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Abstract
Exploitation of non–renewable resources is an intensively studied field of environmental economics in the last century. Since the influential Hotelling's paper a huge progress is made in the depletable resources literature. Although a variety of methodologies is used in that problem's solutions a basic question of time inconsistency arises in the solution process. We show the sources of dynamical time inconsistency in a leader – follower game for which the buyer leads while the extractor follows and the players employ open loop strategies. Also we make use of Markovian informational structure, in a non – renewable resource Nash game, in order to extract strategies that are time consistent. Finally we enlarge the utility function space from the logarithmic utility to the utility functions that exhibits relative risk aversion with the same, with respect to time consistency, strategies.

Keywords: Non-renewable resources; time consistency; Markovian strategies; leader-follower.

JEL classification codes: Q30, C60, C61, C62, C70, C72

1. Introduction

Economists have given much attention to the role of natural resources in the operation of an economic system. The main interests in environmental history have been, at least in the beginning, the scarcity and exhaustion of natural resources. The systematic allocation of resources and the importance of markets were emphasized by the classical economists in the 19th century. Adam Smith (1776) gave attention to the dynamic effects of market. For Smith nature was generous and agriculture capable of offering outputs in excess of inputs and he did not consider as an obstacle to growth the resources scarcity problem. Malthus (1798, 1820) and Ricardo (1973) considered the land of a country as one of the main characteristics of its economy. Malthusian scarcity considers natural resources as homogeneous in quality while Ricardian considers them as varying in quality (Barnett and Morse, 1963).

In the first neoclassical models there is an absence in the production functions of any natural resources. Natural resources are introduced into neoclassical models of economic growth in the 1970s with the systematic investigation of optimal resource depletion. Marshall (1890) and the new neoclassical economists adopted an optimistic view of natural resource scarcity which holds till 1960s. Modern Marxism has also seen natural resource scarcity as a potential growth constraint without proposing any alternative view.

After the Victorian economists and in the beginning of the last century economists showed little or no concern for resource exhaustion. Modern theories of natural resources scarcity in the 1960s and 1970s were proposed by Hotteling (1931) and Ramsey (1928). There is no big difference between Marshall's and Hotteling's views on exhaustible resource depletion but we may say that Hotteling's theory is more completely developed (Halkos, 2007).

In Economics of nonrenewable resources, it is well known that when there is a fixed stock of an exhaustible resource, Pareto optimality requires the difference between price and marginal extraction cost rise at the interest rate. The above Hotelling condition is true only in competitive market equilibrium. A monopolist, however, supplies the resource in such a way that the difference between marginal revenue and marginal extraction cost rises at the interest rate, as well. This is a type of modified Hotelling condition in monopolistic situations. As a result, a monopolist will not supply in general the resource efficiently.

The analysis of the dynamics of an economy with natural resources (exhaustible and non) demands attention to the nature of the group of consumption (extraction) paths available in this economic system. This requires thoughtful consideration of the technology of resource use. An emerging conclusion is that if a natural resource is necessary and un-substitutable and at the same time is available in a finite amount then in every feasible path extraction must decline to zero.

In the present paper we propose a simple nonrenewable resource extraction model and we find the precise analytic forms of the Markovian equilibrium resource exploitation strategies that are strongly time consistent. As a consequence the closed loop Markov perfect Nash equilibrium is by definition a robust one. We enlarge the utility function space from the usual logarithmic utility function to the wider class of utility functions that exhibit relative risk aversion. In this way, we conclude about the relationships of the utility functions and discount rates and the number of players of the induced dynamic game. Also we use the most recent modern perspective of dynamic economic analysis that is the Hamilton – Jacobi – Bellman (hereafter HJB) equation of dynamic programming.

The structure of the paper is the following. Section 2 reviews the existing literature and states the problem into consideration while section 3 discusses the leader-follower formulation and time inconsistency in non-renewable resource economics. Section 4 present the proposed model and section 5 discusses the utility function in terms of risk aversion. The last section concludes the paper.

2. Literature review and statement of the problem.

In the Economics of non-renewable resources arisen by the famous article of Hotelling (1931) every resource that is mined without the possibility to regenerate, also including the forests, is meant as exhaustible resources. The discussion on the Hotelling's paper is based on the problem to find "the optimum social value of the resource" under full competitive extraction. That is in a time instant this quantity is defined by $u(q)=\int_0^q p(q)dq$, where the integrant is a decreasing function, while the upper limit of integration is the market consumed quantity. Consequently if one discounts the future utility with the discount rate γ, then the present value will be expressed as: $V=\int_0^T u[q(t)]e^{-\gamma t}dt$. In the same manner in an oligopolistic situation the same problem is set as the choice of the quantity $q(t)$ under the constraint $\int_0^\infty qdt = a$, that is the maximization of the present value $J=\int_0^\infty qp(q)e^{-\gamma t}dt$.

The above problems have been solved with the calculus of variation method and the main conclusion that was extracted of the celebrated Hotelling's paper is that perfect competition is able to induce a time trajectory that is a social optimum one, but the monopoly yields an extraction time path not only more conservative but also socially sub–optimal.

In the same model Stiglitz (1976) adopting a demand function of the form $p = f(t)q^{a-1}$ for one unit of quantity q, with $0 < a < 1$ and demand elasticity expressed as $\frac{1}{1-a}$, he concludes that in order to maximize the discounted revenues of one firm, extraction will be socially optimal, both under perfect competition and in a monopolistic environment, the Hotelling rule must hold. With Hotelling rule to represent the equation $\frac{\dot{p}}{p} = r$ holds, where r is the discount rate and $\frac{\dot{p}}{p}$ is the rate of the price increment. Substituting in the demand function the Hotelling condition yields finally the following necessary conditions that must hold under perfect competition and for monopolistic firms, as well:

$$\frac{\dot{q}}{q} = \frac{r - \frac{f'}{f}}{a-1}$$

together with the exhaustion condition of the resource:

$$\int_0^\infty q(t)\,dt \le S_0,$$

with S_0 to express the total stock of the resource.

However if the demand elasticity increases in time, that is an expected fact, the original Hotelling condition takes the following form:

$$\frac{\dot{p}}{p} = r - \frac{a'}{a}, \text{ with } a' > 0$$

The latter equation leads us to conclude that the price increment rate will be slower in the monopolistic firms than in the competitive ones. Consequently the latter implies that if the resource exhaustion condition $\int_0^{\infty} q(t)\,dt \le S_0$ holds then the resource extraction rate for the monopoly will be lower if the monopolistic firms follow more conservative extraction policies. The next figure compares the two extractions time paths for monopolistic and competitive firms respectively.

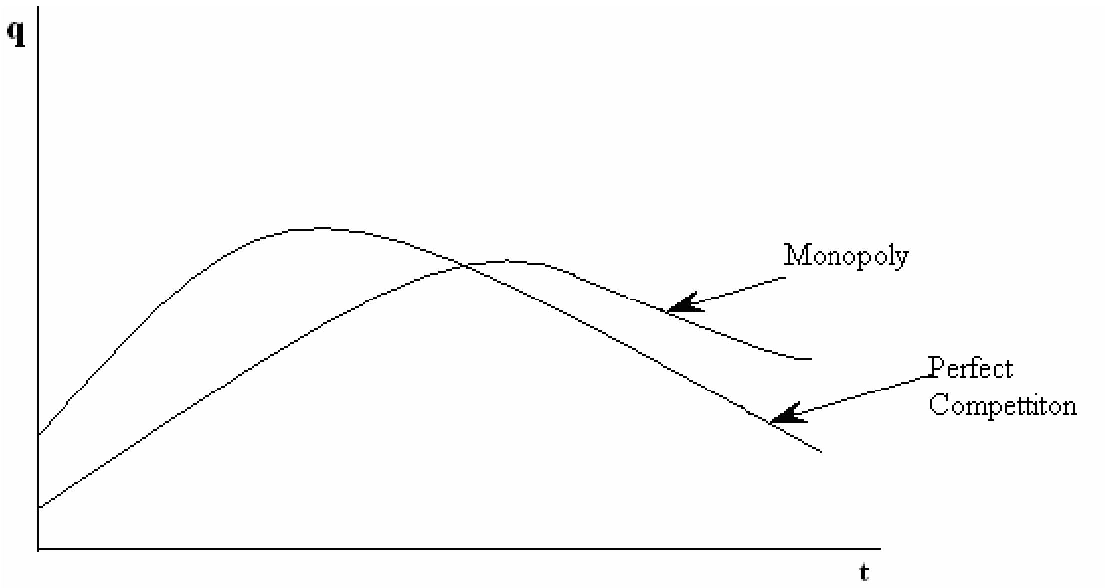

Figure 1: Extraction time paths under monopoly and perfect competition with an incremental demand elasticity.

A similar biased situation takes place for monopolistic extraction firms that follow a conservative extraction policy if the extraction costs are entered into the utility function. Assuming constant costs per extracted unit and decreasing with respect to time cost functions, that is $g(t)$ is the cost function and $g'(t) \le 0$, then monopolistic revenues will be $\int_0^{\infty} \left(fq^a - gq \right) e^{-rt}\,dt$ and after simplifications and rearrangement of the terms the Hotelling condition now becomes:

$$\frac{\dot{p}}{p}=r\left(1-\gamma_m\right)+\frac{\dot{g}}{g}\gamma_m$$

where $\gamma_m=\frac{g}{ap}<1$ is the quotient of the division of the extraction cost by the marginal revenue. Clearly the quantity γ_m is less than unit and if extraction cost decays quickly so does the market price.

The competitive market solution presupposes condition $\frac{\dot{p}}{p}=r\left(1-\gamma_c\right)+\frac{\dot{g}}{g}\gamma_c$ to hold, with $\gamma_c=\frac{g}{p}$. Consequently in any time instant t the corresponding price $p(t)$ will be the same in both competitive and monopolistic markets, since $\gamma_c(t)<\gamma_m(t)$ and the rate of price increment $\frac{\dot{p}}{p}$ for the competitive market will be higher from the monopolistic. The price curves have only one intersection point and the monopolist follows a more conservative extraction policy. The above extraction policies are depicted in figure 2.

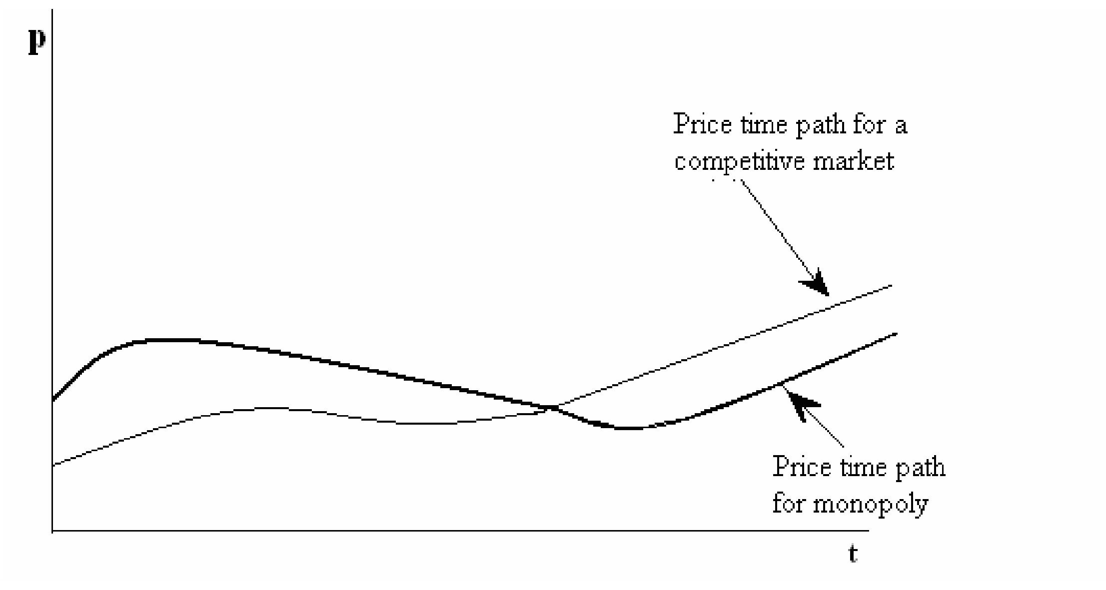

Figure 2: Price time paths for competitive market and monopoly.

3. Leader–follower formulation and time inconsistency in non-renewable resources economics

A considerable literature exists on the behavior of monopolistic or oligopolistic sellers of a non–renewable resource (Bergstrom et al., 1981; Dasgupta and Heal, 1979; Lewis et al. 1979; Ulph and Folie, 1980) but less progress has been made in analyzing imperfections on the buyer's side of the market. Karp (1984) studies the effect of allowing the buyers to exercise market power by using a tariff against sellers that assumed to behave competitively or monopolistically. While Kemp and Long (1980), in the same tariff argument, derive the open loop tariff for a non–renewable there is a considerable objection on the buyer's announced tariff. Precisely an importing country (modeled as a buyer) wants to revise the originally announced tariff. Kemp and Long point out that the open loop extracted tariff, in their model with constant extraction costs, is dynamically inconsistent. Karp (1984) in a different, with respect to extraction costs, model makes the assumption that the seller obtains no utility from consuming the resource, finds that the inconsistency on open loop tariff caused by the assumption of stock – dependence cost.

The well known leader – follower formulation can be adapted in the non – renewable resource modeling. The conflict between buyers (the users of the resource or even the importers) and sellers (the extractors of the resource) can be modeled as a Stackelberg game in which the buyers lead. The buyers choose a tariff and the sellers choose the rate of extraction. If the sellers control the rate of extraction in a non – cooperative Nash game, it would never be optimal for them to extract anything. If they committed themselves to extracting at a certain rate, the buyers could charge an arbitrary high tariff. Conversely if the sellers lead, consequently were allowed to choose the price at which they would sell the buyers would face an infinitely elastic supply curve, and the optimal price charged would be zero.

It is well known that the rationality of the follower, in a leader – follower game, implies that the open loop policy does not in general satisfies the principle of optimality (Kydland and Prescott, 1977), consequently that policy is not a time consistent one. In the non – renewable resource extraction game, for which the buyer leads, it is likely for the seller (the follower) would not believe an inconsistent policy even though the resulting payoff is less than with the open loop policy. On the other hand, if the buyer uses a consistent price charged against a competitive seller, the competitive rate of extraction results and if the buyer uses a consistent price against a monopolist, the resulting extraction path may be either more or less conservative than the extraction path under pure monopoly.

More formally the conflict between buyers and sellers in the resource game can be modeled, in the case of unit price charged, as follows. The buyer's payoff is the discounted stream of the difference between the utility of consuming at rate $x(t)$ and the payment $\left(P(x)-q\right)x$ and in the infinite date at which consumption terminates with discount rate ρ, buyer's (leader's) payoff may be written as

$$J_L = \int_0^\infty \left[u(x)-\left(P(x)-q\right)x\right]e^{-\rho t}dt \tag{1}$$

Taking now the follower's position his payoff maximization can be modeled as follows. Let $c(z)$ be the average cost of extracting a unit of the resource given that stock size is z then the instantaneous extraction cost is $c(z)x$. We assume $c'(z)\leq 0$ and $c''(z)\geq 0$ then the seller's payoff is the following

$$J_F = \int_0^{\infty} \left[P(x) - q - c(z) \right] x e^{-\rho t} dt \tag{2}$$

In this setting together with the assumption that the follower enjoys no utility from consuming the resource, the leader controls the quantity, $q(t)$, that buys and the follower controls the rate of its extraction $x(t)$. Both the players are constrained by the non negativity of the stock $z(t)$ with the justification that the resource reduces with the extraction rate that is the constraint:

$$\dot{z}(t) = -x(t),\ z(0) = z_0 \text{ given, } z(t) \geq 0 \quad \forall t \tag{3}$$

In the price path determination, the buyer takes into consideration the reaction of the seller.

In the case of a competitive seller open loop prices charged results in time inconsistent policies except when costs are constant. Indeed if the follower takes the announced leader's control variable $q(t)$ has to solve his problem for which the Hamiltonian formed as follows:

$H_F = \left[P(x) - q - c(z) \right] x e^{-\rho t} - \lambda(t) x$ and the resulting first order conditions

$$\frac{\partial H_F}{\partial x} = 0 \Rightarrow \left[P(x) - q - c(z) \right] e^{-\rho t} - \lambda(t) = 0 \tag{4}$$

(with respect to control) and

$$-\frac{\partial H_F}{\partial z} = \dot{\lambda} \Rightarrow \dot{\lambda} = e^{-\rho t} c'(z) x \tag{5}$$

(with respect to state variable), with the co–state $\lambda(t)$ to denote the follower's marginal utility of an additional unit of stock at time t. It is worth noting that a prediction of the arisen time inconsistency is that the follower's co–state variable $\lambda(t)$ is dependent upon leader's control variable $q(t)$.

Following Simaan and Cruz (1973) leader's problem can be converted into an optimal control problem that he solves. The intuition behind this approach is straightforward. The leader treats the follower's first order condition as a constraint and the follower's co–state $\lambda(t)$ as a state (with equation of motion given by the second of the two first order conditions). The elimination of the leader's control using the first of the first order conditions yields $q = P(x) - c(z) - e^{\rho t}\lambda$ and substitution into the leader's payoff (1) the payoff can be written as

$$J_L = \int_0^{\infty} \left[\left[u(x) - c(z)x \right] e^{-\rho t} - \lambda x \right] dt \tag{6}$$

Note the last term in equation (6) gives the present value of the total instantaeous rent which the leader pays. In order to find the source of time inconsistency the next step is to solve the leader's optimal control problem which is defined by maximization of its payoff, given by (6) subject to the original resource constraint $\dot{z}(t) = -x(t)$ and the follower's constraint given by its maximization problem, that is the constraint $\dot{\lambda} = e^{-\rho t} c'(z) x$. The Hamiltonian and first order conditions of the latter problem are given by the following equations:

$$H_L = e^{-\rho t} \left[u(x) - c(z)x \right] - \lambda x - \mu_1 x + \mu_2 e^{-\rho t} c'(z) x \tag{7}$$

$$\frac{\partial H_L}{\partial x} = e^{-\rho t} \left[P(x) - c(z) \right] - \lambda - \mu_1 x + \mu_2 e^{-\rho t} c'(z) = 0 \tag{8}$$

$$-\frac{\partial H_L}{\partial z} = \dot{\mu}_1 \Rightarrow e^{-\rho t} \left[c'(z) - \mu_2 c''(z) \right] x = \dot{\mu}_1 \tag{9}$$

$$-\frac{\partial H_L}{\partial \lambda} = \dot{\mu}_2 \Rightarrow x = \dot{\mu}_2 \tag{10}$$

with μ_1, μ_2 to denote the co–states of the states z and λ respectively. Setting the boundary condition $\mu_2(0) = 0$, equation (10), which gives the marginal value to the leader of an increase of the follower's rent, has the solution $\mu_2(t) = \overline{z} - z(t) > 0$. The last condition implies that the leader would like to increase (revise) the follower's rent and the dynamic time inconsistency arises because for the open loop information structure a commitment against deviations had made.

In the next sections we expose a simple nonrenewable resources extraction model in the lines of the previous research. For the proposed model the players of the Nash game follows Markovian strategies, that are by definition time consistent, and the aim is to extract some useful conclusions with respect to the player's risk aversion and for his maximized utility as well. The methodology used is that of the conjectured value function and strategies and of the maximized Hamiltonian with an auxiliary variables system.

4. The simple model

Assume an absolutely free natural resource, for example a type of fuel that is extracted simultaneously from N firms or N countries (players of the game). With $x(t)$ we denote the resource stock at time t and $c_i(t)$ denotes the extraction rate for player i at the same time. As it is normal we assume that the extraction rate at every time except the bootstrap of the game (zero time) is positive, $c_i(t) \geq 0$, and if the resource stock at time t is zero, $x(t) = 0$, the extraction process ends up, that is $c_i(t) = 0$.

The nonrenewable resource exhaustion equation is described by the following differential equation of motion which simply claims the nonrenewable resource stock to exhaust with its extraction rate:

$$\dot{x}(t) = -\sum_{i=1}^{N} c_i(t) \tag{11}$$

Every player enjoys a utility function $u(c_i)$, well defined for every $c_i > 0$ which is concave and increment with the property $u(0) = -\infty$. The utility function is discounted with the discount rate $\rho > 0$, so every player's objective function is given as:

$$J_i = \int_0^{\infty} e^{-\rho t} u(c_i(t))\, dt \tag{12}$$

In order to extract every player's Markovian strategy we make use of the conjectured method for strategies and the value functions for problem (12) under constraint (11).

Proposition 1

In the symmetric resource extraction game with N firms and utility functions given in logarithmic form, $u(c_i(t)) = \ln c_i$, *every player follows linear Markovian strategies that are independent of the number of players.*

Proof

Supposing that all rivals of the arbitrary $i \in N$ player use the stationary Markovian strategy ϕ the HJB equation for the arbitrary $i \in N$ player is given by

$$\rho V_i(x) = \max_{c_i} \left[\ln(c_i) + V_i'(x)\left(-c_i - (N-1)\phi(x)\right) \right] \tag{13}$$

In order to find player's i best response we maximize the right hand side of (13) with respect to extraction rate c_i, that is

$$\frac{\partial\left[\ln(c_i)+V_i'(x)\left(-c_i-(N-1)\phi(x)\right)\right]}{\partial c_i}=0\Rightarrow\frac{1}{c_i}=V_i'(x)\Rightarrow$$
$$\Rightarrow c_i=\phi(x)=\left[V_i'(x)\right]^{(-1)} \tag{14}$$

We conjecture value functions of the form

$$V_i(x)=A\ln x+B \tag{15}$$

and
$$V_i'(x)=\frac{A}{x}\Rightarrow\left[V_i'(x)\right]^{(-1)}=\frac{x}{A} \tag{16}$$

Combining (14) and (16) the equilibrium Markovian strategies are given by the expression

$$c_i=\phi(x)=\left[V_i'(x)\right]^{(-1)}=\frac{x}{A} \tag{17}$$

which is clearly independent of the number of players N.

Substitution of the strategy given by (14) into the maximized HJB equation (13) yields

$$\rho V_i(x)=\left[\ln(c_i)+V_i'(x)\left(-c_i-(N-1)\phi(x)\right)\right] \tag{18}$$

Further substitution into (18) of the equilibrium strategy (17) yields the following differential equation

$$\rho V_i(x)=\left[\ln\frac{x}{A}+V_i'(x)\left(-\frac{x}{A}-(N-1)\frac{x}{A}\right)\right]=\left[\ln\frac{1}{V'(x)_i}-N\right] \tag{19}$$

with solution

$$V(x) = \frac{\ln(x\rho) - N}{\rho} - \Omega \tag{20}$$

Where Ω is the integration constant. Clearly the value function is dependent upon the number of the players and on the extraction rate.

5. Utility function relative to risk aversion

We discus now the same model entering into the class of games where utility functions of the symmetric players exhibit the relative risk aversion. For this reason we consider the model as described by equation (12) under constraint (11). The present value of the corresponding Pontryagin function will be:

$$P_i(u, \lambda_i, c) = P_i(c, \lambda_i, u_1, ..., u_n) = u_i(c_i) - \lambda_i \sum_{i=1}^{n} c_i \tag{21}$$

Extraction rates are supposed to be non-negative $c_i \geq 0$, so the maximization of function (21) yields

$$c_i = \upsilon_i(\lambda_i) = \begin{cases} \lambda_i = u_i'(c_i) = \dfrac{du_i(c_i)}{dc_i} & c_i > 0 \\ \lambda_i = 0 \quad c_i = 0 \end{cases} \tag{21a}$$

and the latter can be solved for $c_i = \upsilon_i(\lambda_i)$. Substitution into (21) gives the following Hamiltonian function of the game:

$$H_i(\lambda) = u_i(\upsilon_i(\lambda)) - \lambda_i \sum_{j=1}^{n} \upsilon_j(\lambda_j) \tag{22}$$

For symmetric utility functions the simplified Hamiltonian takes the form

$$H(\lambda) = u(\upsilon(\lambda)) - N\lambda\upsilon(\lambda) \tag{23}$$

Player's i value function is $V_i = \int_0^{\infty} e^{-\rho t}\left[u_i\left(c_i\right) - \sum_{i=1}^{n} c_i\right]dt$, which shows the player's fee for the sub–game that begins at time zero. If the value function is differentiable with respect to the nonrenewable resource remainder stock x, then the above function satisfies the Hamilton – Jacobi equation[1] $\rho V_i = H_i\left(x, u_i^*, \hat{u}^*, \frac{dV_i}{dx}\right) \geq H_i\left(x, u_i, \hat{u}^*, \frac{dV_i}{dx}\right)$ with H_i to be given by equation (22) for the non symmetric case and by equation (23) for the symmetric one.

Discussion in the Dockner and Wagener (2008) paper reveals that differentiation of the Hamilton – Jacobi equation yields the following condition to hold:

$$\frac{\partial H}{\partial \lambda}\lambda'(x) = \rho\lambda - \frac{\partial H}{\partial x} \tag{24}$$

with the derivatives $\frac{\partial H}{\partial \lambda} = u'\upsilon' - N\upsilon - N\lambda\upsilon'$ and $\frac{\partial H}{\partial x} = 0$. Substituting back the (23) (the symmetric case) into (24) gives

$$\lambda'(x)\left[u'\upsilon' - N\upsilon - N\lambda\upsilon'\right] = \rho\lambda \tag{25}$$

Using the maximization condition as given by $(21a)$ we have $\lambda_i = u_i'\left(c_i\right) = \frac{du_i\left(c_i\right)}{dc_i}$ and $\upsilon'\left(\lambda\right) = \frac{1}{u''\left(\upsilon\left(\lambda\right)\right)}$, (25) finally takes the form:

$$\begin{aligned} &\lambda'(x)\upsilon\frac{u'\upsilon' - N\upsilon - Nu'\upsilon'}{\upsilon} = \rho\lambda \Rightarrow \lambda'(x)\upsilon\left[N + (N-1)\frac{u'\upsilon'}{\upsilon}\right] = -\rho\lambda \Rightarrow \\ &\Rightarrow \lambda'(x)\upsilon\left[N + (N-1)\frac{u'}{u''\upsilon}\right] = -\rho\lambda \end{aligned} \tag{26}$$

We now make use of the Arrow – Pratt relative measure of risk aversion[2] in order to simplify (26) even more. The Arrow – Pratt measure is given by $\Xi(c) = -\frac{u''(c)c}{u'(c)}$ and the reverse measure is given by $\Theta(c) = \frac{1}{\Xi(c)} = -\frac{u'(c)}{u''(c)c}$. Taking into account that $\lambda'(x) = u''(c(x))c'(x)$ substitution of the latter into (26) yields finally

$$
\begin{aligned}
& u''(c)c'(x)u(c)\left[N + (N-1)\frac{u'(c)}{u''(c)u(c)}\right] = -\rho u'(c) \Rightarrow \\
& \Rightarrow u(c) \Rightarrow \frac{dc}{dx}\left[N + (1-N)\Theta(c)\right] = -\rho\frac{u'(c)}{u''(c)} \Rightarrow \\
& \Rightarrow u(c)\frac{dc}{dx}\left[N + (1-N)\Theta(c)\right] = \rho\Theta(c)c \Rightarrow \\
& \frac{dc}{dx} = \frac{\rho\Theta(c)c}{\left[N + (1-N)\Theta(c)\right]u(c)}
\end{aligned}
\tag{27}
$$

We introduce the auxiliary variable ξ making the following assignments

$$
\begin{aligned}
& \frac{dc}{d\xi} = \rho\Theta(c)c \\
& \frac{dx}{d\xi} = \left[N + (1-N)\Theta(c)\right]u(c)
\end{aligned}
\tag{28}
$$

and finally:
$$
\frac{dc}{dx} = \frac{\frac{dc}{d\xi}}{\frac{dx}{d\xi}} = \frac{\rho\Theta(c)c}{\left[N + (1-N)\Theta(c)\right]u(c)}
$$

Moreover we assume that all players enjoy a utility function of the form $u_i(c_i) = -e^{-a_i c_i}$, where $a_i > 0$ is a constant. It can be shown (Varian 1982) that this utility form is linear with respect to mean and variance of the c_i parameter that is $u_i(\bar{c}, \sigma_c^2) = \bar{c} - \frac{a}{2}\sigma_c^2$. Moreover the same utility form exhibits constant absolute risk aversion as the following simplifications reveals

$$\Theta(c_i)=\frac{1}{\Xi(c_i)}=-\frac{u'(c_i)}{u''(c_i)c_i}=-\frac{e^{-a_ic_i}a_i\ln(e)}{-e^{-a_ic_i}a_i^2\ln(e)^2c_i}=\frac{1}{a_ic_i}$$

while for symmetric players the latter simplifies to

$$\Theta(c)=\frac{1}{ac} \tag{29}$$

Substituting (29) into (27) yields

$$\frac{dc}{dx}=\frac{\rho\Theta(c)c}{\left[N+(1-N)\Theta(c)\right]u(c)}\Rightarrow\frac{dc}{dx}=\frac{\rho}{\left[N+(1-N)\frac{1}{ac}\right]au(c)} \tag{30}$$

The integration of (30) and taking into account the initial condition $c(0)=0$ the symmetric extraction rates are given by the following expression:

$$c(x)=\frac{(N-1)+\sqrt{(1-N)^2+2\rho Nax}}{Na} \tag{31}$$

The previous reasoning leads us to next corollary.

Corollary 1.

If all players of the nonrenewable resource extraction game enjoy the same utility function that exhibits constant relative absolute risk aversion, then in the symmetric case extraction rates are dependent upon the total number of extractors and upon the remainder stock. Extraction strategies are given by (31).

Alternatively we can use the auxiliary system (28) which under the assumptions of the constant relative risk aversion is modified as:

$$\begin{aligned}\frac{dc}{d\xi}&=\frac{\rho}{a}\\ \frac{dx}{d\xi}&=Nu(c)+\frac{(1-N)}{a}\end{aligned} \tag{32}$$

The integration of the first yields the solution $c(\xi)=c_0+\frac{\rho}{a}\xi$, while second's solution is: $x(\xi)=x_0+\left[Nc(x)+\frac{1-N}{a}\right]\xi$. Taking the initial conditions $x_0=0, c_0=0$ the solutions turn out to the simplified $c(\xi)=\frac{\rho}{a}\xi$ and $x(\xi)=\left[Nc(x)+\frac{1-N}{a}\right]\xi$.

We consider now the utility function of the form:

$$u(c)=\frac{c^{1-a}}{1-a} \tag{33}$$

with $0<a<1$. The above function exhibits constant relative risk aversion. Indeed calculating the reverse relative risk aversion for symmetric player's we have:

$$\Theta(c)=\frac{1}{\Xi(c)}=-\frac{u'(c)}{u''(c)c}=-\frac{c^{(1-a)}}{c^2\left[\frac{c^{(1-a)}(1-a)}{c^2}-\frac{c^{(1-a)}}{c^2}\right]}=-\frac{c^{(1-a)}}{c^{(1-a)}[1-a-1]}=\frac{1}{a} \tag{34}$$

Substituting equation (34) into (27) is transformed into

$$\frac{dc}{dx}=\frac{\rho\Theta(c)}{\left[N+(1-N)\Theta(c)\right]}\Rightarrow\frac{dc}{dx}=\frac{\rho}{\left(N+\frac{1-N}{a}\right)a} \tag{35}$$

And the solution of (35) (with direct integration) is

$$c(x)=\frac{\rho x}{(1-a)N-1}+\Omega \text{ , } \Omega \text{ is the integration constant}$$

Setting zero extraction rates at time zero that is $x(0)=0$, the extraction strategy finally takes the following linear form

$$c(x)=Ax \tag{36}$$

with

$$A=\frac{\rho}{(1-a)N-1} \tag{37}$$

The previous reasoning leads to the following corollary.

Corollary 2

Under the assumption of the utility function that exhibits constant relative risk aversion for the symmetric players of the non renewable resource game, all the game players follow Markovian linear strategies of the form (36) *proportional to the discount rate and decrement to the number of players.*

To that end we reexamine the model under the assumption of a logarithmic utility function, according to the relative risk aversion approach. Further we assume the logarithmic utility function $u(c)=\ln(c)$. The reverse function of relative risk aversion now is:

$$\Theta(c)=\frac{1}{\Xi(c)}=-\frac{u'(c)}{u''(c)c}=-\frac{\frac{1}{c}}{-\frac{1}{c^2}c}=1$$

and substituting the latter into (37) gives

$$\frac{dc}{dx}=\frac{\rho\Theta(c)}{\left[N+(1-N)\Theta(c)\right]}\Rightarrow\frac{dc}{dx}=\frac{\rho}{(N+1-N)}=\rho \qquad (38)$$

Direct integration of (38) yields the equilibrium strategy

$$c(x)=\rho x \qquad (39)$$

The form of (39) verifies the conjectured strategy that is used in order to obtain the equilibrium Markovian strategies for the symmetric case of Proposition 1 for $A=\frac{1}{\rho}$.

6. Conclusions

In this paper we propose a model of nonrenewable resource extraction along the lines of the classical Hotelling model and his successors. We adopt the dynamic programming techniques in order to extract the equilibrium Markov strategies that the players of the proposed differential game must follow. First we make use of a conjectured strategy method and conjectured value functions of the Hamilton – Jacobi equation. This conjecture is verified at the end of the paper. Moreover we enlarge the utility function space from the logarithmic space to the utility functions that exhibits relative risk aversion. This enlargement case reveals interesting results. Specifically the reverse measure of Arrow – Pratt relative risk aversion is given by the expression $\Theta(c) = -\frac{u'(c)}{u''(c)c}$.

Consequently for utility functions of the form $u_i(c_i) = -e^{-ac_i}$ for every symmetric player of the game we are able to derive the analytic forms of the equilibrium strategies for every player. These analytic forms of equilibrium strategies are dependent on the number of the players and on the discount rate, as well. A second application of the shadow price system is the case of constant relative risk aversion that is induced for the utility functions of the form $u_i(c_i) = \frac{c_i^{(1-a)}}{1-a}$ $0 < a < 1$. Applying the reverse measure of relative risk aversion, that is expression $\Theta(c) = \frac{1}{\Xi(c)} = \frac{1}{a}$ then we are able to calculate the analytic forms of the Markovian solution strategies in equilibrium.

The Markovian equilibrium strategies for this case were linear expressions dependent upon discount rate and decrement with respect to the number of the players. To that end the usage of the auxiliary shadow price system verified the initial conjecture about linear equilibrium strategies and value functions for logarithmic utility functions $u_i(c_i) = \ln(c_i)$. Precisely Markovian strategies are given by linear expressions of the form $c(x) = \frac{x}{A}$, where $A = \frac{1}{\rho}$.

Notes

1. For a full exposition of differential games see Olsder and Basar (1998)

2. Microeconomic Analysis Varian (1982), page 189

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