The sugar-pie game

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Abstract:

Playing a bargaining game the players are trying to enlarge their share of a sugar-pie. However, HE is not very keen on sweets and does not prefer a piece of the pie if the size of the pie is too small or too large. In HIS view, too small or too large pies are not of a reasonable quality. In contrast, SHE, the second actor, likes sweets what ever they are. HE is a soft negotiator but SHE is a tough negotiator. The paper addresses the problem: what should be HIS power of negotiations if an equal ½-division of the pie is desirable.

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Key words: bargaining power, bargaining game

Introduction. The players at the bargaining table usually are trying to split an economic surplus in a rational and efficient way. To be sure, the main problem, what the players try to solve during negotiations, is the slicing of the pie. Slicing depends upon characteristics and expectations of bargainers. Given that the expectations of players are non-conforming, i.e., single peaked for the first in contrast to the other, the traditional bargaining procedure may be put differently—no longer a *modus operandi* of how to split the pie. The procedure, instead of slices, can be resettled, then, to proceed on distinct levels of one parameter—parametrical interval of the size, which turns to be the *scope of negotiations*. In fact, Cardona and Ponsattí (2007: 628) noticed that "*the bargaining problem is not radically different from negotiations to split a private surplus,*" when all in the bargain have the same, conforming expectations. This is even true when the expectations of the second player are principally non-conforming, not single peaked but concave. Indeed, in the case of non-conforming expectations, related to individual rationality, (Roth 1977), also known as "*well defined bargaining problem*" or *"bargaining set,"* the scope of negotiations allows dropping the axiom of "Pareto Efficiency." Thus, combined with the *breakdown* point the well-defined problem, instead of slices, can be solved inside parametrical interval of the size of the pie.

With these remarks in mind, the procedures of negotiating on slices and the sizes can be perceived as two sides of the same bargain portfolio. Therefore, it is irrelevant whether the players are bargaining on slices of the pie, or trying to agree on size. Hereby, the main advantage of parametric procedure is exhibited: it brings about a number of different patterns of interpretations of outcomes in the game, linking an outcome, for example, to a suitable size fitting with the resources of an economy, etc., all as indicators of most desirable solutions.

The game. The game demonstrates how a sugar-pie is fairly sliced between two players: HE is a soft negotiator, not very keen on sweets, but with emphasis on quality; SHE is a tough negotiator and prefers sweets. [1]

The axiomatic bargaining theory finds the asymmetric Nash solution by maximizing the product of players' expectations above the disagreement point $d = \langle d_1, d_2 \rangle$:

$$\arg\max_{0 \le x+y \le 1} f(x, y, \alpha) = (u(x) - d_1)^{\alpha} \cdot (g(y) - d_2)^{1-\alpha},$$

the asymmetric variant (Kalai, 1977).

Although the answer may be known to game theory purists, the questions often asked by many include: *What are* x, y, α, $u(x)$ *and* $g(y)$? *What does the point* $\langle d_1, d_2 \rangle$ *mean? How the* $\arg\max$ *formula is used?* The simple answer can be given as:

x is HIS slicing the pie, and α is HIS bargaining power, $0 \le x \le 1$, $0 \le \alpha \le 1$;
$u(x)$ is HIS expectation, for example $u(x) \equiv x$, of HIS x slicing the pie;
y is HER slicing the pie, and $1 - \alpha$ is HER bargaining power, $0 \le y \le 1$;
$g(y)$ is HER expectation, for example $g(y) \equiv \sqrt{y}$, of HER y slicing the pie.

Based on the widely accepted nomenclature, we call $s = \langle u(x), g(y) \rangle$ the utility pair. The disagreement point $d = \langle d_1, d_2 \rangle$ is what HE and SHE collect if they disagree on how to slice the pie. The sugar-pie disagreement point is $d = \langle d_1, d_2 \rangle = \langle 0, 0 \rangle$, whereby the players collect nothing. Further, we believe that expectations from the pie are more valuable for SHE, indicating HER desire $g(½) = \sqrt{½} = 0.707$, which is greater than HIS desire $u(½) = 0.5$.

Now considering the $\arg\max$ formula of $f(x, y, \alpha)$, one may ask a new question: *What is the standard that will help to redesign bargaining power* α *facilitating* HIS *negotiations to obtain a desired half of the pie?* SHE may only accept or reject the proposal. A technical person can shed light on the solution. We can start by replacing $u(x)$ with x, $y = 1 - x$, $g(y)$ with $\sqrt{1-x}$, and taking the derivative of the result $f(x, 1-x, \alpha)$ with respect to the variable x by evaluating $f'_x(x, 1-x, \alpha)$. Finally, with $x = ½$, the equation $f'_x(½, ½, \alpha) = 0$ can be solved for α; indeed the equation $f'_x(½, ½, \alpha) = 0$ resolves for $\alpha = 1/3$.

In general, one might feel comfort in the following judgment: "*Even in the face of the fact that SHE is twice as tough a negotiator,* [2] *to count on the half of the pie is a realistic attitude toward HIS position of negotiations. Surely, rather sooner than later, since HE revealed that SHE prefers sweets whatever they are, HE would have HER agree to a concession.*" This attitude might well be the standard of redesigning the power of HIS negotiation abilities if half of the pie is desirable as a specific outcome of negotiations.

[1] Note that for the purpose of the game we do not ignore the size of the pie but put this issue temporarily aside.
[2] Let us say that SHE pays HER solicitor twice as much of that of HE does.

Next, it will be assumed that in the background of HIS judgment, the quality of the pie first increases, when the size is small, but reaching the peak point it starts to decline: the quality, one will say, is single peaked upon the size. For HER, the pie is always desirable. The above can be restated, then, with the condition that HE seeks an efficient size z of the pie. Although HE is not committed to size z whether or not to accept the recommendation of HER, HE is committed, however, to slice x aligned in eventual agreement. Let the utility pair $\langle u,g\rangle$ of HIS and HER expectations be given by:

$$u(\mathrm{z},x)=\mathrm{z}\cdot[(1+x/2)-\mathrm{z}],\ g(\mathrm{z},y)=\mathrm{z}\cdot\sqrt{y},\ \mathrm{z}\in[0,1],\ x,y\in[0,1].$$

Define efficient slices to the size z as a curve $x(\mathrm{z})$, which solves $\mathrm{u}'_{\mathrm{z}}(\mathrm{z},x)=0$ for x. Evaluating x from $\mathrm{u}'_{\mathrm{z}}(\mathrm{z},x)=0$ and subsequently replacing $x(\mathrm{z})$ into $u(\mathrm{z},x)$ and $g(\mathrm{z},x)$, yields $u(\mathrm{z})=\mathrm{z}^2$ and $g(\mathrm{z})=\mathrm{z}\cdot\sqrt{3-4\cdot\mathrm{z}}$. Hereby, the bargaining problem $\langle \boldsymbol{S},d\rangle$ reschedules, then, into parametric space $\boldsymbol{S}_{\mathrm{b}}=\langle u(\mathrm{z}),g(\mathrm{z})\rangle$ of the size parameter $\mathrm{z}\in[\frac{1}{2},\frac{3}{4}]\subset[0,1]$. Given the scope of negotiations—the interval $[\frac{1}{2},\frac{3}{4}]$—the root $\frac{1}{2}$ resolves $\mathrm{u}'_{\mathrm{z}}(\mathrm{z}|_{\mathrm{z}=\frac{1}{2}},0)=0$, and the root $\frac{3}{4}$ resolves $\mathrm{u}'_{\mathrm{z}}(\mathrm{z}|_{\mathrm{z}=\frac{3}{4}},1)=0$ for z accordingly. In HIS view the pie must fit the size, since outside the interval $[\frac{1}{2},\frac{3}{4}]$ it is too small—not useful at all, or too large—and offers a low quality. Therefore, the disagreement occurs at $d=\langle u(\frac{1}{2}),g(\frac{3}{4})\rangle=\langle\frac{1}{4},0\rangle$. Assuming that the size of the sugar-pie remained fixed (*fiscally safe*) during the delivery to its end destinations, the Nash symmetric solution to the game is found at $\mathrm{z}=0.69$, $x=0.74$. On the other hand, HIS asymmetric power 0.212 is adequate to negotiate with HER about the half of the pie. The size $\mathrm{z}=0.62$, fits, for example, the necessary power 0.212 capacities of a bakery for the pie preparation. [3]

References.

1. Kalai, E., 1977. Nonsymmetric Nash Solutions and Replications of 2-Person Bargaining, International Journal of Game Theory 6, 129-133.

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[3] Once again, to find the Nash symmetric solution a technically minded person must resolve the equation $f'_{z}(\mathrm{z},\alpha)=0$ for z, where $f(\mathrm{z},\alpha)=(u(\mathrm{z})-\frac{1}{4})^{\alpha}\cdot g(\mathrm{z})^{1-\alpha}$ when $\alpha=\frac{1}{2}$; $\mathrm{z}=0.69$ resolves the equation. Then, resolve the equation $\mathrm{u}'_{\mathrm{z}}(0.69,x)=0$ for x, and find that $x=0.74$. To find the power of asymmetric solution, first resolve the equation $\mathrm{u}'_{\mathrm{z}}(\mathrm{z},\frac{1}{2})=0$ for z, $\mathrm{z}=0.62$, $x=\frac{1}{2}$. Then, resolve $f'_{z}(0.62,\alpha)=0$ for α and find that the power of HIS that is equivalent to $\alpha=0.212$, and is suitable to negotiate with HER when the equal, ½-slicing of the pie is desirable.