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Abstract

We introduce the sequential equal surplus division for sharing the total welfare resulting form the cooperation of agents along a river with a delta. This allocation rule can be seen as a generalization of the contribution vectors introduced by Ju, Borm and Ruys [2007] in the context of TU-games. We provide two axiomatic characterizations of the sequential equal surplus division.

Keywords: Amalgamation – Consistency – Fairness – Sequential Equal Surplus Division – Sharing a river.

JEL Classification number: C71, D63, D74, Q25, Q56.

1 Introduction

$$L^{\rightarrow}$$

In this article, we consider the distribution of a scarce resource among a set of agents that are partially ordered. This kind of situations is particularly well illustrated by the problem of sharing water from a river with a delta among a group of agents located along this river. Between each pair of neighbors along the river there is an additional flow of water that flows from upstream to downstream. Water inflow at the territory of downstream agents cannot be consumed by upstream agents and agents located at other branches of the river. The benefit of each agent depends on the amount of water he consumes at his location. An optimal water allocation maximizes the total benefits. Cooperation among the riparian agents is often necessary to achieve the optimal allocation of water since the upstream agents may have to let pass some water to the downstream agents if the latter have a higher marginal benefit from consuming water. Monetary transfers among the agents can sustain the optimal allocation of water. A welfare distribution yields a utility for each agent, which is equal to his benefit from the water consumption plus his monetary transfer. Historically, distributing the optimal welfare that arises from the river water consumption has been at the center of international tensions and conflicts. The international law suggests several equity notions as guidelines for resolving international water disputes. Unfortunately, as explained in Kilgour and Dinar (1995)

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and Wolf (1999), these principles are often difficult to apply and even contradictory, especially for river, which constrain transfers because of the direction of the flow.

One approach to the problem of finding a fair welfare distribution is to model the situation as a cooperative digraph TU-game as in Ambec and Sprumont (2002) and Khmelnitskaya (2010). The river with a delta and the location of each agent along the river are highlighted by a digraph on the set on riparian agents. The worth of each connected coalition is given by the social welfare that it can secure for itself without the cooperation of the agents located upstream and at different branches of the river. The worth of each other coalition is defined by the sum of the worths of their maximally connected parts.

Different allocation rules for distributing the social welfare resulting from the optimal water allocation in digraph TU-games are discussed in the literature. These rules are characterized by translating into axioms the above-mentioned principles of international law. The views expressed in the three doctrines advocated in international disputes are the efficiency of water use, the Absolute Territorial Sovereignty (ATS for short) and Territorial Integration of all Basin States (TIBS for short). Efficiency of the water use means that the optimal social welfare is fully shared among the agents. The ATI doctrine states that a country has absolute sovereignty over the area of any river basin on its territory. The TIBS doctrine attributes an equal use of water to all agents whatever their contribution to the flow. Within the cooperative game theoretical framework, efficiency of the water use is obviously the usual property of efficiency of the allocation rule. The combination of efficiency and the ATS doctrine has been very often translated into the property that the welfare distribution should be core-stable. The various game-theoretical interpretations of TIBS accords different shares of the surplus resulting from the cooperation of a coalition of agents.

In this article we introduce an allocation rule, the sequential equal surplus division as we call it, which relies on a weaker interpretation of the ATS doctrine and a more flexible interpretation of the TIBS doctrine. In particular, we would like to insist on four features of the sharing river problem.

First, although the core-stability is a nice requirement for an allocation rule, it is too strong if one sticks to the original statement of the ATS doctrine. In fact, the ATS doctrine only requires a stability condition for every singleton coalitions. The sequential equal surplus division satisfies the weaker interpretation of the ATS doctrine that each individual agent should get a payoff at least equal to the welfare he can secure without the cooperation of any other agent. The sequential equal surplus division is not core-stable but satisfies the intuitive stability property that a coalition containing an agent and all the agents located downstream obtains at least the welfare it can achieve without the cooperation of the other agents. This property is very similar to the subsidy-free property used in Aadland and Kolpin (1998).

Second, we believe that a minimal requirement that an allocation rule should satisfies is to coincide with the standard solution in the two-agent case. This means that if the river flows through two countries, then each of them should get the welfare resulting from consuming the inflow entering its territory plus a half of the welfare surplus resulting from the cooperation of the two countries. The reason is that the participation of each country is equally important in the achievement of the optimal water consumption whatever its location on the river. The allocation rules proposed so far in the literature, namely the downstream incremental distribution introduced in Ambec and Sprumont (2002), the upstream incremental distribution studied in van den Brink *et al.* (2007) for rivers without a delta, their generalization to rivers with a delta or multiple springs examined by Khmelnitskaya (2010) and the weighted hierarchical solutions studied in van den Brink *et al.* (2012), violate this very basic and intuitive requirement. On the contrary, it is satisfied by the sequential equal surplus division introduced in this article.

Third, we aim at generalizing the well-accepted principle of the standard solution to n-agent sharing river situations. The construction of the sequential equal surplus division is done sequentially by following the natural direction of the river's flow. At the spring/root of the river, the water can possibly flow through several branches. Together with the groups of agents located on each of these branches, the agent located at the spring of the river achieves some surplus compared to the situation in which each branch would behave selfishly. This surplus is precisely the difference between the total welfare and the sum of the welfare achieved separately by the rooted agent and each of the downstream branches. In order to distribute this surplus, the rooted agent as well as the cooperating coalitions corresponding to the downstream branches of the river can be considered as single entities. As in the two-agent case, the participation of each entity is necessary for the production of the welfare surplus. Therefore, it seems natural to give to each entity an equal share of the surplus in addition to the welfare it can guarantee in the absence of cooperation. This is equivalent to reward each entity by the well-known equal surplus division for TU-games. For each remaining branch of the river, the obtained total payoff is what remains to be shared among its members. This payoff is in the spirit of the individual standardized remainder vector introduced by Ju *et al.* (2007) for the class of TU-games. The sequential equal surplus division then consists in applying repeatedly the above step to all agents located at the roots of the remaining branches considered as single entities. The distribution of the surplus of cooperation between two or more coalitions of agents is also at the heart of the construction of many of the solutions proposed in the literature. Either this surplus is fully allocated to only one agent as in the downstream incremental distribution, the upstream incremental distribution and their generalizations or it is shared on a more egalitarian basis as in the equal loss property solution studied in van den Brink *et al.* (2007). In any of these solutions, only the most upstream agent of the downstream coalition and the most downstream agent of the upstream coalition get a share of the surplus. Contrary to the sequential equal surplus division, these solutions somehow ignore the crucial role of the other participating agents in the coalitions located on the downstream branches.

Fourth, since the sharing river problem is represented by a digraph TU-game, we think that it is desirable to adapt several well-known principles used in cooperative (graph) TU-games to account for one-directionality of the river's flow. Such desirable principles include consistency, fairness and amalgamation considerations as studied, among others, by Thomson (2011), Myerson (1977) and Lehrer (1988) respectively. The first and the third principles are left out of the existing literature[1]. In this article, they are adapted to digraph TU-games and are employed to provide two axiomatic characterizations of the sequential equal surplus division.

The rest of the article is organized as follows. Section 2 is devoted to the game-theoretical definitions. The river sharing problem and its associated digraph TU-game are defined in section 3. Section 4 introduces the sequential equal surplus division and states a first result which describes some of its properties. Several properties for an allocation on the class of all digraph TU games are presented in section 5. Section 6 contains the two axiomatic characterizations.

[1]Ansink and Weikard (2012) is an exception, but they deal with the sharing river problem from a bankruptcy point of view. Furthermore, they only consider rivers without a delta, *i.e.* rivers shaped like a line.

2 Preliminaries : TU-games on directed graphs

For any finite set A, the notation $|A|$ stands for the cardinality of A. The inclusion relation is denoted by $\subseteq$ and its strict part by $\subset$.

A *cooperative game* with transferable utility (TU-game) is a pair (N, v) consisting of a finite player (or agent) set N of size $n \in \mathbb{N}$ and a coalition function $v : 2^N \longrightarrow \mathbb{R}$ satisfying $v(\emptyset) = 0$. An element S of 2^N is a coalition, and $v(S)$ is the maximal worth that the members of S can obtain by cooperating. The class of all TU-games is denoted by $\mathcal{C}$. For each nonempty coalition $S \in 2^N$, the *subgame* of (N, v) induced by S is the TU-game $(S, v_{|S})$ such that for any $T \in 2^S$, $v_{|S}(T) = v(T)$. A TU-game (N, v) is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for any pair of disjoint coalitions S and T. In TU-game $(N, v) \in \mathcal{C}$ each player $i \in N$ may receive a payoff $z_i \in \mathbb{R}$. A *payoff vector* $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ lists a payoff z_i for each $i \in N$. For any nonempty coalition $S \in 2^N$ the notation z_S stands for $\sum_{i \in S} z_i$. An *allocation rule* Φ on $\mathcal{C}$ is a map that assigns to each TU-game $(N, v) \in \mathcal{C}$ a payoff vector $\Phi(N, v) \in \mathbb{R}^n$.

The cooperation structure on the player set N is specified by a graph. A *graph* on N is a subset L of all unordered pairs $\{\{i, j\} \subseteq N : i \neq j\}$. Each element $\{i, j\}$ of L is called a link with ends i and j. We will use the shorthand notation ij to designate a link $\{i, j\} \in L$. The links of the graphs represent the communication possibilities between pairs of players and thus affect the formation of coalitions. A *path of length* k in (N, L) is a finite sequence of distinct players $(i_0, i_1, \ldots, i_k)$ if $i_q i_{q+1} \in L$ for each $q \in \{0, \ldots, k-1\}$. A *cycle* in (N, L) is a finite sequence of players $(i_0, i_1, \ldots, i_k, i_0)$ such that $k \geq 2$, $(i_0, i_1, \ldots, i_k)$ is a path and $i_k i_0 \in L$. A graph (N, L) is *acyclic* if it does not contain any cycle. For each subset of players $C \subseteq N$, the subset of links $L(C) \subseteq L$ represents the set of communication links between players in C, and the pair $(C, L(C))$ is the *subgraph* of (N, L) induced by C. The induced subgraph $(C, L(C))$ of a graph (N, L) is *connected* if there exists at least one path between any pair of distinct players in C. By extension we say that coalition C is connected in (N, L). The induced subgraph $(C, L(C))$ is maximally connected if it is connected and for each $i \in N \backslash C$ the subgraph $(C \cup \{i\}, L(C \cup \{i\}))$ of (N, L) is not connected. If $(C, L(C))$ is a maximally connected subgraph of (N, L), C is a *component* of (N, L). Denote by N/L the set of components of (N, L) and by $C/L(C)$ the set of components of the subgraph $(C, L(C))$ of (N, L) induced by $C \subseteq N$. Note that if (N, L) is acyclic, then there is exactly one path between any pair of distinct players who belong to the same component. An acyclic graph is called a *forest*. If the forest has exactly one component, it is called a *tree*. Therefore, each component of a forest induces a tree. It follows from these remarks that the class of acyclic graphs on N is closed under link deletion, *i.e.* if (N, L) is a forest or a tree and $ij \in L$, then the graph $(N, L \backslash ij)$ is a forest on N.

A digraph $\vec{L}$ on N is a graph L on N in which each link is assigned a direction, one end being designed its tail and the other its head. An edge with tail i and head j is called a directed link and is denoted by $i \to j$. We also say that i is a *predecessor* of j and j is a *successor* of i in $\vec{L}$. A digraph reflects the idea that two players incident to a communication link do not have equal access or control to that link.

Given a forest (N, L), a *rooted tree* $\vec{L}^r(C)$ on the induced subgraph $(C, L(C))$ is a digraph on C that arises from a component $C \in N/L$ by selecting player $r \in C$, called the *root*, and directing all links of $L(C)$ away from the root r. Note that each agent $r \in C$ is the root of exactly one rooted tree $\vec{L}^r(C)$. Note also that for any rooted tree $\vec{L}^r(C)$ on (N, L) and any agent $i \in C \backslash \{r\}$, there is exactly one directed link $j \to i$; agent j is the unique predecessor of i and i is a successor of j in $\vec{L}^r$. Denote by $p_r(i)$ the unique predecessor of agent $i \neq r$ and by

$s_r(i)$ the possibly empty set of successors of player i in $\vec{L}^r$. We will also use the notation $s_r[i]$ to denote the union of $s_r(i)$ and $\{i\}$. All players having the same predecessor are called brothers. We denote by $B_r(i)$ the set of all brothers of i in $\vec{L}^r(C)$. A player j is a *subordinate* of i in $\vec{L}^r(C)$ if there is a directed path from i to j, *i.e.* if there is a sequence of distinct agents $(i_0, i_1, \ldots, i_l)$ such that $i_0 = i$, $i_l = j$ and for each $q \in \{0, 1, \ldots, l-1\}$, $i_{q+1} \in s_r(i_q)$. The set $S_r[i]$ denotes the union of the set of all subordinates of i in $\vec{L}^r$ and $\{i\}$. So, we have $s_r(i) \subseteq S_r[i] \backslash \{i\}$. If a player j is a subordinate of i, then we say that i is a *superior* of j. The set $P_r[i]$ denotes the union of the set of all superiors of i in $\vec{L}^r$ and $\{i\}$. So, we have $p_r(i) \in P_r[i] \backslash \{i\}$ for each $i \neq r$. The *depth* of an agent $i \in C$ is the length of the directed path from r to i. The *depth of the tree* is the depth of its deepest agents. For an illustration of these definitions, see Figure 1 on Section 3.1. For the sake of notations, we will use $\vec{L}$ to denote a digraph in which each component is a rooted tree and we will precise the designated component C and root r when necessary.

A digraph TU-game is a TU-game $(N, v^{\vec{L}}) \in \mathcal{C}$ in which the coalition function $v^{\vec{L}}$ takes explicitly into account the restrictions on coalition formation induced by $\vec{L}$. More precisely, $v^{\vec{L}}$ is the graph-restricted game introduced by Myerson (1977) where the worth of each non connected coalition is equal to the sum of the worths of its components. Formally, $v^{\vec{L}}$ assigns to each connected coalition S a worth $v^{\vec{L}}(S)$, and for each other coalition S, $v^{\vec{L}}(S) = \sum_{T \in S/L(S)} v^{\vec{L}}(T)$. Denote by $\mathcal{C}_0$ the class of all digraph TU-games where the underlying digraph is a collection of rooted trees.

3 The river sharing problem

3.1 Sharing a river with a delta

Consider a finite set N of agents of size $n \in \mathbb{N}$ located along an international river. At the location of each agent $i \in N$, rainfall and inflow from tributaries increase the total river flow by $e_i \in \mathbb{R}_+$. Water inflow at the territory of downstream agents cannot be consumed by upstream agents. Following Khmelnitskaya (2010), we assume that the river can have a delta. Under these assumptions, the river is shaped like a rooted tree. More precisely, consider first an undirected tree as in Figure 1 (a). A rooted-tree arises from it by selecting an agent, called the root, and directing all edges away from this root. Then add the inflow of water on each agent location in order to obtain a river with delta as in Figure 1 (b) where the root 1 is the spring of the river.

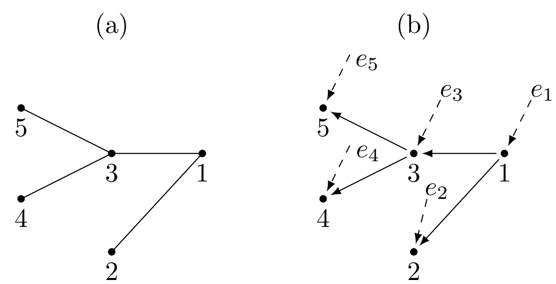

Figure 1: *(a) An undirected tree L – (b) A river with a delta $\vec{L}$.*

Example 1 *To illustrate the different notions introduced in Section 2 consider the rooted tree of Figure 1 (b). The set of successors of agent 1 is $s_1(1) = \{2, 3\}$, the set of its subordinates* *including himself is* $S_1[1] = \{1,2,3,4,5\}$*, whereas the set of successors of agent 3 is* $s_1(3) = \{4,5\}$ *and* $S_1[3] = s_1[3]$*. Agent 1 has no predecessor and the predecessor of agent 3 is agent 1:* $p_1(3) = 1$*. The set of superiors of agent 4 including himself is* $P_1[4] = \{1,3,4\}$ *and set of superiors of agent 5 including himself* $P_1[5] = \{1,3,5\}$*. Agents 4 and 5 are brothers since they have the same predecessor:* $p_1(4) = p_1(5) = 3$*. We thus have* $B_1(4) = \{5\}$ *and* $B_1(5) = \{4\}$*. This is also true for agents 3 and 2. We have* $B_1(2) = \{3\}$ *and* $B_1(3) = \{2\}$*. The deepest agents in the rooted tree are 4 and 5. Their depth is equal to 2.* □

Each agent $i \in N$ is endowed with a benefit function $b_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$ that assigns to each amount $x_i \in \mathbb{R}_+$ of water consumed the benefit $b_i(x_i) \in \mathbb{R}$.

Assumption 1 (A1). The water inflow e_1 is strictly positive. Each benefit function b_i is strictly increasing, strictly concave, continuously differentiable on $\mathbb{R}_{++}$, $b_i(0) = 0$, and the first derivative $b_i^{(1)}(x_i)$ tends to $+\infty$ as x_i vanishes.

A consumption plan $x = (x_i)_{i \in N} \in \mathbb{R}^n_+$ satisfies the following constraints:

$$\forall k \in N, \sum_{j \in P_r[k]} x_j \leq \sum_{j \in P_r[k]} e_j, \text{ and } \sum_{j \in P_r[k] \cup B_r(k)} x_j \leq \sum_{j \in P_r[k] \cup B_r(k)} e_j. \tag{1}$$

The first constraints in (1) indicate that each agent $k \in N$ consumes at most the sum of the water inflow at his location and the water inflows not consumed by his upstream agents $l \in N$. The second constraints in (1) indicate that the set of brothers of any agent $k \in N$ including himself do not consume more water than the total inflow of water available to them. A consumption plan x induces the social welfare $\sum_{i \in N} b_i(x_i)$; x is optimal if it maximizes the social welfare. Under (A1) there is a unique optimal consumption plan.

In order to reach the optimal consumption plan, some agents may refrain from consuming water. In turn, monetary compensations can be set up for these agents, which allow Pareto improvements. More precisely, money is available in unbounded quantities to perform side-payments. A compensation for agent $i \in N$ is an amount of money $t_i \in \mathbb{R}$. If $t_i > 0$, then agent i is a net beneficiary of monetary transfers; if $t_i < 0$, then agent i is a net contributor to monetary transfers. A compensation scheme is a profile $t = (t_i)_{i \in N} \in \mathbb{R}^n$ of monetary transfers satisfying the budget constraint $\sum_{i \in N} t_i \leq 0$. A budget constraint is balanced if transfers add to zero. Agents value both water and money and are endowed with a quasi-linear utility function $u_i : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ that assigns to each pair $(x_i, t_i) \in \mathbb{R}_+ \times \mathbb{R}$ the utility $u_i(x_i, t_i) = b_i(x_i) + t_i$. A welfare distribution is a pair (x, t) where x is a consumption plan and t is a compensation scheme. The welfare distribution (x^*, t) is Pareto optimal if and only if x^* is the optimal consumption plan and the budget constraint is balanced. Each agent i receives the payoff $z_i = u_i(x_i^*, t_i)$, and the sum of these payoffs is equal to the optimal social welfare $\sum_{i \in N} b_i(x_i^*)$. The problem is to find a fair distribution of this social welfare.

3.2 The river TU-game

One way to solve a river sharing problem is by treating it as a digraph TU-game. Ambec and Sprumont (2002) and Khmelnitskaya (2010) construct the following digraph TU-game from a river sharing problem. The player set N is the set of riparian agents and the digraph $\vec{L}$ is the set of arcs which connect the agents along the river. Due to the direction of the river's flow, each arc $i \to j$ is controlled by the tail i. Indeed, by refusing to pass water to j, agent i does not cooperate with j. The definition of the coalition function $v^{\vec{L}}$ takes into account this fact. Since the total worth to distribute is equal to the optimal social welfare, the worth of the grand coalition N is given by $v^{\vec{L}}(N) = \sum_{i \in N} b_i(x_i^*)$. Further a group of agents on the river can cooperate if and only if it is connected in the undirected tree L. The worth of each connected coalition S is given by the social welfare that it can secure for itself without the cooperation of the agents who are not a member of this coalition. Therefore, for any connected coalition S in $\vec{L}$, the worth $v^{\vec{L}}(S)$ is given by:

$$v^{\vec{L}}(S) = \sum_{k \in S} b_k(x_k^S),$$

where $x^S = (x_i^S)_{i \in S}$ solves

$$\max \sum_{k \in S} b_k(x_k) \text{ s.t. } \forall k \in S,$$

$$\sum_{j \in P_r[k] \cap S} x_j \leq \sum_{j \in P_r[k] \cap S} e_j, \qquad \sum_{j \in P_r[k] \cup B_r(k) \cap S} x_j \leq \sum_{j \in P_r[k] \cup B_r(k) \cap S} e_j. \tag{2}$$

Assumption (A1) ensures that there is a unique solution to this program. If the coalition is not connected, then its worth is the sum of the worths of its connected components. If $S \in 2^N$ is not connected, then it admits a unique partition into components. Agents belonging to $N \backslash S$ do not cooperate with S and act non-cooperatively. Since each benefit function is strictly increasing in the water consumed, each agent located between two components of S will consume all the water inflow entering at his location. Therefore, a component of S will never receive the water left over by another component located upstream. It follows that the worth of a non connected coalition S is the sum of the worths of its components and so the river TU-game $(N, v^{\vec{L}})$ belongs to $\mathcal{C}_0$. Note also that a river TU-game is superadditive.

By allocating a payoff z_i to each $i \in N$, we determine a compensation scheme t as follows: for each $i \in N$, $t_i = z_i - b_i(x_i^*)$. If $z_N = v^{\vec{L}}(N)$, then the optimal social welfare is integrally allocated among the agents and t is budget balanced. The difficulty is thus to find an agreement on the allocation of the social welfare resulting from an optimal consumption plan, which in turn determines monetary compensations.

4 The sequential equal surplus division

Sharing river water has often been at the center of upstream-downstream tensions and conflicts. Many principles of international law have been developed to prevent or resolve international water disputes. Unfortunately, they are not easy to apply and often are contradictory because the effects are one-way and the property rights over water are not well defined (Wolf, 1999, Kilgour and Dinar, 1995). The first principle is the efficiency of water use. It indicates that the total welfare resulting from an implementation of the optimal consumption plan is fully redistributed among the riparian agents. The second principle listed in Kilgour and Dinar (1995) is the doctrine of "Absolute Territorial Sovereignty" (ATS for short), often initially claimed by upstream agents. This doctrine argues that an agent has absolute rights to water flowing through its territory. If we extend this principle to each group of agents, this doctrine implies that each group of agents receive a total payoff at least equal to the worth they can obtain by agreeing to cooperate. Translated into game theoretical terms, the combination of water use efficiency and the extended interpretation of the ATS doctrine implies core stability. The third principle listed in Kilgour and Dinar (1995) is the doctrine of "Territorial Integration of all Basin States" (TIBS). Symmetrically, this principle favors downstream agents to which it accords "equal" use, without regard to their contribution to the flow. Ambec and Sprumont (2002) apply an extreme case of the TIBS doctrine, called the "Absolute Territorial Integrity" (ATI). It aims at protecting downstream agent by stating that each agent has the right to all upstream water. Translated into game theoretical terms, this doctrine indicates that each coalition of agents should get a total payoff as least equal to the worth that it can secure for itself when the upstream agents let pass all the water flow. In the context of a river without a delta – the river is shaped like a line – Ambec and Sprumont (2002) show that there is a unique efficient allocation rule that satisfies both the ATS and ATI doctrines. van den Brink *et al.* (2007) provide an axiomatic characterization of this solution. They also discuss this solution and note that it favors too much the downstream agents. Indeed, upstream agents can stop the cooperation with the downstream agents by consuming their total water inflow and so can claim much more than they get under the solution designed by Ambec and Sprumont (2002).

In this article we look at solutions that can be motivated by the ATS and TIBS doctrines, but with a more flexible interpretation of the TIBS doctrine. The general TIBS doctrine states that given an agent, this agent and his downstream neighbors are entitled to receive a share of the surplus they generate when they decide to cooperate. A very popular allocation rule that satisfies these two principles in a broader sense for the the two-agent situation is the standard solution. It assigns first to an agent his stand-alone worth and then distributes an equal share of the left surplus created by the cooperation. Rephrased in terms of the river sharing problem, the standard solution assigns first to each agent the benefit of consuming the water inflow entering his own territory and then allocates half of the surplus that can be created by the two agents when they coordinate their water consumption. That is, agent $i \in \{1, 2\}$ gets:

$$v^{\vec{L}}(\{i\}) + \frac{v^{\vec{L}}(\{12\}) - v^{\vec{L}}(\{2\}) - v^{\vec{L}}(\{1\})}{2}$$

Since the river TU-game is superadditive, $v^{\vec{L}}(\{12\}) - v^{\vec{L}}(\{2\}) - v^{\vec{L}}(\{1\}) \geq 0$, which implies that each agent gets at least his stand-alone payoff: the ATS principle is well satisfied. On the other hand, since the surplus resulting from cooperation is equally distributed among them, the TIBS principle is also satisfied. That's why we think that a desirable property for a solution on the class of river TU-games is that it implements the standard solution in the two-agent situation. In section 6, we will translate this requirement into a property.

We would like to generalize this principle from the two-agent situation to the situation where n agents are located along a river with a delta. There is a major difference between a river shaped like a line (Ambec and Sprumont, 2002) and a river with a delta (Khmelnitskaya, 2010). When the river is shaped like a line, agents are totally ordered by the direction of the river's flow. In a river with a delta, the direction of the river's flow generates a partial order over the set of agents. Any pair of brothers in a rooted tree have the same position in the sense neither is the subordinate of the other. Therefore, the sequential nature of the individual decisions is more complex. We incorporate this characteristic to our solution.

So, assume that a set of agents N of size $n > 2$ are located along a river with a delta. Denote by r its spring or root. Pick any agent $i \in N$ and focus on the welfare produced by this agent and all his subordinates. The worth $v^{\vec{L}}(S_r[i])$ is jointly created by agent i and the existing coalitions $S_r[j]$ for each $j \in s_r(i)$. The reason is obvious: on the one hand, agent i can threaten to consume all the water inflow available to him instead of letting some water flows out toward his subordinates; but on the other hand, his subordinates may need to get extra water from i in order to achieve an optimal consumption plan among themselves. In other words, agent i and each coalition of agents $S_r[j]$ for each $j \in s_r(i)$ have to cooperate in order to produce the total worth $v^{\vec{L}}(S_r[i])$. In the absence of cooperation, the total welfare achieved by these agents reduces to:

$$v^{\vec{L}}(\{i\}) + \sum_{j \in s_r(i)} v^{\vec{L}}(S_r[j]).$$

If one considers each coalition $S_r[j]$ as a single entity in the problem of distributing the total welfare $v^{\vec{L}}(S_r[i])$, then it is natural to apply the equal surplus division to such a $|s_r[i]|$-agent situation. We have:

$$v^{\vec{L}}(\{i\}) + \frac{1}{|s_r[i]|}\left(v^{\vec{L}}(S_r[i]) - v^{\vec{L}}(\{i\}) - \sum_{j \in s_r(i)} v^{\vec{L}}(S_r[j])\right)$$

to player i and, for each $j \in s_r(i)$,

$$v^{\vec{L}}(S_r[j]) + \frac{1}{|s_r[i]|}\left(v^{\vec{L}}(S_r[i]) - v^{\vec{L}}(\{i\}) - \sum_{j \in s_r(i)} v^{\vec{L}}(S_r[j])\right).$$

The allocation rule that we construct consists in a sequential application of the equal surplus division principle from upstream to downstream, which means that for agent i and his subordinates, it might remain more than $v^{\vec{L}}(S_r[i])$ to share. Before defining formally this allocation rule, we provide an example.

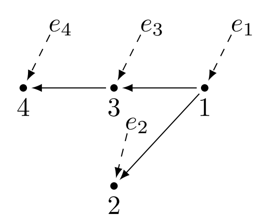

Figure 2: *A river with a delta $\vec{L}$.*

Example 2 *Consider the river represented by the rooted tree $\vec{L}$ in Figure 2. Initially the four agents cooperate with each other and the welfare $v^{\vec{L}}(\{1,2,3,4\})$ is to be distributed. The agents receive their payoff sequentially according to their distance to the spring of the river. We compute the payoff of agent 1 first on the basis of what would happen if he refuses to cooperate. In such a case, the grand coalition would partition into three coalitions: $\{1\}$, $\{2\}$ and $\{3,4\}$, where $\{3,4\}$ acts as single entity. How much should agent 1 get from the worth $v^{\vec{L}}(\{1,2,3,4\})$ as he refuses to cooperate? No doubt these three entities should first get the total worth they can secure without the cooperation of the other agents, i.e. $v^{\vec{L}}(\{1\})$, $v^{\vec{L}}(\{2\})$ and $v^{\vec{L}}(\{3,4\})$, respectively. As for the surplus $v^{\vec{L}}(\{1,2,3,4\}) - v^{\vec{L}}(\{1\}) - v^{\vec{L}}(\{2\}) - v^{\vec{L}}(\{3,4\})$, one can argue that since agent 1 is now negotiating with coalition $\{2\}$ and with coalition $\{3,4\}$ as a whole and the surplus is jointly created by these three parties, the equal surplus division* *for 3-agent TU-games can be applied so that each party should get a third of the joint surplus. Consequently, agent 1 obtains his individual worth plus a third of the the joint surplus:*

$$v^{\vec{L}}(\{1\})+\frac{1}{3}\Big(v^{\vec{L}}(\{1,2,3,4\})-v^{\vec{L}}(\{1\})-v^{\vec{L}}(\{2\})-v^{\vec{L}}(\{3,4\})\Big).$$

The welfare shares remaining for coalitions $\{2\}$ and $\{3,4\}$, which we call the egalitarian remainders for $\{2\}$ and $\{3,4\}$, are given respectively by:

$$v^{\vec{L}}(\{2\})+\frac{1}{3}\Big(v^{\vec{L}}(\{1,2,3,4\})-v^{\vec{L}}(\{1\})-v^{\vec{L}}(\{2\})-v^{\vec{L}}(\{3,4\})\Big) \tag{3}$$

and

$$v^{\vec{L}}(\{3,4\})+\frac{1}{3}\Big(v^{\vec{L}}(\{1,2,3,4\})-v^{\vec{L}}(\{1\})-v^{\vec{L}}(\{2\})-v^{\vec{L}}(\{3,4\})\Big).$$

Then, the above procedure has to be repeated on each branch of the river with respect to the direction of the river's flow. Thus both remaining coalitions $\{2\}$ and $\{3,4\}$ behave independently on their own branch of the river in order to negotiate the allocation of their respective remainder. On the one hand, agent 2 is the unique agent on his branch of the river so that the final payoff of 2 is given by (3). On the other hand, agent 3 can now be considered as the spring of the branch of the river on which he is located. As above, the egalitarian remainder for agent $\{4\}$ will be:

$$v^{\vec{L}}(\{4\})+\frac{1}{2}\bigg(v^{\vec{L}}(\{3,4\})+\frac{1}{3}\Big(v^{\vec{L}}(\{1,2,3,4\})-v^{\vec{L}}(\{1\})-v^{\vec{L}}(\{2\})-v^{\vec{L}}(\{3,4\})\Big)-v^{\vec{L}}(\{3\})-v^{\vec{L}}(\{4\})\bigg),$$

while agent 3 will get:

$$v^{\vec{L}}(\{3\})+\frac{1}{2}\bigg(v^{\vec{L}}(\{3,4\})+\frac{1}{3}\Big(v^{\vec{L}}(\{1,2,3,4\})-v^{\vec{L}}(\{1\})-v^{\vec{L}}(\{2\})-v^{\vec{L}}(\{3,4\})\Big)-v^{\vec{L}}(\{3\})-v^{\vec{L}}(\{4\})\bigg).$$

□

Extending this argument to each n-agent river TU-game, we obtain a solution on the class of all river TU-games with a delta, called the *sequential equal surplus division*. For convenience, we define this solution on the domain of all digraph TU games $\mathcal{C}_0$. This means that if the digraph has several components, then we treat each component as an independent river. The sequential equal surplus division on $\mathcal{C}_0$, denoted by Φ^e, is defined as follows. Pick any $(N, v^{\vec{L}}) \in \mathcal{C}_0$, any component $C \in N/L$, any $i \in C$, and define the *egalitarian remainder* α for coalition $S_r[i]$ as:

$$\alpha(S_r[i]) = \begin{cases} v^{\vec{L}}(C) & \text{if } i = r, \\ v^{\vec{L}}(S_r[i]) + \dfrac{\alpha(S_r[p_r(i)]) - \sum_{j\in s_r(p_r(i))} v^{\vec{L}}(S_r[j]) - v^{\vec{L}}(\{p_r(i)\})}{|s_r[p_r(i)]|} & \text{otherwise.} \end{cases} \tag{4}$$

Note that by definition $i \in s_r(p_r(i))$ for each $i \neq r$. The sequential equal surplus division assigns to each $i \in C$, $C \in N/L$, the payoff given by:

$$\Phi^e_i(N, v^{\vec{L}}) = v^{\vec{L}}(\{i\}) + \frac{1}{|s_r[i]|}\Big(\alpha(S_r[i]) - \sum_{j\in s_r(i)} v^{\vec{L}}(S_r[j]) - v^{\vec{L}}(\{i\})\Big). \tag{5}$$

The allocation rule Φ^e generalizes the individual standardized remainder vectors proposed by Ju, Borm and Ruys (2007) for the class of TU-games. Each individual standardized remainder vector is constructed from a bijection over the player set. Because a bijection induces a total order on the player set, it is isomorphic to a line. In this special case, the resulting individual standardized remainder vector coincides with (4). The allocation rule defined by Ju, Borm and Ruys (2007) and called the consensus value, is the average, over the set of bijections on the player set, of the individual standardized remainder vectors. There are two major differences with our allocation rule. First, the river's flow defines a unique partial order on the player set. Secondly, more than two coalitions negotiate to share a surplus when the river continues on different branches.

Let us verify that the sequential equal surplus division is an efficient allocation rule and provides a compromise between the ATS and TIBS doctrines.

We show below that for each coalition formed by an agent and his subordinates, the egalitarian remainder is fully redistributed among its members, *i.e.* for each $i \in N$, it holds that $\Phi^e_{S_r[i]}(N, v^{\vec{L}}) = \alpha(S_r[i])$. For each component $C \in N/L$ with root r, we have $S_r[r] = C$, $\alpha(C) = v^{\vec{L}}(C)$ by (4) and so $\Phi^e_C(N, v^{\vec{L}}) = v^{\vec{L}}(C)$, which proves (component) efficiency.

The ATS doctrine prescribes that each agent i has the right to all water on this territory, *i.e.* $\Phi^e_i(N, v^{\vec{L}}) \geq v^{\vec{L}}(\{i\})$. This doctrine can be extended to all coalitions. In our case this doctrine is guaranteed only for coalitions formed by an agent and his subordinates, *i.e.* for each $i \in N$, it holds that $\Phi^e_{S_r[i]}(N, v^{\vec{L}}) \geq v^{\vec{L}}(S_r[i])$. Therefore, a coalition consisting of any agent and all his subordinates obtains a share of the total welfare that is at least as large as the welfare they can achieve without the cooperation of their superiors in the river. Note that this property is similar to the subsidy-free property used in Aadland and Kolpin (1998) for the related problem of sharing the cleaning costs of a polluted river.

The TIBS doctrine states that each agent has the right to all water flowing along the component of the river where he lives, no matter where it enters in this component. This means that each agent has the right to a certain share of the total welfare resulting from the cooperation of all members of this component. This share is measured by the non-negative difference $\Phi_i(N, v^{\vec{L}}) - v^{\vec{L}}(\{i\}) \geq 0$ given by (5).

These properties are summarized in the following proposition and stated for all digraph TU-games.

Proposition 1 *For each digraph TU-game $(N, v^{\vec{L}}) \in \mathcal{C}_0$, each component $C \in N/L$ and each agent $i \in C$, it holds that:*
(i) $\Phi^e_{S_r[i]}(N, v^{\vec{L}}) = \alpha(S_r[i])$.
If, furthermore, $(N, v^{\vec{L}}) \in \mathcal{C}_0$ is superadditive, then it holds that:
(ii) $\alpha(S_r[i]) \geq v^{\vec{L}}(S_r[i])$;
(iii) $\Phi_i(N, v^{\vec{L}}) \geq v^{\vec{L}}(\{i\})$.

Proof. Pick any river TU-game $(N, v^{\vec{L}}) \in \mathcal{C}_0$, any component $C \in N/L$ with root $r \in N$,and any $i \in N$. The proof of part (i) is by induction on the number of subordinates of i.
Initial step: Assume that agent i has no subordinates, *i.e.* $S_r[i] = \{i\}$. Thus, $s_r(i) = \emptyset$ and (5) imply:

$$\Phi^e_{S_r[i]}(N, v^{\vec{L}}) = \Phi^e_i(N, v^{\vec{L}}) = \alpha(\{i\}).$$

Induction hypothesis: Assume that the assertion holds when $S_r[i]$ contains at most $q < |C|$ elements.

INDUCTION STEP: Assume that $S_r[i]$ contains $q+1$ elements. Then:

$$\Phi^e_{S_r[i]}(N, v^{\vec{L}}) = \Phi^e_i(N, v^{\vec{L}}) + \sum_{j \in s_r(i)} \Phi^e_{S_r[j]}(N, v^{\vec{L}}).$$

Since $j \in s_r(i)$, each $S_r[j]$ contains at most q elements. By the induction hypothesis, the right-hand side of the above equality is equivalent to:

$$\Phi^e_i(N, v^{\vec{L}}) + \sum_{j \in s_r(i)} \alpha(S_r[j]).$$

Using the definition (4) of the egalitarian remainder and the fact that for each $j \in s_r(i)$, $p_r(j) = i$, the previous expression can be rewritten as follows:

$$\begin{aligned}
& v^{\vec{L}}(\{i\}) + \frac{1}{|s_r[i]|}\Big(\alpha(S_r[i]) - \sum_{j \in s_r(i)} v^{\vec{L}}(S_r[j]) - v^{\vec{L}}(\{i\})\Big) \\
& + \sum_{j \in s_r(i)} \Big(v^{\vec{L}}(S_r[j]) + \frac{1}{|s_r(i)|+1}\Big(\alpha(S_r[i]) - \sum_{k \in s_r(i)} v^{\vec{L}}(S_r[k]) - v^{\vec{L}}(\{i\})\Big)\Big) \\
= \;& v^{\vec{L}}(\{i\}) + \frac{1}{|s_r[i]|}\Big(\alpha(S_r[i]) - \sum_{j \in s_r(i)} v^{\vec{L}}(S_r[j]) - v^{\vec{L}}(\{i\})\Big) \\
& + \sum_{j \in s_r(i)} v^{\vec{L}}(S_r[j]) + \frac{|s_r(i)|}{|s_r[i]|}\Big(\alpha(S_r[i]) - \sum_{k \in s_r(i)} v^{\vec{L}}(S_r[k]) - v^{\vec{L}}(\{i\})\Big) \\
= \;& \alpha(S_r[i]).
\end{aligned}$$

Therefore, we get:

$$\begin{aligned}
\Phi^e_{S_r[i]}(N, v^{\vec{L}}) &= \Phi^e_i(N, v^{\vec{L}}) + \sum_{j \in s_r(i)} \alpha(S_r[j]) \\
&= \alpha(S_r[i]),
\end{aligned}$$

as desired.

The proof of part (ii) is by induction on the depth q of the agents in $(C, \vec{L}(C))$.
INITIAL STEP: Assume that $q = 0$ so that i is the root r of the directed tree on C, *i.e.* $S_r(r) = C$. By (4), we get $\alpha(C) = v^{\vec{L}}(C)$, as desired.
INDUCTION HYPOTHESIS: Assume that the assertion is true for each $i \in C$ such that his depth is inferior or equal to $q < d_C$ where d_C is the depth of the rooted tree $(C, \vec{L}(C))$.
INDUCTION STEP: Pick any $i \in C$ such that his depth is $q+1$. It follows that the number of links on the path from the root r to $p_r(i)$ is equal to q. By the induction hypothesis and superadditivity of $v^{\vec{L}}$, we get:

$$\begin{aligned}
\alpha(S_r(i)) &= v^{\vec{L}}(S_r[i]) + \frac{1}{|s_r[p_r(i)]|}\Big(\alpha(S_r[p_r(i)]) - \sum_{j \in s_r(p(i))} v^{\vec{L}}(S_r[j]) - v^{\vec{L}}(\{p_r(i)\})\Big) \\
&\geq v^{\vec{L}}(S_r[i]) + \frac{1}{|s_r[p_r(i)]|}\Big(v^{\vec{L}}(S_r[p_r(i)]) - \sum_{j \in s_r(p_r(i))} v^{\vec{L}}(S_r[j]) - v^{\vec{L}}(\{p_r(i)\})\Big) \\
&\geq v^{\vec{L}}(S_r[i]),
\end{aligned}$$

as desired.

The proof of part (iii) follows from (ii), the definition (5) of Φ^e and the superadditivity of $v^{\vec{L}}$. It suffices to note that the collection of coalitions $S_r[j]$, $j \in s_r(i)$, and the singleton $\{i\}$ constitute a partition of $S_r[i]$. ■

It is interesting to compare our solution to the solutions provided by Ambec and Sprumont (2002), van den Brink *et al.* (2007) for a river shaped like a line and their extension to a river with a delta provided by Khmelnitskaya (2010). Assume first that the river is shaped like a rooted tree, then the solution suggested by Khmelnitskaya (2010) is the so-called hierarchical outcome introduced by Demange (2004) and defined by:

$$\forall i \in N, \quad h_i(N, v^{\vec{L}}) = v^{\vec{L}}(S_r[i]) - \sum_{j \in s_r(i)} v^{\vec{L}}(S_r[j]). \tag{6}$$

Each agent's payoff is equal to his marginal contribution to all his subordinates when he joins them. In case the river is shaped like a line, each agent other than agent n located at the sink of the river has exactly one successor $i+1$, so that $S_r[i] = \{i, i+1, \ldots, n\}$ and $S_r[i+1] = \{i+1, i+2, \ldots\}$ for $\{i+1\} = s_r(i)$. The hierarchical outcome reduces to:

$$h_n(N, v^{\vec{L}}) = v^{\vec{L}}(\{n\}) \text{ and for } i \in N \backslash \{n\},\ h_i(N, v^{\vec{L}}) = v^{\vec{L}}(\{i, \ldots, n\}) - v^{\vec{L}}(\{i+1, \ldots, n\}).$$

As shown by Demange (2004), van den Brink *et al.* (2007) and Khmelnitskaya (2010), this solution possesses several advantages. First it is efficient and satisfies also point (ii) in Proposition 1 not only for coalitions formed by an agent and all his subordinates but for all coalitions, which means that this solution is core-stable. It also gives an incentive for an agent to cooperate with his subordinates in the sense that all the surplus resulting from this cooperation is allocated to his agent. Indeed, this surplus is given by:

$$v^{\vec{L}}(S_r[i]) - \sum_{j \in s_r(i)} v^{\vec{L}}(S_r[j]) - v^{\vec{L}}(\{i\})$$

and i exactly gets

$$v^{\vec{L}}(\{i\}) + v^{\vec{L}}(S_r[i]) - \sum_{j \in s_r(i)} v^{\vec{L}}(S_r[j]) - v^{\vec{L}}(\{i\}).$$

This solution is the opposite of the solution suggested by Ambec and Sprumont (2002) for a river shaped like a line, which distributes all the surplus resulting from the cooperation of an agent with his superiors to this agent:

$$\text{the root gets } v^{\vec{L}}(\{r\}) \text{ and each other agent } i \text{ gets } v^{\vec{L}}(\{1, 2, \ldots, i\}) - v^{\vec{L}}(\{1, 2, \ldots, i-1\}).$$

One main drawback of these solutions is that they rely on two extreme interpretations of the TIBS doctrine. As a consequence, these solutions do not coincide with the standard solution for the two-agent case. On the contrary, our procedure is a comprise between the ATS and TIBS doctrines, without relying on an extreme interpretation of the latter. Of course, it is still possible to average between several solutions that do not violate the TIBS doctrine. But averaging between different solutions makes the procedure of allocation often unclear and artificial. On this point, we refer the reader to van den Brink *et al.* (2007), van den Brink *et al.* (2012) and Béal *et al.* (2010, 2012).

Another way to compare (6) with our solution is to use point (i) of Proposition 1. Indeed, for each $i \in N$, we have:

$$\begin{aligned}\Phi^e_i(N, v^{\vec{L}}) &= \Phi^e_{S_r[i]}(N, v^{\vec{L}}) - \sum_{j \in s_r(i)} \Phi^e_{S_r[j]}(N, v^{\vec{L}}) \\ &= \alpha(S_r[i]) - \sum_{j \in s_r(i)} \alpha(S_r[j]).\end{aligned}$$

Thus, each agent's payoff is equal to his contribution to the egalitarian remainder when he joins all his subordinates.

As we will see below, our solution satisfies several natural principles such as two deletion link principles, a consistency principle, and an amalgamation principle. The rest of the article is devoted to the axiomatic study of the sequential equal surplus division. We provide two alternative characterizations of it on the class $\mathcal{C}_0$.

5 Properties for resolving water disputes

In this section, we define some properties of allocation rules Φ on the class $\mathcal{C}_0$. The first property states that the allocation rule should coincide with the standard solution if the sharing river TU-game involves only two agents.

Standardness (S) For each $(N, v^{\vec{L}}) \in \mathcal{C}_0$ and each $C \in N/L$ such that $C = \{1, 2\}$, it holds that:

$$\forall i \in \{1, 2\}, \quad \Phi_i(N, v^{\vec{L}}) = v^{\vec{L}}(\{i\}) + \frac{v^{\vec{L}}(\{1, 2\}) - v^{\vec{L}}(\{1\}) - v^{\vec{L}}(\{2\})}{2}.$$

The principle of the efficiency of the water use is translated into the following property.

Component Efficiency (CE) For each $(N, v^{\vec{L}}) \in \mathcal{C}_0$ and each $C \in N/L$, it holds that:

$$\Phi_C(N, v^{\vec{L}}) = v^{\vec{L}}(C).$$

We consider two deletion link properties of solutions on $\mathcal{C}_0$ that measure changes in payoffs as a result of deleting links from the digraph. In our context, removing a directed link means that the tail of the directed link does not want to cooperate with the head of this link. This interpretation is consistent with the fact that water flows from upstream to downstream, which implies that the tail controls the directing link by refusing to give up water for higher-value uses in downstream locations. But it could be the case that the head does not want to cooperate with the tail of the link in which case he will not consume the water inflow coming from his superiors, for instance by constructing a dam or by wasting water resources. We consider these two possibilities in the following properties. The first property measures changes in payoffs when an agent removes all his links. It states that deleting all links of an agent yields for this agent and each coalition consisting of any of his successors and their subordinates the same total change in payoff. The second property applies to a river shaped like an outward-pointing star, *i.e.* the root of a tree is linked to all other agents. The second property requires that if the root breaks the connection with only one of his successors, the change in payoff for the root and for each of his successors other than the head of the deleted link is the same. Since these two properties are related to fairness as introduced by Myerson (1977) for graph TU-games, we refer to them as downstream fairness and star fairness respectively.

Consider any TU-game $(N, v^{\vec{L}}) \in \mathcal{C}_0$, any component $C \in N/L$ with root r and any directed link $i \to j$. We denote by $(S_r[j], v^{\vec{L}}_{|S_r[j]}) \in \mathcal{C}_0$ the TU-game induced by $S_r[j]$ and $v^{\vec{L}}$, *i.e.* the player set is $S_r[j] \subseteq C$, the digraph is described by the induced subtree $(S_r[j], \vec{L}(S_r[j]))$ whose root is j and $v^{\vec{L}}_{|S_r[j]}$ is the subgame of $v^{\vec{L}}$ induced by $S_r[j]$. The interpretation follows: if the connection $i \to j$ is broken, then the cooperation between the members of $S_r[j]$ and their superiors in the digraph is no longer possible. Thus, the members of $S_r[j]$ consider the worths they can secure for themselves without the cooperation of agents located upstream to j. We will also consider the subgame $(\{i\}, v^{\vec{L}}_{|\{i\}})$ which describes the situation where agent i removes all his links.

Downstream Fairness (DF) For each $(N, v^{\vec{L}}) \in \mathcal{C}_0$, each $C \in N/L$ with at least two agents and each $i \in C$ such that $s_r(i) \neq \emptyset$, it holds that:

$$\forall j \in s_r(i), \quad \Phi_i(N, v^{\vec{L}}) - \Phi_i(\{i\}, v^{\vec{L}}_{|\{i\}}) = \Phi_{S_r[j]}(N, v^{\vec{L}}) - \Phi_{S_r[j]}(S_r[j], v^{\vec{L}}_{|S_r[j]}).$$

In particular, this property implies that deleting all links incident to player i yields the same total change in payoff for each coalition consisting of a successor of i and all his subordinates, *i.e.*

$$\forall j, k \in s_r(i), \quad \Phi_{S_r[j]}(N, v^{\vec{L}}) - \Phi_{S_r[j]}(S_r[j], v^{\vec{L}}_{|S_r[j]}) = \Phi_{S_r[k]}(N, v^{\vec{L}}) - \Phi_{S_r[k]}(S_r[k], v^{\vec{L}}_{|S_r[k]}).$$

Downstream fairness has the flavor of (weighted) Component Fairness (CF) introduced by Herings *et al.* (2008) and Béal *et al.* (2012) respectively and applied to river TU-games by van den Brink *et al.* (2012). Component fairness states that deleting a link between two players in a forest yields the same average change in payoff in the two new components that result from deleting the link. The subtree $(S_r[j], \vec{L}(S_r[j]))$ is the new component containing agent j when the directed link $i \to j$ is deleted. Nevertheless, DF differs from CF in three respects: DF is concerned with the total change in payoff in $S_r[j]$ and not with the average change in payoff in $S_r[j]$; DF evaluates changes in payoffs when an agent deletes all his links simultaneously and not sequentially; DF takes into account the direction of the river's flow since the total change in payoff in the set containing i's superiors is not described.

In order to define the second property of fairness we need the notion of outward-pointing star. Consider a component $C \in N/L$. We say that $(C, \vec{L}(C))$ is an outward-pointing star with root $r \in C$ if $\vec{L}(C) = \{(r, i) : i \in C \setminus \{r\}\}$.

Star Fairness (SF) For each $(N, v^{\vec{L}}) \in \mathcal{C}_0$, each $C \in N/L$ such that $|C| \geq 3$ and $(C, \vec{L}(C))$ is an outward-pointing star with root $r \in C$, and each directed link $r \to i$, $i \in C \backslash \{r\}$, it holds that:

$$\forall j, k \in C\backslash\{i\}, \quad \Phi_j(N, v^{\vec{L}}) - \Phi_j(C\backslash\{i\}, v^{\vec{L}}_{|C\backslash\{i\}}) = \Phi_k(N, v^{\vec{L}}) - \Phi_k(C, \backslash\{i\}, v^{\vec{L}}_{|C\backslash\{i\}}).$$

SF indicates that if the root breaks the connection with agent i, then the change of payoff between any pair of remaining players is the same. Note that the property above applies to outward-pointing stars with at least three members. Otherwise, we can apply S if the outward-pointing star contains two members and CE if it is a singleton.

The next property incorporates a consistency principle. Informally, a consistency principle states the following. Fix a solution for a class of TU-games. Assume that some agents leave the game with their payoffs and examine the reduced problem that the remaining agents face. The solution is consistent if for this reduced game, there is no need to re-evaluate the payoffs of the remaining agents. As noted by Thomson (2011), the consistency principle has been examined in the context of a great variety of concrete problems of resource allocation. Since our TU-game is endowed with rooted trees, it is natural to limit our attention to the coalitions formed by an agent and his subordinates.

Pick any digraph TU-game $(N, v^{\vec{L}}) \in \mathcal{C}_0$, any $C \in N/L$ with root $r \in C$ and $i \in N$. Assume that the payoffs have been distributed according to the payoff vector $z \in \mathbb{R}^n$ and that the players $C \backslash S_r[i]$ leaves the game with their component of the vector z. Let us re-evaluate the situation of $S_r[i]$ at this point. To do this we define the reduced game they face. The worths of coalitions in the reduced games depend on (i) the worths that these coalitions could earn on their own in the original game, (ii) what these coalitions could earn with the leaving players and (iii) the payoff with which the leaving player left the game. Our *reduced digraph TU-game* $(S_r[i], v^{\vec{L}}_{z,i}) \in \mathcal{C}_0$ induced by $S_r[i]$ and z is defined as follows: $v^{\vec{L}}_{z,i}(S_r[i]) = v^{\vec{L}}(C) - z_{C\backslash S_r[i]}$ and for each other coalition $S \subset S_r[i]$, $v^{\vec{L}}_{z,i}(S) = v^{\vec{L}}(S)$. Thus, $v^{\vec{L}}_{z,i}(S_r[i])$ is the total worth left for the remaining agents who interact according to $(S_r[i], v^{\vec{L}}_{z,i})$ on the induced rooted tree $(S_r[i], \vec{L}(S_r[i]))$. Note that for a superadditive digraph game, its reduction is not necessarily superadditive. This depends on the worth $v^{\vec{L}}(C) - z_{C\backslash S_r[i]}$. If we consider Φ^e to define the reduced digraph TU-games, then each $(S_r[i], v^{\vec{L}}_{\Phi^e,i})$ remains superadditive by point (ii) of Proposition 1. Consequently, if the underlying digraph TU-game is superadditive, as for the river TU-games, we can restrict the analysis to allocations z that preserve this superadditivity property. In this way, the following consistency property can be applied to the class of all superadditive digraph TU-games.

Downstream Consistency (DC) For each $(N, v^{\vec{L}}) \in \mathcal{C}_0$, each $C \in N/L$ and each $i \in C$, it holds that:

$$\forall j \in S_r[i], \quad \Phi_j(N, v^{\vec{L}}) = \Phi_j(S_r[i], v^{\vec{L}}_{\Phi,i}).$$

DC is a robustness requirement guaranteeing that a coalition of subordinates respect the recommendations made by Φ when the other agents have already received their payoffs according to the solution Φ.

The last property incorporates an amalgamation principle. This principle says something about the changes in payoffs when two or more agents are amalgamated to act as if they were a single agent. It states that if a set of agents are amalgamated into one agent, then the payoff of this agent in the new game coincides with the sum of the payoffs of the amalgamated agents in the original game. This principle has been used by Lehrer (1988), Albizuri (2001) and Albizuri, Aurrekoetxea (2006) in order to characterize various Banzhaf-Coleman indexes. Applying this idea to digraph TU-games we only allow sets of subordinates to be amalgamated.

Pick any digraph TU-game $(N, v^{\vec{L}}) \in \mathcal{C}_0$, any $C \in N/L$ with root $r \in C$ and $i \in C$. Assume that i's subordinates are amalgamated in a way that respects the direction of the river's flow. Precisely, for each $j \in s_r(i)$, the members of $S_r[j]$ are amalgamated into one player denoted by $\overline{S_r[j]}$. From this operation of amalgamation, we define a new digraph TU-game as follows. The player set N^i is given by:

$$N^i = \left[N \backslash \left(\bigcup \{S_r[j], j \in s_r(i)\}\right)\right] \bigcup \{\overline{S_r[j]}, j \in s_r(i)\}.$$

The rooted tree $\vec{L}_i$ on N^i contains the directed links $i \to \overline{S_r[j]}$ for each $j \in s_r(i)$ plus all the original directed links between pairs of agents that belong either to a component other than C or to $C \backslash \left(\bigcup \{S_r[j], j \in s_r(i)\}\right)$. Since the members of $S_r[j]$ behave as a single player, the coalitions contained in $S_r[j]$ as well as their links are not taken into account in the description of the new coalition function. Therefore, we define $(N^i, v^{\vec{L}_i}) \in \mathcal{C}_0$ as follows: for each $S \in 2^{N^i}$,

$$v^{\vec{L}_i}(S) = \begin{cases} v^{\vec{L}}(S) & \text{if } S \cap S_r[j] = \emptyset, \\ & \text{and } j \in s_r(i), \\ & \\ v^{\vec{L}}\left(\left(S \backslash \{\overline{S_r[j]} : j \in s_r(i)\}\right) \bigcup \{S_r[j] : j \in s_r(i), \overline{S_r[j]} \in S\}\right) & \text{otherwise.} \end{cases}$$

In particular, for each player $\overline{S_r[j]}$, we have $v^{\vec{L}_i}(\{\overline{S_r[j]}\}) = v^{\vec{L}}(S_r[j])$. Of course, if $s_r(i)$ is empty, then $(N^i, v^{\vec{L}_i})$ coincides with $(N, v^{\vec{L}})$.

Downstream Amalgamation (DA) For each $(N, v^{\vec{L}}) \in \mathcal{C}_0$, each $C \in N/L$ with root $r \in C$ and each $i \in C$ such that $s_r(i) \neq \emptyset$, it holds that:

$$\forall j \in s_r(i), \quad \Phi_{S_r[j]}(N, v^{\vec{L}}) = \Phi_{\overline{S_r[j]}}(N^i, v^{\vec{L}_i}).$$

A similar principle has been used in Ansink and Weikard (2012) in the context of a river shaped like a line. But, these authors depart from the TU-game approach by assuming that each agent along the river has a claim to the river's flow and that the sum of downstream claims exceeds the sum of downstream endowments at each location. They transform the full problem into an ordered list of two-agent interdependent river sharing problems, each of which is formally equivalent to a bankruptcy problem and is solved through a bankruptcy rule.

van den Brink (2009) also considers the possibility that neighbors collude in a digraph TU-game. His approach is closer to the idea developed by Haller (1994). Here, collusion describes a situation where an agent, say i, becomes a proxy for another agent j, in the sense that j does not contribute anything to any coalition that does not contain i, and when i enters it is as if i and j enter together. So, the new digraph TU-game contains the same player set than the original game whereas the number of players is smaller when some of them have been amalgamated in our sense. The property defined by Haller (1994) and generalized by van den Brink (2009) to graph TU-game states that the sum of the payoffs distributed to two neighbors in the graph should not change when they collude in their sense. This collusion neutrality property is used to characterize the hierarchical outcome defined in (6).

6 Two characterizations of Φ^e

This section contains the main results of this article. We first show that Φ^e satisfies all the properties listed in the previous section.

Proposition 2 *The allocation rule Φ^e satisfies S, CE, DF, SF, DC and DA on $\mathcal{C}_0$.*

Proof. From (4) and (5), it is easily verified that Φ^e satisfies S on $\mathcal{C}_0$. The fact that Φ^e satisfies CE on $\mathcal{C}_0$ follows directly from part (i) in Proposition 1 and (4). To verify that Φ^e satisfies DF, pick any digraph TU-game $(N, v^{\vec{L}})$, any component $C \in N/L$ with root $r \in C$ and any agent $i \in C$ such that $s_r(i) \neq \emptyset$. For each $j \in s_r(i)$, we have:

$$\begin{aligned}
\Phi^e_{S_r[j]}(N, v^{\vec{L}}) - \Phi^e_{S_r[j]}(S_r[j], v^{\vec{L}}_{|S_r[j]}) &= \Phi^e_{S_r[j]}(N, v^{\vec{L}}) - v^{\vec{L}}(S_r[j]) \\
&= \alpha(S_r[j]) - v^{\vec{L}}(S_r[j]) \\
&= \frac{1}{|s_r[i]|}\bigg(\alpha(S_r[i]) - \sum_{k \in s_r(i)} v^{\vec{L}}(S_r[k]) - v^{\vec{L}}(\{i\})\bigg) \\
&= \Phi^e_i(N, v^{\vec{L}}) - v^{\vec{L}}(\{i\}) \\
&= \Phi^e_{\{i\}}(N, v^{\vec{L}}) - \Phi^e_{\{i\}}(\{i\}, v^{\vec{L}}_{|\{i\}}),
\end{aligned}$$

where first equality follows from the fact that Φ^e satisfies CE on $\mathcal{C}_0$. The second equality follows from part (i) in Proposition 1. The third and four equalities come from $p_r(j) = i$ and (4)-(5) respectively. Therefore, Φ^e satisfies DF on $\mathcal{C}_0$.

Next, consider a situation where the digraph TU-game $(N, v^{\vec{L}}) \in \mathcal{C}_0$ possesses a component $C \in N/L$ containing at least three agents and shaped like an outward-pointing star with root $r \in C$. Pick any directed link $r \to i$. In order to verify that Φ^e satisfies SF, we first compute $\Phi^e_j(N, v^{\vec{L}})$ for each $j \in C$. By construction and definition of the egalitarian remainder, we have: $S_r[r] = C$, $\alpha(C) = v^{\vec{L}}(C)$ and for each $j \in C\backslash\{r\}$, $p_r(j) = r$ and also $S_r[j] = \{j\}$. Consequently, from (4) and (5) we get:

$$\forall j \in C, \quad \Phi^e_j(N, v^{\vec{L}}) = v^{\vec{L}}(\{j\}) + \frac{1}{|C|}\bigg(v^{\vec{L}}(C) - \sum_{k \in C} v^{\vec{L}}(\{k\})\bigg). \tag{7}$$

By a similar computation on the subgame $(C\backslash\{i\}, v^{\vec{L}}_{|C\backslash\{i\}})$, we get that for each $j \in C\backslash\{i\}$:

$$\Phi^e_j(C\backslash\{i\}, v^{\vec{L}}_{|C\backslash\{i\}}) = v^{\vec{L}}_{|C\backslash\{i\}}(\{j\}) + \frac{1}{|C|-1}\bigg(v^{\vec{L}}_{|C\backslash\{i\}}(C\backslash\{i\}) - \sum_{k \in C\backslash\{i\}} v^{\vec{L}}_{|C\backslash\{i\}}(\{k\})\bigg). \tag{8}$$

From (7)-(8) and the fact that $v^{\vec{L}}(\{j\}) = v^{\vec{L}}_{|C\backslash\{i\}}(\{j\})$ for each $j \in C\backslash\{i\}$, we get:

$$\forall j, k \in C\backslash\{i\}, \quad \Phi^e_j(N, v^{\vec{L}}) - \Phi^e_j(C\backslash\{i\}, v^{\vec{L}}_{|C\backslash\{i\}}) = \Phi^e_k(N, v^{\vec{L}}) - \Phi^e_k(C\backslash\{i\}, v^{\vec{L}}_{|C\backslash\{i\}}),$$

which proves that Φ^e satisfies SF on $\mathcal{C}_0$.

In order to verify that Φ^e satisfies DC on $\mathcal{C}_0$, we first compare the payoffs $\Phi^e_i(N, v^{\vec{L}})$ and $\Phi^e_i(S_r[i], v^{\vec{L}}_{\Phi^e,i})$ for any $(N, v^{\vec{L}}) \in \mathcal{C}_0$, any $C \in N/L$ and $i \in C$. By definition of $(S_r[i], v^{\vec{L}}_{\Phi^e,i})$, we have $v^{\vec{L}}_{\Phi^e,i}(S_r[i]) = v^{\vec{L}}(C) - \Phi_{C\backslash S_r[i]}(N, v^{\vec{L}})$. Because Φ^e satisfies CE and by part (i) of Proposition 1, we get:

$$v^{\vec{L}}_{\Phi^e,i}(S_r[i]) = \Phi_{S_r[i]}(N, v^{\vec{L}}) = \alpha(S_r[i]).$$

Using the definition of $(S_r[i], v^{\vec{L}}_{\Phi,i})$, it follows that:

$$\begin{aligned}\Phi^e_i(S_r[i], v^{\vec{L}}_{\Phi^e,i}) &= v^{\vec{L}}_{\Phi^e,i}(\{i\}) + \frac{1}{|s_r[i]|}\left(v^{\vec{L}}_{\Phi^e,i}(S_r[i]) - \sum_{j\in s_r(i)} v^{\vec{L}}_{\Phi^e,i}(S_r[j]) - v^{\vec{L}}_{\Phi^e,i}(\{i\})\right)\\ &= v^{\vec{L}}(\{i\}) + \frac{1}{|s_r[i]|}\left(\alpha(S_r[i]) - \sum_{j\in s_r(i)} v^{\vec{L}}(S_r[j]) - v^{\vec{L}}(\{i\})\right)\\ &= \Phi^e_i(N, v^{\vec{L}}),\end{aligned}$$

as desired for player i. From this, we deduce that the egalitarian remainder for each $j \in s_r(i)$ is the same in $(S_r[i], v^{\vec{L}}_{\Phi^e,i})$ and $(N, v^{\vec{L}})$. So, using the definition of $(S_r[i], v^{\vec{L}}_{\Phi^e,i})$, we also have $\Phi^e_j(S_r[i], v^{\vec{L}}_{\Phi^e,i}) = \Phi^e_i(N, v^{\vec{L}})$ for each $j \in s_r(i)$. Continuing in this fashion for each subordinate of i, we reach the desired conclusion.

Finally, consider any digraph TU-game $(N, v^{\vec{L}}) \in \mathcal{C}_0$, any component $C \in N\backslash L$, any $i \in C$ such that $s_r(i) \neq \emptyset$. Pick any $j \in s_r(i)$. On the one hand, by part (i) of Proposition 1, we have $\Phi^e_{S_r[j]}(N, v^{\vec{L}}) = \alpha(S_r[j])$. On the other hand, from (5) and definition of the digraph TU-game $(N^i, v^{\vec{L}_i})$, we deduce that $\Phi^e_k(N^i, v^{\vec{L}_i}) = \Phi^e_k(N, v^{\vec{L}})$ for each player k who does not belong to the set of i's subordinates. To understand this equality, note that the egalitarian remainder for such a k in $(N, v^{\vec{L}})$ and $\Phi^e_k(N, v^{\vec{L}})$ do not rely on proper coalitions of $S_r[j]$, $j \in s_r(i)$ so that the computations from $v^{\vec{L}}$ give the same results as the computations from $v^{\vec{L}_i}$. Therefore, for each $j \in s_r(i)$, the egalitarian remainder for $S_r[j]$ in $(N, v^{\vec{L}})$ coincides with the egalitarian remainder for $\overline{S_r[j]}$ in $(N^i, v^{\vec{L}_i})$. Because each player $\overline{S_r[j]}$, $j \in s_r(i)$, has no successor in $(N^i, \vec{L}_i)$, we easily conclude that $\Phi^e_{\overline{S_r[j]}}(N^i, v^{\vec{L}_i}) = \alpha(S_r[j]) = \Phi^e_{S_r[j]}(N, v^{\vec{L}})$. This proves that Φ^e satisfies DA in $\mathcal{C}_0$. ■

Combining CE and DF we obtain a characterization of the sequential equal surplus division on $\mathcal{C}_0$.

Proposition 3 *The sequential equal surplus division Φ^e is the unique allocation rule that satisfies CE and DF on $\mathcal{C}_0$.*

Proof. By Proposition 2, Φ^e satisfies CE and DF. Next, consider an allocation rule Φ that satisfies CE and DF on $\mathcal{C}_0$. To show: $\Phi^e = \Phi$ on $\mathcal{C}_0$. Pick any digraph TU-game $(N, v^{\vec{L}})$ and any $C \in N/L$. First note that, for each $i \in C$, we have:

$$\Phi_i(N, v^{\vec{L}}) = \Phi_{S_r[i]}(N, v^{\vec{L}}) - \sum_{j\in s_r(i)} \Phi_{S_r[j]}(N, v^{\vec{L}}).$$

It remains to prove that, for each $i \in C$, $\Phi_{S_r[i]}(N, v^{\vec{L}})$ is uniquely determined by CE and DF. We proceed by induction on the depth q of the agents in $(C, \vec{L}(C))$.
Initial step: If $q = 0$, then $i = r$. By CE, we get:

$$\Phi_{S_r[r]}(N, v^{\vec{L}}) = \Phi_C(N, v^{\vec{L}}) = v(C).$$

Thus, $\Phi_{S_r[r]}(N, v^{\vec{L}})$ is uniquely determined.

Induction hypothesis: Assume that the assertion is true for each $i \in C$ such that his depth is inferior or equal to $q < d_C$ where d_C is the depth of the rooted tree $(C, \vec{L}(C))$.
Induction step: Pick any $i \in C$ such that his depth is equal to $q+1$. By DF, we have:

$$\Phi_{p_r(i)}(N, v^{\vec{L}}) - v^{\vec{L}}(\{p_r(i)\}) = \Phi_{S_r[i]}(N, v^{\vec{L}}) - v^{\vec{L}}(S_r[i]).$$

Denote this quantity by δ. By DF and CE, we obtain:

$$\forall j \in s_r(p_r(i)), \quad \Phi_{S_r[j]}(N, v^{\vec{L}}) = v^{\vec{L}}(S_r[j]) + \delta.$$

We also have:

$$\Phi_{p_r(i)}(N, v^{\vec{L}}) = v^{\vec{L}}(\{p_r(i)\}) + \delta.$$

Summing all these equalities, we get:

$$\Phi_{S_r[p_r(i)]}(N, v^{\vec{L}}) = \sum_{j \in s_r[p_r(i)]} v^{\vec{L}}(S_r[j]) + |s_r[p_r(i)]|\delta.$$

By the induction hypothesis, $\Phi_{S_r[p_r(i)]}(N, v^{\vec{L}})$ is uniquely determined. As a consequence, the parameter δ is uniquely determined. This gives the result for player i since $\Phi_{S_r[i]}(N, v^{\vec{L}}) = v^{\vec{L}}(S_r[i]) + \delta$. ■

The next proposition provides an alternative characterization of the sequential equal surplus division.

Proposition 4 *The sequential equal surplus division Φ^e is the unique allocation rule that satisfies CE, S, SF, DC and DA on $\mathcal{C}_0$.*

In order to prove the above statement, we need an intermediary result establishing that if an allocation rule satisfies CE, S and SF on the class of outward-pointing star TU-games, *i.e.* situations in which the rooted trees are shaped like outward-pointing stars, then it coincides with Φ^e.

Lemma 1 *If an allocation rule Φ satisfies CE, S and SF on $\mathcal{C}_0$, then for each $(N, v^{\vec{L}}) \in \mathcal{C}_0$ and each $C \in N/L$ such that $\vec{L}(C)$ is shaped like an outward-pointing star, it holds that $\Phi_j(N, v^{\vec{L}}) = \Phi^e_j(N, v^{\vec{L}})$ for each $j \in C$.*

Proof. By Proposition 2, Φ^e satisfies CE, S and SF on $\mathcal{C}_0$. Consider any allocation rule Φ that satisfies CE, S and SF on $\mathcal{C}_0$ and pick any $(N, v^{\vec{L}}) \in \mathcal{C}_0$ such that there is $C \in N/L$ where $\vec{L}(C)$ is shaped like an outward-pointing star with root $r \in C$. We proceed by induction on the number of elements of C.
Initial step: If $C = \{j\}$ for some $j \in N$, then by CE, $\Phi_j(N, v^{\vec{L}}) = \Phi^e_j(N, v^{\vec{L}})$. If $C = \{i, j\}$, then by S, $\Phi_i(N, v^{\vec{L}}) = \Phi^e_i(N, v^{\vec{L}})$ and $\Phi_j(N, v^{\vec{L}}) = \Phi^e_j(N, v^{\vec{L}})$.
Induction hypothesis: Assume that the statement is true for C with at most $q \in \mathbb{N}$ elements.
Induction step: Assume that C contains $q+1$ elements and delete the directed link $r \to i$ for some $i \in C \backslash \{r\}$. By DF and the induction hypothesis we have:

$$\begin{aligned}
\forall j, k \in C\backslash\{i\}, \quad \Phi_j(N, v^{\vec{L}}) - \Phi_k(N, v^{\vec{L}}) &= \Phi_j(C\backslash\{i\}, v^{\vec{L}}_{|C\backslash\{i\}}) - \Phi_k(C, \backslash\{i\}, v^{\vec{L}}_{|C\backslash\{i\}}) \\
&= \Phi^e_j(C\backslash\{i\}, v^{\vec{L}}_{|C\backslash\{i\}}) - \Phi^e_k(C, \backslash\{i\}, v^{\vec{L}}_{|C\backslash\{i\}}) \\
&= \Phi^e_j(N, v^{\vec{L}}) - \Phi^e_k(N, v^{\vec{L}})
\end{aligned}$$

Thus there is a constant $c \in \mathbb{R}$ such that:

$$\forall j \in C\backslash\{i\}, \quad \Phi_j(N, v^{\vec{L}}) - \Phi^e_j(N, v^{\vec{L}}) = c.$$

Because these equalities remain valid whatever the chosen deleted link, we have:

$$\forall j \in C, \quad \Phi_j(N, v^{\vec{L}}) - \Phi^e_j(N, v^{\vec{L}}) = c.$$

By CE, $\Phi_C(N, v^{\vec{L}}) = \Phi^e_C(N, v^{\vec{L}}) = v(C)$ and so $c = 0$. This shows that, for each $j \in C$, $\Phi_j(N, v^{\vec{L}}) = \Phi^e_j(N, v^{\vec{L}})$. ■

Proof. (of Proposition 4). By Proposition 2, Φ^e satisfies CE, S, SF, DC and DA on $\mathcal{C}_0$. Next, pick any allocation rule Φ that satisfies CE, S, SF, DC and DA on $\mathcal{C}_0$ and consider any $(N, v^{\vec{L}}) \in \mathcal{C}_0$ and any $C \in N/L$. To show: for each $i \in C$, $\Phi_i(N, v^{\vec{L}})$ is uniquely determined. We proceed by induction on the depth of the subtree $(C, \vec{L}(C))$ rooted at $r \in C$.
Initial step: Assume that the depth of the rooted tree $(C, \vec{L}(C))$ is equal to zero or one. In such case, the rooted tree is shaped like an outward pointing star. By Lemma 1, the payoffs $(\Phi_i(N, v^{\vec{L}}))_{i\in C}$ are uniquely determined.
Induction hypothesis: Assume that the payoffs $(\Phi_i(N, v^{\vec{L}}))_{i\in C}$ are uniquely determined for each $(C, L(\vec{C}))$ whose depth is $d_C < |C| - 1$.
Induction step: Assume that the depth of $(C, \vec{L}(C))$ is $d_C + 1 > 1$. From $(N, v^{\vec{L}})$ and r, construct the digraph TU-game $(N^r, v^{\vec{L}_r})$ where player r's subordinates are amalgamated. By construction, r is now the root of an outward-pointing star in the corresponding component of $(N^r, \vec{L}_r)$, *i.e.* we have $r \to \overline{S_r[j]}$ for each $j \in s_r(r)$. Since by assumption there is at least one agent with no successor whose depth is strictly greater than one, $(N^r, v^{\vec{L}_r})$ does not coincide with $(N, v^{\vec{L}})$. By Lemma 1, for each $j \in s_r(r)$, the payoff $\Phi_{\overline{S_r[j]}}(N^r, v^{\vec{L}_r})$ is uniquely determined. By DA, we get:

$$\forall j \in s_r(r), \quad \Phi_{\overline{S_r[j]}}(N^r, v^{\vec{L}_r}) = \Phi_{S_r[j]}(N, v^{\vec{L}}),$$

which means that the total payoff of each coalition $S_r[j]$, $j \in s_r(r)$, is uniquely determined in the original game $(N, v^{\vec{L}})$. Using CE, we obtain that agent r's payoff is uniquely determined in $(N, v^{\vec{L}})$:

$$\Phi_r(N, v^{\vec{L}}) = v^{\vec{L}}(C) - \sum_{j\in s_r(r)} \Phi_{S_r[j]}(N, v^{\vec{L}}).$$

It remains to show that the payoff of each other agent in the component is uniquely determined. Consider any $j \in s_r(r)$ and construct the reduced digraph TU-game $(S_r[j], v^{\vec{L}}_{\Phi,j})$. By definition of the reduced digraph TU-game, we have:

$$\forall S \subset S_r[j], \quad v^{\vec{L}}_{\Phi,j}(S) = v^{\vec{L}}(S) \text{ and } v^{\vec{L}}_{\Phi,j}(S_r[j]) = v^{\vec{L}}(C) - \Phi_{C\backslash S_r[j]}(N, v^{\vec{L}}).$$

Note that $\Phi_{C\backslash S_r[j]}(N, v^{\vec{L}})$ is uniquely determined by the previous step so that $(S_r[j], v^{\vec{L}}_{\Phi,j})$ is well defined. By construction, the depth of each directed subtree $(S_r[j], \vec{L}(S_r[j]))$, $j \in s_r(r)$, is at most d_C. By the induction hypothesis, for each $j \in s_r(r)$ and $i \in S_r[j]$, $\Phi_i(S_r[j], v^{\vec{L}}_{\Phi,j})$ is uniquely determined. By DC we get:

$$\forall j \in s_r(r), \forall i \in s_r(j), \quad \Phi_i(S_r[j], v^{\vec{L}}_{\Phi,j}) = \Phi_i(N, v^{\vec{L}}),$$

which means that the payoff of each subordinate of r is uniquely determined in $(N, v^{\vec{L}})$. This completes the proof. ■

7 Concluding remarks

Several concluding remarks are in order.

• On the class of digraph TU-games where each component of the digraph is shaped like a directed line, Φ^e is characterized by S, CE, DA, DC.

• In the definition of DA, all i's subordinates are amalgamated. We could instead amalgamate the subset of i's subordinates located on one branch of the river originating from i. More precisely, pick any digraph TU-game $(N, v^{\vec{L}}) \in \mathcal{C}_0$, any $C \in N/L$ with root $r \in C$ and $i \in C$ such that $s_r(i) \neq \emptyset$. Choose any $j \in s_r(i)$ and amalgamate the members of $S_r[j]$ into one player denoted by $\overline{S_r[j]}$. From this operation of amalgamation, we define a new digraph TU-game as follows. The player set N^{ij} is given by:

$$N^{ij} = (N \setminus S_r[j]) \cup \{\overline{S_r[j]}\}.$$

The rooted tree $\vec{L}_{ij}$ on N contains all the directed edges $l \to k$ of $\vec{L}$ such that $k \notin S_r[j]$, plus the link $i \to \overline{S_r[j]}$ for the chosen $j \in s_r(i)$. We then define $(N^{ij}, v^{\vec{L}_{ij}}) \in \mathcal{C}_0$ as follows: for each $S \in 2^{N^{ij}}$,

$$v^{\vec{L}_{ij}}(S) = \begin{cases} v^{\vec{L}}(S) & \text{if } \overline{S_r[j]} \notin S, \\ \\ v^{\vec{L}}((S \setminus \{\overline{S_r[j]}\}) \cup S_r[j]) & \text{otherwise.} \end{cases}$$

Weak Downstream Amalgamation (WDA) For each $(N, v^{\vec{L}}) \in \mathcal{C}_0$, each $C \in N/L$ with root $r \in C$, each $i \in C$ and $j \in s_r(i)$, it holds that:

$$\Phi_{S_r[j]}(N, v^{\vec{L}}) = \Phi_{\overline{S_r[j]}}(N^{ij}, v^{\vec{L}_{ij}}).$$

Note that DA can be reached by successive applications of WDA. Therefore, WDA is weaker than DA and it is tempting to replace DA by WDA in Proposition 4. But this fails, due to pending agents. We define a pending agent i in a component C as an agent such that $r \to i$ and $s_r(i) = \emptyset$. To see this, consider the allocation rule Φ defined on $\mathcal{C}_0$ a follows: for each $(N, v^{\vec{L}})$ and each $C \in N/L$,
– $\Phi_i(N, v^{\vec{L}}) = \Phi^e_i(N, v^{\vec{L}})$ for each $i \in C$ when $(C, v^{\vec{L}}(C))$ is either an outward-pointing star or has a non-pending agent. In any other case,
– $\Phi_i(N, v^{\vec{L}}) = 0$, if $i \in C$ is a pending agent,
$\Phi_r(N, v^{\vec{L}}) = \Phi^e_r(N, v^{\vec{L}}) + \sum_{\{i \in C:\, i \text{ is a pending agent}\}} \Phi^e_r(N, v^{\vec{L}})$ for the root r and
$\Phi_i(N, v^{\vec{L}}) = \Phi^e_i(N, v^{\vec{L}})$ otherwise.
One easily checks that both Φ^e and Φ satisfy CE, S, SF, DC and WDA.

• The sequential equal surplus division can be expressed by a recursive formula by means of the reduced digraph TU-games. Consider the reduced game $(S_r[i], v^{\vec{L}}_{\Phi^e,i})$. By DC, for each $j \in S_r[i]$, $\Phi_j(S_r[i], v^{\vec{L}}_{\Phi^e,i}) = \Phi^e_j(N, v^{\vec{L}})$. By point (i) of Proposition (1), $\Phi^e_{S_r[i]}(N, v^{\vec{L}}) = \alpha(S_r[i])$. Therefore substituting $\alpha(S_r[i])$ by $\Phi^e_{S_r[i]}(S_r[i], v^{\vec{L}}_{\Phi^e,i})$ in (5) we get:

$$\Phi^e_i(N, v^{\vec{L}}) = v^{\vec{L}}(\{i\}) + \frac{1}{|s_r[i]|}\Bigg(\Phi^e_{S_r[i]}(S_r[i], v^{\vec{L}}_{\Phi^e,i}) - \sum_{j \in s_r(i)} v^{\vec{L}}(S_r[j]) - v^{\vec{L}}(\{i\})\Bigg).$$

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