there is no reason a priori to assume that this relation holds. The same reasoning applies to cases (vii), (viii) and (ix).

In order to study the behaviour of the system in a neighbourhood of the points (i), (ii), (iii), (iv) and (v) let us apply the Hartman-Grobman theorem which states that if x^* is a hyperbolic[6] fixed point of a non linear dynamical system then it is topologically equivalent to the fixed point of the linearization of the system [See Gintis (2000)]. The Jacobian of the system formed by expressions (8)'' and (12)'' is given by:

$$J = \begin{bmatrix} (1-2n_f)f(\eta_f) & n_f(1-n_f)f_{\eta_f}(\eta_f) \\ \eta_f(1-\eta_f)g_{n_f}(n_f) & (1-2\eta_f)g(n_f) \end{bmatrix} \tag{16}$$

The characteristic equation of the Jacobian matrix at each equilibrium points is given by: $\lambda^2 - S_1\lambda + S_2 = 0$. The following propositions sum up the local behaviour of the system.

Proposition 1:

If both profits in the formal and informal sector are positive, namely $y_i - w_i > \psi e$ and $(1-\gamma)y_f > w_f$ then: (i) (0,0) is a locally stable point, (ii) (0,1) is a locally unstable point, (iii) is a locally unstable point, (iv) (1,1) is a locally stable point and (v) $\left(\dfrac{y_i - w_i - \psi e}{(1-\gamma)y_f - w_f + y_i - w_i - \psi e}, \dfrac{u[w_i - \rho m]}{u[(1-\tau)w_f] + u[w_i - \rho m]}\right)$ is a saddle point.

Proof. **(i)** Evaluating the Jacobian matrix in equilibrium $\eta_f = n_f = 0$, it yields:

$$J = \begin{bmatrix} -u[w_i - \rho m] & 0 \\ 0 & -(y_i - w_i - \rho m) \end{bmatrix} \tag{16'}$$

In this case $S_1 = trJ = -u[w_i - \rho m] - (y_i - w_i - \rho m)$. If $y_i - w_i - \rho m > 0$ then $S_1 = trJ < 0$. In this case $S_2 = \det J = u[w_i - \rho m)](y_i - w_i - \rho m) > 0$ and we have a stable node.

(ii) Evaluating the Jacobian matrix at $n_f = 0$ and $\eta_f = 1$ it yields:

[6] A system is hyperbolic if every eigenvalue of the Jacobian matrix has nonzero real part.

$$J = \begin{bmatrix} u[(1-\tau)w_f] & 0 \\ 0 & y_i - w_i - \rho m \end{bmatrix} \qquad (16)''$$

Hence $trJ = (1-\gamma)w_f + [y_i - w_i - \psi e] > 0$ and $\det J = u[(1-\gamma)w_f][y_i - w_i - \psi e] > 0$ and we have undoubtedly an unstable node. □

(iii) In the third equilibrium (iii) $n_f = 1$ and $\eta_f = 0$ the Jacobian matrix is given by:

$$J = \begin{bmatrix} u[w_i - \rho m] & 0 \\ 0 & (1-\gamma)y_f - w_f \end{bmatrix} \qquad (16)'''$$

In this case, $S_1 = trJ = u[w_i - \rho m] + (1-\gamma)y_f - w_f > 0$ and $S_2 = \det J = (w_i - \rho m)[y_f - (1+\gamma)w_f] > 0$. In this case we also have an unstable node. □

(iv) The fourth case is given by: (iv) $n_f = 1$, $\eta_f = 1$; the Jacobian in this case is given by:

$$J = \begin{bmatrix} -u[(1-\tau)w_f] & 0 \\ 0 & -[(1-\gamma)y_f - w_f] \end{bmatrix} \qquad (16)''''$$

In this case $S_1 = trJ < 0$ and $S_2 = \det J = u[(1-\tau)w_f][(1-\gamma)y_f - w_f] > 0$, which gives rise to a stable node.

(v) The fifth case is given by: $f(\eta_f) = 0$ and $g(n_f) = 0$. In this case it is possible to obtain the values of η_f and n_f, respectively as: $n_f = \dfrac{y_i - w_i - \psi e}{(1-\gamma)y_f - w_f + y_i - w_i - \psi e}$ and $\eta_f = \dfrac{u[w_i - \rho m]}{u[(1-\tau)w_f] + u[w_i - \rho m]}$. The Jacobian in this case is given by:

$$J = \begin{bmatrix} 0 & (1-n_f)u[w_i - \rho m] \\ n_f(1-n_f)(y_i - w_i - \rho m) & 0 \end{bmatrix} \qquad (16)'''''$$

In this case we have a saddle point since $S_1 = trJ = 0$ and $S_2 = -n_f(1-n_f)^2(y_i - w_i - \psi e)u[w_i - \rho m] < 0$. □

Although the study of the local behaviour of the system around the equilibrium points has provided us important information in order to proceed to a better characterization of the dynamics of the labour market let us use the Lyapunov theorem.

4. Lyapunov Stability

As we have shown in the previous section the behaviour of the dynamical system depends crucially on the assumptions made on the magnitudes and signs of the profits both in the formal and informal sector. If the profits are positive then we have five possible equilibrium, two stable, two instable and a saddle point. One of aims this section is to prove that points (0,0) and (1,1) are asymptotically stable equilibrium of dynamical system (8)' and (12)'. In order to accomplish this task it is necessary to prove the Lyapunov stability of these points. The method used to prove this is the Lyapunov who consists in finding a function that satisfies the conditions of the Lyapunov theorem. This theorem requires the existence of an open neighbourhood of $(\bar{n}_f, \bar{\eta}_f)$, namely Ω, with the following properties [See Gintis (2000)]:

a) $V(\bar{n}_f, \bar{\eta}_f) = 0$;

b) $V(n_f, \eta_f) > 0$, for all $(n_f, \eta_f) \in \Omega$;

c) $\dfrac{dV}{dt}(n_f, \eta_f) \leq 0$, for all $(n_f, \eta_f) \in \Omega$.

If these conditions are met the equilibrium is Lyapunov stable in Ω. According to Takeuchi (1996), the Lyapunov function for the system formed by (8)'' to (14) around point (1,1) is properly given by:

$$V(n_f, \eta_f) = \delta_1 (n_f - \bar{n}_f - \bar{n}_f \ln \frac{n_f}{\bar{n}_f}) + \delta_2 (\eta_f - \bar{\eta}_f - \bar{\eta}_f \ln \frac{\eta_f}{\bar{\eta}_f}) \tag{17}$$

Then we can prove the following:

Proposition 2:

If profits in formal and informal sectors are positive, namely $y_i - w_i > \psi e$ and $(1-\gamma)y_f > w_f$, then the dynamic system (8)'' and (12)'' is Lyapunov stable at (1,1) in the set A defined by:

$$A = \left\{ (n_f, \eta_f) \in [0,1]\times[0,1]; n_f > \frac{y_i - w_i - \rho m}{y_f - (1+\gamma)w_f + y_i - w_i - \rho m}, \eta_f > \frac{w_i - \rho m}{(1-\gamma)w_f + w_i - \rho m} \right\}$$

Proof.

The requirement a) of the Lyapunov theorem is easily satisfied at (1,1), namely $V(1,1) = 0$. In order to prove condition b) it is sufficient to show that: $n_f - \ln n_f > 1$ and $\eta_f - \ln \eta_f > 1$ for all (n_f, η_f) in a neighbourhood Ω of (1,1). But this result holds for every $0 < n_f < 1$ and $0 < \eta_f < 1$. In order to prove c) let us take the time derivative of expression (17) which yields:

$$V' = \delta_1 \frac{(n_f - \bar{n}_f)}{n_f} \dot{n}_f + \delta_2 \frac{(\eta_f - \bar{\eta}_f)}{\eta_f} \dot{\eta}_f \tag{18}$$

Substituting expressions (8)'' and (12)'' into the expression above we obtain:

$$V' = \delta_1 \frac{(n_f - \bar{n}_f)}{n_f} n_f (1-n_f) f(\eta_f) + \delta_2 \frac{(\eta_f - \bar{\eta}_f)}{\eta_f} \eta_f (1-\eta_f) g(n_f) \tag{19}$$

After some algebraic manipulation and considering that $(\bar{n}_f, \bar{\eta}_f) = (1,1)$ it is possible to show that the expression above may be written as:

$$V' = -\delta_1 (1-n_f)^2 f(\eta_f) - \delta_2 (1-\eta_f)^2 g(n_f) \tag{20}$$

A possible way of proving requirement c) is to show that (1,1) is a local maximum of the function V'. Note that V'(1,1)=0 and if we prove that V' is definite negative then V'<0 for all points in a neighbourhood Ω of (1,1). In order to show this let us rewrite expression (20) as:

$$V' = -\delta_1(1-n_f)^2\{(\eta_f - 1 + 1)u[(1-\tau)w_f] + (1-\eta_f)u[w_i - \rho m]\} - \\ -\delta_2(1-\eta_f)^2\{(n_f - 1 + 1)[y_f - (1+\gamma)w_f] + (1-n_f)[y_i - w_i - \psi e]\} \qquad (20)'$$

Where we add and subtract 1 to n_f and η_f in order to evaluate the expression around the point (1,1). After some algebraic manipulation this expression reduces to:

$$L(n_f,\eta_f) = \delta_1(1-n_f)^2(1-\eta_f)\{u[(1-\tau)w_f] + u[w_i - \rho m]\} - \delta_1(1-n_f)^2 u[(1-\tau)w_f] + \\ + \delta_2(1-\eta_f)^2(1-n_f)\{[y_f - (1+\gamma)w_f] + [y_i - w_i - \psi e]\} - \delta_2(1-\eta_f)^2[y_i - w_i - \psi e] \qquad (20)''$$

Where $L(n_f,\eta_f) = V'$. By taking the partial derivatives of the above expression in relation to n_f and η_f we obtain:

$$L_{n_f}(n_f,\eta_f) = -2\delta_1(1-n_f)(1-\eta_f)\{u[(1-\tau)w_f] + u[w_i - \rho m]\} + 2\delta_1(1-n_f)u[(1-\tau)w_f] - \\ -\delta_2(1-\eta_f)^2\{[y_f - (1+\gamma)w_f] + [y_i - w_i - \psi e]\}$$

(28)

$$L_{\eta_f}(n_f,\eta_f) = -\delta_1(1-n_f)^2\{u[(1-\tau)w_f] + u[w_i - \rho m]\} - \\ -2\delta_2(1-\eta_f)(1-n_f)\{[y_f - (1+\gamma)w_f] + [y_i - w_i - \psi e]\} + 2\delta_2(1-\eta_f)[y_i - w_i - \psi e] \qquad (21)$$

It is important to note that (1,1) is a critical point of the function $L(n_f,\eta_f)$ since $L_{n_f}(1,1) = L_{\eta_f}(1,1) = 0$. Besides, taking the second partial derivatives of $L(n_f,\eta_f)$ it is possible to conclude that:

$$L_{n_f n_f}(n_f,\eta_f) = 2\delta_1(1-\eta_f)\{u[(1-\tau)w_f] + u[w_i - \rho m]\} - 2\delta_1 u[(1-\tau)w_f] \qquad (22)$$

$$L_{\eta_f\eta_f}(n_f,\eta_f) = 2\delta_2(1-n_f)\{[y_f - (1+\gamma)w_f] + [y_i - w_i - \psi e]\} - 2\delta_2[y_i - w_i - \psi e] \qquad (23)$$

$$L_{n_f\eta_f}(n_f,\eta_f) = L_{\eta_f n_f}(n_f,\eta_f) = 2\delta_1(1-n_f)\{u[(1-\tau)w_f] + u[w_i - \rho m]\} - \\ -2\delta_2(1-\eta_f)^2\{[y_f - (1+\gamma)w_f] + [y_i - w_i - \psi e]\} \qquad (24)$$

Evaluating the matrix Hessian at the point (1,1) it yields:

$$\mathrm{H}(1,1) = \begin{bmatrix} -2\delta_1 u[(1-\tau)w_f] & 0 \\ 0 & -2\delta_2[y_i - w_i - \psi e] \end{bmatrix} \qquad (25)$$

Thus $\det \mathrm{H}_1(1,1) = -2\delta_1 u[(1-\tau)w_f] < 0$ and $\det \mathrm{H}(1,1) = 4\delta_1 u[(1-\tau)w_f]\delta_2[y_i - w_i - \psi e] > 0$ provided that $y_i - w_i - \psi e > 0$, which is our assumption. The Hessian matrix is then shown to be definite negative at (1,1). Then it is possible to conclude that (1,1) is a local maximum of the function $L(n_f,\eta_f)=V'(n_f,\eta_f)$ and since $L(1,1)=V'(1,1)=0$ then: $V'(n_f,\eta_f)<0$ in a neighbourhood Ω of (1,1) as we wanted to prove. □

This result shows that the equilibrium (1,1) is not only locally stable in the region defined by the set *A* but it is also Lyapunov stable. Then following the classification of Vega-Redondo (1996) it is possible to say that (1,1) is an asymptotically stable equilibrium of the dynamic system (8)' and (12)'. In order to prove that the system is also asymptotically stable in (0,0) let us consider the following Lyapunov function suggested by Nani and Freedman (2000):

$$V(n_f,\eta_f)=\frac{1}{2}\delta_1(n_f-\bar{n}_f)^2+\frac{1}{2}\delta_2(\eta_f-\bar{\eta}_f)^2 \qquad (26)$$

Then we can prove the following:

Proposition 3:

If $y_i-w_i>\psi e$ and $(1-\gamma)y_f>w_f$ then the dynamic system (8)'' and (12)'' is Lyapunov stable at (0,0) in the set *B* defined by:

$$B=\left\{(n_f,\eta_f)\in[0,1]\times[0,1];n_f<\frac{y_i-w_i-\rho m}{y_f-(1+\gamma)w_f+y_i-w_i-\rho m},\eta_f<\frac{w_i-\rho m}{(1-\gamma)w_f+w_i-\rho m}\right\}$$

Proof.

Note that $V(0,0)=0$. Besides $V(n_f,\eta_f)>0$ for all $(n_f,\eta_f)\in[0,1]\times[0,1]$ and $(n_f,\eta_f)\neq(0,0)$. A possible way of proving requirement c) is to show that (0,0) is a local maximum of the function *V*'. Note that *V*'(0,0)=0 and if we prove that *V*' is definite negative then *V*'<0 for all points in a neighbourhood Ω of (0,0). Taking the derivative of the Lyapunov function we conclude that:

$$V'=\delta_1(n_f-\bar{n}_f)\dot{n}_f+\delta_2(\eta_f-\bar{\eta}_f)\dot{\eta}_f \qquad (27)$$

Hence by substituting (8)'' and (12)'' into the expression above we obtain:

$$V' = \delta_1(n_f - \overline{n}_f)n_f(1-n_f)f(\eta_f) + \delta_2(\eta_f - \overline{\eta}_f)\eta_f(1-\eta_f)g(n_f) \quad (28)$$

By considering that $(\overline{n}_f, \overline{\eta}_f) = (0,0)$ expression (28) reduces to:

$$\begin{aligned} L(n_f,\eta_f) &= \delta_1 n_f^2(1-n_f)\{\eta_f u[(1-\tau)w_f] + (1-\eta_f)u[w_i - \rho m]\} + \\ &+ \delta_2\eta_f^2(1-\eta_f)\{n_f[y_f - (1+\gamma)w_f] + (1-n_f)[y_i - w_i - \psi e]\} \end{aligned} \quad (28)'$$

Where $L(n_f,\eta_f) = V'$. By taking the partial derivatives of the above expression in relation to n_f and η_f we obtain:

$$\begin{aligned} L_{n_f}(n_f,\eta_f) &= \delta_1 n_f(2-3n_f)\{\eta_f u[(1-\tau)w_f] + (1-\eta_f)u[w_i - \rho m]\} + \\ &+ \delta_2\eta_f^2(1-\eta_f)\{[y_f - (1+\gamma)w_f] - [y_i - w_i - \psi e]\} \end{aligned} \quad (29)$$

$$\begin{aligned} L_{\eta_f}(n_f,\eta_f) &= +\delta_1 n_f^2(1-n_f)\{u[(1-\tau)w_f] - u[w_i - \rho m]\} + \\ &+ \delta_2\eta_f(2-3\eta_f)\{n_f[y_f - (1+\gamma)w_f] + (1-n_f)[y_i - w_i - \psi e]\} \end{aligned} \quad (30)$$

Note that (0,0) is a critical point of the function *L*. By taking the second partial derivatives of *L* one obtains:

$$L_{n_f n_f}(n_f,\eta_f) = \delta_1(2-6n_f)\{\eta_f\{u[(1-\tau)w_f] + (1-\eta_f)u[w_i - \rho m]\} \quad (31)$$

$$L_{\eta_f \eta_f}(n_f,\eta_f) = \delta_2(2-6\eta_f)\{n_f[y_f - (1+\gamma)w_f] + (1-n_f)[y_i - w_i - \psi e]\} \quad (32)$$

$$\begin{aligned} L_{n_f \eta_f}(n_f,\eta_f) &= \delta_1 n_f(2-3n_f)\{u[(1-\tau)w_f] - u[w_i - \rho m]\} + \\ &- \delta_2\eta_f(2-3\eta_f)\{[y_f - (1+\gamma)w_f] - [y_i - w_i - \psi e]\} \end{aligned} \quad (33)$$

Evaluating the matrix Hessian at the point (0,0) it yields:

$$\mathrm{H}(0,0) = \begin{bmatrix} -2\delta_1 u[w_i - \rho m] & 0 \\ 0 & -2\delta_2[y_i - w_i - \psi e] \end{bmatrix} \quad (34)$$

Thus $\det \mathrm{H}_1(0,0) = -2\delta_1 u[w_i - \rho m] < 0$ and $\det \mathrm{H}(0,0) = 4\delta_1 u[w_i - \rho m]\delta_2[y_i - w_i - \psi e] > 0$ provided that $y_i - w_i - \psi e > 0$, which is our assumption. The Hessian matrix is then shown to be negative definite at (0,0). □

Then it was also proven that the point (0,0) is an asymptotically stable equilibrium of the dynamical system (8)' and (12)'. Note that these results –

Propositions 2 and 3 – depend crucially on the assumptions made in relation to the profits of firms in the formal and informal sector.

In this vein the dynamics of model is best suited to explain the growth experience of the labour market when it is assumed that the profits are positive and in this case by proving the local stability and the Lyapunov stability it was possible to prove that the equilibrium in which firms and workers choose the formal sector or the informal sector are asymptotically stable equilibrium of the dynamical system derived from an evolutionary game theory model of the labour market.

In this case the dynamic analysis of the model allows us to conclude for the existence of path dependence, namely the initial conditions play a decisive role in the determination of the configuration of the equilibrium in the labour market. According to Wirl and Feichtinger (2005, p. 391) path dependence in a one dimensional model means that there exists a threshold value such that the steady state outcome depends on whether one starts by historical incidence either to the left or the right of this threshold. In the present treatment as we are dealing in the plane it is possible to identify not a threshold point but a threshold curve or set that defines sets that give rise to poverty traps. In the literature such a threshold set become known as Skiba threshold – or points or sets – in honour of the pioneering work of Skiba (1978). [see Deissenberg et al (2003)].

If for instance, $\left(n_f(0), \eta_f(0)\right) \in B$ then the final outcome of the model is the low level equilibrium (0,0) and the government can do nothing to change this situation. But the basin of attraction is affected by the choice of tax, fine and the probabilities of catching firms and workers in the informal sector. This means that the government is able to determine the size of the set B and consequently of set A – defined in the proposition 1 – by choosing properly these variables as policy tools but once they are chosen the model presents path dependency[7]. A similar result in terms of phase diagrams was obtained by Hiller (2010) by studying workers' behaviour and labour contracts in an evolutionary set up. He has found multiple equilibrium with a saddle path interior solution and two unstable, namely (0,1) and (1,0), and two stable, namely (0,0) and (1,1) points. Besides the size of basin of attraction is affected by one of the parameters of the model and the final outcome of the model depends on the initial conditions which is evidence of path dependence.

Another example is Vega-Redondo (1996, p.109) who considers an evolutionary model that exhibits trading complementariness similar to the one we consider here: populations of two separated islands may decide to be 'employed' or 'unemployed' and then they are matched in pairs. If occurs the matching of two employed individuals, they exchange their goods and they both have a positive utility. If two 'unemployed' individuals are matched they have zero utility but if an 'employed' individual of one island matches an unemployed individual of the other island the 'employed' individual receives a negative pay-off since she has worked to produce the good but can neither consume nor exchange its good while the 'unemployed' worker has zero utility since it didn't made any effort. The final outcome of this evolutionary game is that equilibrium (0,0) and (1,1) in which populations of both islands chooses (employed, employed) or (unemployed, unemployed) are asymptotically evolutionary stable.

5. Conclusion

In order to modelling labour market evolution, in this paper we have adopted an evolutionary methodology, in which agents choices are evaluated, may it be workers or

[7] See Apendix I for an exposition of the phase diagrams that show a variation in the basins of attraction due to changes in the parameters of the model such as the taxation on wages and profits.

firms, considering the payoffs associated to each strategy: be formal or informal; and the mean payoff of the other agents. This methodology has yielded a system of differential equation which has multiple equilibriums.

We then have studied the local and Lyapunov stability of the differential system to show that both the equilibrium in each all labour force and all firms operate in the formal sector and the case in which all the labour force and all firms operate in the informal sector are asymptotically stable. The former case is probably the situation that best describe developed economies in which the underground economy is negligible. The latter case may describe the case of some underdeveloped countries.

The economic meaning of these results go beyond the findings that informality arises as the optimal response of workers and firms in response of rigid labour legislation, high taxes and deficient enforcement frameworks, results that are well established in the literature. The existence of multiple equilibriums in the labour market points to a correspondence between the labour market and the stage of economic development. It is also shown that the government plays a central role in the determination of mixed equilibrium but the final position of a country depends on the initial position, that is, on the fraction of workers and firms that are skilled and operate in the formality respectively.

Appendix I

The figures below show that the choice of the parameters of the model determines the size of the basin of attractions. In order to perform this analysis let us consider linear utility functions such as: $u_f = (1-\tau)w_f$ and $u_i = w_i - \rho m$. The instantaneous profit functions are given by: $\pi_f = (1-\gamma)y_f - w_f$ and $\pi_i = y_i - w_i - \psi e$. By choosing $\tau = 0{,}2$; $w_f = 0{,}625$; $w_i = 0{,}6$; $m = 3$; $\rho = 0{,}1$; $y_f = 1{,}25$; $y_i = 1$; $w_i = 0{,}3$; $\psi = 0{,}2$; $e = 0{,}2$

and $\gamma = 0{,}18$ we obtain: $u_f = 0.5; u_i = 0.3; \pi_f = 0.8$ and $\pi_i = 0.5$. In the second case let us change the values of taxations of wages and profits to $\tau = 0{,}52$ and $\gamma = 0{,}18$. This yields the following values for the utility and profit functions: $u_f = 0.3; u_i = 0.3; \pi_f = 0.4$ and $\pi_i = 0.5$. In order to illustrate the path dependence issue let us consider the following initial conditions in the table below. The second and third lines of the table show the final position of each initial condition. Note that in the first case the number of equilibrium (1,1) is larger than (0,0). This case is related to smaller taxation of the formal sector. In case II equilibrium (0,0) is ubiquitous as the final outcome indicating that a higher taxation may induce to the low income equilibrium.

Initial Condition	$n_f(0)=0{,}4$ $\eta_f(0)=0{,}4$ orange	$n_f(0)=0{,}5$ $\eta_f(0)=0{,}8$ black	$n_f(0)=0{,}7$ $\eta_f(0)=0{,}7$ red	$n_f(0)=0{,}25$ $\eta_f(0)=0{,}25$ green	$n_f(0)=0{,}6$ $\eta_f(0)=0{,}125$ brown	$n_f(0)=0{,}56$ $\eta_f(0)=0{,}3$ maroon	$n_f(0)=0{,}7$ $\eta_f(0)=0{,}7$ blue
Case I	(1,1)	(1,1)	(1,1)	(0,0)	(0,0)	(1,1)	(1,1)
Case II	(0,0)	(1,1)	(1,1)	(0,0)	(0,0)	(0,0)	(1,1)

Table me:

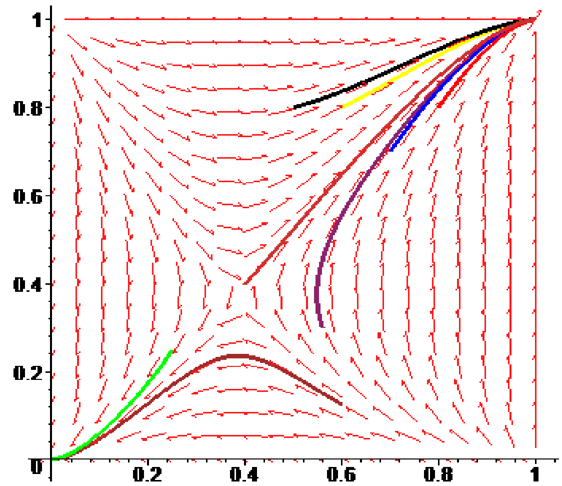

Figure 1:

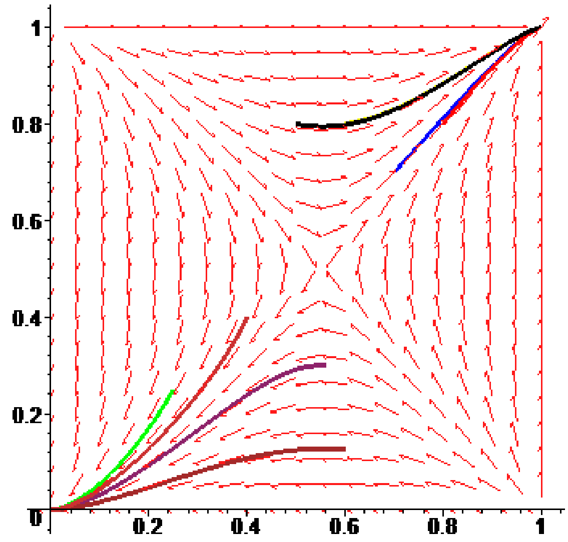

Figure 2

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