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Abstract The present study analyses the dynamics of a nonlinear Cournot duopoly with managerial delegation and bounded rational players. Problems concerning strategic delegation (based on relative performance evaluations) have recently received in depth attention in both the theoretical and empirical industrial economics literatures. In this paper, we take a dynamic view of this problem and assume that the owners of both firms hire a manager and delegate output decisions to him. Each manager receives a fixed salary plus a bonus offered in a publicly observable contract. The bonus entitled to the manager hired by the owner of every firm is based on relative (profit) performance. Managers of both firms may collude or compete. In such a context, we find, in either cases of collusion and low degree of competition, that synchronised dynamics takes place. However, when the degree of competition increases the dynamics can undergo symmetry-breaking bifurcations that may cause relevant global phenomena. In particular, on-off intermittency and blow-out bifurcations are observed for several parameter values. Moreover, coexistence of attractors may also occur. The global behaviour of the noninvertible map is investigated through the study of the transverse Lyapunov exponent and the folding action of the critical curves of the map. These phenomena are impossible under profit maximisation.

Keywords Cournot; Managerial delegation; Nonlinear dynamics; Oligopoly; Relative profits

JEL Classification C62; D43; L13

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1. Introduction

The attempt of inquiring about firms' objectives different from pure profit maximisation dates back at least to Baumol (1958). Indeed, when ownership and management are separated between each other (see Fama and Jensen, 1983), it seems reasonable to assume that managers are driven by motives different from the maximisation of profits, so that owners may try to motivate them through adequate incentive contracts to get a competitive advantage in the market (see Jensen and Murphy, 1990, for empirical evidence). This is the case of large companies, where governance is at all different than that of (small) perfectly competitive firms. Indeed, as clearly pointed out by Miller and Pazgal (2002, p. 51): "The complexity of managerial decisions and the clear separation between ownership and management are frequently cited as the main obstacles to the achievement of true profit maximization."

The strategic use of managerial incentive contracts has been introduced in the theoretical literature by the pioneering papers of Vickers (1985), Fershtman (1985), Fershtman and Judd (1987) and Sklivas (1987) (VFJS henceforth). According to them, owners have the opportunity to compensate their managers with a bonus based on a weighted sum of profits and sales ("sales delegation"), in order to direct him to a more aggressive behaviour on the market.[1] Moreover, a more recent literature argues that managerial delegation can also allow for the possibility of incentive contracts based on market shares delegation and relative (profit) performance evaluations, see Salas Fumas (1992), Miller and Pazgal (2002), Jansen et al. (2007), van Witteloostuijn et al. (2007) and Jansen et al. (2009). In the latter case, the general idea is that in oligopoly models, the performance of every firm (profit) is evaluated in comparison with the performance of competitors.

In particular, managerial contracts based on relative performance are of particular importance, as stressed by both the empirical and theoretical literatures. Examples are provided by Gibbons and Murphy (1990) and Aggarwal and Samwick (1999), as regards empirical evidence, while Salas Fumas (1992) and Miller and Pazgal (2002), offer theoretical attempts on this issue. Relative profit delegation contracts are essentially established in such a way that every manager receives a fixed salary plus a bonus offered in a publicly observable contract, which represents the incentive scheme that drives her behaviour on the market. The bonus is computed as a weighted sum of profits of the firm where the manager is delegated and profits of the rival, where the weight denotes the manager's attitude (degree) to collude or compete with the manager delegated in the other firm. However, the literature above mentioned concentrates exclusively on the study of static two-stage games played between owners and managers, and compares the results (output) of an economy where managerial delegation exists with respect to the standard case of profit maximisation.

In addition, there exists a burgeoning literature on dynamic oligopolies (see Bischi et al., 2010), that analyses several aspects of local and global dynamic events (stability) that can indeed arise in the study of this class of models.

The novelty of the present paper is the introduction of managerial delegation schemes based on relative (profits) performance in a dynamic nonlinear duopoly game à la Cournot (1838), in both cases of collusion and competition between managers. As

[1] Indeed, in a relevant work, Grossman and Stiglitz (1977) attempt to unify the theories of conventional profit maximisation objective of firms and the managerial literature.

is known, a duopolistic market with standard linear demand and cost functions is stable if expectations of every firm are "naïve" (i.e. both firms expect that the current period rival's production equal that of the previous period), as shown by Theocharis (1960). However, if expectations of one or both firms are of the type of those recently suggested by the dynamic oligopoly literature, and, in particular, they follow the myopic rule introduced by Dixit (1986), i.e. firms are assumed to be "bounded rational" and increase/decrease outputs in the current period according to the information given by marginal profits in the last period, then the pure strategy Nash equilibrium in a Cournot duopoly with linear or isoelastic demand functions can be destabilised, either through a flip bifurcation or Neimark-Sacker bifurcation, when the intensity of the reaction of every firm to the strategy played by the rival becomes high and, in particular, complex dynamics can also occur (see, e.g., Puu, 1991, 1998; Kopel, 1996; Bischi et al., 1998; Bischi et al., 1999; Tramontana, 2010).

However, despite the widespread use of managerial incentive contracts à la VFJS, a stability analysis of a nonlinear dynamic duopoly with quantity competition and relative performance has not been so far tackled on. The present studies aims to fill this gap by assuming *homogeneous* players with bounded rationality. The dynamic system in our model is therefore characterised by a two-dimensional symmetric map. In this context, we study the cases of both collusion and competition between managers. It is interesting to note that the introduction of an objective function (i.e., the manager bonus based on a contract that includes relative performance evaluations as an incentive device) different from firms' profit causes relevant dynamic phenomena, which are usually observed in models with *heterogeneous* players.

Indeed, we recall that in a classical profit-maximising dynamic Cournot game with linear demand and cost functions, the dynamic evolution becomes symmetric after few iterations, and this in turn implies that synchronised dynamics are governed by a one-dimensional map, which summarises the common behaviour of the two players. In contrast with this, in a managerial delegation game with relative profit incentive contracts and *homogeneous* players with identical production costs, we obtain results similar to those found by Bischi et al. (1998, 1999) in models with *heterogeneous* players with different production costs. In particular, in the case of competition between managers, we can observe: (*i*) symmetry-breaking bifurcations. Though the two players are homogeneous because of the assumptions of identical production costs and managers' attitudes, they can either coordinate their behaviours only after a very long transient (on-off intermittency), or non-coordinate themselves in the long run (blow out); (*ii*) coexistence of several attractors and their basins of attraction.

In the opposite case of collusion, we find that when a chaotic set exists on the diagonal, the dynamics tend to synchronise more rapidly in such a case. From a technical point of view, there exists a threshold value of collusion beyond which the attractor from being weak stable (i.e., stable in the Milnor sense) becomes Lyapunov stable. In other words, collusion makes coordination between managers delegated in both firms easier, while competition is responsible of dramatically more complex dynamic events. We will investigate these phenomena through the study of the transverse stability of the diagonal, where symmetric trajectories may take place, while also using the critical curves technique to delimitate areas of non-synchronised dynamics and study global bifurcations of the basins of attraction.

The remainder of the paper is organised as follows. In Section 2 we build on a Cournot duopoly with relative performance and concentrate on the market stage of the game where managers make decisions on the quantity to be produced. Section 3 introduces expectations and studies both the local and global properties of the dynamic system. In particular, Section 3 shows in either cases of collusion and low degree of competition between managers, that synchronised dynamics takes place. However, when the degree of competition increases, the dynamics may undergo symmetry breaking which can indeed cause relevant global phenomena. In particular, on-off intermittency, blow-out bifurcations are observed for several parameter values. Moreover, coexistence of attractors can also occur.

2. The model

The model is outlined in accordance with the pioneering papers by VFJS. In particular, as regards the assumptions of both the normalised inverse demand and manager's bonus, we strictly follow the structure of the recent papers by van Witteloostuijn et al. (2007) and Jansen et al. (2009).

We consider a Cournot duopoly for a single homogeneous product with normalised linear inverse demand given by $p=1-Q$,[2] where p is the market price of product Q, which is the sum of outputs q_1 and q_2 produced by firm 1 and firm 2, respectively. The (average and marginal) cost of producing an additional unit of output is $0<c<1$ for every firm. This hypothesis implies that firm i produces through a production function with constant (marginal) returns to labour, that is $q_i = L_i$, where L_i represents the labour force employed by the ith firm (see, e.g., Correa-López and Naylor, 2004).

We assume that owners of both firms hire a manager and delegate output decisions to him. Each manager receives a fixed salary plus a bonus offered in a publicly observable contract. In particular, the bonus entitled to the manager hired by firm $i=\{1,2\}$ is given by the normalised linear combination $U_i = \Pi_i - \beta_i \Pi_j$, where $\Pi_i = (p-c)q_i$ denotes profits of ith firm (i.e., the firm where the manager is delegated) and $-1<\beta_i<1$ denotes the manager's attitude ("type") and captures the relative weight of profits of firm i (Π_i) with respect to those obtained by the rival (Π_j) in manager i's objective.[3] Negative values of β_i imply that the owner of firm i is interested in collusive behaviour with manager of firm j (collusion).[4] Positive values of β_i describe the case under which the manager of the ith firm is more appreciated

[2] Note that the standard inverse demand $p' = A - BQ'$ can be normalised by using $p = p'/A$ and $Q = (B/A)Q'$.

[3] This formulation of the manager's bonus is equivalent to assuming the existence of weight w attached to her own profits (Π_i) and weight $1-w$ attached to the difference between her own profits and those obtained by the rival ($\Pi_i - \Pi_j$), where $0<w<2$ (see Miller and Pazgal, 2002). Indeed, $U_i = w_i\Pi_i - (1-w_i)(\Pi_i - \Pi_j) = \Pi_i - (1-w_i)\Pi_j = \Pi_i - \beta_i\Pi_j$, where $\beta_i := 1-w_i$. We can reasonably avoid to take $|\beta_i|>1$ into account, as the manager of firm i would be more concerned with profits of the rival than with profits of her own firm in such a case.

[4] It is important to note that in a context without managerial delegation, problems concerning partial cooperation in oligopolies have been studied, amongst others, by Cyert and DeGroot (1973) in a static model, and by Kopel and Szidarovszky (2006) in a dynamic model.

by the owner the higher are profits of the firm where she is delegated (competition). Given the assumption of uniform manager's type $\beta_i = \beta_j = \beta$, the bonus of manager of firm i can be expressed as $U_i = (1 - Q - c)(q_i - \beta q_j)$. This is the rule that drives the behaviour of managers in this simple economy. We note that the relative weight β is assumed to be (i) exogenously given for each single manager and, (ii) not chosen by the owner of firm i to maximise profits. This because the main purpose of the present study is to inquire about both the local and global dynamic effects of β in a nonlinear Cournot oligopoly. Indeed, in the industrial economics literature, it is usual to assume that β_i is an endogenous variable chosen by the owner of firm i to maximise profits in the contract-stage of the game (i.e., the stage played by owners and managers to choose the size of weight β_i in the managerial contract), while being taken as given by managers in the market-stage of the game (i.e., the stage at which the manager makes decisions about the quantity to be produced – under Cournot competition) (see Miller and Pazgal, 2002; Jansen et al., 2009).[5]

Therefore, in order to find the equilibrium in the market stage played by manager 1 and manager 2, we compute the following system of first-order conditions, obtained from the maximisation of managers' objectives (U_i) with respect to the quantity produced by the firm where each of them is delegated, by taking β as given, that is:

$$\frac{\partial U_1}{\partial q_1} = 0 \Leftrightarrow 1 - 2q_1 - q_2(1 - \beta) - c = 0 \ , \tag{1.1}$$

$$\frac{\partial U_2}{\partial q_2} = 0 \Leftrightarrow 1 - 2q_2 - q_1(1 - \beta) - c = 0 \ . \tag{1.2}$$

The solutions of the Eqs. (1.1) and (1.2) for q_1 and q_2 respectively, define the best-reply of player i, that is the manager hired by firm i, given the strategy played by the rival, that is the manager hired by firm $j \neq i$. Then, we find the following reaction functions:

$$q_1(q_2) = \frac{1}{2}[1 - q_2(1 - \beta) - c], \tag{2.1}$$

$$q_2(q_1) = \frac{1}{2}[1 - q_1(1 - \beta) - c]. \tag{2.2}$$

3. Expectations and dynamics

In this section we introduce dynamic elements into the Cournot duopoly with relative performance studied in the previous section. In order to do so, we have first to inquire (given the objective function of every manager) about the quantity q_i that the ith firm will produce in the future depending on expectations that the manager delegated in such a firm has about the quantity that will be produced by the rival in such a period. Second, we need to specify the type of expectations formation mechanism of both managers. Third, we study both the local and global stability of the dynamic system.

As is known, the Nash equilibrium in a duopoly game with quantity competition and standard linear demand and cost functions is stable if expectations of each firm

[5] Alternatively, β_i can be chosen as the solution of a Nash bargaining played between both the owner and manager in the contract-stage of the game (see, e.g., van Witteloostuijn et al., 2007).

are of the "naïve" type (i.e., each firm expects that the output produced today by the rival equals the output produced in the previous period), as shown by Theocharis (1960). The use of static expectations, however, has been criticised because it overestimates the importance of past values. Indeed, the rational expectations[6] revolution, which has initiated in macroeconomics with the works by, amongst others, Lucas (1972) and Sargent and Wallace (1973), prescribes that (*i*) agents form their expectations on the value take on a certain variable in the future by using in the most efficient way the information they have, and (*ii*) the expectations of a single economic agent on a specific variable (i.e., the subjective probability distribution of the events) tend to be distributed in accordance with the prediction of the prevailing economic theory (i.e., the objective probability distribution of the events). However, also the hypothesis of rational expectations has been subject to some criticisms (see, e.g., Burmeister, 1980), because it seems to overestimate the ability of agents to predict the behaviour of prices and quantities.

The burgeoning interest in nonlinear dynamic models has therefore renewed the interest in the use of expectations formation mechanisms at all different from the scheme of rational expectations. Indeed, as claimed by Agliari, Chiarella and Gardini (2006, p. 527), "When one takes into account the fact that nonlinear dynamical systems can produce dynamic paths that are not so regular and predictable, one of the major arguments against adaptive expectations does not seem so strong.", because linear models represent an approximation of nonlinear models (see, amongst others, Chiarella, 1986, 1990; Bischi et al., 2010).[7]

Therefore, if expectations formation mechanisms of one or both firms on the quantity produced by the rival in the future are those of the type suggested in the most part of the recent dynamic oligopoly literature, i.e. firms are assumed the have bounded rational expectations (see Simon, 1957, for the notion of bounded rationality), and then increase/decrease outputs in the future according to a certain degree or speed of adjustment depending on information given by the marginal profit in the current period, then the Nash equilibrium in a standard Cournot duopoly with may be destabilised when the speed of adjustment of each firm's output is fairly high and, in particular, complex dynamics can also be observed, as shown by, e.g., Puu, 1991, 1998; Kopel, 1996; Agiza and Elsadany, 2003, 2004; Zhang et al., 2007; Tramontana, 2010).

We now turn to the study of the dynamic duopoly with relative performance. Let $q_i(t)$ be firm i's quantity produced at time $t = 0,1,2,\ldots$. Then, $q_i(t+1)$ is obtained through the following optimisation programmes:

$$q_1(t+1) = \arg\max_{q_1(t)} U_1\left(q_1(t), q^e{}_2(t+1)\right), \tag{3.1}$$

$$q_2(t+1) = \arg\max_{q_2(t)} U_2\left(q^e{}_1(q+1), q_2(t)\right). \tag{3.2}$$

where $q^e{}_i(t+1)$ represents the quantity that the rival (i.e., the firm where manager j is delegated) today (time t) expects will be produced at time $t+1$ by the firm where manager i is delegated. In the present study, we assume that every manager has bounded rational expectations about the quantity to be produced in the future by the

[6] See Muth (1961) for the notion of rational expectations.

[7] From the consumer perspective in a microeconomic model, Naimzada and Tramontana (2010) show that the behaviour of bounded rational consumers that learn from the past can converge in the long run to the rational behaviour as regards the optimal choice of the consumption bundle. For an analysis of macroeconomic models with bounded rationality, see Sargent (1993).

rival, i.e. players have homogeneous expectations.[8] Therefore, each player uses information on the current manager's objective in such a way to increase or decrease the quantity produced at time $t+1$ depending on whether manager i's marginal bonus (i.e., $\partial U_i / \partial q_i(t)$) is either positive or negative. Following the general idea by Dixit (1986),[9] we now adopt the adjustment mechanism (of quantities over time) proposed by Bischi and Naimzada (2000) in a discrete-time oligopoly model. Then, production at time $t+1$ of the ith bounded rational player follows the rule

$$q_i(t+1) = q_i(t) + \alpha_i \, q_i(t) \frac{\partial U_i(\bullet)}{\partial q_i(t)}, \tag{4}$$

where $\alpha_i > 0$ is a coefficient that captures the speed of adjustment of firm i's quantity with respect to a marginal change in the manager's bonus when q_i varies at time t, and $\alpha_i \, q_i(t)$ is the intensity of the reaction of every bounded rational player. Therefore, through Eq. (4), and using the marginal bonus as given by Eqs. (1.1) and (1.2), the system that characterises the dynamics of this simple duopoly game with quantity competition can easily be expressed as follows:

$$T = T(q_1, q_2) : \begin{cases} q_1' = q_1 + \alpha q_1 [1 - 2q_1 - q_2(1-\beta) - c] \\ q_2' = q_2 + \alpha q_2 [1 - 2q_2 - q_1(1-\beta) - c] \end{cases}. \tag{5}$$

where $\alpha_1 = \alpha_2 = \alpha$ and q_i' is the unit-time advancement of variable q_i (used to simplify notation). For any given value of α and knowing that $-1 < \beta < 1$, from Eq. (5) it is easy to see that when $\beta > 0$ (competition), a rise in β has the straightforward effect to increase the marginal bonus of manager i (of an amount exactly equal to the quantity of goods and services produced by the rival), and then to raise the quantity produced by firm i at time $t+1$. Alternatively, when $\beta < 0$ (collusion) a rise in the absolute value of β reduces the marginal bonus of manager i, and then it tends to reduce the quantity produced by firm i at time $t+1$. For this reason β plays an important role in determining the dynamic properties of the symmetric map, while being responsible of the existence of local and global phenomena which are usually observed in asymmetric maps (e.g., different production costs, see Bischi et al., 1998). From a mathematical point of view, it is relevant to stress the presence of the parameter β (which we recall it represents the relative degree of collusion or competition between managers in our context) which multiplies the mixed term $q_1 q_2$ of map T. This implies that, different from Bischi et al. (1998) and related literature, by changing β, we can indeed vary the weight of mixed term without affecting the pure quadratic terms q_i^2, with $i = \{1, 2\}$. Moreover, when $\beta = 1$ the map becomes totally uncoupled.

The (symmetric) map defined by Eq. (5) is characterised by the existence of four fixed points: $E_0 = (0,0)$, $E_1 = \left(0, \frac{1-c}{2}\right)$ and $E_2 = \left(\frac{1-c}{2}, 0\right)$ located on the invariant coordinate axes,[10] and

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[8] There are several examples in the economic literature on nonlinear dynamic duopolies where players are assumed to have homogeneous expectations (see, e.g., Kopel, 1996; Agiza, 1999; Bischi and Naimzada, 2000; Bischi and Kopel, 2001; Agliari et al., 2005; Agliari, Gardini and Puu, 2006).

[9] We note that Dixit (1986) introduces a time-adjustment mechanism of quantities in a continuous-time Cournot model.

[10] Testing for the invariance of axis i is straightforward: if we start from $q_i = 0$, we get $q_i' = 0$. Thus,

$$E^* = \left(\frac{1-c}{3-\beta}, \frac{1-c}{3-\beta} \right), \tag{6}$$

which represents the unique pure strategy Cournot-Nash equilibrium for this duopoly game with relative performance, where $q^*_1 = q^*_2 = q^*$.[11] Following Bischi et al. (1998), we note that even if the axes are not economically meaningful, they play an important role in the determining the boundary of the basins of attraction of the finite distance attractors. In Section 3.2 we will investigate this question in depth.

One important characteristic of the map defined by Eq. (5) is that diagonal $\Delta = \{(q_1, q_2) : q_1 = q_2\}$ is an invariant manifold, that is $T(\Delta) \subseteq \Delta$. In fact, map T has a symmetric form, so that it does change if variables q_1 and q_2 are swapped, that is $T \circ S = S \circ T$, where $S : (q_1, q_2) \to (q_2, q_1)$. This implies that if the two players start with the same initial conditions $q_1(0) = q_2(0)$ the dynamics lies on Δ for every t. In this case, the behaviour of the dynamic system is described by the restriction of map T on Δ, and the synchronised trajectories (i.e., $q_1(t) = q_2(t)$, for every t) are governed by $T_\Delta : \Delta \to \Delta$, where

$$T_\Delta : \quad q' = q + \alpha q[1 - q(3-\beta) - c]. \tag{7}$$

This map is conjugated to the logistic map

$$z' = \mu z(1-z). \tag{8}$$

by the linear transformation

$$q := \frac{1 + \alpha(1-c)}{\alpha(3-\beta)} z. \tag{9}$$

with $\mu := 1 + \alpha(1-c)$. It follows that the dynamics on Δ can be obtained from the well known behaviour of the standard logistic map by a homeomorphism. The main properties of T_Δ are listed in the following proposition.

Proposition 1. *Let* $q^* = \frac{1-c}{3-\beta}$ *be the unique interior fixed point of the map. If* $0 < \alpha(1-c) < 2$, *then* q^* *is asymptotically stable and its basin of attraction is given by* $\left(0, \frac{1+\alpha(1-c)}{\alpha(3-\beta)}\right)$. *When* $\alpha(1-c) = 2$, q^* *undergoes a flip bifurcation, and for* $2 < \alpha(1-c) < 6$, *an attracting cycle of period two exists around* q^*. *Starting from* $\alpha(1-c) = 2$, *an increase in* $\alpha(1-c)$ *causes other flip bifurcations according to which attracting cycles of periods* $4, 8, \ldots, 2^n$ *appear.*

Proof. We refer to Devaney (1989) for details of the proof.

To be more precise, for $\alpha(1-c) > \upsilon^* - 1$ ($\upsilon^* \cong 3.57$) the fate of a generic trajectory

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the dynamics lies on axis i for every t.

[11] We note that equilibrium profits are given by $\Pi_i = \Pi_j = \Pi^* = \frac{(1-c)^2(1-\beta)}{(3-\beta)^2} > 0$ for every $-1 < \beta < 1$.

starting in the interval $\left(0, \frac{1+\alpha(1-c)}{\alpha(3-\beta)}\right)$ is either an attracting cycle or cyclic-invariant chaotic intervals, or a Cantor set belonging to trapping intervals bounded by critical points. Finally, for $\alpha(1-c) > 3$, the generic trajectory of map (5) is divergent.

It is interesting to note that β does not affect the dynamics of the map on the diagonal. Indeed, as can easily be ascertained by looking at the first order conditions (1.1) and (1.2), since both players are identical, when $q_1(t) = q_2(t)$, for every t, then every manager behaves as if she was not interested in the behaviour of the rival, that is the effects of the parameter β on the reactions of both managers are completely sterilised.

Given the above analysis, we are now wondering about whether an attractor for the one-dimensional map Eq. (7) also represents an attractor for the two dimensional map Eq. (5). In order to study this issue, we have to investigate the transverse attractivity of attractors located on Δ. Therefore, we consider the following Jacobian matrix associated to Eq. (5):

$$J(q_1, q_2) = \begin{pmatrix} 1+\alpha-4\alpha q_1+\alpha q_1(\beta-1)-\alpha c & \alpha q_1(\beta-1) \\ \alpha q_2(\beta-1) & 1+\alpha-4\alpha q_2+\alpha q_1(\beta-1)-\alpha c \end{pmatrix}, \tag{10}$$

which can be computed in a point (q,q) on the diagonal to obtain:

$$J(q,q) = \begin{pmatrix} l(q) & m(q) \\ m(q) & l(q) \end{pmatrix}, \tag{11}$$

where $l(q) = 1+\alpha-\alpha c+\alpha q(\beta-5)$ and $m(q) = \alpha q(\beta-1)$. The eigenvalues associated to a generic point on Δ are the following:

$$\lambda_{\parallel} = l(q)+m(q) = 1+\alpha-2\alpha q(3-\beta)-\alpha c, \tag{12}$$

with eigenvector $(1,1)$ and

$$\lambda_{\perp} = l(q)-m(q) = 1+\alpha-4\alpha q-\alpha c, \tag{13}$$

with eigenvector $(1,-1)$. The eigenvalue $\lambda_{\parallel}$ is related to the invariant manifold Δ and coincides with the multiplier of the restriction of the map on Δ. The eigenvector associated with the eigenvalue $\lambda_{\perp}$ is always orthogonal to Δ regardless of q.

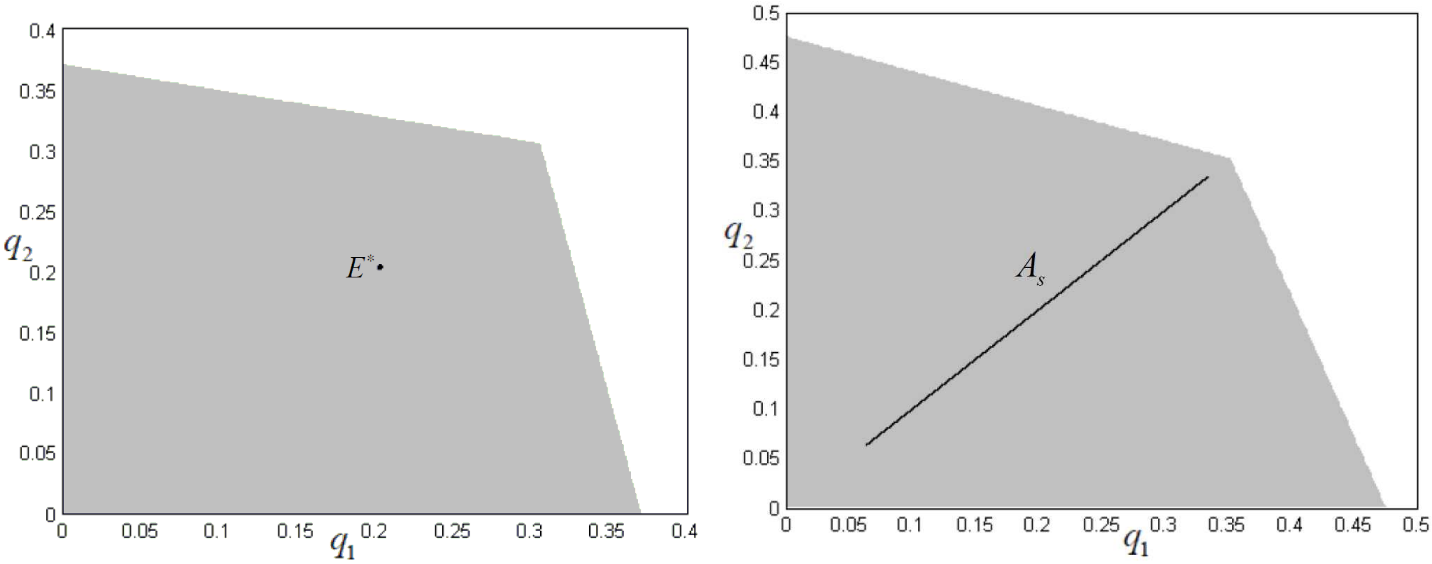
At this point, by using the coordinates of E^* the dynamic properties around the Cournot-Nash equilibrium are summarised in the following proposition.

Proposition 2. *The Cournot-Nash equilibrium E^* is asymptotically stable if and only if $\alpha < \frac{2}{1-c}$. For $\alpha = \frac{2}{1-c}$, the Cournot-Nash equilibrium E^* is destabilised by a supercritical flip bifurcation.*[12]

Proof. By the symmetry of the map it follows that eigenvalues defined by Eqs. (12) and (13) are real numbers (i.e., a Neimark-Sacker bifurcation cannot occur). Then, by substituting out the coordinates of E^* in the generic expression of the eigenvalues of a point on Δ, we have $\lambda_{\parallel}(E^*) = 1-\alpha(1-c) < \lambda_{\perp}(E^*) = 1-\frac{\alpha(1-c)(1+\beta)}{3-\beta} < 1$. It follows that the

[12] We note that in this context (with a linear demand function), the Nash equilibrium cannot be destabilised through a Neimark-Sacker bifurcation.

fixed point E^* is asymptotically stable if and only if $\lambda_{\parallel}(E^*) > -1 \Leftrightarrow \alpha < \frac{2}{1-c}$. **Q.E.D.**



(a) (b)

Figure 1. (a) Case $\alpha < \frac{2}{1-c}$ (see Proposition 2): the Cournot-Nash equilibrium E^* is the unique attractor of the system (parameter values: $\alpha = 4$, $\beta = 0.575$ and $c = 0.51$). (b) Case $\alpha > \frac{2}{1-c}$ (see Proposition 2): the Cournot-Nash equilibrium E^* is unstable and a chaotic set A_s exists along the diagonal. The dynamic properties of the chaotic set A_s will be analysed in Section 3.3 (parameter values: $\alpha = 4$, $\beta = 0.3$ and $c = 0.3$). Note that in both figures, the grey region denotes the basin of attraction of the unique finite distance attractor, while the white region represents the set of unfeasible trajectories.

It is important to stress that β does not matter even on the local dynamic properties in the neighbourhood of E^*. From an economic point of view, this implies that if we start from an initial condition close to the Nash equilibrium, then every manager does not take the behaviour of the rival into account.

Starting from Eqs. (12) and (13), it is possible to inquire about the stability of the cycles that can occur on Δ. Indeed, for a k-cycle $\{(x_1, x_1), (x_2, x_2), ..., (x_k, x_k)\}$ of Eq. (5), corresponding to cycle $\{x_1, x_2, ..., x_k\}$ of the one-dimensional map Eq. (7), the two multipliers are:

$$\lambda_{\parallel}^{(k)} = \prod_{i=1}^{k} (l(x_i) + m(x_i)) = \prod_{i=1}^{k} (1 + \alpha - 2\alpha x_i (3 - \beta) - \alpha c), \tag{14}$$

with eigenvector $(1,1)$, and

$$\lambda_{\perp}^{(k)} = \prod_{i=1}^{k} (l(x_i) - m(x_i)) = \prod_{i=1}^{k} (1 + \alpha - 4\alpha x_i - \alpha c), \tag{15}$$

with eigenvector $(1,-1)$, which is independent of β. This implies that β is a *normal parameter* (see Ashwin et al., 1996; Bischi and Gardini, 2000). The conditions for local stability and local bifurcations along Δ (that is, those related to the eigenvalue $\lambda_{\parallel}^{(k)}$ associated to the eigenvector $(1,1)$) are those briefly listed in Proposition 1. Hence, we can focus attention only to transverse stability. In particular, the problem is trivial when on the diagonal Δ a cycle of finite period exists, as only the eigenvalue $\lambda_\perp^{(k)}$ associated to the trajectory should be evaluated in such a case. The problem becomes interesting when the dynamics restricted to the invariant sub-manifold are embedded in a chaotic set A_s. In order to study this issue, we now introduce the definition of transverse Lyapunov exponent by which the "average" local behaviour of the trajectories in a neighbourhood of the invariant set $A_s \subset \Delta$ may be classified.

Definition 1. *Let* $\{x_i = f^i(x_0), i \geq 0\}$ *be a trajectory of Eq. (7) embedded in* $A_s \subset \Delta$. *Then,*

$$\Lambda_\perp = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \ln|\lambda_\perp(x_i)| = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \ln|1+\alpha-4\alpha x_i-\alpha c|, \tag{16}$$

is the transverse Lyapunov exponent.

In particular, if x_0 belongs to a k-cycle, then $\Lambda_\perp = \ln\left|\lambda_\perp^{(k)}\right|$ and if $\Lambda_\perp < 0$, then the k-cycle is transversely stable. Otherwise if x_0 belongs to a generic aperiodic trajectory embedded inside the chaotic set and if $\Lambda_\perp < 0$, then such a trajectory is transversely stable. From the consideration above, the following classical definition of attractiveness can be established:

Definition 2. *A is an asymptotically stable attractor (or topological attractor) if it is Lyapunov stable, i.e. for every neighbourhood U of A there exists a neighbourhood V of A such that $T^n(V) \subset U$ for every $n \geq 0$ and the basin of attraction $B(A)$ contains a neighbourhood of A.*

On the one hand, we note that the definition of an asymptotically stable attractor given in Definition 2, is difficult to be proved analytically, as it requires the evaluation of the Lyapunov exponent over an infinity of trajectories, so that it is sometimes tested through numerical simulations. On the other hand, it is easier to show whether a chaotic set along the diagonal *is not* an attractor in the Lyapunov sense, because it is sufficient to find a trajectory (for instance, a cycle) with respect to which the condition $\Lambda_\perp < 0$ is violated.

3.1. Critical curves

An important feature of map Eq. (5) is that it is a noninvertible endomorphism. Indeed, for a given (q_1', q_2'), the rank-1 preimage (that is, the backward iterate defined as T^{-1}) may not exist or may be multivalued. By considering the map Eq. (5), if we want to compute (q_1, q_2) in terms of (q_1', q_2'), we have to solve a fourth degree algebraic system that may have four, two or no solutions. Thus, we can divide the plane in regions Z_0, Z_2, Z_4, according to the number of preimages (where the subscript in Z indicates their number). A direct consequence of this fact is that, if we let (q_1', q_2') vary in the plane R^2, the number of rank-1 preimages changes as the point (q_1', q_2') crosses the boundary that separates these regions. Such boundaries are generally characterised by the existence of two coincident preimages. In this regard, following the notation used by Mira et al. (1996), we introduce the definition of critical curves.

The critical curve of rank-1, denoted by LC (from the French "Ligne Critique"), is defined as the locus of points with two (or more) coincident rank-1 preimages located on a set called LC_{-1}. It is quite intuitive to interpret: (*i*) the set LC as the two-dimensional generalization of the notion of critical value, defined as the value corresponding to either the local minimum or maximum of a one-dimensional map, and (*ii*) LC_{-1} as the generalisation of the notion of critical point. Thus, arcs of LC separate the regions of the plane characterised by a different number of real preimages (see Mira et al., 1996 for details). Since the map defined by Eq. (5) is continuously differentiable, LC_{-1} belongs to the locus of points where the Jacobian determinant of T vanishes (i.e. the points where T is not locally invertible), i.e.: $LC_{-1} \subseteq \{(q_1, q_2) \in R^2 : Det(T) = 0\}$, and LC is the rank-1 image of LC_{-1} under T, i.e. $LC = T(LC_{-1})$. From direct computations, we have that:

$$\begin{aligned} Det(T) = 0 \Leftrightarrow\ & 4\alpha^2 q_1^2(\beta - 1) + 4\alpha^2 q_2^2(\beta - 1) + 16\alpha^2 q_1 q_2 + \alpha q_1(5 - \beta)(\alpha(c - 1) - 1) \\ & + \alpha q_2(5 - \beta)(\alpha(c - 1) - 1) + (\alpha(c - 1) - 1)^2 = 0 \end{aligned}, \qquad (17)$$

By studying the invariants of the previous quadratic form, it is easy to recognise that Eq. (17) is the equation of a hyperbola in the plane (q_1, q_2). Thus, LC_{-1} is formed by two branches, denoted by $LC_{-1}^{(a)}$ and $LC_{-1}^{(b)}$, respectively, whose explicit equations can be found by using standard geometric transformation and curve parametrization (see, e.g., Gibson, 2003). This also implies that LC and subsequent iterates of critical curves may be interpreted as the union of two different branches indexed by a and b. In Figure 2, the critical curves $LC_{-1}^{(a)}$, $LC_{-1}^{(b)}$, $LC^{(a)}$ and $LC^{(b)}$ are depicted. Each branch of critical curve LC separates the phase plane of T into regions whose points have the same number of distinct rank-1 preimages. Specifically, $LC^{(b)}$ separates region Z_0 from region Z_2, and $LC^{(a)}$ separates region Z_2 from region Z_4.

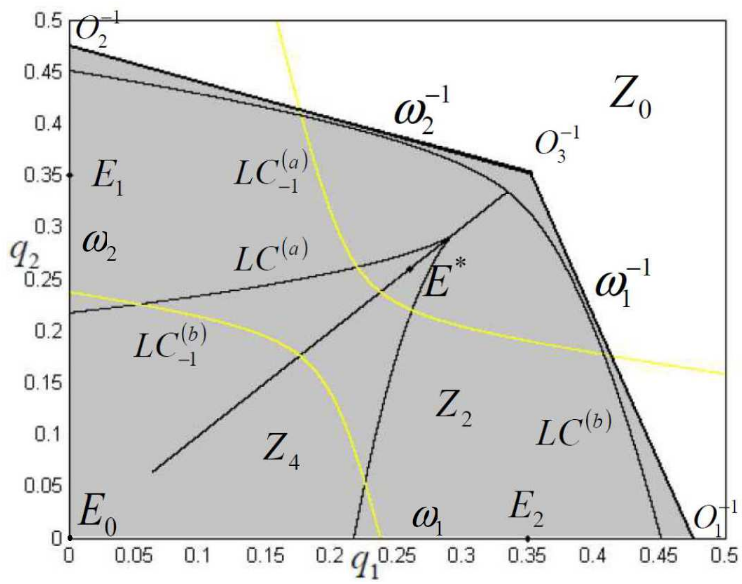


Figure 2. Critical curves are represented for the parameter set $\alpha = 4$, $\beta = 0.3$ and $c = 0.3$ (that is, the same parameters used in Figure 1.b). The yellow lines $LC_{-1}^{(a)}$ and $LC_{-1}^{(b)}$ are the two branches of the LC_{-1} set. The portion of the plane that lies between the axes and $LC^{(a)}$ defines region Z_4; the portion of the plane that lies between $LC^{(a)}$

and $LC^{(b)}$ defines region Z_2; the portion of the plane beyond $LC^{(b)}$ defines region Z_0. Moreover E_0 and its preimages O_1^{-1}, O_2^{-1} and O_3^{-1} define the (grey) region of feasible trajectories. The region of unfeasible trajectories is white-coloured. Notice that the sides $\omega_1^{-1} = [O_1^{-1}, O_3^{-1}]$ and $\omega_2^{-1} = [O_2^{-1}, O_3^{-1}]$ are the preimages of $\omega_1 = [E_0, O_1^{-1}]$ and $\omega_2 = [E_0, O_2^{-1}]$, respectively.

The forward images of rank-k of LC_{-1} give the critical sets of rank-k, denoted by $LC_{k-1} = T^k(LC_{-1}) = T^{k-1}(LC)$. Portions of critical curves of increasing rank can be used to obtain the boundary of absorbing and chaotic areas of non-invertible maps of the plane (see Mira et al., 1996 for details). We recall here some basic definitions and we refer to Agliari et al. (2001) for further details:

Definition 3. *An absorbing area A is a bounded region of the plane whose boundary is given by portions of critical curves and their images of increasing order, such that a neighbourhood $U \supset A$ exists whose points enter A after a finite number of iterations and then never escape it since $T(A) \subseteq A$.*

Definition 4. *A chaotic area is an absorbing area A such that $T(A) = A$ and chaotic dynamics occurs inside A.*

The boundary of chaotic area A will be shown in Figure 5.b and it is obtained by following the procedure described by Mira et al. (1996) and Bischi and Gardini (1998). First, we take subsequent images of the two segments of LC_{-1} included in A until a chaotic area is defined. Then, we repeat the procedure by taking only the images of $\gamma_i = A \cap LC_{-1}^i$, $i = \{a, b\}$ (this last step shows that the union of the first four iterates of γ_i covers the whole boundary of chaotic area A, and this ensures that such a selected area is invariant, i.e. $T(A) = A$).

3.2. Basins of attraction

In defining the economic problem in Section 2, we have assumed as a natural condition to study map T defined by Eq. (5) on the positive orthant of R^2. However, it is important to note that the map generates unbounded (economically not feasible) trajectories if the initial condition for the dynamic system of Eq. (5) is taken with at least a negative coordinate or far enough from the origin, i.e., it is taken in a suitable neighbourhood of ∞ (which is always an attractor of the system). In fact, if we take fairly high values of the initial conditions of q_1 and q_2, then the first iterate of the map defined by Eq. (5) gives negative values of q_1 and q_2 so that the subsequent iterates give negative and decreasing values as well. This implies that any attractor at finite distance cannot be globally attracting in R^2. It follows that an important feature related to the study of such a model is the delimitation of the feasible set. Following Bischi et al. (1998), it is possible to show that coordinate axes and their preimages of any rank form the boundary of the feasible set. As regards the case under scrutiny, we note that the dynamics of the map restricted to one of the axis is governed by the following one-dimensional system:

$$q_i' = q_i + \alpha q_i (1 - 2q_i - c). \tag{18}$$

This map is conjugate to the logistic map (see Devaney, 1989) by means of the linear transformation $\mu := 1 + \alpha(1-c)$, $q_i = \frac{1+\alpha(1-c)}{2\alpha z_i}$. Bounded trajectories along the invariant axes are obtained when $1 < \alpha(1-c) < 3$ provided that the initial value of the map lies on the segments

$$\omega_i = \left[O, O_i^{-1}\right], \tag{19}$$

where O_i^{-1} is the rank-1 preimage of the origin on the corresponding axis with non-null coordinate equals $\frac{1+\alpha(1-c)}{2\alpha}$. Instead, negatively divergent trajectories along the invariant axis are obtained starting from an initial condition out of the segment. Furthermore, from the computation of the eigenvalues of the cycles that belong to ω_i we have that the direction transverse to the coordinate axes is always repelling. From these arguments, it follows that ω_i belongs to $\partial B(\infty)$ as well as to their preimages of any rank. Now, by following Bischi et al. (1998), the next proposition gives an exact delimitation of $\partial B(\infty)$.

Proposition 3. *Let $0 < \alpha(1-c) < 3$ and ω_i, $i = \{1,2\}$ be the segment lines defined in Eq. (19). Then,*

$$\partial B(\infty) = \left(\bigcup_{n=0}^{+\infty} T^{-n}(\omega_1) \right) \cup \left(\bigcup_{n=0}^{+\infty} T^{-n}(\omega_2) \right). \tag{20}$$

Proof. We refer to Bischi et al. (1998) for analytical details of the proof.

In contrast to Bischi et al. (1998), however, for all the parameter sets that we have taken into account, the segments ω_i^{-1} lie on region Z_0, so that the boundary of the basin of attraction is exactly made up by $\omega_i \cup \omega_i^{-1}$, $i = \{1,2\}$ (see Figure 2).

3.3. Intermittency, synchronisation and blow-out

In the first part of Section 3 we have analysed simple dynamic events where a fixed point, a cycle, or a chaotic attractor on the diagonal is the unique attractor of the dynamic system, which, according to the definition of deterministic chaos by Li and Yorke (1975), captures almost all non-diverging orbits.

In this section we show that the dynamics of our simple model present more interesting and complicated phenomena. From an economic point of view, one of the most important result is given by the fact that even if the structure the economy is characterised by symmetric homogeneous players (managers), we observe coordination failure when (in the case of competition between managers) the size β increases and the intensity of competition becomes higher.

We note that in the standard Cournot model with no strategic delegation ($\beta = 0$), linear demand and cost functions, and homogeneous players, Bischi et al. (1998) have shown that, starting from initial conditions $q_i(0)$ of output outside the diagonal, coordination failure may occur only along the transient (even if sometimes the synchronisation process may take a very long time). This is the case of the phenomenon known as on-off intermittency. Departing from the standard Cournot model and introducing strategic delegation, we can have collusion ($-1<\beta<0$) or competition ($0<\beta<1$) between managers. Then, situations characterised by coordination failure along the transient (on-off intermittency) are sterilised when collusion between managers exists and the intensity of such a collusion is strong enough. In contrast, in the case of competition between managers we note that: (*i*) coordination failure phenomena are observed only along the transient (on-off intermittency), as in the basic model ($\beta=0$), see Bischi et al. (1998), or (*ii*) coordination failure phenomena become persistent (blow-out and multistability). It is important to stress that blow-out and multistability are indeed usually observed: in models where players are heterogeneous (for instance, because of the existence of asymmetric production costs), or in models the mathematical structure presents strong nonlinearities as, e.g., in the works by Bischi and Gardini (2000), Kopel et al. (2000), and Bischi and Kopel (2003), where the dynamic system is described by maps with denominators.

Now, in order to make the subsequent analysis clearer and self-contained, we now recall both definitions and properties that allow us to describe and characterise the dynamics of our model. We start by noting that the study of dynamic systems with chaotic trajectories embedded into an invariant sub manifold of lower dimensionality than the total phase space, have recently captured attention of researchers (see, e.g., Mosekilde et al., 2002 for a survey on the subject). Moreover, the analysis of how synchronised attractors and their basins of attraction undergo qualitative changes when some basic parameters vary has also been taken into account. Of particular importance in the study of these phenomena is the use of the Lyapunov exponent (see, e.g., Bischi et al., 1998; Maistrenko et al., 1998; Bischi and Gardini, 2000).

Consistently with these arguments, we now take the analysis of our Cournot model with relative performance into account, and concentrate on the case under which a chaotic set A_s exists on the diagonal (that is, when $\alpha(1-c)>2.57$). Then, we let β vary and study the dynamic properties of the map. First, we observe that in the case of collusion between managers with $|\beta|$ sufficiently high, there exists numerical evidence such that the diagonal is Lyapunov stable: given any initial condition $q_i(0)$ of output within the quadrilateral of vertices $OO_1^{-1}O_3^{-1}O_2^{-1}$, as defined in Figure 2, this means that after a few iterations both managers coordinate their behaviours on the diagonal. Then, moving from negative to positive values of β, we find that a threshold value $\beta=\beta_{\mathrm{M}}$ exists (whose size depends on the other parameters of the model), beyond which the dynamic behaviour of the system markedly modifies. Since a chaotic set Δ on the main diagonal exists in such a case, it follows that infinitely many cycles, all unstable along $(1,1)$ are embedded inside the chaotic set A_s. This implies that this object cannot be classified as an attractor in the Lyapunov sense. This type of loss of stability, however, may not be recognised by looking at Figure 1.b, because in the long run all feasible trajectories synchronise on the diagonal in such a case.

Now, in order to distinguish and classify the dynamic phenomena that can be observed for $\beta>\beta_{\mathrm{M}}$, it is useful to define a spectrum of transverse Lyapunov exponents

$$\lambda_{\perp}^{\min}<....<\lambda_{\perp}^{nat}<...<\lambda_{\perp}^{\max}, \tag{21}$$

each of which is associated with a specific trajectory, where $\lambda_{\perp}^{nat}$ is the Lyapunov exponent evaluated on a generic trajectory taken in the chaotic attractor. Roughly speaking, $\lambda_{\perp}^{nat}$ may be interpreted as a sort of "weighted balance" amongst the Lyapunov exponents associated to different cycles. Now, if a set is a Lyapunov attractor, it follows that $\lambda_{\perp}^{\max} < 0$ (see Definition 2) and also $\lambda_{\perp}^{nat} < 0$. Notwithstanding, by considering positive values of β beyond β_{M}, it is possible to find that some cycles embedded in chaotic set A_s become transversely unstable. This implies that the transverse multiplier is $\lambda_{\perp}^{\max} > 0$, while $\lambda_{\perp}^{nat}$ is still negative (see Maistrenko et al., 1998; Bischi and Gardini, 2000). In this case, A_s is no longer Lyapunov stable, but it continues to attract a positive (Lebesgue) measure set of points of the two dimensional phase space. If the set A_s has the property to attract a positive (Lebesgue) measure set of points of the two dimensional phase space, it is said to be *weak stable or stable in Milnor sense*. To this purpose, Figure 1.b has already shown this phenomenon for a positive value of β: a generic trajectory that starts in the grey region is attracted by the one-dimensional Milnor attractor on the diagonal.

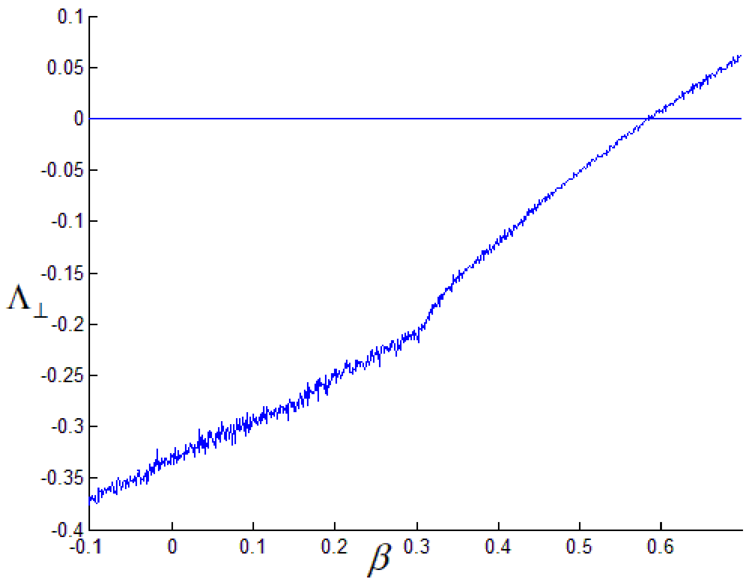
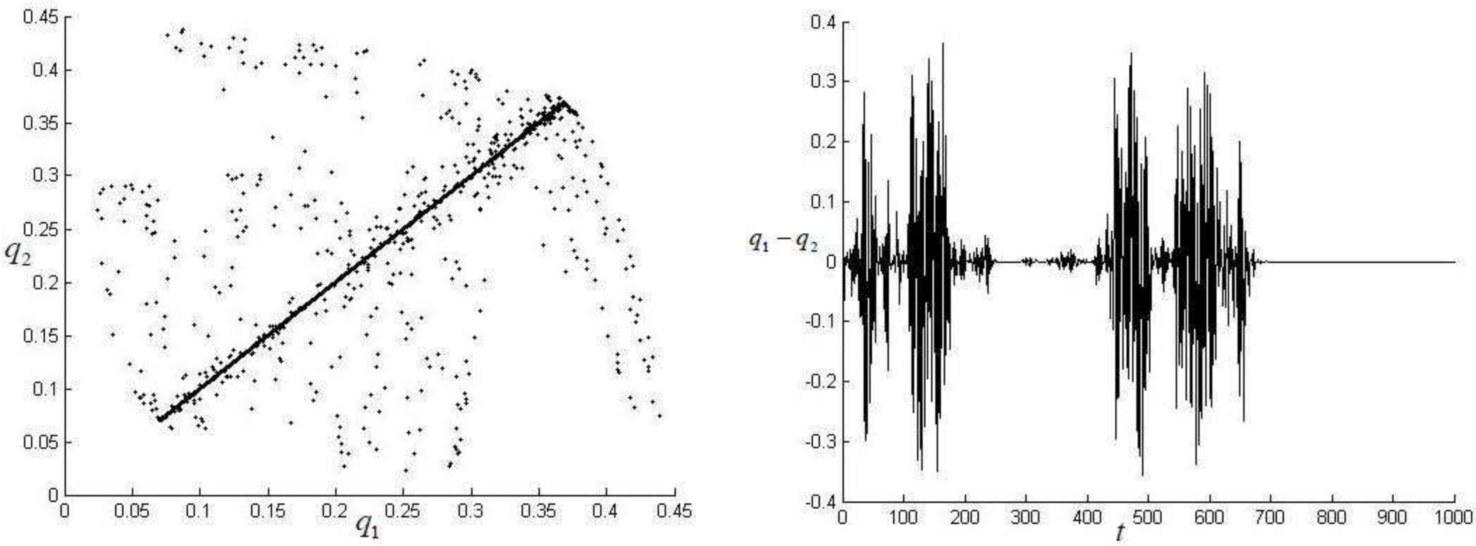


Figure 3. Natural transverse Lyapunov exponent $\lambda_{\perp}^{nat}$ as a function of β belonging to the interval $[-0.1, 0.7]$ (parameter set: $\alpha = 0.4$ and $c = 0.3$). The figure is obtained by dividing the interval $[-0.1, 0.7]$ into 1000 equidistant points. Starting from a generic initial condition on the diagonal, each point has been iterated 10000 times through map T (to eliminate the transient), and then averaging over subsequent 50000 iterations.

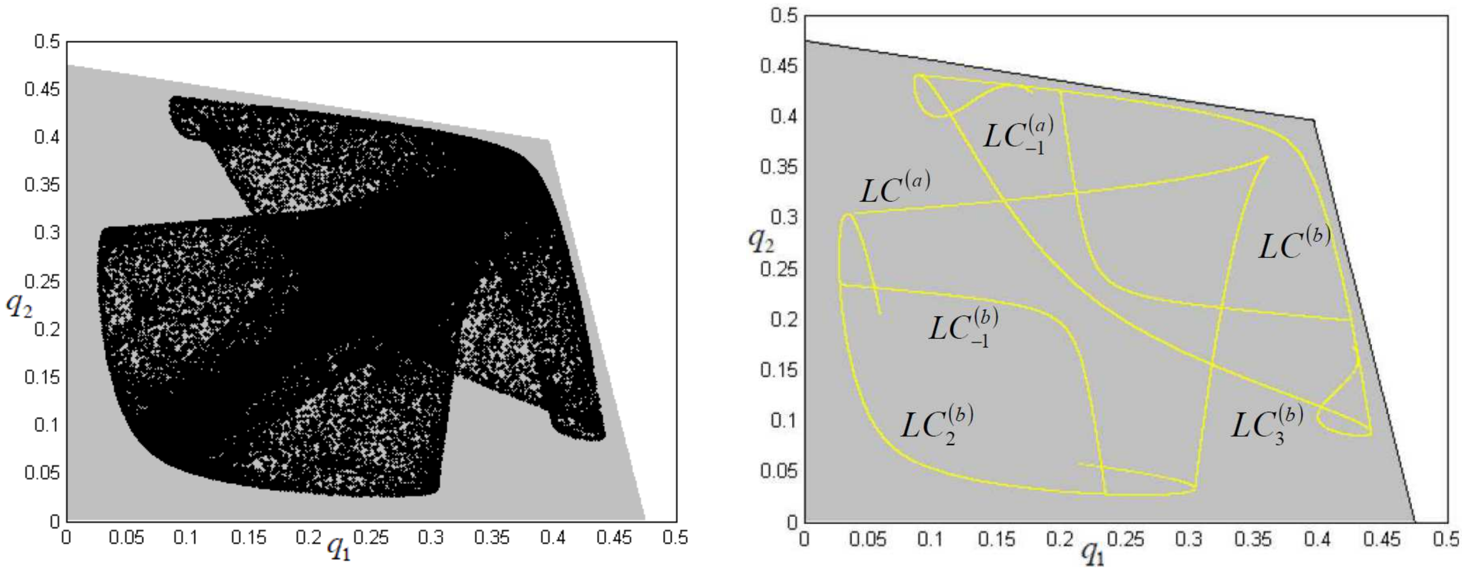
In the numerical example illustrated in Figures 4.a and 4.b no other attractors than A_s exist, so that trajectories are first move away from the main diagonal and then they approach towards A_s, and the dynamics are characterised by some bursts away from Δ before synchronising on it (on-off intermittency). The synchronisation takes a very long time because $\lambda_{\perp}^{nat}$ is negative but close to zero.



(a) (b)

Figure 4. Parameter values: $\alpha = 0.4$, $c = 0.3$ and $\beta = 0.556$. (a) The figure shows a typical trajectory that starts in the basin of attraction of the finite distance (Milnor) attractor. (b) The figure shows the phenomenon of intermittency by displaying the difference between q_1 and q_2 versus time: the convergence towards the unique chaotic (Milnor) attractor of the system embedded in the diagonal occurs only after a very long transient.

If we let β increase (ceteris paribus), also $\lambda_{\perp}^{nat}$ becomes positive due to the fact that the transversely unstable periodic orbits embedded in A_s weights more than the transversely attracting ones (see Figures 5.a and 5.b). In this case, A_s becomes a chaotic saddle and we observe an explosion of the attractor which is now no more confined on the diagonal, the so-called blow-out bifurcation occurs and a two dimensional attractor is observed.



(a) (b)

Figure 5. Parameter values: $\alpha = 0.4$, $c = 0.3$ and $\beta = 0.6$. (a) The figure shows the effect of a rise in β on the long run dynamics: the one dimensional Milnor attractor, as shown in Figure 1.b for $\beta = 0.3$ and in Figure 4.a for $\beta = 0.556$, has been replaced by a two dimensional attractor. (b) Boundary of the chaotic area (simulated in Figure 5a) obtained by arcs of the critical curves LC, LC_1, LC_2 and LC_3.

In this case we can talk about coordination failure or symmetry-breaking of the dynamics. We note that even if the long-run dynamics are no more restricted on the diagonal, by using the critical curves and their iterates we may delimitate the borders of the phase plane in which the asymptotic dynamics are bounded. It is important to note that the same technique may also be applied to give an upper bound to the transient trajectories of the map when intermittency occurs.

3.4 Multistability and related basins of attraction

A dramatic change in the long run dynamics is observed if we take different parameter sets and let beta vary. In particular, by assuming $\alpha = 3.8$ and $c = 0.3$ we observe some interesting events which are detailed below when β belongs to the interval $[0.165, 0.5379]$. In this case, we have the birth (through local bifurcations) of new stable equilibria outside the diagonal. This phenomenon is tangled in several ways with the set A_s. In the following numerical experiment, it is possible to observe an example of the phenomenon known as riddled basins. The set A_s is a Milnor attractor and it is characterised by $\lambda_{\perp}^{nat} > 0$, as in Section 3.3: but with this parameter set several trajectories move away from the diagonal and approach another attractor.

In Figure 6.a ($\beta = 0.45$) we note that a four-period cycle exists together with the diagonal. If β increases, the cycle undergoes a Neimark-Sacker bifurcation and an attractor formed by four smooth curves coexists with the diagonal (see Figure 6.b, where $\beta = 0.46$). Further increases in β first cause the loss of smoothness of the attractor formed by four curves (see Figure 6.c, where $\beta = 0.485$), and then cause the birth of a four-piece chaotic attractor (see Figure 6.d, where $\beta = 0.53$).

The subsequent basin boundary (final) bifurcation for $\beta \cong 0.5379$ causes the death of the attractor and only the two-piece Milnor attractor exists as the unique attractor of the system in such a case (we note that this phenomenon is not reported in any figures).

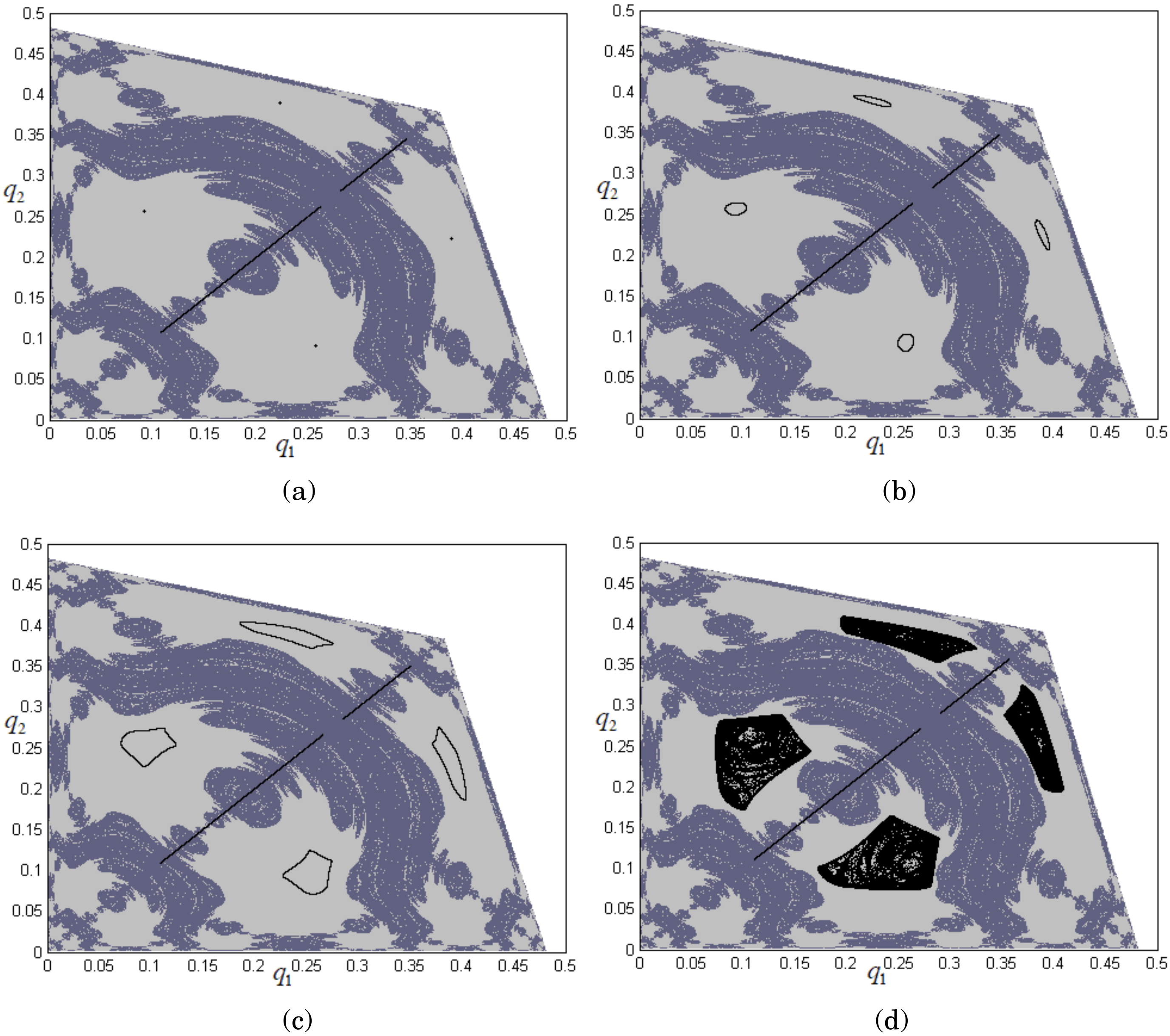
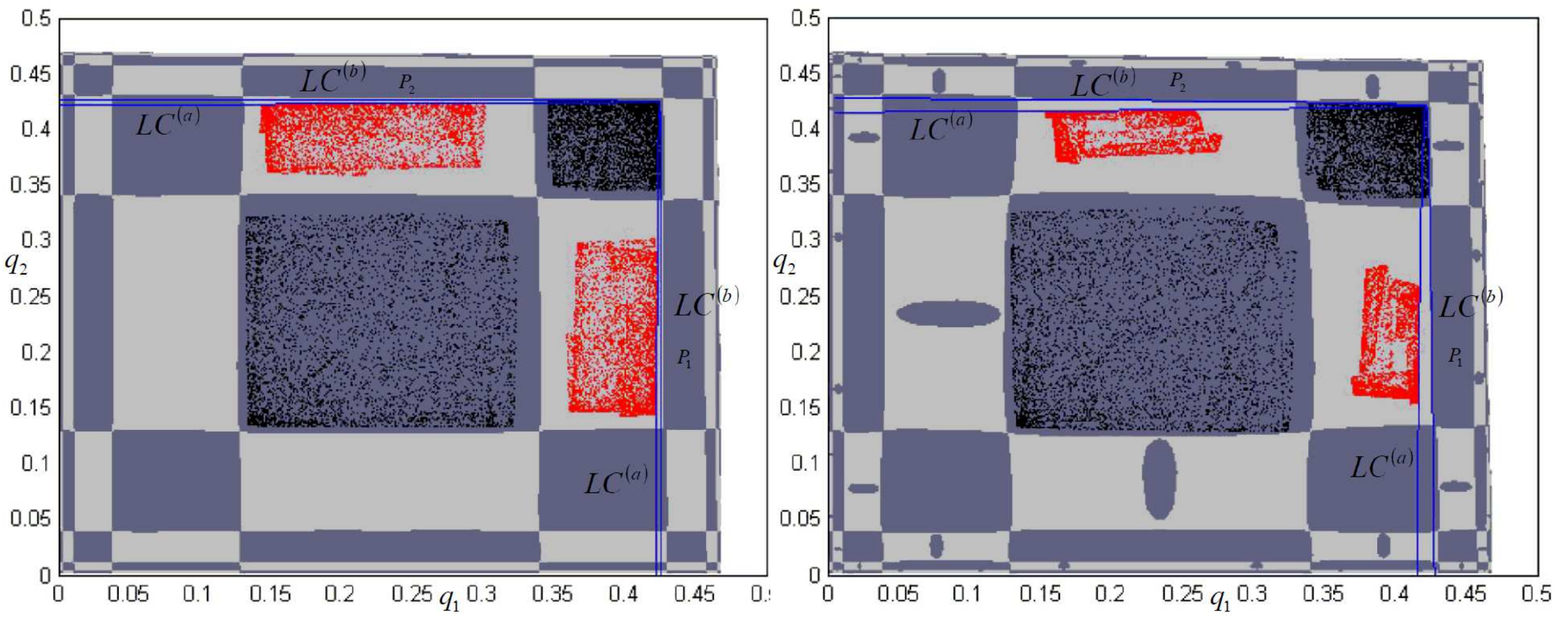
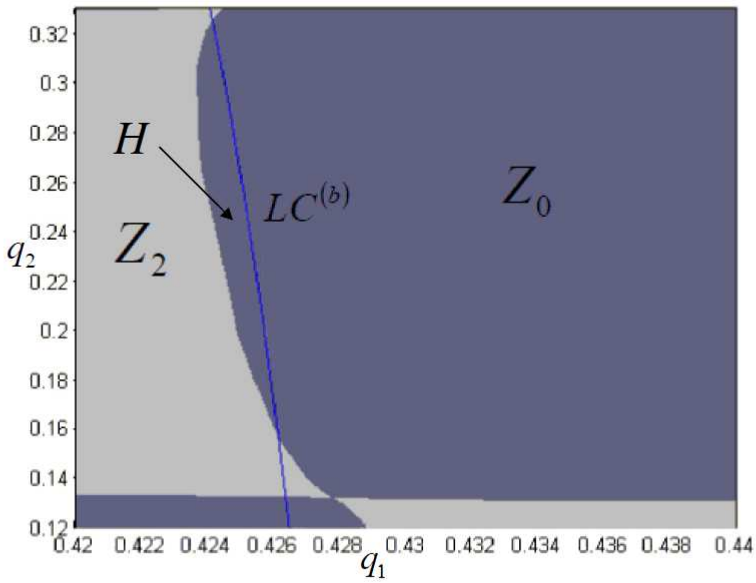


Figure 6. Parameter values: $\alpha = 3.8$ and $c = 0.3$. (a) For $\beta = 0.45$, an attracting four-period cycle exists together with the two-piece Milnor attractor on the diagonal. (b) For $\beta = 0.46$, the four-period cycle has undergone a Neimark-Sacker bifurcation and an attractor formed by four smooth curves exist together with the two-piece Milnor attractor. (c) For $\beta = 0.485$, the attractor formed by the four smooth curves has become non-smooth. (d) For $\beta = 0.53$, a four-piece chaotic attractor exists outside the diagonal.

Other interesting phenomena can be observed by slightly changing the basic parameters of the model. In particular, multistability occurs even if along the diagonal an attractive set does not exist. We start by fixing $\alpha = 3.9$ and $c = 0.32$. Then, in order to properly shows this phenomenon we choose the following two fairly high values of β: $\beta = 0.987$ and $\beta = 0.97$. In the former case ($\beta = 0.987$), two chaotic two-piece attractors exist whose basins of attraction are separated by some invariant manifolds associated to the unstable cycles. This can be seen by looking at Figure 7.a, where the dark-grey (light-grey) region denotes the basin of attraction of the black-coloured (red-coloured) chaotic attractor). A more complicated structure of the basins of attraction is obtained by slightly reducing the parameter β, i.e. $\beta = 0.97$ (see Figure 7.b). Indeed, by looking at Figure 7.b, we can now observe new portions of the basin of attraction of the black-coloured chaotic attractor embedded in the basin of attraction of the red-coloured chaotic attractor. We now make use of critical curves in order to explain such a phenomenon. In Figure 7.c, an enlargement of Figure 7.b is presented where it is shown that a global bifurcation of the basins of attraction is just occurred. Indeed, starting from high values, we can observe that a reduction in beta causes an enlargement of the distance between the two critical curves $LC^{(a)}$ and $LC^{(b)}$, so that the branches of rank-1 critical curve $LC^{(b)}$, which separates Z_0 from Z_2, move towards the regions denoted by P_1 and P_2 in Figure 7.a. This implies that a threshold value of β (β_T) does exist such that a tangency between $LC^{(b)}$ and the boundary of P_1 and P_2 occurs simultaneously, and this last property is due to the symmetry of the map. Then, as evidenced in Figures 7.b and 7.c (which is an enlargement of Figure 7.b referred to the region around P_1 only), a portion of P_1 and P_2 (labelled as H in Figure 7.c) enters the Z_2 area. Since P_1 and P_2 belong to the basin of attraction of the black-coloured attractor, then also their preimages belong to such a basin. This creates infinitely many dark-grey *lakes* (see Mira et al., 1996) embedded in the light-grey region inside the quadrilateral of vertices $OO_1^{-1}O_3^{-1}O_2^{-1}$. In particular, after the bifurcation (tangency) two main lakes lie inside Z_4. Hence, both have further preimages which form lakes of smaller dimension within the light-grey region. These lakes lie inside the quadrilateral in the region complementary to Z_0. This phenomenon is iterated indefinitely (see Mira et al., 1996 for an analysis and deep discussion of this type of basin bifurcation).



(a) (b)



(c)

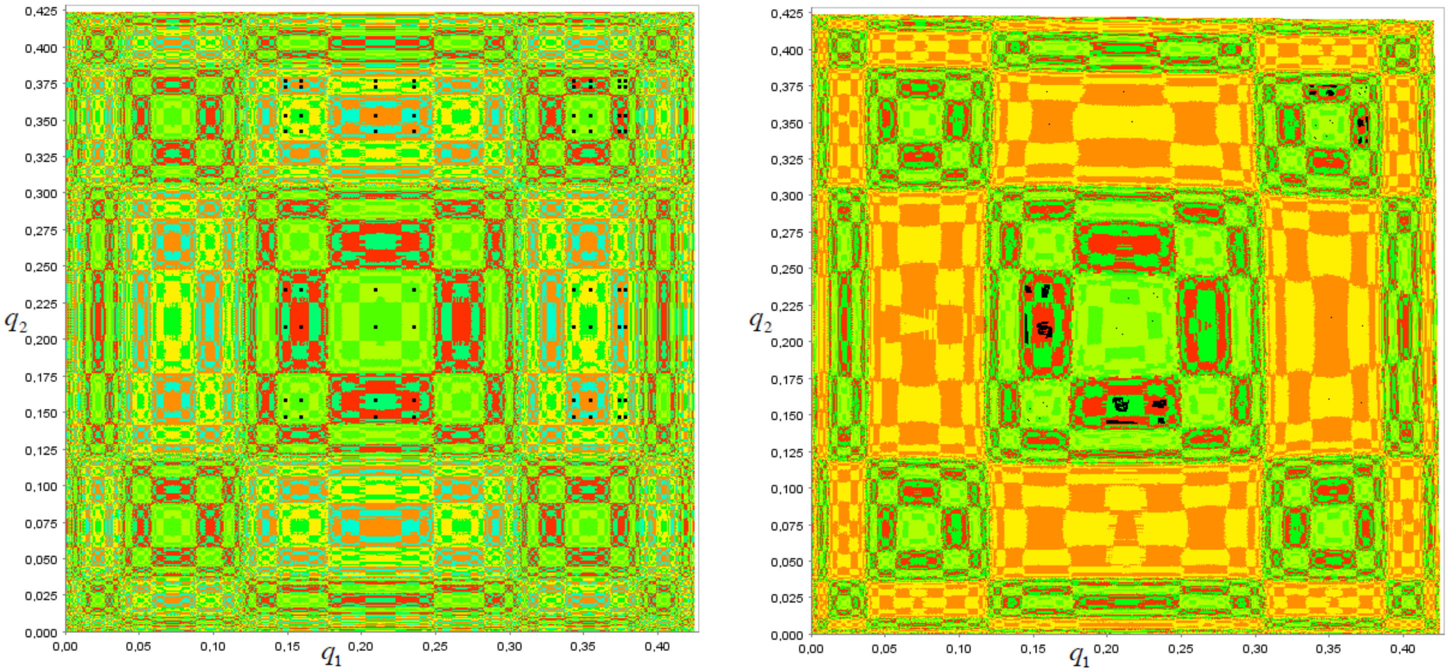
Figure 7. Parameter values: $\alpha = 3.9$ and $c = 0.32$ (a) For $\beta = 0.987$, we observe the coexistence of two chaotic attractors, black-coloured and red-coloured, whose basins of attraction are represented by the dark-grey region and the light-grey region, respectively. (b) For $\beta = 0.97$, a global bifurcation is just occurred and infinitely many dark-grey lakes embedded in the light-grey region are born. In both Figures 7.a and 7.b also the critical curves $LC^{(a)}$ and $LC^{(b)}$, which separate the zones with number of preimages, are depicted. (c) An enlargement view of Figure 7.b., showing a portion of the phase plane after the tangency between $LC^{(b)}$ and the boundary of P_1 and P_2 has occurred. The birth of the region between $LC^{(b)}$ and the boundary of P_1 (denoted by H) is also shown.

It is also important to note that in this model multistability can be recognised not only when there are chaotic attractors, but also when several distinct attractors (with their corresponding basins of attraction) are in existence. In order to show this behaviour of map T, the parameter set is slightly changed to $\alpha = 4.2$ and $c = 0.39$. Then we let β reduce from 1 (see Figure 8.a), even if this case is not economically feasible because profits of both firms are zero in such a case, to 0.98 (see Figure 8.b). Indeed, by fixing $\beta = 1$, map T (see Eq. 5) collapses to:

$$T = T(q_1, q_2): \begin{cases} q_1' = q_1 + \alpha q_1 [1 - 2q_1 - c] \\ q_2' = q_2 + \alpha q_2 [1 - 2q_2 - c] \end{cases}. \tag{22}$$

Eq. (22) is topologically conjugated to the double logistic map whose properties have been studied by, e.g., Shin (2003), where, in particular, the global pattern of the basins of attraction of the map is classified. We recall that Eq. (22) describes a totally uncoupled map, that is $q_i' = f(q_i)$, with $i = \{1, 2\}$.

Then, for $\beta = 1$ we observe that six attractors exist and their basins of attraction are composed of an infinite number of rectangles. This result persists if we let β decrease to an economically feasible value ($\beta = 0.98$). It is also interesting to note that in this latter case: (*i*) non-synchronised attractors exist along the diagonal, and (*ii*) some of the attractors have undergone local bifurcations.



(a) (b)

Figure 8. Parameter values: $\alpha = 4.2$ and $c = 0.39$. (a) For $\beta = 1$, we observe a plethora of attractors with the corresponding basins of attractions. In particular, a four-period cycle exists on the diagonal. (b) For $\beta = 0.98$, several attractors have undergone to local bifurcations. No attractors exist anymore on the diagonal. Perhaps, some of the attractors are chaotic.

4. Conclusions

This paper originated from a twofold reason: on the one hand, the increasing interest for a refined analysis of the burgeoning dynamic oligopoly literature (e.g., Tramontana et al., 2009; Bischi et al., 2010); on the other hand, the importance, emphasised by both the theoretical and empirical literatures (e.g., Gibbons and Murphy, 1990; Miller and Pazgal, 2002) of compensation practices to managers based on sales, market share and relative profit delegation schemes, has raised an established literature that extends the standard oligopoly (static) model with profit-maximising firms by including managerial incentive contracts (e.g., Fershtman and Judd, 1987; Miller and Pazgal, 2002; Jansen et al., 2009). The novelty of the present paper is the introduction of managerial incentive contracts, based on relative performance evaluations, in a dynamic duopoly game with quantity competition and homogeneous players.

While the existing literature à la VFJS clarified the ranking between equilibrium outcomes in models with sales delegation, market share delegation and relative performance evaluations (and compared them with the outcome of the standard Cournot model with profit-maximising firms), we showed that the existence of managerial delegation with relative profit schemes may cause interesting local and global dynamic events when either the degree of competition or collusion between managers becomes higher. In particular, when the basic parameter of the model are such that a chaotic set exists on the diagonal, the collusion between managers may favour synchronisation of the dynamics in a few iterations. In contrast, when competition exists we observe a dramatic change in the dynamic events, because relevant phenomena such as on-off intermittency, blow out phenomena and multistability are more likely to occur. This means that competition between managers induces a lack of coordination which is the main cause of all these dynamic phenomena that are usually observed when players are heterogeneous (Bischi et al., 1998; 1999).

This paper can be extended in several directions, for instance by introducing managerial delegation contracts in dynamic duopoly games with quantity competition and isoelastic demand functions, or in dynamic duopoly games with price competition and horizontal and/or vertical differentiation between products.

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