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An Unbalanced Two-sector Growth Model with Constant Returns:
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Abstract

Recent industry-based empirical studies among countries demonstrate that individual industry's per capita capital stock and output grow at industry's own steady state growth rate. The industry growth rate is highly correlated to industry's technical progress measured by total factor productivity (TFP) of the industry, which exhibits large difference across industries as reported recently by Syverson (2011). Let us refer to this phenomenon as "unbalanced growth among industries." Very few researches concerned with this phenomenon have been done yet. Some exceptions are Echevarria (1997), Kongsamut, Rebelo and Xie (2001), and Acemoglu and Guerrieri (2008) among others. However their models and analytical methods are different from mine. Applying the theoretical method developed by McKenzie and Scheinkman in turnpike theory, I now construct a two-sector optimal growth model with an industry specific Hicks-neutral technical progress and show that each sector's per capita capital stock and output grow at the rate of the sector's technical progress (the sector's TFP growth rate).

JEL Classification: O14,O21,O24,O41

1. INTRODUCTION

Since the seminal papers by Romer (1986) and Lucas (1988), economics has witnessed a strong revival of interest in growth theory under the name of "Endogenous growth theory." Neoclassical optimal growth models have been applied as benchmarks and studied intensively since the late 1960. However, these analytical models have a serious drawback: they are based on highly aggregated macro-production functions and cannot explain the important empirical evidence that I discuss in the following section. Recent industry-based empirical studies among countries clearly revealed that growth in an individual industry's per capita capital stock and output grow at industry's own growth rate, which is closely related to its technical progress measured by total factor productivity (TFP) of the industry. For example, per capita capital stock and output of the agriculture industry grow at 5% per annum along its own steady-state, whereas they grow at 10% annually in the manufacturing industry, also paralleling the industry's steady state. Syverson (2011) has recently reviewed these arguments discussed above. Let us refer this phenomenon as "**unbalanced growth among industries**." The attempt to understand this phenomenon has generated a strong theoretical demand for constructing a multi-sector growth model, yet very little progress has been made so far. Some exceptions are Echevarria (1997), Kongsamut, Rebelo and Xie (2001), and

Acemoglu and Guerrieri (2008). However their models and analytical methods are different from mine. Setting up an optimal growth model with three sectors: primary, manufacturing and service, Echevarria (1997) has applied a numerical analysis to solve the model. Kongsamut, Rebelo and Xie (2001) has constructed the similar model to the one of Echevarria (1997), while they have investigated the model under a much stronger assumption than her: each sector produces goods with the same technology. On the other hand, Acemoglu and Guerrieri (2008) has studied the model with two intermediate-goods sectors and single final-goods sector. Note that the last two models will share a common character: consumption goods and capital goods are identical. Contrastingly, my model presented here exhibits a sharp contrast with them. Since I assume that each good is produced with a different technology, consumption goods and capital goods are completely different goods. As I will demonstrate later, this feature of the model will make the characteristics of the model far complicated.

The optimal growth model with heterogeneous capital goods has been studied intensively since the early 1970' under the title of turnpike theory by McKenzie (1976, 1982, 1983 and 1986) and Scheinkman (1976). Turnpike theory shows that any optimal path converges asymptotically to the corresponding optimal steady state path without initial stock sensitivity. In other words, the turnpike property implies that the per capita capital stock and output of each industry eventually converge to an industry-specific constant ratio. Therefore, turnpike theory too, cannot explain the empirical phenomenon: unbalanced growth among industries. McKenzie (1998) has articulated this point: "Almost all the attention to asymptotic convergence has been concentrated on convergence to balanced paths, although it is not clear that optimal balanced growth path will exist. This type of path is virtually impossible to believe in, if the model is disaggregated beyond the division into human capital and physical capital, and new goods and new methods of production appear from time to time." An additional point is that the turnpike result established in a reduced form model has not been fully applied to a structural neoclassical optimal growth model. A serious obstacle in applying the results from the reduced form model is that the transforming of a neoclassical optimal growth model into a reduced form model will not yield a strictly concave reduced form utility function, but just a concave one. In this context, McKenzie (1983) has extended the turnpike property to the case in which the reduced form utility function is not strictly concave, that is, there is a flat segment on the surface, which contains an optimal steady state. This flat segment is often referred to as the Neumann-McKenzie facet. Yano (1990) has studied a neoclassical optimal growth model with heterogeneous capital goods in a trade theoretic context. However, in case of the Neumann-Mckenzie facet with a positive dimension, Yano explicitly assumed the "dominant diagonal block condition" concerned with the reduced form utility function (see Araujo and Scheinkman (1978) and McKenzie (1986)). Thus, he still did not fully exploit the structure of the neoclassical optimal growth model, especially the dynamics of the path on the Neumann-McKenzie facet, to obtain the turnpike property.

By applying the theoretical method developed in turnpike theory, this study seeks to fill the gap between the results derived by the theoretical research explained above and the empirical evidence from recent studies at the industry level among countries. First, I will set up a multi-industry optimal growth model, in which each industry exhibits the Hicks-neutral technical progress with an industry specific rate. This model will be regarded as a multi-industry optimal growth version of the Solow model with the Hicks neutral technical progress. Second, I will rewrite the original model into a per capita efficiency unit model. Third, I will transform the efficiency unit model into a reduced form model, after which the method developed in turnpike theory will be applied. The neighborhood turnpike theorem demonstrated in McKenzie (1983) indicates that any optimal path will be trapped in a neighborhood of the corresponding optimal steady state path when discount factors are sufficiently close to 1, and the neighborhood can be minimized by choosing a discount factor arbitrarily close to 1. I will demonstrate the local stability theorem by applying the logic used by Scheinkman (1976): a stable manifold extends over today's capital stock plane. As we see later, the dynamics of the Neumann-McKenzie facet are important in demonstrating both the theorems. Combining the neighborhood turnpike and the local stability produces the complete turnpike property: any optimal path converges to a corresponding optimal steady state when discount factors are sufficiently close to 1. For establishing both theorems, we assume generalized capital intensity conditions, which are the generalized versions of those in a two-sector model. The complete turnpike property means that each sector's optimal per capita capital stock and output converge to its own steady state path with the rate of technical progress determined by the industry's TFP.

The paper is organized in the following manner: In Section 2, I will provide a several empirical facts based on the recent database at the industry level among countries. In Section 3, the model and assumptions are presented and show some existence theorem. In Section 4, the Neumann-McKenzie facet is introduced and the Neighborhood Turnpike Theorem is demonstrated. The results obtained in Section 4 will be used repeatedly in the proofs of main theorems. In Section 5, I show the complete turnpike theorem. Some comments are given in Section 6.

2. SOME EMPIRICS

In Takahashi, Mashiyama and Sakagami (2011), we have empirically examined the Post-war Japanese economy and other OECD countries based on a two-sector growth model setting originally investigated by Uzawa. We have found the following facts among others.

- Through the observation period (1955-1995), per-capita capital stocks grew exponentially at a sector specific constant rate. Furthermore, the per-capita capital stock of the consumption sector grew much faster than the investment sector (see Figure 1).

<Insert Figure 1 here>

- During the High-speed Growth Era (1960-1975), the investment sector was more capital-intensive than the consumption good sector. After the 1973 oil-shock, the consumption good sector turned out to be more capital-intensive than the investment good sector. In other words, "the capital-intensity reversal" took place (see Figure 2).

<Insert Figure 2 here>

- In other OECD countries, the capital-intensity reversal cannot be observed as shown in Figure 3, and the consumption good sector is more capital intensive than the investment good sector over the observation period (1970-1990).

<Insert Figure 3 here>

The first fact implies that each sector's steady state has a sector specific positive growth rate. Following Baumol (1967), we may call this phenomenon "unbalanced growth." The second and the third facts imply that in the long-run, the consumption sector generally turns out to be more capital intensive than the investment sector.

Let us compare our explanation of the post-war Japanese economy based on the two-sector growth model with that of the one-sector growth model by Valdes (2005). His exposition has totally based on his observation of the long-run Japanese real per-capita GDP data reported in Maddison (1995), which is depicted as Figure 4.

<Insert Figure 4 here>

He has applied a theoretical framework of the Solow-Swan growth model with a technical progress to the post-war Japanese economy for understanding the basis of Japan's high-growth period. Superimposing three trend lines of steady-state paths denoted by A, B and C in Figure 4, he has identified the fact that the high-growth period was regarded as a transition process and the Japanese economy converged from the old steady-state to the new one, which had a higher per-capita GDP level than the old one (called a level-effect), but has a similar positive slope as that of the old one. Based on the Solow-Swan growth framework, he has concluded that the high saving rate caused the level-effect and a slowdown of the technical progress brought about the end of the transition process. Contrastingly, we have measured the two-sector capital intensities in the post-war Japanese economy and found several characteristic facts. One of the striking facts is that a capital-intensity reversal had occurred around 1975, and simultaneously the Japanese high-speed growth ended. We have accommodated the two-sector growth framework to identify the cause of the High-speed Growth Era as the magnification effect arising from differences in capital-intensity between sectors during the transition process. Clearly, one-sector growth models failed to account for this phenomenon.

We applied the same method to the post-war Korean economy and found empirical evidence such that, by 1995, the two-sector capital-intensity ratio had reached 0.96. This may imply that, sooner or later, we could observe the appearance of a capital-intensity reversal in Korea, too. Unfortunately, we don't have enough data to estimate after 1995. Because the capital stock data of Taiwan and China were not obtained at this time, we gave up estimating the capital intensities of both countries.

From the empirical evidence examined above, we may therefore conclude that the two-sector model that we will set up in Section 2 should satisfy the following three important characteristics:

1) The per-capita capital stocks of the consumption and investment sectors grow at a sector specific growth rate, which is closely related to a sector's TFP growth rate.

2) The per-capita growth rate of the consumption sector is greater than that of the investment good sector.

3) Along the steady state path, the consumption good sector is more capital-intensive than the capital good sector.

In Section 3, we will set up a two-sector growth model which satisfies the above properties.

3. The Model

Our model is a discrete two-sector optimal growth model studied by Uzawa (1964) with the sector-specific Hicks-neutral technical progress, which will be measured as sector-specific total factor productivity. Our model is the following:

$$Max\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^{t}u(C(t))$$

$$subject\ to: K(0)=\overline{K}\ and\ L(0)=\overline{L},$$

$$Y(t)+(1-\delta)K(t)-K(t+1)=0, \qquad (1)$$

$$C(t) = A_0(t)F^0(K_0(t), L_0(t)), \tag{2}$$

$$Y(t) = A_1(t)F^1(K_1(t), L_1(t)), \tag{3}$$

$$L_0(t) + L_1(t) = L(t), \tag{4}$$

$$K_0(t) + K_1(t) = K(t) \tag{5}$$

, and the notation is as follows:

r	: a subjective rate of discount, $r \geq g$,
$C(t) \in \mathbb{R}_+$	: the total goods consumed at t,
$Y(t) \in \mathbb{R}_+$	: the t^{th} period capital output of the capital goods sector,
$K(t) \in \mathbb{R}_+$	: the total capital goods at t,
$K(0) \in \mathbb{R}_+$	: the initial total capital goods,
$K_i(t) \in \mathbb{R}_+$	: the t^{th} period capital stock of the i^{th} sector,
$F^0(\cdot) : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$	: a production function of the consumption goods sector,
$F^1(\cdot) : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$	: a production function of the capital goods sector,
$L(t) \in \mathbb{R}_+$	: the total labor input at t,
$L(0) \in \mathbb{R}_+$	: the initial total labor input,
$L_i(t) \in \mathbb{R}_+$	: the t^{th} labor input of the i^{th} sector,
δ	: the depreciation rate,
$A_i(t)$	: the t^{th} period Hicks neutral technical-progress of the i^{th}

sector.

, where $i=0$ and $i=1$ indicated the consumption goods sector and the capital goods sector respectively.

Assumption 1.

1) The utility function u(・) is defined on $\mathbb{R}_+$ as the following:

$$u(C(t))=\begin{cases}\dfrac{C(t)^{\tau}}{\tau}, if\ \tau\in(-\infty,1)\\ \\ \log C(t), if\ \tau=0.\end{cases}$$

2) $L(t)=(1+g)^t L(0)$, where g is a rate of population growth.

3) $A_i(t)=(1+\alpha_i)^t A_i(0)$, where α_i is a rate of output-augmented (the Hicks-neutral) technical-progress of the i^{th} sector and given as $|\alpha_i|<1$.

Note that 3) of Assumption 1 means that the sectoral TFP is measured by the sectoral output-augmented technical progress (the Hicks-neutral technical progress), which is externally given.

Assumption 2.

1) All the goods are produced non-jointly with the production functions $F^i(\cdot)$ $(i=0,1)$ which are defined on $\mathbb{R}^2_+$, homogeneous of degree one, strictly quasi-concave and continuously differentiable for positive inputs.

2) Any good i $(i = 0,1)$ cannot be produced unless $K_i = 0$ *or* $L_i = 0$.

Dividing all the variables by $A_i(t)L(t)$, we will transform the original model into per-capita efficiency unit model. Firstly, let us transform the capital goods sector's production function as follows; dividing both side of Eq.(1.3) by $A_1(t)L(t)$, we have:

$$\frac{Y(t)}{A_1(t)L(t)} = F^1\left(\frac{K_1(t)}{L(t)}, \frac{L_i(t)}{L(t)}\right),$$

And similarly, for the consumption goods sector's production function, we obtain¥

$$\frac{C(t)}{A_0(t)L(t)} = F^0\left(\frac{K_0(t)}{L(t)}, \frac{L_0(t)}{L(t)}\right).$$

Now let us define the following normalized variables:

$$y(t) = \frac{Y(t)}{A_1(t)L(t)}, c(t) = \frac{C(t)}{A_0(t)L(T)}, k_1(t) = \frac{K_1(t)}{L(T)}, k_0(t) = \frac{K_0(t)}{L(T)},$$

$$\ell_1(t) = \frac{L_1(t)}{L(T)}, \ell_0(t) = \frac{L_0(t)}{L(T)}.$$

By normalizing the production functions with respect to $A_i(t)L(T)$ $(i = 0,1)$,

$$y(t) = f^1(k_1(t), \ell_1(t)) \text{ and } c(t) = f^0(k_0(t), \ell_0(t)).$$

Moreover normalize the accumulation equation similarly, we have

$$\frac{Y(t)}{A_1(t)L(t)} + (1-\delta)\frac{K(t)}{A_1(t)L(t)} - \frac{K(t+1)}{A_1(t)L(t)} = 0.$$

Substituting the following relation into this:

$$\frac{K(t+1)}{A_1(t)L(t)} = \frac{(1+\alpha_1)(1+g)K(t+1)}{[(1+\alpha_1)A_1(t)]\{(1+g)L(t)]} = (1+\alpha_1)(1+g)\tilde{k}(t+1)$$

where

$$\tilde{k}(t) = \frac{K(t)}{A_1(t)L(t)} \text{ and } \tilde{k}(t+1) = \frac{K(t+1)}{A_1(t+1)L(t+1)}.$$

We have finally the following normalized accumulation equation:

$$\tilde{y}(t) + (1-\delta)\tilde{k}(t) - (1+\alpha_1)(1+g)\tilde{k}(t+1) = 0.$$

We can also rewrite the objective function in terms of per-capita as follows: Substituting the following relation into the objective function yields:

$$\tilde{c}(t) = \frac{C(t)}{A_0(t)L(t)} = \frac{C(t)}{(1+\alpha_0)^t(1+g)^t A_0(0)L(0)} = \frac{C(t)}{(1+\alpha_0)^t(1+g)^t}$$

where we assume that $A_0(0)L(0) = 1$.

Finally, it follows that

$$\sum_{t=0}^{\infty}\left[\frac{(1+\alpha_0)^{\tau}(1+g)^{\tau}}{(1+r)}\right]^{t}\frac{\tilde{c}^{\tau}(t)}{\tau}=\sum_{t=0}^{\infty}\rho^{t}u(\tilde{c}(t))$$

where

$$\rho=\frac{(1+\alpha_0)^{\tau}(1+g)^{\tau}}{(1+r)}.$$

Now the original model can be rewritten as the per-capita efficiency unit model as shown below:

-The Per-capita Efficiency Unit Model-

$$Max\sum_{t=0}^{\infty}\rho^{t}u(\tilde{c}(t))$$
$$s.t.\,\tilde{k}(0)=\overline{k},$$

$$\tilde{c}(t)=f^{0}(k_0(t),\ell_0(t)), \tag{6}$$

$$\tilde{y}(t)=f^{1}(k_1(t),\ell_1(t)), \tag{7}$$

$$\tilde{y}(t)+(1-\delta)\tilde{k}(t)-(1+\alpha_1)(1+g)\tilde{k}(t+1)=0, \tag{8}$$

$$\ell_0(t)+\ell_1(t)=\ell(t), \tag{9}$$

$$\frac{k_0(t)+k_1(t)}{A_1(t)}=\tilde{k}(t). \tag{10}$$

We may add the following extra assumption and prove the basic property below:

Assumption 3. $A_0(0)L(0)=1\,and\;0<\rho<1$.

Remark1. The value of ρ consists of four parameters; the coefficient of relative risk averse $(1-\tau)$, the rate of population growth (g), the rate of subjective discount rate (γ) and the rate of technical progress in consumption goods sector (α_0). Note that the rate of population growth could be negative. For example, we may consider the case where $\tau = 0.5,\ g = -0.2,\ \gamma = 0.2\ and\ \alpha_0 = 0.2.$

Lemma 1. Under Assumption 2, Equations (6)-(10) except Equation (8) are summarized as the social production function $\tilde{c}(t) = T(\tilde{y}(t), \tilde{k}(t))$ which is continuously differentiable in the interior of $\mathbb{R}_+^2$ and concave.

Proof. Solving the following problem (*) we can derive the social production function:

$$(*)\begin{cases} Max\ f^0(k_0(t), \ell_0(t)) \\ \\ s.t.\ \tilde{y}(t) = f^1(k_1(t), \ell_1(t)), \\ \ell_0(t) + \ell_1(t) = 1\ and\ \dfrac{k_0(t) + k_1(t)}{A_1(t)} = \tilde{k}(t). \end{cases}$$

See in detail Benhabib and Nishimura (1979).■

If x and z indicate initial and terminal capital stocks respectively, the reduced form utility function $V(x,z)$ and the feasible set D can be defined as follows:

$$V(x,z) = u(T[(1+\alpha_1)(1+g)z - (1-\delta)x, x])$$

and

$$D = \left\{(x,z) \in \mathbb{R}_+ \times \mathbb{R}_+ : T[(1+\alpha_1)(1+g)z - (1-\delta)x \geq 0\right\}$$

where $x = \tilde{k}(t)\ and\ z = \tilde{k}(t+1).$ Note that we eliminate time index for simplicity.

Finally, the per-capita efficiency-unit model will be summarized as the following standard reduced form model, which have been studied in detail by McKenzie (1986) and Scheinkman (1976).

-The Reduced Form Model-

$$\begin{aligned} &Max \sum_{t=0}^{\infty} \rho^t V(\tilde{k}(t), \tilde{k}(t+1)) \\ &s.t. \\ &(\tilde{k}(t), \tilde{k}(t+1)) \in D \ for\ t \geq 0\ and\ \tilde{k}(0) = \overline{k}. \end{aligned}$$

Also note that any interior optimal path must satisfy the following Euler equation, which shows an intertemporal efficiency allocation:

$$V_z(\tilde{k}(t-1), \tilde{k}(t)) + \rho V_x(\tilde{k}(t), \tilde{k}(t+1)) = 0 \ for\ all\ t \geq 0 \tag{11}$$

where the partial derivatives mean that

$$V_x(\tilde{k}(t), \tilde{k}(t+1)) = \frac{\partial V(\tilde{k}(t), \tilde{k}(t+1))}{\partial \tilde{k}(t)}, \ V_z(\tilde{k}(t-1), \tilde{k}(t)) = \frac{\partial V(\tilde{k}(t-1), \tilde{k}(t))}{\partial \tilde{k}(t)}.$$

Note that under the differentiability assumptions, all the prices will be obtained by the following relations:

$$q = \frac{du(\tilde{c})}{d\tilde{c}} = 1,\ p = -q\frac{\partial T(\tilde{y}, \tilde{k})}{\partial \tilde{y}},\ w = q\frac{\partial T(\tilde{y}, \tilde{k})}{\partial \tilde{k}},\ w_0 = q\tilde{c} + p\tilde{y} - w\tilde{k},$$

where we normalize the price of consumption goods as 1.

Definition. An optimal steady state path (OSS) k^ρ is an optimal path which solves the per-capita efficient unit model and $\tilde{k}^{\rho} = \tilde{k}(t) = \tilde{k}(t+1)$ *for all* $t \geq 0$.

Due to the homogeneity assumption of each sector's production function, it is often convenient to express a chosen technology as a technology matrix. Now let us define the technology matrix as follows:

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$$

where $a_{00} = \frac{\ell_0}{\tilde{c}}$, $a_{01} = \frac{k_0}{\tilde{c}}$, $a_{10} = \frac{\ell_1}{\tilde{y}}$, $a_{11} = \frac{k_1}{\tilde{y}}$.

Assumption 4 (Viability). For a given $r\,(\geq g)$, a chosen technology coefficient $\overline{a}_{11}^{\gamma}$ satisfy $1-(r+\delta)\ \overline{a}_{11}^{\gamma} > 0$.

The following extra assumption will be made.

Assumption 5. $1 > \alpha_0 > \alpha_1 > 0$

Remark 2. This assumption means that the TFP growth rate in the consumption goods sector is the highest one among both sectors. As we have examined in Section 2, Takahashi, Mashiyama and Sakagami (2011) have observed the empirical evidence such that in the Post-war Japanese economy, along the steady state path, the consumption sector has exhibited a higher per-capita output growth rate than that of the capital goods sector.

McKenzie (1983,1984) has demonstrated the existence theorem for both an optimal and an optimal steady state paths in the reduced form model. Applying a same logic as that of McKenzie's, we can prove the following existence theorem under our Assumptions 1 through 5.

Existence Theorem (McKenzie). Under Assumptions 1 through 5, there exists an optimal steady state path $\tilde{k}^{\rho}$ for $\rho \in (0,1]$ and an optimal path $\left\{\tilde{k}^{\rho}(t)\right\}^{\infty}$ from any sufficient initial stock $\tilde{k}(0)$ [1].

Proof. We need to demonstrate that under Assumptions 1 through 3, all the conditions listed below in Theorem 1 of McKenzie (1983) and McKenzie (1984) are satisfied.

McKenzie's Conditionds:

1) $V(x,z)$ is defined on a closed convex set D.

2) There is a $\eta > 0$ such that $(x,z) \in D$ *and* $|z| < \xi < \infty$ implies $|z| < \eta < \infty$.

3) If $(x,z) \in D$, *then* $\left(\tilde{x},\tilde{z}\right) \in D$ *for all* $\tilde{x} \geq x$ *and* $0 \leq \tilde{z} \leq z$. Moreover $V(\tilde{x},\tilde{z}) \geq V(x,z)$.

4) There is $\varsigma > 0$ *such that* $|x| > \varsigma$ *implies for any* $(x,y) \in D, |z| < \lambda |x| (0 < \lambda < 1)$.

5) There is $(\bar{x},\bar{z}) \in D$ *such that* $\rho\bar{z} > \bar{x}$.

It is straightforward to show that Assumptions 1 through 3 satisfy McKenzie's Conditions 1) through 4). We will show that Assumptions 4 and 5 satisfies Condition 5)

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[1] A capital stock x is "**sufficient**" if there is a finite sequence (k(0),k(1),…,k(T)) such that x=k(0), (k(t), k(T+1)) $\in$ D and k(T) is expansible. K(T) is **"expansible"** if there is k(T+1) such that k(T+1)>>k(T) and (k(T),k(T+1)) $\in$ D.

as follws:

$$z-\rho^{-1}x=\frac{1}{(1+\alpha_1)(1+g)}y+\frac{1-\delta}{(1+\alpha_1)(1+g)}x-\rho^{-1}x$$

$$>\frac{1}{(1+\alpha_1)(1+g)}\left\{1+\left[(1-\delta)-\frac{(1+\alpha_1)(1+g)}{\rho}\right]\left(\bar{a}_{11}^{\gamma}\right)\right\}\tilde{y},$$

$$\textit{by the fact that } \frac{1-\delta}{(1+\alpha_1)(1+g)}<1<\rho^{-1}$$

$$=\frac{1}{(1+\alpha_1)(1+g)}\left\{1+\left[(1-\delta)-(1+r)\frac{(1+\alpha_1)}{(1+\alpha_0)^{1-\gamma}(1+g)^{\gamma-1}}\right]\left(\bar{a}_{11}^{\gamma}\right)\right\}\tilde{y},$$

$$>\frac{1}{(1+\alpha_1)(1+g)}\left\{1+\left[(1-\delta)-(1+r)\frac{(1+\alpha_1)}{(1+\alpha_0)}\right]\left(\bar{a}_{11}^{\gamma}\right)\right\}\tilde{y}\ \ (by\ \gamma=1),$$

$$>\frac{1}{(1+\alpha_1)(1+g)}\left\{1+\left[(1-\delta)-(1+r)\right]\left(\bar{a}_{11}^{\gamma}\right)\right\}\tilde{y}\ \ (by\ Assumption\ 5),$$

$$=\frac{1}{(1+\alpha_1)(1+g)}\left\{1+\left[(1-\delta)-(1+r)\right]\left(\bar{a}_{11}^{\gamma}\right)\right\}\tilde{y}$$

$$=\frac{1}{(1+\alpha_1)(1+g)}\left\{\left[1-(r+\delta)\right]\left(\bar{a}_{11}^{\gamma}\right)\right\}\tilde{y}>0.\ (by\ Assumption\ 4)$$

Therefore y will be chosen so that $z-\rho^{-1}x\geq 0$ where $(x,z)\in D$.■

Remark 3. It should be noticed that since $\tilde{k}^{\rho}=\frac{K^{\rho}(t)}{A_1(t)L(t)}=\frac{k^{\rho}(t)}{A_1(t)}$, it follows that

$k^{\rho}(t) = (1+\alpha_1)^t A_1(0)\tilde{k}^{\rho}$. Hence the original series of the sector's optimal per-capita stock $k^{\rho}(t)$ is growing at the rate of its own sector's TFP growth rate α_1.

Suppose that $\tilde{k}^{\rho}$ is an interior OSS in the efficiency unit with a given ρ, it must also satisfy the Euler equation:

$$V_z(\tilde{k}^{\rho}, \tilde{k}^{\rho}) + \rho V_x(\tilde{k}^{\rho}, \tilde{k}^{\rho}) = 0.$$

Note that the following relations hold:

$$V_x(\tilde{k}^{\rho}, \tilde{k}^{\rho}) = p^{\rho}(1-\delta) + w^{\rho},\ and\ V_z(\tilde{k}^{\rho}, \tilde{k}^{\rho}) = -(1+g)(1+\alpha_1)p^{\rho}.$$

Substituting above relations into the Euler equation will lead to the following equation:

$$\rho[w^{\rho} + p^{\rho}(1-\delta)] - (1+g)(1+\alpha_1)p^{\rho} = 0.$$

We will not prove the following important theorem, but it has been established by applying the well-known non-substitution theorem in a multi-sector model by Takahashi (2011).

Lemma 2. When $\rho \in (0,1]$, there exists a unique OSS $\tilde{k}^{\rho}$ >0 with the corresponding unique positive prices $p^{\rho} > 0$ *and* $w^{\rho} > 0$.

From this lemma, along the OSS with ρ, the nonsingular technology matrix A^{ρ} will be chosen, and the cost-minimization and the full-employment conditions will be expressed as follows:

$$(1, p^{\rho}) = (w_0^{\rho}, w_1^{\rho})A^{\rho} \quad and \quad (1, k^{\rho}) = A^{\rho}(c^{\rho}, y^{\rho}).$$

If A^{ρ} has an inverse matrix B^{ρ}, solving above equations yields,

$$p^{\rho} = w^{\rho}\left(a_{11}^{\rho} - \frac{1}{a_{00}^{\rho}} a_{10}^{\rho} a_{01}^{\rho} \right) + \frac{a_{01}^{\rho}}{a_{00}^{\rho}} = w^{\rho} (b_{11}^{\rho})^{-1} + \frac{a_{01}^{\rho}}{a_{00}^{\rho}},$$

and

$$\tilde{k}^{\rho} = \left(a_{11}^{\rho} - \frac{1}{a_{00}^{\rho}} a_{10}^{\rho} a_{01}^{\rho} \right) \tilde{y}^{\rho} + \frac{a_{01}^{\rho}}{a_{00}^{\rho}} = (b_{11}^{\rho})^{-1} \tilde{y}^{\rho} + \frac{a_{01}^{\rho}}{a_{00}^{\rho}},$$

where b_{11}^{ρ} is the element of the matrix B^{ρ} defined as follows:

$$B^{\rho} = \left(A^{\rho} \right)^{-1} = \begin{pmatrix} b_{00}^{\rho} & b_{01}^{\rho} \\ b_{10}^{\rho} & b^{\rho} \end{pmatrix}.$$

From now on, we are concentrated on the OSS with $\rho = 1$ denoted by $\tilde{k}^{*}$. We will also use the symbol "*" to indicate the elements and variables evaluated at $k*$.

Definition (Capital-intensity Conditions). When $a_{11}^{\rho} / a_{01}^{\rho} < a_{10}^{\rho} / a_{00}^{\rho}$ is established, the consumption goods sector is ***capital intensive*** in comparison with the capital goods sector. For $a_{11}^{\rho} / a_{01}^{\rho} > a_{10}^{\rho} / a_{00}^{\rho}$, the consemption goods sector is ***labor intensive*** in comparison with the capital goods sector.

As we have examined in Section 2 based on Takahashi, Mashiyama and Sakagami (2011), we have found that there exists a firm evidence that any country exhibit that the consumption goods sector is capital intensive. So it will be justified to make the following assumption:

Assumption 6. $a_{11}^{\rho} / a_{01}^{\rho} < a_{10}^{\rho} / a_{00}^{\rho}$.

Note that under Assumption 6, it follows that

$$(b^{\rho})^{-1} = a_{11}^{\rho} - \frac{1}{a_{00}^{\rho}} a_{10}^{\rho} a_{01}^{\rho} < 0.$$

Now we can show the following Lemma 5.

Lemma 5. Under Assumption 5, there exists a positive $\overline{\rho}$ such that for $\rho \in [\overline{\rho}, 1]$, the OSS k^{ρ} is unique and is a continuous function of ρ, namely $\tilde{k}^{\rho} = k(\rho)$.

Proof. From the Euler equation, its Jacobian can be calculated as

$$J(k,\rho) = \rho V_{xx}(k,k) + \rho V_{xz}(k,k) + V_{zx}(k,k) + V_{zz}(k,k).$$

Evaluating it at k^* yields

$$J(\tilde{k}^*,1) = V_{xx}(\tilde{k}^*,\tilde{k}^*) + V_{xz}(\tilde{k}^*,\tilde{k}^*) + V_{zx}(\tilde{k}^*,\tilde{k}^*) + V_{zz}(\tilde{k}^*,\tilde{k}^*).$$

Applying the same logic used in Takahashi (2011), we finally obtain

$$\begin{aligned} J(\tilde{k}^*,1) &= [b^* - ((1+g)(1+\alpha_1) + (\delta - 1)]^2 [(b^*)^{-1}]^2 T_{zz} \\ &= [b^* - (g\alpha_1 + \delta)]^2 [(b^*)^{-1}]^2 T_{zz} < 0 \; (\neq 0), \end{aligned}$$

where T_{zz} is an element of the Hessian matrix of the social transformation function $T(x,z)$ and negative. Thus the result follows from the Implicit Function Theorem.■

4. Neumann-McKenzie Facet and Turnpikes

Now we will introduce the Neumann-McKenzie Facet (denoted by "NMF" for short), which plays an important role in stability arguments regarding neoclassical growth models as studied in Takahashi (1985) and Takahashi (1992), and has been intensively studied in the reduced-form models by McKenzie (1983). The NMF will be defined in the reduced form model as follows:

Definition. The Neumann-McKenzie Facet of an OSS, denoted by $F(\tilde{k}^{\rho}, \tilde{k}^{\rho})$, is defined as:

$$F(\tilde{k}^{\rho}, \tilde{k}^{\rho}) = \left\{ (x, z) \in D : u(c) + \rho p^{\rho} z - p^{\rho} x = u(\tilde{c}^{\rho}) + \rho p^{\rho} \tilde{k}^{\rho} - p^{\rho} \tilde{k}^{\rho} \right\},$$

where $\tilde{k}^{\rho}$ is the OSS and p^{ρ} is the supporting price of the OSS when a subjective discount rate ρ is given.

Due to this definition, if a path will happen to deviate from the NMF, then a value-loss will take place, which will be defined as follows:

Definition. The value-loss δ is defined as,

$$\delta = (u(c) + \rho p^{\rho} z - p^{\rho} x) - (u(\tilde{c}^{\rho}) + \rho p^{\rho} \tilde{k}^{\rho} - p^{\rho} \tilde{k}^{\rho}),\ for\ (x, z) \in D.$$

Let make the following assumption in order to make the NMF have non-zero dimension.

Assumption 7. $u(\tilde{c})$ is linear in the neighborhood of the OSS k^{ρ} $(u(\tilde{c}^{\rho}) = \tilde{c}^{\rho})$.

By the definition above, the NMF is a set of capital stock vector (x, z) which arise from the exact same net benefit as that of the OSS when it is evaluated by the prices of the OSS. Also, the NMF is the projection of a flat segment on the surface of the utility function V that is supported by the price vector $(-p^{\rho}, \rho p^{\rho}, 1)$ onto the (x, z)-space. In the two-sector model, we can examine the NMF diagrammatically. Based on the assumption that the consumption goods sector is capital intensive, it is possible to draw a graph on the coordinates $(y(t), c(t))$, which is often used in trade theory.

<Insert Figure 5 here>

Here, $(\tilde{y}^{\rho}, \tilde{c}^{\rho})$ is a production vector corresponding to the OSS and is written as a point of intersection for the labor-constraint line and the capital-constraint line. Also, the fact that the labor-constraint line intersects the capital-constraint line from the above is due to the capital intensity assumed above. Note that production specialization occurs at points A and B. Now, suppose that $\bar{k}(t)(<1/a_{10}^{\rho})$, which is greater than OSS, $\{\tilde{k}^{\rho}\}$ were given. Then, if we leave the capital-constraint line and the price vector as they are, then move upward along the labor-constraint line, a new point of intersection "E" is obtained. Also, the corresponding production vector $(\bar{y}(t), \bar{c}(t))$ is obtained at point E. Also, by substituting this value for accumulation equation (1.2), the next period's capital stock

$\bar{k}(t+1)$ can be attained. The capital stock pairs $(\bar{k}(t), \bar{k}(t+1))$ obtained in this manner can be plotted as point E on plane (x, z). By further altering $k(t)$ and repeatedly conducting the similar procedure, line AB can be drawn on plane (x, z) in the manner of figure 6. Now, it can be understood that labor-constraint line AB on the production plane (y, c) directly corresponds to line AB on plane (x, z). The portions of this line AB excluding the ends are the von Neumann-McKenzie facet as dipicted in Figure 6.

<Insert Figure 6 here>

Since k^{ρ} *and* p^{ρ} are continuous functions of ρ as shown in Lemma 5, based on the above discussion, we may observe the following:

Lemma 6. The NMF is a continuous correspondence of $\rho \in [\bar{\rho}, 1)$.

Proof. A formal proof will leave for Takahashi (1985).■

This property will guarantee that if a path will be away from a ε-neighborhood of the NMF, then there exists a $\bar{\delta}$ such that $\delta \geq \bar{\delta} > 0$. $\bar{\delta}$ is referred to as the uniform value loss .

Based on the above discussion again, it is possible to redefine NMF in a neoclassical model as follows.

Definition (Characterization of NMF).

$F(\tilde{k}^{\rho}, \tilde{k}^{\rho}) \equiv \{(\tilde{k}(t), \tilde{k}(t+1)) \in D:$ There exist $c(t) \geq 0$ and $y(t) \geq 0$ such that they satisfy following conditions (1) through (5): (1) $1 = w_0^\rho a_{00}^\rho + w^\rho a_{10}^\rho$, (2) $p^\rho = w_0^\rho a_{01}^\rho + w^\rho a_{11}^\rho$, (3) $1 = a_{00}^\rho c(t) + a_{10}^\rho y(t)$, (4) $\tilde{k}(t) = a_{01}^\rho \tilde{c}(t) + a_{11}^\rho \tilde{y}(t)$, (5) $\tilde{k}(t+1) = \tilde{y}(t) + (1-\delta)\tilde{k}(t))\}$, where the consumption good's price is normalized as 1; also, for simplification, the population growth rate has been postulated as zero.

Equations (1) and (2) are cost minimization conditions; equations (3) and (4) are equilibrium conditions for labor and capital goods. Also, (5) is a capital good accumulation equation.

Based on (3) and (4),

$$\tilde{y}(t) = b\tilde{k}(t) + b_{01}^\rho \tag{12}$$

Here, b^ρ and b_{10}^ρ are defined before as elements of the matrix B^ρ. Also, based on accumulation equation (5) and equation (12), we obtain the following difference equation:

$$\tilde{k}(t+1) = [b^\rho + (1-\delta)]\tilde{k}(t) + b_{10}^\rho. \tag{13}$$

By defining $\eta(t+1) = \tilde{k}(t) - \tilde{k}^\rho$, difference equation (3.2) is rewritten as

$$\eta(t+1) = [b^\rho + (1-\delta)]\eta(t) \tag{14}$$

It is clear that the behaviors of the path on the NMF can be obtained by investigating this differential equation.

Now, by making a suitable selection of units, it is possible to normalize the element

b^{ρ} of the matrix B^{ρ} as follows.

$$\left|b^{\rho}\right| = \left|a^{\rho}_{00} / (a^{\rho}_{00}a^{\rho}_{11} - a^{\rho}_{01}a^{\rho}_{10})\right| < 1 .$$

Since $b^{\rho} < 0$, it follows that $-1 < \left|b^{\rho} + (1-\delta)\right| < 1$. Therefore, the difference equation (14) will exhibit stability and the NMF will become a linear stable manifold. Let us introduce the following definition:

Definition. The NMF is ***stable*** if there are no cyclic paths on it.

Thus we have proved that the NMF is stable and we may prove the following turnpike property, which I leave for McKenzie (1983).

Theorem 1(Neighborhood Turnpike Theorem). Provided that the NMF is stable. Then for any $\varepsilon > 0$, there exists a $\overline{\rho}$ such that for $\rho \in [\overline{\rho}, 1)$ and the corresponding $\varepsilon(\rho)$, any optimal path $k^{\rho}(t)$ with a sufficient initial capital stock $k(0)$ eventually lies in the $\varepsilon-$*neighborhood* of k^{ρ}. Furthermore, as $\rho \to 1$, $\varepsilon(\rho) \to 0$.

The neighborhood turnpike theorem means that any optimal path must be trapped in a neighborhood of the corresponding OSS and the neighborhood can be taken as small as possible by making ρ sufficiently close to 1.

Note that by expanding the Euler equation around k^{ρ}, we have the following characteristic equation:

$$\left| V_{xz}^{\rho} \lambda^2 + (V_{xx}^{\rho} + V_{zz}^{\rho})\lambda + V_{zx}^{\rho} \right| = 0. \tag{15}$$

The local stability will be determined by this equation and the following property concerned with it is well-known.

Lemma . Provided that $V_{xz}^{\rho} \neq 0$, the characteristic equation (15) has λ as a root, then it also has $\dfrac{1}{\rho\lambda}$.

Proof. See Levhari and Liviatan (1976)■.

Since the NMF is stable, it implies that the one characteristic root along the NMF has an absolute value less than 1. In the two-sector model, V_{12}^{ρ} is calculated as follows (see in detail Benhabib and Nishimura (1985)).

$$V_{xz}^{\rho} = -((b^{\rho})^{-1})^2 \left(\frac{\partial w}{\partial k} \right) \left[b^{\rho} + (1-\delta) \right] \tag{16}$$

Based on this, Equation (16) does not become zero other than in $b^{\rho} = -(1-\delta)$. By assuming that $b^{\rho} \neq -(1-\delta)$, we can apply the Lemma and obtain that the both characteristics of Equation (15) have absolute values less than 1. This implies that the OSS $\tilde{k}^{\rho}$ is locally stable; any path near the OSS will converge to the OSS. Thus we have proved the following local stability concerned with the OSS:

Theorem 2 (Local Stability). The OSS $\tilde{k}^{\rho}$ *for* $\rho \in [\overline{\rho}, 1)$ satisfies the local stability.

Suppose that the optimal path $\tilde{k}^{\rho}$ would satisfy the neighborhood turnpike theorem; for any $\varepsilon > 0$, there exists a $\bar{\rho}$ such that for $\rho \in [\bar{\rho}, 1)$ and the corresponding $\varepsilon(\rho)$, any optimal path $\tilde{k}^{\rho}(t)$ with a sufficient initial capital stock $\tilde{k}(0) = \bar{k}$ eventually lies in the $\varepsilon - neighborhood$ of k^{ρ} and $\rho \to 1, \varepsilon(\rho) \to 0$. By choosing ρ close enough to 1 such that the local stability will also hold. We have now proved the following complete turnpike theorem:

Complete Turnpike Theorem. There is a $\bar{\bar{\rho}} > 0$ close enough to 1 such that for any $\rho \in [\bar{\bar{\rho}}, 1)$, an optimal path $\tilde{k}^{\rho}(t)$ with the sufficient initial capital stock $\tilde{k}(0) = \bar{k}$ will asymptotically converge to the optimal steady state $\tilde{k}^{\rho}$.

Note that the complete turnpike means that each sector's optimal path will converge to its own optimal steady state; $(\tilde{c}^{\rho}(t), \tilde{y}^{\rho}(t)) \to (\tilde{c}^{\rho}, \tilde{y}^{\rho})\ as\ t \to \infty$. It follows that in original model of series, sector's per-capita capital stock and output grow at the rate of sector's TFP growth: $c^{\rho}(t) = (1+\alpha_0)^t \tilde{c}^{\rho}\ and\ y^{\rho}(t) = (1+\alpha_1)^t \tilde{y}^{\rho}$. Thus our original purpose, stated below as the proposition, have accomplished by demonstrating the complete turnpike theorem.

Proposition (Unbalanced Growth). *Under our assumptions, each sector's optimal path converges to the own optimal steady state with a sector-specific TFP growth rate.*

5. Conclusion

We have demonstrated turnpike property under two types of generalized capital intensity conditions. As I mentioned before, the complete turnpike property means that each industry's per capita capital stock and output converge to the industry-specific optimal steady state paths with the rate of technical progress determined by industry's TFP. It means that, the per-capita capital stock of the agriculture industry grows at its own rate of technical progress along its optimal steady state and another industry, say the manufacturing industry grows at its own rate of technical progress along its own optimal steady state. A similar explanation can be applicable to other industries. Therefore, our established theoretical results are consistent with the evidence obtained in recent empirical research.

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Fig.1 Capital Intensities in Two sectors

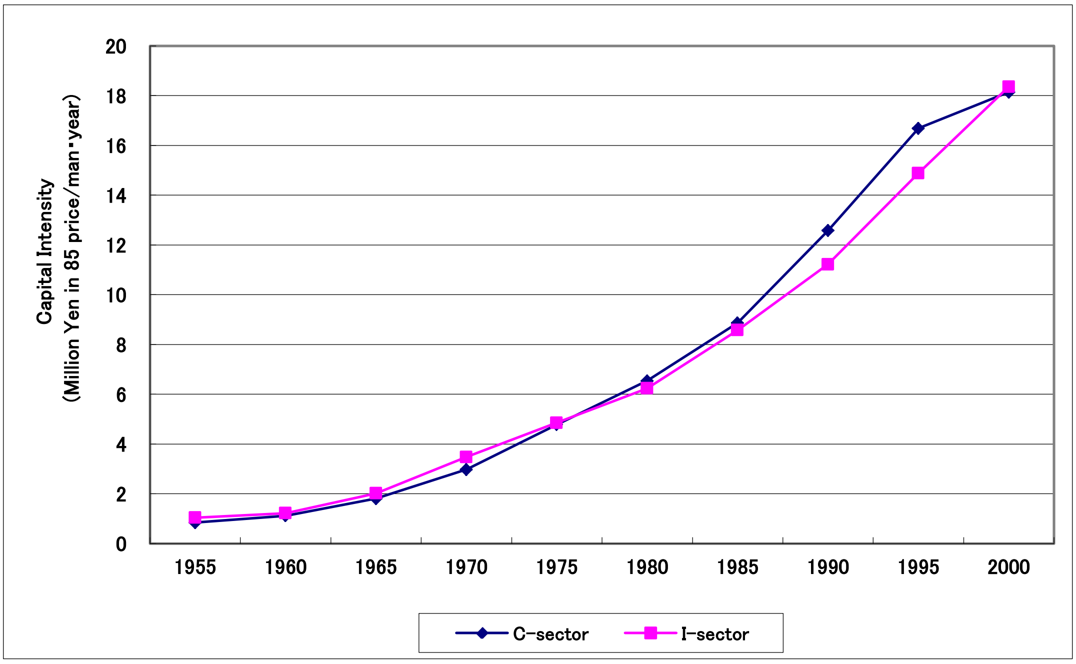


Fig.2: Capital Intensity Ratio in the Postwar Japanese Economy

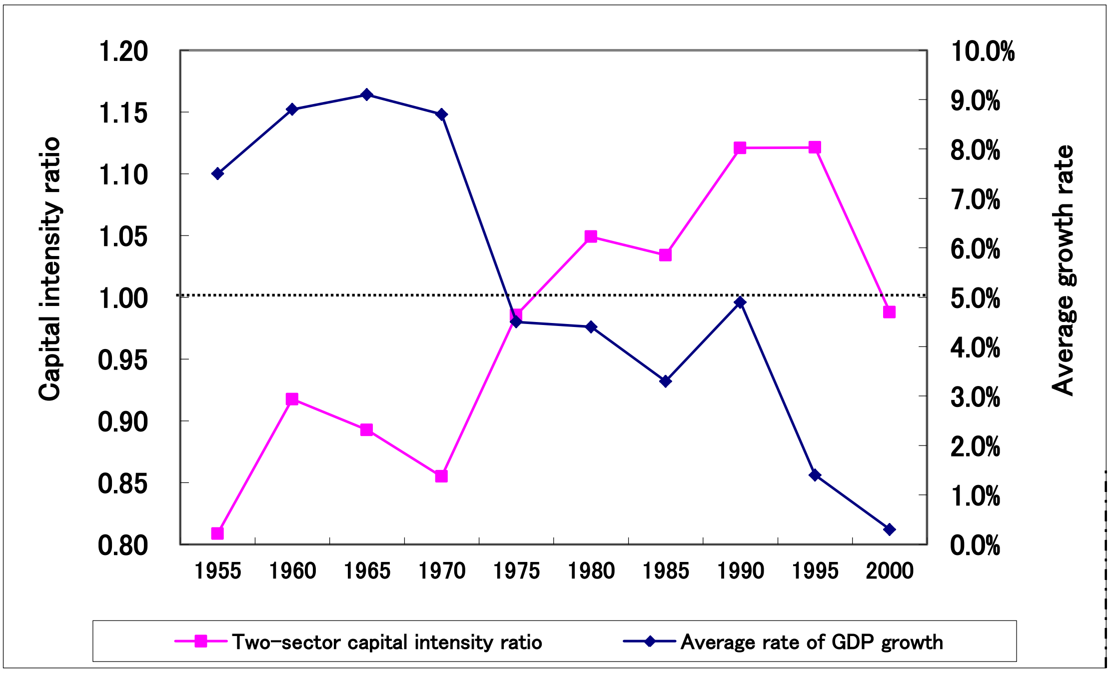


Fig.3: Capital-intensity Ratios in OECD Countries

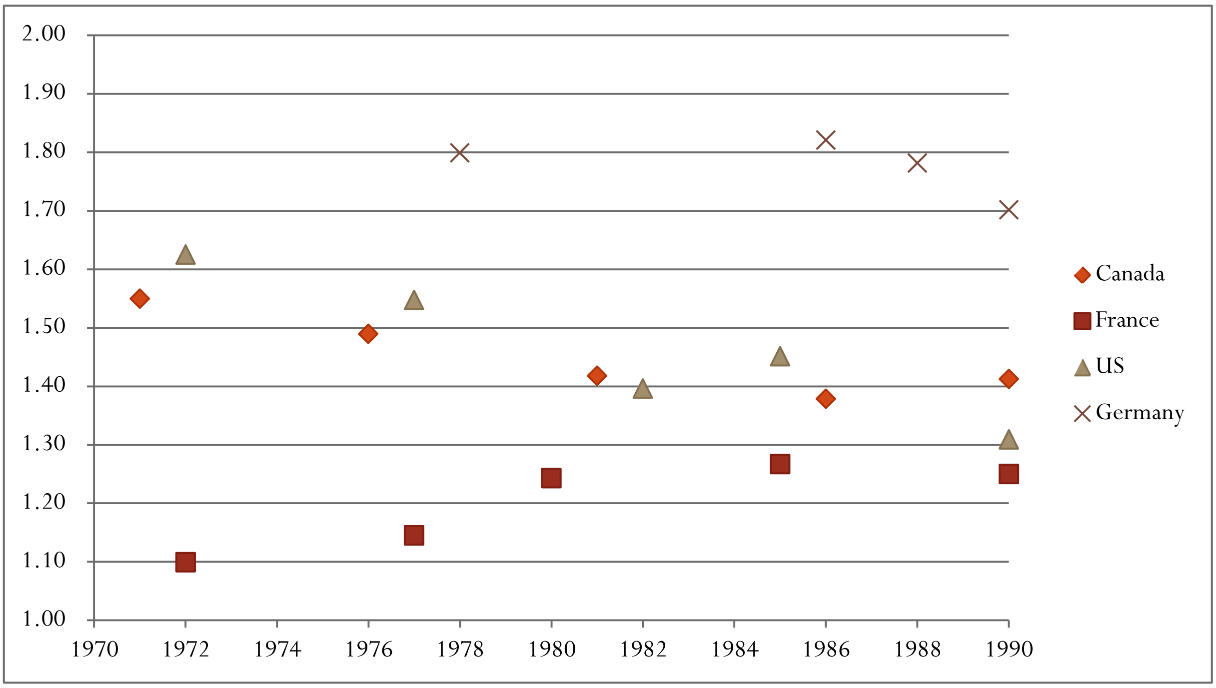
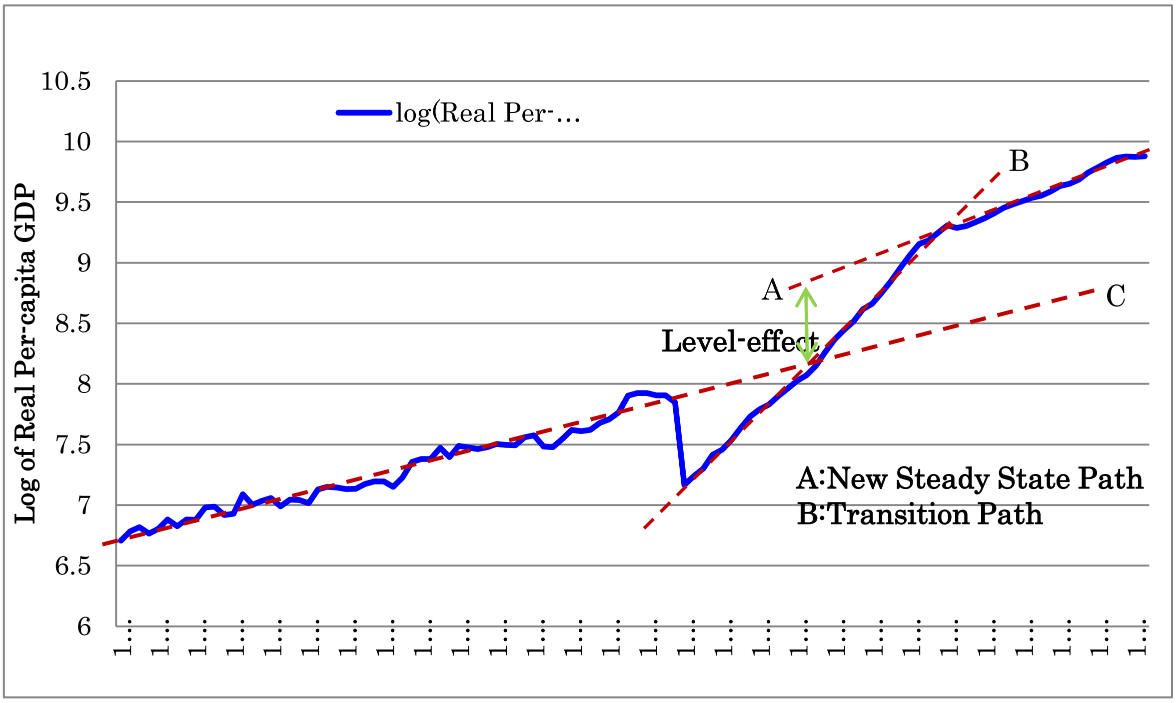
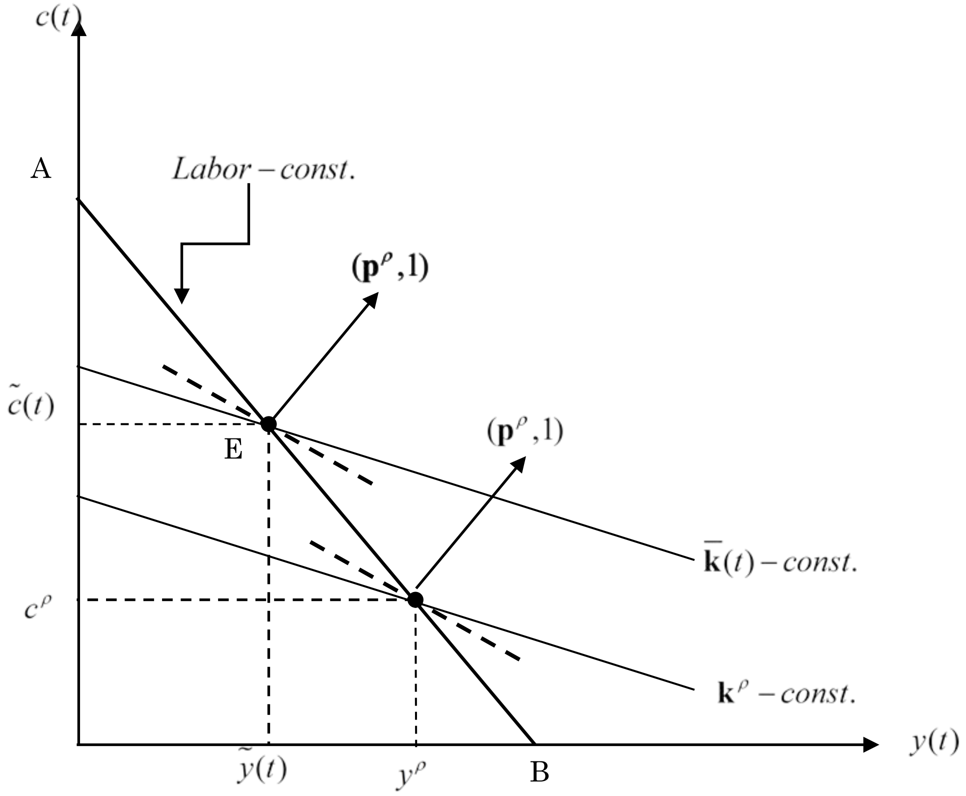


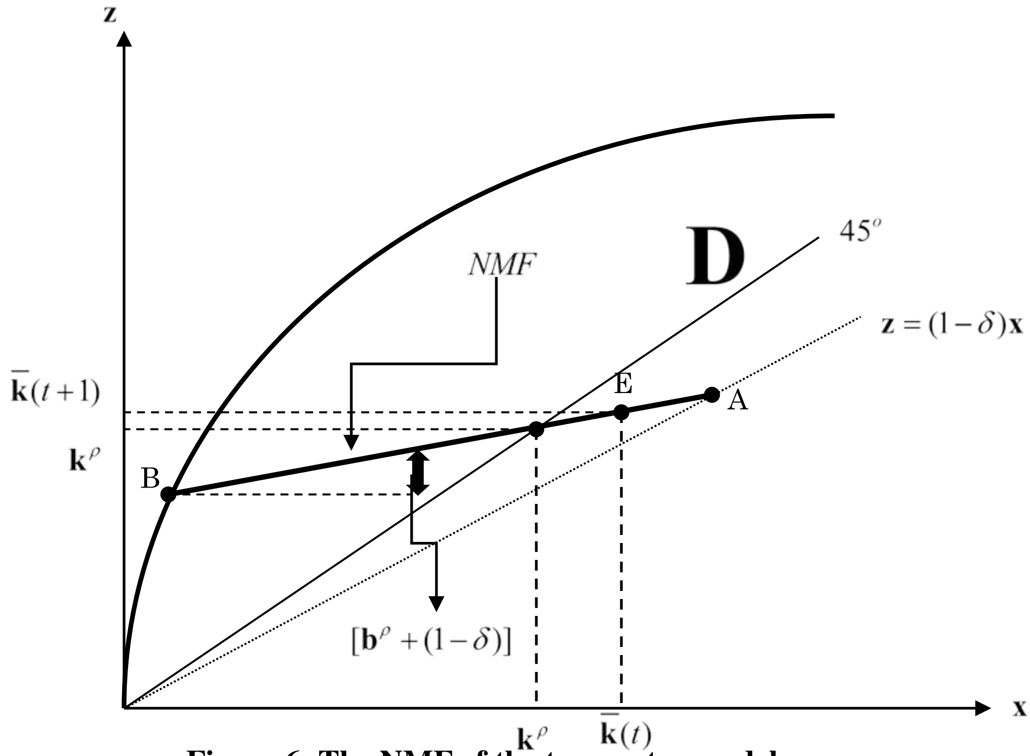
Fig. 4: Japanese Per-capita Real GDP (1885-1994)



Source: Data from Maddison (1995, Table C-18)



-Figure 5: Derivation of the NMF in the two-sector model-



-Figure 6: The NMF of the two-sector model-