# Monopolistic competition, indeterminacy and growth

Benhabib, Jess and Perli, Roberto and Xie, Danyang

New York University, New York University, Hong Kong University of Science and Technology

1994

*Ricerche Economiche* (1994) **48**, 279–298

# Monopolistic competition, indeterminacy and growth

JESS BENHABIB AND ROBERTO PERLI†

*Department of Economics, New York University*

DANYANG XIE

*Department of Economics, The Hong Kong University of Science and Technology*

**Summary**

In this paper we analyse the dynamics of both Romer's original model of endogenous growth and of a modified version where the level of labour and human capital are determined endogenously. We find that the original model can have an indeterminate Balanced Growth Path (BGP) if there is some degree of complementarity between the intermediate inputs, and if agents have a high intertemporal elasticity of substitution of consumption. Once we allow for the endogenous determination of labour and of total human capital, we find that equilibrium can be indeterminate with a much lower elasticity of intertemporal substitution of consumption. Moreover, if some modest increasing returns are introduced into the production function for human capital, the issue of global as opposed to local indeterminacy arises: this refers to situations when there exist multiple determinate BGPs, but where the global dynamics is still indeterminate from given initial conditions.

**J.E.L. Classification:** E00, E3, O40.
**Keywords:** Endogenous growth, indeterminacy, transitional dynamics.

## 1. Introduction

Recently, a number of papers on the endogenous growth literature have explored the possibility of the indeterminacy of equilibrium, that is the possibility that there exists a whole continuum of

† For correspondence.



0035–5054/94/040279 + 20 $08.00/0 

equilibria associated with given initial conditions (see, for example, Benhabib & Farmer, 1994; Benhabib & Perli, 1994; Boldrin & Rustichini, 1994; Xie, 1994; see also Farmer & Guo, 1994; Gali, 1994, in the context of business fluctuations). Furthermore, these papers study and often confirm not only the theoretical possibility but also the empirical plausibility of indeterminacy. In this paper we study indeterminacy in the model of Romer (1990), in which endogenous growth is driven by the production of new designs (knowledge) purchased by monopolistically competitive firms that produce intermediate goods. We find in Section 2.2 that indeterminacy can obtain in this model, modified to allow for some complementarity between the intermediate inputs, with a markup rate of 25% in the monopolistically competitive sector, for very standard parameter values except for the intertemporal elasticity of substitution in consumption which is too high.

In Section 3 we take our analysis one step further and modify the model to allow the total level of human capital present in the economy to be chosen endogenously by the optimizing agents. To keep the analysis as simple as possible, we assume that human capital is produced using only a fraction of the non-leisure time supplied by the agents, which is thus drawn out of the time dedicated to supply raw labour to the consumption sector. This modification causes the Balanced Growth Path (BGP) to be indeterminate for even more plausible parameter values; in particular, the intertemporal elasticity of consumption gets much closer to a realistic range, although it still remains a bit high at around 1·4. We should note, however, that endogenizing leisure time as well should allow for intertemporal elasticities for both labour and consumption to easily fall well within the realistic and commonly used ranges, as shown by Benhabib and Perli (1994) in the context of the Lucas (1988) model.

Finally, in Section 4 we introduce a small externality into the research sector: in this case there may exist two BGPs, both of which may be locally determinate for low values of the intertemporal elasticity of substitution. Nevertheless, there will be a global indeterminacy, in the sense that from given initial conditions there will be equilibrium paths that converge to either BGP. Even more disturbing, it appears that one of the BGPs is degenerate, i.e. it has a zero growth rate. There is not enough human capital along that path, and all of it is used in the sector that produces the consumption good. As a consequence, no new designs are produced in the research sector, and the economy is deprived of its engine of growth. The model seems, therefore, to indicate that a country may (though need not) get caught in a growth trap not because of "fundamentals", but simply because of the agents coordinating their expectations on the "bad" equilibrium.

## 2. The standard Romer model

In this section we analyse the transitional dynamics of a version of Romer's model of endogenous growth (Romer, 1990). With respect to the original paper, we allow for some complementarity between the different inputs in the production of the final consumption good, as is fairly standard in models involving imperfect competition.

### 2.1. DESCRIPTION OF THE MODEL

The economy is divided into three sectors. At the top there is a research sector, which uses knowledge $A$ and human capital $H$, and which produces new knowledge, or "designs" for new intermediate goods. The intermediate sector uses the designs and capital to produce intermediate goods used as inputs by the final good sector. The last sector, therefore, uses these intermediate inputs together with human capital and labour to produce the final, unique, consumption good. In this section the total labour supply $L$ and the total level of human capital $H$ are assumed to be fixed and supplied inelastically to the market. The sum of all the intermediate goods available at a certain date $t$ constitutes the total quantity of capital available at that date. Assuming that $\eta$ units of foregone consumption are needed to produce one unit of an intermediate good, and assuming, for analytical convenience, that there is a continuum of such goods, it is therefore possible to write total capital as $K=\eta\int_0^A x(a)\mathrm{d}a$, and the production function of the final good as:

$$Y=H_Y^{\alpha}L^{\beta}\left(\int_0^A x(a)^{\frac{\gamma}{\zeta}}\mathrm{d}a\right)^{\zeta},$$

where $\gamma\equiv 1-\alpha-\beta$ and where $\zeta\geq 1$ is a parameter that captures the degree of complementarity between the inputs (if $\zeta=1$ we have no complementarity). Moreover, since everything in this model is symmetric, all the available intermediate stock of capital are supplied at the same level $x$. This implies that the total stock of capital can be written as $K=\eta Ax$, and then the production function becomes:

$$Y=\eta^{-\gamma}K^{\gamma}A^{\zeta-\gamma}H_Y^{\alpha}L^{\beta}.$$

As is standard, total capital accumulation is simply given by the difference between total output and consumption; therefore:

$$\dot{K} = Y - C = \eta^{-\gamma} K^{\gamma} A^{\zeta-\gamma} H_Y^{\alpha} L^{\beta} - C. \tag{1}$$

It is assumed that the firms in the final sector are perfect competitors. Therefore, they choose the level of intermediate goods needed for production, and for which they have to pay the price $p(a)$, that maximizes their profits. Their problem is thus:

$$\max_x \left\{ H_Y^{\alpha} L^{\beta} \left( \int_0^A x(a)^{\gamma/\zeta} \mathrm{d}a \right)^{\zeta} - \int_0^A p(a) x(a) \mathrm{d}a \right\}. \tag{2}$$

On the other hand, since the intermediate firms use knowledge, which is a non-rival good, as an input to their production, they cannot be perfect competitors; indeed they must have some market power [see section II in Romer (1990) for a detailed discussion]. From the solution of equation (2) we can obtain an expression for $p(a)$, which gives the inverse demand function for the intermediate good $x(a)$:

$$p(a) = \gamma H_Y^{\alpha} L^{\beta} \left( \int_0^A x(a)^{\gamma/\zeta} \mathrm{d}a \right)^{\zeta-1} x(a)^{(\gamma/\zeta)-1}. \tag{3}$$

This demand function is taken as given by the intermediate firm which produces $x(a)$, which also maximizes its profits. Since, by assumption, $\eta$ units of $x(a)$ are needed to produce one unit of the same $x(a)$, the problem of the typical intermediate firm is:

$$\max_x \left\{ \gamma H_Y^{\alpha} L^{\beta} \left( \int_0^A x(a)^{\gamma/\zeta} \mathrm{d}a \right)^{\zeta-1} x(a)^{\gamma/\zeta} - r\eta x(a) \right\},$$

where $r$ is the interest rate. The solution to this problem allows us to express the interest rate $r$ as a function of the other variables of the model:

$$r = \frac{\gamma^2}{\eta\zeta} H_Y^{\alpha} L^{\beta} \left( \int_0^A x(\mathrm{a})^{\gamma/\zeta} \mathrm{d}a \right)^{\zeta-1} x(a)^{(\gamma/\zeta)-1} = \frac{\gamma^2 \eta^{-\gamma}}{\zeta} K^{\gamma-1} A^{\zeta-\gamma} H_Y^{\alpha} L^{\beta}. \tag{4}$$

Comparing equation (3) with equation (4), we see that the price $p(a)$ that the intermediate sector firm charges for $x(a)$ is just a markup over the marginal cost $r$, i.e. $p(a) = (\eta\zeta/\gamma)r$. This fact will be useful later when we will have to choose a calibrated value for the parameters. In this way the profit of the monopolistic firm is therefore:

$$\pi = p(a)x - rx = \frac{\eta(\zeta-\gamma)}{\gamma} rx. \qquad (5)$$

The firms that produce designs in the research sector are also competitive. Each of them has access to the entire stock of knowledge, which is perfectly plausible, since knowledge is a non-rival good. Therefore, the production function of each firm $a$ is $f_A(a) = \delta H(a)A$, where $H(a)$ is the amount of human capital used by firm $a$. Aggregating across firms we obtain the following law of motion for $A$:

$$\dot{A} = \delta H_A A = \delta(H - H_Y)A, \qquad (6)$$

where the last equality follows from the fact that the sum of the human capital used in the research sector, $H_A$, and in the final sector, $H_Y$, must be equal to the total human capital available in the economy, $H$.

Since the firms in the research sector are competitive, the price of each design at time $t$, $P_A(t)$, is equal to the present value of the stream of profits that each intermediate firm (which buys the design) can extract. Hence:

$$P_A = \int_0^\infty \pi(\tau) \mathrm{e}^{-\int_t^\tau r(s)\mathrm{d}s} \mathrm{d}\tau. \qquad (7)$$

The evolution of $P_A$ over time is immediately obtained by differentiating equation (7) with respect to $t$:

$$\dot{P}_A = rP_A - \pi. \qquad (8)$$

Using equations (6) and (8) we can find the expression that governs the evolution of $H_Y$ over time. We also have to note that the rental rate of human capital must be equal in the two sectors where it is used, i.e. the research and the final sectors. This implies that:

$$P_A \delta A = \alpha \eta^{-\gamma} K^\gamma A^{\zeta-\gamma} H_Y^{\alpha-1} L^\beta,$$

since the final sector is competitive, and therefore:

$$P_A = \frac{\alpha \eta^{-\gamma}}{\delta} K^\gamma A^{\zeta-\gamma-1} H_Y^{\alpha-1} L^\beta. \qquad (9)$$

From this equation we get, taking logarithms and differentiating with respect to time:

$$\frac{\dot{P}_A}{P_A}=\gamma\eta^{-\gamma}K^{\gamma-1}A^{\zeta-\gamma}H_Y^{\alpha}L^{\beta}-\gamma\frac{C}{K}+\delta(\zeta-\gamma-1)(H-H_Y)+(\alpha-1)\frac{\dot{H}_Y}{H_Y}. \tag{10}$$

On the other hand, from (8) and (9), we get also that:

$$\frac{\dot{P}_A}{P_A}=r-\frac{\delta\gamma(\zeta-\gamma)}{\alpha\zeta}H_Y=r-\frac{\delta}{\Lambda}H_Y, \tag{11}$$

where $\Lambda\equiv\alpha\zeta/[\gamma(\zeta-\gamma)]$. Equating (10) and (11) we obtain an expression for $\dot{H}_Y/H_Y$:

$$\begin{aligned}\frac{\dot{H}_Y}{H_Y}=&\frac{\gamma\eta^{-\gamma}(\gamma-\zeta)}{\zeta(\alpha-1)}K^{\gamma-1}A^{\zeta-\gamma}H_Y^{\alpha}L^{\beta}+\frac{\delta\Lambda(\zeta-\gamma-1)-\delta}{\Lambda(\alpha-1)}H_Y\\&+\frac{\gamma}{\alpha-1}\frac{C}{K}-\frac{\delta(\zeta-\gamma-1)}{\alpha-1}H.\end{aligned} \tag{12}$$

To complete the model, an expression for the evolution of consumption over time is needed. Such an expression can be readily obtained by considering the representative consumer's problem, which consists in the maximization of its lifetime utility function subject to the intertemporal budget constraint:

$$\max_{C(t)}\int_0^{\infty}\frac{C^{1-\sigma}-1}{1-\sigma}e^{-\rho t}dt$$

subject to

$$\int_0^{\infty}(C-w_HH-w_LL-rK)e^{-\int_0^t r(s)ds}dt=0.$$

The Hamiltonian for this system can be written as:

$$\mathscr{H}=\frac{C^{1-\sigma}-1}{1-\sigma}e^{-\rho t}+\lambda(rK+w_HH+w_LL-C)e^{-\int_0^t r(s)ds}dt=0.$$

The first-order necessary conditions for this problem are:

$$C^{-\sigma}e^{-\rho t}=\lambda e^{-\int_0^t r(s)ds},$$
$$\dot{\lambda}=0,$$

and then, since the Hamiltonian is concave, we see that the optimal path for consumption must obey the following law of motion:

$$\frac{\dot{C}}{C}=\frac{r(t)-\rho}{\sigma}=\frac{\gamma^2\eta^{-\gamma}}{\zeta\sigma}H_Y^{\alpha}L^{\beta}K^{\gamma-1}A^{\zeta-\gamma}-\frac{\rho}{\sigma}. \tag{13}$$

The model is now completely characterized by the four differential equations (1), (6), (12) and (13) in $K$, $A$, $H_Y$ and $C$.

### 2.2. TRANSITIONAL DYNAMICS

Since one of our goals is to study the transitional dynamics implied by the model, we reduce the dimensionality of the problem from four to three by a change of variable very similar to those used in Mulligan and Sala-i-Martin (1993) and in Benhabib and Perli (1994). Hence we define $y\equiv\eta^{\gamma/(\gamma-1)}KA^{(\zeta-\gamma)/(\gamma-1)}$ and $q\equiv C/K$; note that $y$ would be equal to a multiple of $x$ if there was not complementarity between the inputs ($\zeta=1$). Since $\dot{y}/y=\dot{K}/K-(\zeta-\gamma)/(\gamma-1)\dot{A}/A$ and $\dot{q}/q=\dot{C}/C-\dot{K}/K$, we have:

$$\frac{\dot{y}}{y}=y^{\gamma-1}H_Y^{\alpha}L^{\beta}+\frac{\delta(\zeta-\gamma)}{\gamma-1}(H-H_Y)-q, \tag{14}$$

$$\begin{aligned}\frac{\dot{H}_Y}{H_Y}=&\frac{\gamma(\gamma-\zeta)}{\zeta(\alpha-1)}y^{\gamma-1}H_Y^{\alpha}L^{\beta}+\frac{\delta\Lambda(\zeta-\gamma-1)-\delta}{\Lambda(\alpha-1)}H_Y\\&+\frac{\gamma}{\alpha-1}q-\frac{\delta(\zeta-\gamma-1)}{\alpha-1}H,\end{aligned} \tag{15}$$

$$\frac{\dot{q}}{q}=\left(\frac{\gamma^2}{\zeta\sigma}-1\right)y^{\gamma-1}H_Y^{\alpha}L^{\beta}+q-\frac{\rho}{\sigma}. \tag{16}$$

This is a reduced three-dimensional system in $y$, $H_Y$ and $q$ only; its dynamics is equivalent to that of the original four-dimensional system in the sense that its steady states correspond to the BGPs of the original four-dimensional model. It turns out that this model has a unique BGP, which can easily be found in the following way. For notational convenience define $z\equiv y^{\gamma-1}H_Y^{\alpha}L^{\beta}$. Then, from equation (16), we can find an expression for $q$ as a function of $z$:

$$q=\frac{\rho}{\sigma}-\frac{\gamma^2-\zeta\sigma}{\zeta\sigma}z. \tag{17}$$

If we substitute (17) into equation (15) we get the following expression for $z$ as a function of $H_Y$:

$$z=\frac{\zeta[\delta\sigma(\zeta-1)H-\gamma(\rho+\delta\sigma H)]}{\gamma^2(\sigma-\gamma)}-\frac{\zeta\sigma[\delta\Lambda(\zeta-\gamma-1)-\delta]}{\gamma^2\Lambda(\sigma-\gamma)}H_Y. \tag{18}$$

Finally, substituting (17) and (18) into (14), we obtain an expression for $H_Y$ as a function of the parameters of the model:

$$\mathrm{H}_Y^* = \frac{\Lambda}{\delta} \cdot \frac{\delta H[\sigma(\gamma-\zeta)-(1-\zeta)]-\rho(1-\gamma)}{\Lambda[\sigma(\gamma-\zeta)-(1-\zeta)]-(1-\gamma)}. \tag{19}$$

Values for $y^*$ and $q^*$ can then be found using (18) and (17). Since there is a unique $H_Y^*$ there are also unique $y^*$ and $q^*$, and the steady state is therefore unique. This implies a rate of growth of knowledge for this economy of:

$$\mu_A = \delta(H - H_Y^*) = \frac{(1-\gamma)(\delta H - \Lambda\rho)}{(1-\gamma)+\Lambda(1-\zeta)-\sigma\Lambda(\gamma-\zeta)},$$

which, if $\zeta = 1$, is, of course, the same found by Romer.

We now look at the conditions needed for $H_Y^*$ to lie between zero and its maximum feasible value of $H$. It is easy to show that this is true if the parameters of the model lie in either one of the following two sets

$$\Theta_1 \equiv \left\{ \rho \geq \frac{\delta}{\Lambda} H \text{ and } \sigma < \frac{\rho(1-\gamma)+\delta(1-\zeta)H}{\delta(\gamma-\zeta)H} \right\},$$

$$\Theta_2 \equiv \left\{ \rho \leq \frac{\delta}{\Lambda} H \text{ and } \sigma > \frac{\rho(1-\gamma)+\delta(1-\zeta)H}{\delta(\gamma-\zeta)H} \right\}.$$

If the parameters are such that they do not lie in either one of these two sets, we have a corner solution; from now on, however, we assume that either $\Theta_1$ or $\Theta_2$ contain the parameters of the model.

Having established that this model has a unique steady state, we can move on and study its dynamic properties. An equilibrium is defined here as a collection of functions of time $\{K(t), A(t), C(t), H_Y(t)\}$ consistent with the maximization problems of each class of agents and with market clearing. Moreover, since we want to focus on balanced growth paths, the functions are such that, as $t$ goes to infinity, $K$, $A$ and $C$ grow at a constant rate, and $H_Y$ is constant. The initial values of physical capital and knowledge, $K(0)$ and $A(0)$, respectively, are exogenously given, while the agents are free to choose the initial values of consumption, $C(0)$, and human capital used in the production of the final good, $H_Y(0)$. In terms of the three-dimensional model, this means that $y(0)$ is fixed, while $H_Y(0)$ and $q(0)$ are free. It is then interesting to find out whether this model has a unique equilibrium, completely determined by the given initial conditions, or whether there exists a continuum of such equilibria, making it impossible to predict which equilibrium the economy will select, based on the available information. In the latter case we say that the equilibrium is indeterminate.

Indeterminacy of the equilibrium can be useful to interpret some of the available empirical evidence on different growth experiences of apparently similar countries, as discussed in Benhabib and Perli (1994), without relying on differences in fixed effects. It is known from Boldrin and Rustichini (1994) that it appears under very mild conditions in two-sector exogenous and endogenous growth models in *discrete* time with only one choice variable (typically consumption). However, there are not comparable theoretical results for *continuous* time models, such as the one at issue here, even though in the literature there are quite a few examples of such models that exhibit indeterminacy, both with one sector and two choice variables (e.g. Benhabib & Farmer, 1994) and with two sectors and one or two choice variables (e.g. Benhabib & Perli, 1994; Xie, 1994). The next proposition shows that Romer's model considered here may also exhibit an indeterminate equilibrium. For notational convenience define $\theta$ as the vector of all the parameters of the model, i.e. $\theta \equiv (\alpha, \beta, \gamma, \delta, \eta, \sigma, \rho, \zeta)$. Then we can prove the following:

PROPOSITION 1: *suppose that the model has an interior steady state; then a necessary condition for the equilibrium of Romer's model to be indeterminate is that* $\theta \in \Theta_1$.

PROOF: our strategy is to linearize the three-dimensional system of equations (14)–(16) around its steady state, and show that the Jacobian matrix evaluated at the steady state can have two negative roots only if $\theta \in \Theta_1$.

The Jacobian matrix evaluated at $(y^*. H_Y^*, q^*)$ can be written as:

$$J^* = \begin{pmatrix} J_{11} & \dfrac{y^*}{H_Y^*}\left(\dfrac{\alpha}{\gamma-1}J_{11} + \delta H_Y^*\right) & -y^* \\[2ex] \dfrac{\gamma(\gamma-\zeta)}{\zeta(\alpha-1)}\dfrac{J_{11}H_Y^*}{y^*} & \dfrac{\alpha\gamma(\gamma-\zeta)}{\zeta(\alpha-1)}J_{11} - \dfrac{\delta\Lambda(\zeta-\gamma-1)-\delta}{\Lambda(\alpha-1)}H_Y^* & \dfrac{\gamma}{\alpha-1}H_Y^* \\[2ex] \dfrac{\gamma^2-\zeta\sigma}{\zeta\sigma}\dfrac{J_{11}q^*}{y^*} & \dfrac{\alpha(\gamma^2-\zeta\sigma)}{\zeta\sigma(\gamma-1)}\dfrac{J_{11}q^*}{H_Y^*} & q^* \end{pmatrix},$$

where $J_{11} = (\gamma-1)z^*$. The key quantity to understand the transitional dynamics implied by the model is the determinant of $J^*$.

The BGP is locally indeterminate if there are two eigenvalues with negative real parts and one positive, which implies that the determinant must be positive. After some passages we see that the determinant of the Jacobian matrix can be written as:

$$\mathrm{Det}\,J^* = \frac{\delta\gamma^2[\zeta\Lambda(\sigma-1)+1+\Lambda-\gamma(1+\sigma\Lambda)]}{\zeta\sigma\Lambda(\alpha-1)(\gamma-1)}\,J^*_{11}H^*_Y q^*.$$

We see then that $\mathrm{Det}\,J^*$ is positive if:

$$\sigma < \frac{(1-\gamma)+\Lambda(1-\zeta)}{\Lambda(\gamma-\Lambda)}.$$

Since:

$$\frac{(1-\gamma)+\Lambda(1-\zeta)}{\Lambda(\gamma-\Lambda)} > \frac{\rho(1-\gamma)+\delta(1-\zeta)H}{\delta(\gamma-\zeta)H}$$

if $\rho \geq (\delta/\Lambda)H$, we can conclude that we have an interior steady state and a positive determinant when the restriction $\theta \in \Theta_1$ is satisfied. ■

The fact that the determinant is positive when $\theta \in \Theta_1$ is not sufficient to guarantee that the equilibrium is indeterminate in the sense defined above. In principle it is possible that the Jacobian matrix has three positive roots, which would imply that the steady state is dynamically unstable. This does not mean that in this case, however, the model predicts explosive growth: since the determinant does not become zero and remains always finite, the only way in which the dynamic properties of the steady state can change is through a Hopf Bifurcation, which implies that cyclical trajectories emerge, the stability properties of which remain to be investigated.

On the other hand, the restriction $\theta \in \Theta_1$ automatically excludes the possibility that the BGP is saddle-path stable, and hence the possibility that the BGP is determinate; there can be a unique equilibrium, however, but if so it has necessarily to be a cycle. An immediate corollary of proposition 1 is that it is necessary for the BGP to be determinate that $\theta \in \Theta_2$. Again, this is not sufficient for determinacy, since we could have a case of complete dynamic stability of the steady state, i.e. three negative eigenvalues. Numerical simulations, however, seem to exclude this case.

We note also that there cannot be indeterminacy if there is not enough complementarity between the intermediate goods, i.e. if $\zeta$ is not sufficiently greater than one. In particular, if $\zeta = 1$, which is the assumption implicit in the original Romer paper (1990), we

TABLE 1 $\alpha=0{\cdot}34$, $\beta=0{\cdot}17$, $\delta=0{\cdot}05$, $\rho=0{\cdot}06$, $\zeta=2$, $\eta=0{\cdot}34$

| $\sigma$ | $H_Y^*$ | $r^*$ | $\mu_A$ | $\mu_K$ | *roots* |
|---|---|---|---|---|---|
| 0·10 | 0·806 | 0·037 | 0·009 | 0·028 | $--+$ |
| 0·15 | 0·739 | 0·039 | 0·013 | 0·038 | $--+$ |
| 0·20 | 0·601 | 0·042 | 0·021 | 0·063 | $--+$ |
| 0·25 | 0·155 | 0·054 | 0·041 | 0·122 | $--+$ |

see that a necessary condition for indeterminacy would be that $\sigma<-\rho/(\delta H)$, which clearly contrasts with the fact that $\sigma$ has to be positive. We now report some results of our simulations. Since this is not a one-sector model with perfect competition, the exponents of $K$, $H_Y$ and $L$ in the production function cannot be interpreted as the shares of capital, human capital and labour, respectively. To obtain plausible values for those parameters, therefore, we use the following criterion: it is known from various other works (see, e.g., Lucas, 1988) that the share of capital in profits is about 0·25 to 0·33. The share of *total* labour, skilled and unskilled, is therefore 0·67 to 0·75. Of this, as, for example, Mankiw, Romer and Weil (1990) point out, about 50% to 70% goes to skilled labour, i.e. to human capital. The shares of the three factors ($K$, $H$ and $L$) are given by the quantity of each factor used in production times its rental rate divided by the total factor remuneration. The latter is, of course, $rK+w_LL+w_HH$, or, substituting for $r$, $w_L$ and $w_H$, $(1-\alpha-\beta)^2/\zeta+\alpha+\beta\equiv D$. Then, the shares of capital, labour and human capital are respectively: $S_K=((1-\alpha-\beta)^2/\zeta)/D$, $S_L=\beta/D$ and $S_H=\alpha/D$. Then we impose the restrictions $S_K=0{\cdot}25$ as in Lucas (1988) and $S_H=2S_L$, which is in the range suggested by Mankiw, Romer and Weil (1990). Imposing, arbitrarily, $\zeta=2$, this implies that $\alpha=0{\cdot}34$, $\beta=0{\cdot}17$ and $\gamma=0{\cdot}49$. As for the other parameters, we also rely on Lucas (1988) in setting $\delta=0{\cdot}05$. We set the discount factor at a level of 0·06, which is in the range of those commonly used in the literature. Finally, we set $\eta=0{\cdot}34$ in order to have a markup of about 25%, and fix the total level of human capital to 1. Table 1 reports the results of the simulations using these parameters and the indicated values of $\sigma$. We see that both the interest rate and the growth rates on the BGP are very realistic. The problem is, of course, that $\sigma$ is quite low. From proposition 1 we know that a necessary condition for indeterminacy is that $\theta\in\Theta_1$; with the above parameter values, this translates into $\rho>0{\cdot}055$ and $\sigma<0{\cdot}26$ (the values are, of course, approximate). With values of $\sigma$ bigger than that, we have a corner solution if we keep $\rho$ fixed. To have an interior solution we have to lower the value of $\rho$, which would bring the parameter vector inside $\Theta_2$, a necessary condition

TABLE 2 $\alpha=0{\cdot}34$, $\beta=0{\cdot}17$, $\delta=0{\cdot}05$, $\rho=0{\cdot}05$, $\zeta=2$, $\eta=0{\cdot}34$

| $\sigma$ | $H_Y^*$ | $r^*$ | $\mu_A$ | $\mu_K$ | roots |
|---|---|---|---|---|---|
| 0·50 | 0·855 | 0·036 | 0·007 | 0·021 | $-++$ |
| 1·00 | 0·958 | 0·033 | 0·002 | 0·006 | $-++$ |
| 1·50 | 0·975 | 0·032 | 0·001 | 0·003 | $-++$ |
| 2·00 | 0·983 | 0·031 | 0·001 | 0·003 | $-++$ |

for determinacy. Table 2 reports the root structure of the Jacobian matrix using the same parameter values as in Table 1, with the exception that $\rho$ is now 0·05.

## 3. Endogenous human capital

As seen above, in the original Romer model, the total levels of human capital and labour in the economy are assumed, for simplicity, to be exogenously given. In this section we extend the model in the direction of making those levels depend on the decisions of the optimizing agents.

### 3.1. DESCRIPTION OF THE MODEL

As in the previous section, we still assume that the agents supply their time inelastically; now, however, we allow them to decide how to use that time: they can either use it to accumulate skills (human capital) or they can use it to supply unskilled labour. In the following model, therefore, human capital is produced using only the part of time the workers decide to devote to its production. We assume that this simplified production function of human capital exhibits non-increasing returns in the time allocated; if we call $T$ the amount of time devoted to the production of skills, the following functional form is analytically very convenient:

$$H=H(T)=T^{\varepsilon} \tag{20}$$

with $\varepsilon\leq 1$. On the other hand, unskilled labour does not need to be produced, and therefore it is assumed to be equal to the amount of time not used to produce human capital, i.e. $L=L(T)=1-T$ if we normalize the total available non-leisure time to unity.

Since the non-leisure time is supplied inelastically, neither labour nor human capital appear in the utility function; the agents maximize the same lifetime discounted utility function used in the previous section, subject to the same intertemporal budget constraint. Now, however, they recognize that human capital and labour are functions of the time that they decide to dedicate to each of them. Formally, their problem is thus:

$$\max_{C(t)} \int_0^\infty \frac{C^{1-\sigma}-1}{1-\sigma} \mathrm{e}^{-\rho t} \mathrm{d}t$$

subject to

$$\int_0^\infty (C - w_H H - w_L L - rK)\mathrm{e}^{-\int_0^t r(s)\mathrm{d}s} \mathrm{d}t = 0.$$

This problem can be solved using the same standard technique as in Section 2. The Hamiltonian is:

$$\mathscr{H} = \frac{C^{1-\sigma}-1}{1-\sigma} \mathrm{e}^{-\rho t} + \lambda(rK + w_H H + w_L L - C)\mathrm{e}^{-\int_0^t r(s)\mathrm{d}s} \mathrm{d}t = 0.$$

And the corresponding first-order necessary conditions are:

$$\begin{aligned} C^{-\sigma}\mathrm{e}^{-\rho t} &= \lambda \mathrm{e}^{-\int_0^t r(s)\mathrm{d}s} \\ \lambda(w_H \varepsilon T^{\varepsilon-1} - w_L)\mathrm{e}^{-\int_0^t r(s)\mathrm{d}s} &= 0, \\ \dot{\lambda} &= 0. \end{aligned}$$

From these conditions we can derive the law of motion of consumption over time, which is again $\dot{C}/C = (r(t) - \rho)/\sigma$, and obtain the following relationship between the rental rates and the time spent accumulating human capital:

$$w_L = \varepsilon w_H T^{\varepsilon-1}. \tag{21}$$

Since both unskilled labour and human capital are employed in the final good sector, which is competitive, their rental rates are equal to their respective marginal products, i.e. $w_L = \beta \eta^{-\gamma} K^{\gamma} A^{\zeta-\gamma} H_Y^{\alpha} L^{\beta-1}$ and $w_H = \alpha \eta^{-\gamma} K^{\gamma} A^{\zeta-\gamma} H_Y^{\alpha-1} L^{\beta}$ (the production function of the consumption good is, of course, still the same as in Section 2). Substituting these two expressions into equation (21), and using the fact that $L = 1 - T$, we can obtain the following expression for $H_Y$:

$$H_Y = \frac{\alpha\varepsilon}{\beta} T^{\varepsilon-1}(1-T). \tag{22}$$

Equation (22) can then be used to eliminate $H_Y$ from the laws of motion of $K$ and $A$. Substituting also the expressions for $H$ and $L$, we can write:

$$\dot{K} = \psi K^{\gamma} A^{\zeta-\gamma} T^{\alpha(\varepsilon-1)}(1-T)^{1-\gamma} - C, \tag{23}$$

$$\dot{A} = \frac{\delta}{\beta} A T^{\varepsilon-1}[\beta T - \alpha\varepsilon(1-T)], \tag{24}$$

where $\psi \equiv \eta^{-\gamma}(\alpha\varepsilon/\beta)^{\alpha}$. As in the previous section, we can obtain an expression for the price of each intermediate good from the typical final sector firm's maximization problem, and use it to write the interest rate as a function of $K$, $A$ and $T$ from the intermediate sector firm's maximization problem. It is easy to see that the expression for $r$ so obtained is again equation (4). We can then use this expression, with (22) substituted for $H_Y$, to write the law of motion of consumption as:

$$\frac{\dot{C}}{C} = \frac{\varphi}{\sigma} K^{\gamma-1} A^{\zeta-\gamma} T^{\alpha(\varepsilon-1)}(1-T)^{1-\gamma} - \frac{\rho}{\sigma}, \tag{25}$$

where $\varphi \equiv (\eta^{-\gamma}\gamma^2/\zeta)(\alpha\varepsilon/\beta)^{\alpha}$.

To obtain an equation for $\dot{T}/T$ we follow exactly the same steps used in Section 2 to get an expression for $\dot{H}_Y/H_Y$. From the equalization of the rental rate of human capital in the final and research sectors, we can write $P_A$, the price of each design, as:

$$P_A = \frac{\alpha\eta^{-\gamma}}{\delta}\left(\frac{\alpha\varepsilon}{\beta}\right)^{\alpha-1} K^{\gamma} A^{\zeta-\gamma-1} T^{(\alpha-1)(\varepsilon-1)}(1-T)^{-\gamma}.$$

Taking logarithms and differentiating with respect to time gives:

$$\frac{\dot{P}_A}{P_A} = \gamma\frac{\dot{K}}{K} + (\zeta-\gamma-1)\frac{\dot{A}}{A} + (\alpha-1)(\varepsilon-1)\frac{\dot{T}}{T} + \gamma\frac{T}{1-T}\frac{\dot{T}}{T}. \tag{26}$$

On the other hand, we can use equation (8) to obtain another expression for $\dot{P}_A/P_A$:

$$\frac{\dot{P}_A}{P_A} = \varphi K^{\gamma-1} A^{\zeta-\gamma} T^{\alpha(\varepsilon-1)}(1-T)^{1-\gamma} - \frac{\delta}{\Lambda} T^{\varepsilon-1}(1-T), \tag{27}$$

where it is easy to verify that now $\Lambda \equiv (\beta\zeta)/[\gamma\varepsilon(\zeta-\gamma)]$. Equating (26) and (27) yields the following expression for $\dot{T}/T$:

$$\begin{aligned}\frac{\dot{T}}{T} = \frac{1}{G(T)}\cdot\Bigg\{&\frac{\gamma(\gamma-\zeta)}{\zeta}\psi K^{\gamma-1} A^{\zeta-\gamma} T^{\alpha(\varepsilon-1)}(1-T)^{1-\gamma} + \gamma\frac{C}{K} \\ &-\delta T^{\varepsilon-1}\left[(\zeta-\gamma-1)T + \frac{\beta-\alpha\varepsilon\Lambda(\zeta-\gamma-1)}{\Lambda\beta}(1-T)\right]\Bigg\},\end{aligned} \tag{28}$$

where $G(T) \equiv [(\alpha-1)(\varepsilon-1)(1-T)+\gamma T]/(1-T)$. In this way our modified model is completely characterized by the four differential equations (23), (24), (25) and (28).

### 3.2. TRANSITIONAL DYNAMICS

Once again, we can reduce the dimension of the system to three using the same change of variables as in Section 2. In particular we define $y \equiv \psi K^{\gamma-1} A^{\zeta-\gamma}$ and $q \equiv C/K$. Then the reduced system is:

$$\begin{aligned}\frac{\dot{y}}{y} = {} & (\gamma-1) y T^{\alpha(\varepsilon-1)}(1-T)^{1-\gamma} - (\gamma-1)q \\ & + \frac{\delta(\zeta-\gamma)}{\beta} T^{\varepsilon-1}[\beta T - \alpha\varepsilon(1-T)],\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\dot{T}}{T} = {} & \frac{1}{G(T)} \cdot \left\{ \frac{\gamma(\gamma-\zeta)}{\zeta} y T^{\alpha(\varepsilon-1)}(1-T)^{1-\gamma} + \gamma q \right. \\ & \left. - \delta T^{\varepsilon-1}\left[(\zeta-\gamma-1)T + \frac{\beta - \alpha\varepsilon\Lambda(\zeta-\gamma-1)}{\Lambda\beta}(1-T)\right]\right\},\end{aligned} \tag{30}$$

$$\frac{\dot{q}}{q} = \omega y T^{\alpha(\varepsilon-1)}(1-T)^{1-\gamma} - \frac{\rho}{\sigma} + q, \tag{31}$$

with $\omega \equiv (\gamma^2 - \zeta\sigma)/(\zeta\sigma)$. The next step consists in finding the steady state of the system of equations (29)–(31). For this purpose, define $z \equiv y T^{\alpha(\varepsilon-1)}(1-T)^{1-\gamma}$; then from equation (31) we can write:

$$q = \frac{\rho}{\sigma} - \omega z$$

Substituting this into equation (29) we can find an expression for $z$ as a function of $T$ only:

$$z = M_1 - M_2 T^{\varepsilon-1}[\beta T - \alpha\varepsilon(1-T)],$$

where, for convenience of notation, we defined the two constants $M_1 \equiv \rho/[\sigma(1+\omega)]$ and $M_2 \equiv [\delta(\zeta-\gamma)]/[\beta(\gamma-1)(1+\omega)]$. In this way both $q$ and $z$ are functions of $T$ alone; we can therefore substitute their expressions into equation (30) and obtain the following function of $T$:

TABLE 3 $\alpha=0{\cdot}3$, $\beta=0{\cdot}3$, $\delta=0{\cdot}08$, $\rho=0{\cdot}05$, $\zeta=4$, $\eta=0{\cdot}125$, $\varepsilon=1$

| $\sigma$ | $L^*$ | $T^*$ | $H^*$ | $H_Y^*$ | $\mu_A$ | $\mu_K$ | *roots* |
|---|---|---|---|---|---|---|---|
| 0·1 | 0·497 | 0·503 | 0·503 | 0·497 | 0·0005 | 0·0033 | $--+$ |
| 0·3 | 0·495 | 0·505 | 0·505 | 0·495 | 0·0008 | 0·0051 | $--+$ |
| 0·6 | 0·484 | 0·516 | 0·484 | 0·516 | 0·0025 | 0·0154 | $--+$ |
| 0·7 | 0·437 | 0·563 | 0·437 | 0·563 | 0·0100 | 0·0600 | $--+$ |
| 0·8 | 0·531 | 0·469 | 0·531 | 0·469 | 0·0000 | 0·0000 | c.s. |

c.s. = corner solution.

TABLE 4 $\alpha=0{\cdot}45$, $\beta=0{\cdot}3$, $\delta=0{\cdot}05$, $\rho=0{\cdot}025$, $\zeta=4$, $\eta=0{\cdot}07$, $\varepsilon=0{\cdot}5$

| $\sigma$ | $L^*$ | $T^*$ | $H^*$ | $H_Y^*$ | $\mu_A$ | $\mu_K$ | *roots* |
|---|---|---|---|---|---|---|---|
| 0·1 | 0·552 | 0·448 | 0·670 | 0·619 | 0·0025 | 0·0127 | $--+$ |
| 0·3 | 0·543 | 0·457 | 0·676 | 0·602 | 0·0037 | 0·0187 | $--+$ |
| 0·6 | 0·467 | 0·533 | 0·723 | 0·480 | 0·0125 | 0·0625 | $--+$ |
| 0·7 | 0·001 | 0·999 | 0·999 | 0·001 | 0·0500 | 0·2500 | $--+$ |
| 0·8 | 0·700 | 0·300 | 0·547 | 0·960 | 0·0000 | 0·0000 | c.s. |

c.s. = corner solution.

$$f(T)=\frac{\gamma M_1(\zeta-\gamma-\zeta\omega)}{\zeta}-\frac{\gamma M_2(\zeta-\gamma-\zeta\omega)}{\zeta}T^{\varepsilon-1}[\beta T-\alpha\varepsilon(1-T)]$$
$$+\frac{\gamma\rho}{\sigma}-\delta T^{\varepsilon-1}\left[(\zeta-\gamma-1)+\frac{\beta-\alpha\varepsilon\Lambda(\zeta-\gamma-1)}{\beta\Lambda}(1-T)\right]. \tag{32}$$

The values of $T$ such that $f(T)=0$ give the steady states of our modified model.

In general, $f(T)$ is impossible to solve analytically, since $T$ appears in it raised to the powers of $\varepsilon$ and $\varepsilon-1$. We will have, therefore, to resort to numerical calculations, some of which are reported in the following tables.

In Table 3 we report the results obtained using a linear production function for human capital, i.e. setting $\varepsilon=1$. The other parameters are $\alpha=0{\cdot}34$, $\beta=0{\cdot}17$ and $\gamma=0{\cdot}49$, as in Section 2.2, $\delta=0{\cdot}05$, $\rho=0{\cdot}05$, $\zeta=2$ and $\eta=0{\cdot}34$. The value for $\eta$ has been chosen in such a way that the resulting markup is 25%, which is a value usually considered plausible. In Table 4 we use instead an $\varepsilon$ of 0·5. The results shown in the two tables are very similar: we obtain an indeterminate steady state only for values of $\sigma$ equal to or less than 0·7, while for higher values we obtain a corner solution. This result resembles very closely what we got, analytically, for the standard Romer model in the previous section. In order to have a determinate BGP we not only need a high $\sigma$, but also a higher discount factor. Note that we cannot say anything, at this point, about the stability properties of the steady state when we have a corner solution.

We could get indeterminacy also for higher values of $\sigma$ if we increased $\zeta$, but in any case $\sigma$ could never be bigger than or equal to one, which seems to be the upper bound for $\sigma$ to have an interior steady state when $\zeta \to \infty$. The problem with a high $\zeta$ is that the discrepancy between the growth rate of knowledge, $\mu_A$, and of physical capital, $\mu_K$, becomes larger and larger as $\zeta$ grows, which suggests that values of $\zeta$ above a certain threshold may not be very realistic. Note that in Benhabib and Perli (1994) two of us were able to obtain indeterminacy in the Lucas (1988) model for larger values of $\sigma$, because the total non-leisure time was also endogenized. We conjecture that a similar situation should occur with this model; given the complexity of the required analysis, however, we leave such investigation for future research.

## 4. Increasing returns in the production of human capital

We have assumed so far that the production function of human capital exhibits non-increasing returns. While this is a realistic hypothesis at the single-agent level, it may not be such at the aggregate level. Agents are usually thought to benefit from the average level of human capital present in the economy, which, in our context, is proportional to the average time spent in the production of human capital. It seems, therefore, appropriate to write equation (20) as:

$$H = T^{\varepsilon} T_{\alpha}^{\theta}.$$

With this alternative specification, following exactly the same steps as in Section 3.1, we get that the amount of human capital used in the consumption good sector is given by:

$$H_Y = \frac{\alpha\varepsilon}{\beta} T^{\varepsilon+\theta-1}(1-T)$$

and that the reduced three-dimensional system is now:

$$\begin{aligned}\frac{\dot{y}}{y} &= (\gamma-1) y T^{\alpha(\varepsilon+\theta-1)}(1-T)^{1-\gamma} - (\gamma-1)q \\ &\quad + \frac{\delta(\zeta-\gamma)}{\beta} T^{\varepsilon+\theta-1}[\beta T - \alpha\varepsilon(1-T)],\end{aligned}$$

TABLE 5 $\alpha=0{\cdot}34$, $\beta=0{\cdot}17$, $\delta=0{\cdot}05$, $\rho=0{\cdot}05$, $\zeta=2$, $\eta=0{\cdot}34$, $\varepsilon=1$

| $\sigma$ | $L^*$ | $T^*$ | $H^*$ | $H_Y^*$ | $\mu_A$ | $\mu_K$ | roots |
|---|---|---|---|---|---|---|---|
| 0·9 | 0·363 | 0·637 | 0·608 | 0·347 | 0·0182 | 0·0924 | $--+$ |
| 1·0 | 0·416 | 0·584 | 0·554 | 0·394 | 0·0127 | 0·0771 | $-++$ |
| 1·5 | 0·471 | 0·529 | 0·496 | 0·442 | 0·0044 | 0·0263 | $-++$ |
| 2·0 | 0·482 | 0·516 | 0·485 | 0·452 | 0·0032 | 0·0159 | $-++$ |

$$\frac{\dot{T}}{T}=\frac{1}{G_1(T)}\cdot\left\{\frac{\gamma(\gamma-\zeta)}{\zeta}yT^{\alpha(\varepsilon+\theta-1)}(1-T)^{1-\gamma}+\gamma q\right.$$
$$\left.-\delta T^{\varepsilon+\theta-1}\left[(\zeta-\gamma-1)T+\frac{\beta-\alpha\varepsilon\Lambda(\zeta-\gamma-1)}{\Lambda\beta}(1-T)\right]\right\},$$
$$\frac{\dot{q}}{q}=\omega yT^{\alpha(\varepsilon+\theta-1)}(1-T)^{1-\gamma}-\frac{\rho}{\sigma}+q,$$

where now $G_1(T)\equiv[(\alpha-1)(\varepsilon+\theta-1)(1-T)+\gamma T]/(1-T)$. Again, this system cannot in general be solved analytically; we therefore have to use numerical techniques. As before we can express $z$ and $q$ as a function of $T$, and obtain the following non-linear equation in $T$ only:

$$f(T)=\frac{\gamma M_1(\zeta-\gamma-\zeta\omega)}{\zeta}-\frac{\gamma M_2(\zeta-\gamma-\zeta\omega)}{\zeta}T^{\varepsilon+\theta-1}[\beta T-\alpha\varepsilon(1-T)]$$
$$+\frac{\gamma\rho}{\sigma}-\delta T^{\varepsilon+\theta-1}\left[(\zeta-\gamma-1)+\frac{\beta-\alpha\varepsilon\Lambda(\zeta-\gamma-1)}{\beta\Lambda}(1-T)\right],\quad(33)$$

where $\omega$, $M_1$ and $M_2$ are defined as in Section 3.2.

We note immediately that the shape of this function of $T$ depends crucially on the magnitude of the exponent $\varepsilon+\theta-1$: if it is smaller than one, we have a situation equivalent to the one described in Section 3.2, with only one steady state. If, however, $\varepsilon+\theta>1$, i.e. if the externality is such that the production of human capital exhibits increasing returns at the aggregate level, $f(T)$ would in general have two steady states. In the following we will concentrate on this last case. The numerical simulations that we conducted seem to indicate that the steady state with a lower $T$ is always a corner solution: the constraint that $H_Y\leq H$ seems to be always binding. The second steady state, however, can either be a corner solution or an interior solution, depending on the parameter values.

The stability properties of the higher steady state for the same parameter values used in Table 3 and a value of the externality $\theta$ of 0·1 are reported in Table 5 for various values of the intertemporal elasticity of substitution of consumption $\sigma$. We see that this steady state is locally indeterminate for values of $\sigma$ smaller than 1 (for some parameter values a BGP may not exist for low $\sigma$), and after that it becomes locally determinate. As far as the local indeterminacy of the equilibrium, these results confirm, of course, those obtained in Section 3.2: the steady state with a higher $T$ is locally indeterminate for relatively small values of $\sigma$. But now, however, while the equilibrium is *locally* determinate for higher values of $\sigma$, there is still no way of deciding which equilibrium (the one with a low $T$ or the one with a high $T$) the economy will select, for a given initial condition, based only on the available information; we have, therefore, a case of *global* indeterminacy. This situation is perhaps even more intriguing than local indeterminacy: while in that case, using local analysis, the model can predict at least the long-run behaviour of the economy (all the infinitely many equilibria have the same growth rate in the end), here nothing can be said at the beginning about where the economy will end up asymptotically. Of course, predictions will be precise once the initial choices of the consumption level and of the time allocated to the production of human capital are made.

As noted in the introduction, the economy can get caught in a growth trap because the agents coordinate their expectations on the low-level equilibrium; a switch to the high-level equilibrium would, therefore, seem to require that all the agents re-coordinate their expectations on the new equilibrium. In such a case the economy would "jump" to the high-level equilibrium and start its development process from there.

## Acknowledgements

The authors are indebted to the participants to the Workshop on Growth, held in Venice, 13–15 January, 1994, and in particular to the discussant, Alfredo Medio. Technical assistance from the C. V. Starr Center for Applied Economics at New York University is gratefully acknowledged.

## References

Benhabib, J. & Farmer, R.E.A. (1994). Indeterminacy and increasing returns. *Journal of Economic Theory*, **63**(1), 19–41.

Benhabib, J. & Perli, R. (1994). Uniqueness and indeterminacy: on the dynamics of endogenous growth. *Journal of Economic Theory*, **63**(1), 113–142.

Boldrin, M. & Rustichini, A. (1994). Indeterminacy of equilibria in models with infinitely lived agents and external effects. *Econometrica*, **62**, 323–342.

Farmer, R.E.A. & Guo, J.T. (1994). Real business cycles and the animal spirit hypothesis. *Journal of Economic Theory*, **63**(1), 42–72.

Gali, J. (1994). Monopolistic competition, business cycles, and the composition of aggregate demand. *Journal of Economic Theory*, **63**(1), 73–96.
Lucas, R.E. (1988). On the mechanics of economic development. *Journal of Monetary Economics*, **22**, 3–42.
Mankiw, N.G., Romer, D. & Weil, D.N. (1990). A contribution to the empirics of economic growth. *Quarterly Journal of Economics*, **429**, 407–437.
Mulligan, C.B. & Sala-i-Martin, X. (1993). Transitional dynamics in two-sector models of endogenous growth. *Quarterly Journal of Economics*, **108**, 739–773.
Romer, P. (1990). Endogenous technological change. *Journal of Political Economy*, **98**, S71–S102.
Xie, D. (1994). Divergence in economic performance: transitional dynamics with multiple equilibria. *Journal of Economic Theory*, **63**(1), 97–112.