Bayesian Equilibrium by Iterative Conjectures: A Theory of Games with Players forming Conjectures Iteratively Starting with First Order Uninformative Conjectures

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Abstract:

This paper introduces a new game theoretic equilibrium, Bayesian equilibrium by iterative conjectures (BEIC). It requires agents to make predictions, starting from first order uninformative predictive distribution functions (or conjectures) and keep updating with statistical decision theoretic and game theoretic reasoning until a convergence of conjectures is achieved. In a BEIC, rationality is achieved for strategies and conjectures. The BEIC approach is capable of analyzing a larger set of games than current Nash Equilibrium based games theory, including games with inaccurate observations, games with unstable equilibrium and games with double or multiple sided incomplete information games. On the other hand, for the set of games analyzed by the current games theory, it generates far lesser equilibriums and normally generates only a unique equilibrium. It also resolves inconsistencies in equilibrium results by different solution concepts in current games theory.

Keywords: new equilibrium concept, iterative conjectures, convergence, Bayesian decision theory, Schelling point

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1. Introduction

Current Nash Equilibrium based games theory solves a game by asking which combinations of strategies constitute equilibriums. The implicit assumption is that agents know the strategies adopted by the other agents and which equilibrium they are in, for otherwise they will not be able to react specifically to the optimal strategies of other agents but must react to the strategies of the other agents they predicted.[1] This implicit assumption reduces the uncertainty facing the agents and simplifies computation. Further refinements such as sub-game perfect equilibrium, Bayesian Nash equilibrium and Perfect Bayesian (Nash) equilibrium, though adding further requirements, do not change this implicit assumption.[2]

The Bayesian equilibrium by iterative conjectures approach, in contrast, solves a game by assuming that the agents do not know the strategies adopted by other agents and have no idea which equilibrium they are in. Therefore, to select a strategy, they need to form predictions or conjectures about the strategies adopted by other agents and the equilibrium they will be in and conjectures about such conjectures, ad infinitum. They do so by starting with first order uninformative predictive probability distribution functions (or conjectures) on the strategy of the other agents. The agents then keep updating their conjectures with game theoretic and statistical decision theoretic reasoning until a convergence of conjectures is achieved. In a BEIC, the convergent conjecture is consistent with the equilibrium it supported. BEIC therefore rules out equilibriums that are based on conjectures that are inconsistent with the equilibriums they supported as well as equilibriums supported by convergent conjectures that do not start with first order uninformative conjectures.

What is the rationale to start with first order uninformative conjectures? Other than the assumption that the agents have no idea about the strategies adopted by other agents and the equilibrium they are in, there is the motive to let the game solves itself and selects its own equilibrium strategies and conjectures. The equilibrium so achieved therefore is not imposed or affected by informative conjectures arbitrarily chosen, but by the underlying structure and elements of the game, including the payoffs of the agents and the information they have.

Harsanyi and Selten (1988) propose a tracing procedure to select the most compelling equilibrium among multiple Nash equilibriums.[3] Their tracing procedure starts with first order uninformative conjectures too. The solution of simultaneous games by Bayesian equilibrium by iterative conjectures (BEIC) is very similar to the tracing procedure of Harsanyi and Selten (1988). However, the BEIC approach does not start its tracing with only Nash equilibriums. It starts with all possible strategies of the players. This is ensured through the enforced use of first order uninformative

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[1] Refer to Nash (1950, 1951).

[2] Refer to Harsanyi (1967, 1968a, 1968b).

[3] See also Harsanyi (1995).

conjectures.

Section 2 presents the BEIC solution of sequential games with incomplete information and inaccurate observation and serves to introduce the general thrust of the new equilibrium concept. Section 3 presents the BEIC approach to sequential games of incomplete but perfect information. Section 4 presents the BEIC approach to simultaneous games. Section 5 concludes the paper.

2. Sequential Games with Incomplete Information and Inaccurate Observation.

In present modeling of incomplete information games, there is either perfect or imperfect information. That is to say, either the action of the first mover is accurately observed by the later movers or it is not observed at all. For instance in a signaling game, the action of the first moving player whose type is unknown is accurately observed by the other players. After observing the action of the player with unknown type, the other players use game theoretic reasoning and the Bayes rule to update their prior beliefs about the type of the player with unknown type. They then choose their optimal strategy given their posterior beliefs about the type of the player with unknown type. The equilibrium so obtained is termed the perfect Bayesian Equilibrium. On the other hand, in an incomplete and imperfect information game, the action of the player whose type is unknown is not observed by the other players at all. The other players choose their optimal strategy given their prior beliefs about the type of the player with unknown type. The equilibrium so obtained is termed Bayesian Nash equilibrium.

The assumption that the action of the first mover is either accurately observed or not observed at all is too restrictive. Given this assumption, there is no statistical inference and decision involved concerning the action of the first mover whose type is unknown. This is despite of the fact that Bayes rule is used to update the belief on the probability of type of the player with unknown type.

Sequential games with incomplete information and inaccurate observation generalizes the current sequential games framework in which there is either perfect information or imperfect information. Here the other player observes inaccurately the action of the player with unknown type. Inaccurate observation means that the other player observes the action of the player with unknown type with a noise term and there is a positive probability that they will make observational error due to the noise term.

In a sequential game with incomplete information and inaccurate observation, the second mover must make statistical inference on the action of the first mover player with unknown type. He does so bases upon two sources of information. One source of information is the inaccurate observations on the action of the player with unknown type. This is the sample data. The other source of information is the evidence which concerns the motive of the player with unknown type constructed through game theoretic reasoning, basing upon knowledge such as the prior distribution function on the type of the player with unknown type and the structure of the game. The information so constructed gives a belief about the probability of possible actions taken by the player with unknown type. This belief is the prior predictive distribution function or conjecture on the action of the player with unknown type.

Given the need for statistical inference and decision, the player has to decide which statistical decision rule to use. Since in games theory, the basic assumption is that the player is rational, that is, he optimizes, the decision rule has to be a Bayes rule. A decision rule is a Bayes rule if it attains the infimum of the expected loss function or the supremum of the expected utility function.[4] Furthermore, given the knowledge a player has about the game, he will form prior predictive distribution function on the possible actions of the other player. There are many ways to construct a prior distribution function. Therefore, in an incomplete information game with inaccurate observation, there could be a lot of equilibriums given that there are many statistical decision rules and many different prior beliefs. Presently in games theory there is no equilibrium concept to solve such games.

This section uses the concept of Bayesian equilibrium by iterative conjectures to solve such games. The conjecture is formed through iterative reasoning, starting with a first order uninformative conjecture or prior predictive distribution function on the action of the player with unknown type and keeps being updated by game theoretic and statistical decision theoretic reasoning until a convergence of the prior predictive distribution function is achieved. Consequently, the convergent conjecture incorporates all available useful information such as the structure of strategic interaction and the prior distribution function on the type of the player with unknown type.

2.1. Example 1: Market Leadership

[4] Other criteria for selecting decision rule include the minimax rule and admissibility. Refer to Berger (1980).

There are two players: Firm 1, the market leader and Firm 2, the market follower. Firm 1 moves first by setting its output level. Firm 2 observes inaccurately the output level of Firm 1 due to a confounding noise term. Firm 2 forms iterative conjectures on the output level of Firm 1 starting with a first order uninformative prior predictive distribution function and keeps updating by statistical decision theoretic and game theoretic reasoning until a convergence of conjectures is achieved and then sets its output level.

The structure of the game is common knowledge. The cost efficiency of Firm 1 which determines the type of Firm 1 is chosen by Nature from a predetermined distribution function which is common knowledge. Once chosen, the type of Firm 1 is private knowledge. The type of Firm 2 is common knowledge. Firm 2 therefore must makes inference on both the type and action of Firm 1. The distribution function of the noise term that confounds the observation by the Firm 2 on the actual output level of Firm 1 is common knowledge.

The Model

q_1, the output level of Firm 1, is the action of Firm 1. q_2, the output level of Firm 2, is the action of Firm 2. Total level of output in the market is $Q = q_1 + q_2$. The inverse demand function is $P = D - Q$. The payoff function of Firm 1 is $\pi_1 = (D - q_1 - q_2 - c_1) q_1$. c_1 is the average and marginal cost of production of Firm 1. c_1 decides the type of Firm 1. Firm 1 knows c_1 but Firm 2 does not know c_1. c_1 has a normal distribution which is common knowledge: $c_1 \sim N\left(\bar{c}_1, \zeta\right)$. The action of Firm 1 is inaccurately observed by Firm 2 with a noise term: $R = q_1 + \varepsilon$. ε is the noise term. ε has a normal distribution: $\varepsilon \sim N(0, \kappa)$. The above leads to the following sampling distribution on R: $R \mid q_1 \sim N(q_1, \kappa)$ and the likelihood function: $q_1 \mid R \sim N(R, \kappa)$.

The game is solved starting with an uninformative first order prior conjecture on q_1. That is, firm 2 solves

$$\max_{q_2} E(\pi_2) = \int_{-\infty}^{\infty} (D - q_1 - q_2 - c_2) q_2 f(q_1 \mid R)\, dq_1 \qquad 1$$

where $f\left(q_1|R\right)$ in equation 1 is the posterior distribution function with an uninformative prior distribution function. The optimal solution is $q_2 = \frac{D-R-c_2}{2}$.

Firm 1, being the first mover, anticipates the stochastic response of firm 2 and solves

$$\max_{q_1} E\left(\pi_1\right) = \int_{-\infty}^{\infty}\left(D-q_1-\frac{D-R-c_2}{2}-c_1\right)q_1 f\left(\varepsilon\right)d\varepsilon \qquad 2$$

The optimal solution is $q_1 = \frac{D+c_2-2c_1}{2}$. Therefore, the second order prior conjecture is $q_1 \sim N\left(\bar{q}_1, \zeta\right)$ with $\bar{q}_1 = \frac{D+c_2-2\bar{c}_1}{2}$. The second order posterior conjecture is $q_1|R \sim N\left(\hat{q}_1, \hat{\rho}\right)$, with $\hat{q}_1 = \frac{\zeta}{\kappa+\zeta}R + \frac{\kappa}{\kappa+\zeta}\bar{q}_1 = \theta R + \left(1-\theta\right)\bar{q}_1$ where $\theta = \frac{\zeta}{\kappa+\zeta}$ and $\hat{\rho} = \frac{\zeta\kappa}{\kappa+\rho}$.

Given the second order conjectures, firm 2 solves

$$\max_{q_2} E\left(\pi_2\right) = \int_{-\infty}^{\infty}\left(D-q_1-q_2-c_2\right)q_2 f\left(q_1|R\right)dq_1 \qquad 3$$

where $f\left(q_1|R\right)$ in equation 3 is the second order posterior conjecture or distribution function and the optimal solution is $q_2 = \frac{D-\hat{q}_1-c_2}{2}$ and $q_2|q_1 \sim N\left(\frac{D-c_2-\left(\theta q_1+\left(1-\theta\right)\bar{q}_1\right)}{2}, \frac{\theta^2}{4}\kappa\right)$.

Anticipating that Firm 1 solves

$$\max_{q_1} E\left(\pi_1\right) = \int_{-\infty}^{\infty}\left(D-q_1-\frac{D-c_2}{2}+\frac{\hat{q}_1}{2}-c_1\right)q_1 f\left(\varepsilon\right)d\varepsilon \qquad 4$$

The optimal solution is $q_1 = \frac{D+c_2+\left(1-\theta\right)\bar{q}_1-2c_1}{2\left(2-\theta\right)}$. Therefore, the third order prior conjecture is $q_1 \sim N\left(\bar{q}_1, \rho\right)$, $\bar{q}_1 = \dfrac{D + c_2 - 2\bar{c}_1}{3-\theta}$ and $\rho = \dfrac{1}{(2-\theta)^2}\zeta$. The third order posterior conjecture is $q_1 | R \sim N\left(\hat{q}_1, \hat{\rho}\right)$, $\hat{q}_1 = \dfrac{\rho}{\kappa+\rho}R + \dfrac{\kappa}{\kappa+\rho}\bar{q}_1 = \theta R + (1-\theta)\bar{q}_1$, $\theta = \dfrac{\rho}{\kappa+\rho}$ and $\hat{\rho} = \dfrac{\rho\kappa}{\kappa+\rho}$.

Given the third order conjectures, firm 2 solves

$$\max_{q_2} E(\pi_2) = \int_{-\infty}^{\infty} (D - q_1 - q_2 - c_2) q_2 f\left(q_1 | R\right) dq_1 \qquad 5$$

where $f\left(q_1 | R\right)$ in equation 5 is the third order posterior distribution function. The optimal solution is $q_2 = \dfrac{D - \hat{q}_1 - c_2}{2}$ and $q_2 | q_1 \sim N\left(\dfrac{D - c_2 - \left(\theta q_1 + (1-\theta)\bar{q}_1\right)}{2}, \dfrac{\theta^2}{4}\kappa\right)$.

Foreseeing that firm 1 therefore solves

$$\max_{q_1} E(\pi_1) = \int_{-\infty}^{\infty} \left(D - q_1 - \frac{D - c_2}{2} + \frac{\hat{q}_1}{2} - c_1 \right) q_1 f(\varepsilon) d\varepsilon \qquad 6$$

The optimal solution is $q_1 = \dfrac{D + c_2 + (1-\theta)\bar{q}_1 - 2c_1}{2(2-\theta)}$. Therefore, the fourth order prior conjecture is $q_1 \sim N\left(\bar{q}_1, \rho\right)$, $\bar{q}_1 = \dfrac{D + c_2 - 2\bar{c}_1}{3-\theta}$ and $\rho = \dfrac{1}{(2-\theta)^2}\zeta$. At this point, the conjectures converge.

At the Bayesian equilibrium by iterative conjectures, firm 2 produces

$$q_2 = \frac{D - c_2}{2} - \frac{\left(\theta(q_1 + \varepsilon) + (1-\theta)\bar{q}_1\right)}{2} \qquad 7$$

Firm 1 produces

$$q_1 = \frac{D + c_2}{3-\theta} - \frac{1}{2-\theta}\left(c_1 + \frac{1-\theta}{3-\theta}\bar{c}_1\right) \qquad 8$$

2.2. Perfect and Complete Information and Indeterminacy.

Now let $\lim_{\rho\to 0}\lim_{\kappa\to\infty}\theta = 0$. In this case, at BEIC,

$$q_1 = \frac{D + c_2 + (1-\theta)\bar{q}_1 - 2c_1}{2(2-\theta)} = \frac{D + c_2 - 2c_1}{3} \tag{9}$$

$$q_2 = \frac{D - c_2}{2} - \frac{\left(\theta(q_1+\varepsilon)+(1-\theta)\bar{q}_1\right)}{2} = \frac{D - 2c_2 + c_1}{3} \tag{10}$$

This is the Cournot solution for the complete and imperfect information game or simultaneous game.

Now let the variance of the type distribution function (ζ) and variance of the noise term (κ) both tend to zero. The variance of the prior conjectural distribution function on q_1 therefore tends to zero as well. The equilibrium q_1 and q_2 when all the three variances tend to zero depend upon the value of $\lim_{\rho\to 0}\lim_{\kappa\to 0}\theta$ which could take on any value from 0 to 1. If $\lim_{\rho\to 0}\lim_{\kappa\to 0}\theta = 0$, then the BEIC has the Cournot solution. When $\lim_{\rho\to 0}\lim_{\kappa\to 0}\theta = 1$, then the BEIC has the Stackelberg solution.[5] That is,

$$q_1 = \frac{D + c_2 + (1-\theta)\bar{q}_1 - 2c_1}{2(2-\theta)} = \frac{D + c_2 + (1-\theta)q_1 - 2c_1}{2(2-\theta)} = \frac{D + c_2 - 2c_1}{2} \tag{11}$$

$$q_2 = \frac{D - c_2}{2} - \frac{\left(\theta(q_1+\varepsilon)+(1-\theta)\bar{q}_1\right)}{2} = \frac{D - c_2}{2} - \frac{\left(\theta(q_1+\varepsilon)+(1-\theta)q_1\right)}{2} \tag{12}$$

$$= \frac{D - c_2 - q_1}{2} = \frac{D - 3c_2 + 2c_1}{4}$$

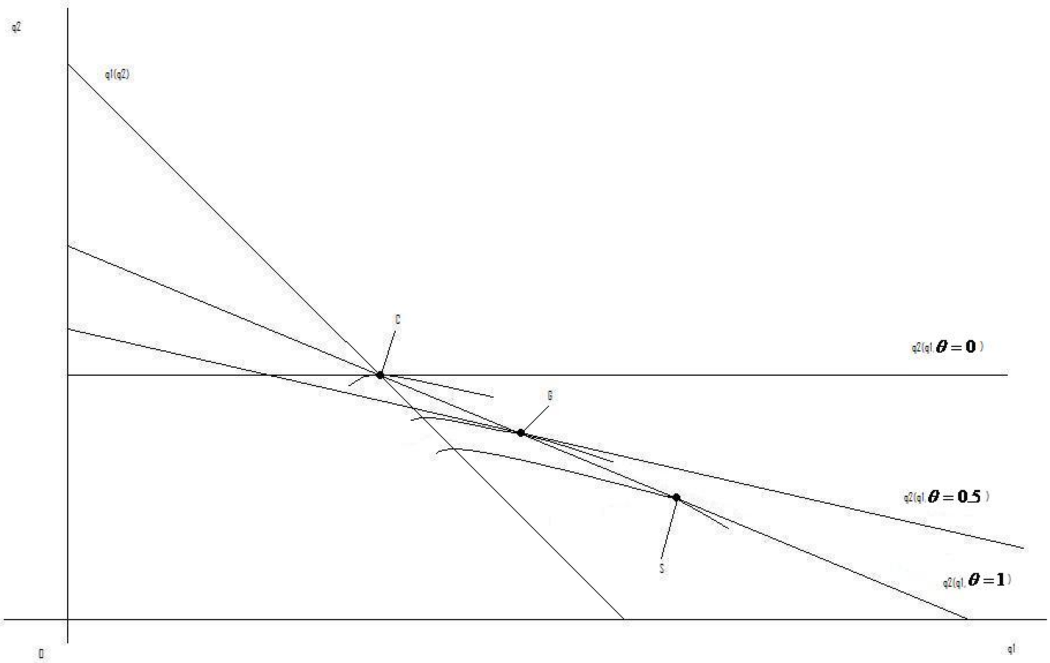
If $\lim_{\rho\to 0}\lim_{\kappa\to 0}\theta = 0.5$, then in the BEIC,

[5] This case assumes that the inferring agent bases his statistical inference and decision entirely on his observation and not prior conjecturing.

$$q_1 = \frac{D + c_2 + (1-\theta)\bar{q}_1 - 2c_1}{2(2-\theta)} = \frac{2(D + c_2 - 2c_1)}{5} \qquad 13$$

$$q_2 = \frac{D - c_2}{2} - \frac{\left(\theta(q_1 + \varepsilon) + (1-\theta)\bar{q}_1\right)}{2} = \frac{2}{5}D - \frac{7}{10}c_2 + \frac{2}{5}c_1 \qquad 14$$

The cases are illustrated in the diagram below:



In the above diagram, C is the solution when $\theta = 0$, S is the solution when $\theta = 1$ and G is the solution when $\theta = 0.5$. This indeterminacy arises for given perfect information and complete information, it begs the question: which is more accurate, the information on action that is perfect or the information on type that is complete.

3. Sequential Games with Incomplete but Perfect Information

In current Nash Equilibrium based games theory, the solution algorithm of sequential games with incomplete and perfect information solves a game by asking which combinations of strategies and posterior beliefs of players constitute equilibriums. Implicit in this solution algorithm is the assumption that the players know which equilibrium they are in and knows the equilibrium strategies and beliefs of the other players. The assumption that players know the equilibrium of the game and strategies and beliefs of the other player removes much of the inherent uncertainty about the strategies of the other players in games of incomplete information.

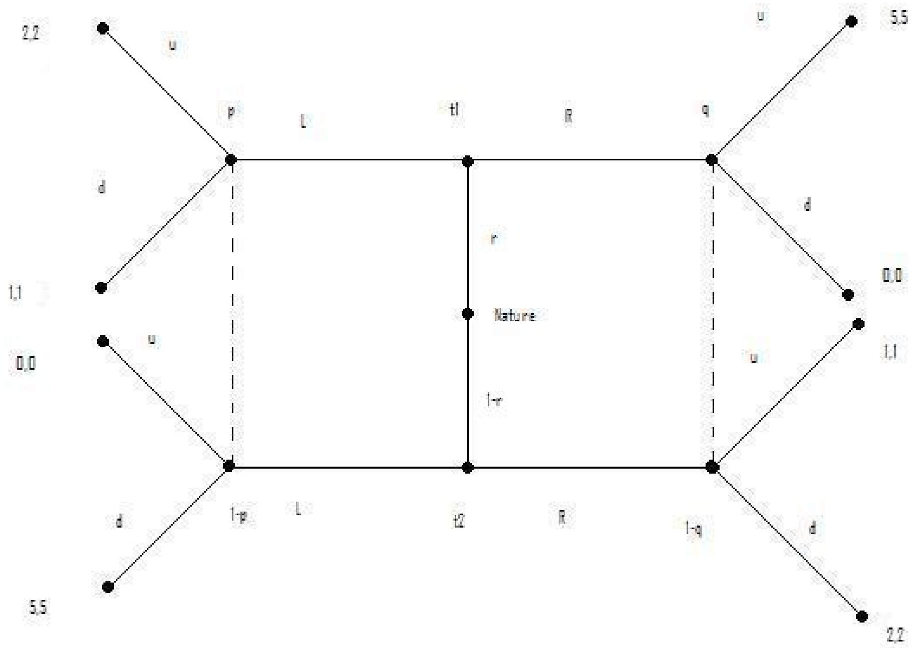
In a sequential game of incomplete and perfect information, there is the uncertainty about the type of some of the players. Therefore, despite the fact that the player observes the action of the player with unknown type perfectly, he must still infer about the strategy of each type of the player with unknown type through game theoretic reasoning. Also, the player with unknown type must also conjecture about the strategy and conjectures of the other player when selecting his strategy. Consequently, unlike a sequential game of complete and perfect information, the player cannot condition his strategy upon the other player's strategy: the player with known type cannot do so for the other player has more than one types and the player with known type observes the other player’s action but not strategy and, the player with unknown type cannot do so for he must infer about the conjectures or beliefs (which he does not observe) and strategies (which depends upon the conjectures) of the other player.

The BEIC approach, in contrast, investigates how the conjectures of players about the strategies of the other players and their conjectures converge. The solution algorithm is exactly the same as that of the previous section, except that in this case the players make perfect observation of action (perfect information) and hence have no need to make statistical inference on action but must make statistical inference and decision on the strategies and types of other players.

In solving sequential games with incomplete and perfect information, the BEIC approach starts from the assumption that players do not know the other player's strategy nor the equilibrium of the game through the use of first order uninformative conjectures, though the players observe perfectly the action of other players. Conjectures are updated using game theoretic reasoning until a convergence emerges which then defines a BEIC. Another important difference between the BEIC approach and the current games theory is that when having pooling equilibrium, the current games theory needs to specify probability beliefs on off equilibrium paths. In contrast, the BEIC approach uses a hierarchy of conjectures, first order uninformative prior conjectures and higher order conjectures, the highest order conjectures being the set of convergent conjectures (if it exists). The convergent conjectures and their corresponding equilibrium, either separating or pooling equilibrium, are supported by lower level conjectures. Therefore, there is no need to specify off equilibrium paths beliefs.

3.1. Example 2: Coordination Game.

Consider the signaling game as depicted in the following diagram.



The probability of player 1 being type 1 and type 2 is r and 1-r.

There are four perfect Bayesian equilibriums:
i. (L, R; u(L), d(R)). This equilibrium is socially suboptimal.
ii. (R, L; d(L), u(R)). This equilibrium is socially optimal.
iii. Pooling equilibrium (R, R; u(L), u(R); r>1/6, p>5/6).
iv. Pooling equilibrium (L, L; d(L), d(R); r<5/6, q<1/6).
The two pooling equilibriums are ruled out by the intuitive criterion. The separating equilibriums do not change as r changes.

Solving by the BEIC approach:

Let the probability that the receiver plays U when observed L be a and the probability that the receiver plays U when observed R be b. Let the probability that the type 1 sender plays L be x and the probability that the type 2 sender plays L be y.

Type 1 sender plays L if $2a+(1-a)>5b$ or $\frac{1}{5}+\frac{a}{5}>b$.

Type 2 sender plays L if $5(1-a)>b+2(1-b)$ or $\frac{3}{5}+\frac{b}{5}>a$.

When L is observed, the receiver plays U if $2xr>xr+5y(1-r)$ or $xr>5y(1-r)$.

When R is observed, the receiver plays U if $5(1-x)r+(1-y)(1-r)>2(1-y)(1-r)$

or $5(1-x)r>(1-y)(1-r)$.

When r<1/6, given the first order conjectures that x=1/2 and y=1/2, the receiver plays D when L is observed and D when R is observed. Anticipating that, type 1 sender plays L and type 2 sender plays L. The receiver updates his conjectures to x=1 and y=1 and plays D(L). Anticipating that, type 1 sender plays L and type 2 sender plays L. The conjectures converge here. The BEIC is (L, L; D(L), D(R)).

When r=1/6, given the first order conjectures that x=1/2 and y=1/2, the receiver plays D when L is observed and is indifferent between U and D when R is observed. Anticipating that, type 1 sender plays R and type 2 sender plays L. The receiver updates his conjectures to x=0 and y=1 and plays D(L) and U(R). Anticipating that, type 1 sender plays R and type 2 sender plays L. At this point the conjectures converge. The BEIC is (R, L; D(L), U(R)).

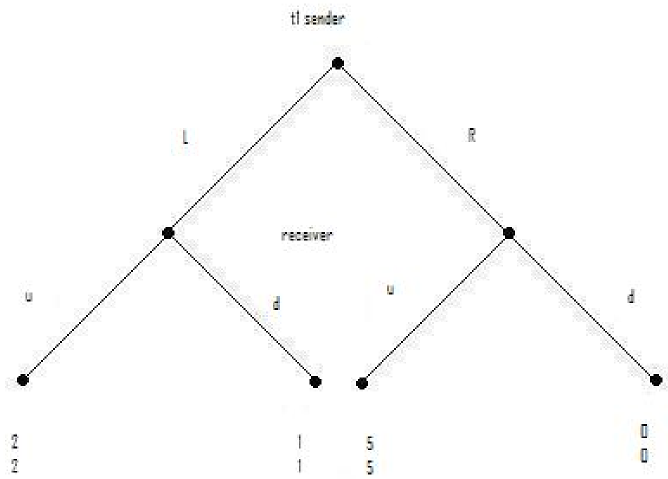
When 1/6<r<5/6, given the first order conjectures that x=1/2 and y=1/2, the receiver plays D when L is observed and U when R is observed. Anticipating that, type 1 sender plays R and type 2 sender plays L. The receiver updates his conjectures to x=0 and y=1 and plays D(L) and U(R). Anticipating that, type 1 sender plays R and type 2 sender plays L. The conjectures converge here. The BEIC is (R, L; D(L), U(R)).

When r=5/6, given the first order conjectures that x=1/2 and y=1/2, the receiver is indifferent between U and D when L is observed and plays U when R is observed. Anticipating that, type 1 sender plays R and type 2 sender plays L. The receiver updates his conjectures to x=0 and y=1 and plays D(L) and U(R). Anticipating that, type 1 sender plays R and type 2 sender plays L. The conjectures converge at this point. The BEIC is (R, L; D(L), U(R)).

When r>5/6, given the first order conjectures that x=1/2 and y=1/2, the receiver plays U when L is observed and U when R is observed. Anticipating that, type 1 sender plays R and type 2 sender plays R. The receiver updates his conjectures to x=0 and y=0 and plays U(R). The conjectures converge here. The BEIC is (R, L; D(L), U(R)).

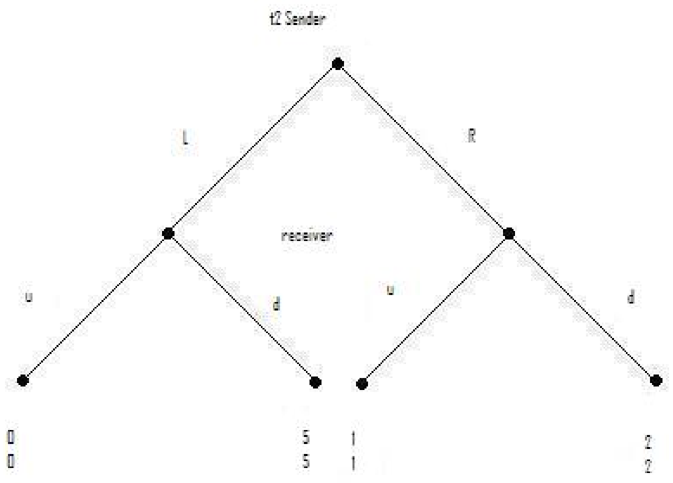
By the BEIC approach, the selection of equilibriums depends on the value of r. In contrast, the PBE approach has separating equilibriums that have nothing to do with r.

The two extreme cases of r approaches one and r approaches zero helps to shed light on this distinction between the two approaches. When r approaches one, the BEIC is (R, R; U(L), U(R)). It is equilibrium iii of the PBE approach which is ruled out by the intuitive criterion. Note that the BEIC for this limiting case agrees with the equilibrium for the sequential complete and perfect information game which is represented by the diagram below:



The equilibrium is (R; u(L), u(R)) which is derived through backward induction.

When r approaches zero, the BEIC is (L, L; D(L), D(R)). It is equilibrium iv of the PBE approach which is ruled out by the intuitive criterion. Note that the BEIC for this limiting case agrees with the equilibrium for the sequential complete and perfect information game which is represented by the diagram below:



The equilibrium is (L; d(L), d(R)) which is derived through backward induction.

The above example illustrates the BEIC approach to solving a game of incomplete and perfect information. It also illuminates the relationship between incomplete and perfect information sequential games and complete and perfect information sequential games. When the variance of type tends to zero, a sequential game with incomplete and perfect information becomes a sequential game with complete and perfect information where the player relies upon the observation totally for his statistical inference and decision and not the prior conjectures. The equilibrium of the latter should equal to the equilibrium of the former in the limiting case. The BEIC approach satisfies this requirement.

4. Simultaneous Games

The way the Nash equilibrium approach solves a simultaneous move game is to get the interaction points of the reaction functions. Implicit in this solution algorithm is that there is perfect information and the moves are sequential. That is what a reaction function means: the reaction of one player to the action of the other player. That implies perfect information for you must observe the action of the other party before you could react to his action. If there is simultaneity in moves and the players do not observe the moves of the other players, then they could not react to the actions of the other players. In this situation, a player would react to his conjectures of the actions of the other players. It is clear that conjecture plays a central role here. The reaction functions of a simultaneous game are therefore not really reaction functions as those of a perfect information sequential game and are best named as virtual reaction functions for differentiation from the real reaction functions of a perfect information sequential game.

In a simultaneous move game, none of the players observed what the other players are doing and they all make their decisions simultaneously and all these are common knowledge. By the very definition of simultaneous move, even if one of players will play a particular equilibrium strategy prescribed by the concept of Nash equilibrium, the other players still do not observe the action of that player. They therefore have to conjecture about the move. Since what the players think or conjecture will affect their decisions, it therefore follows that the players must conjecture about the other player' conjectures, besides conjecturing what the other players are doing or will do.

The BEIC solution of a simultaneous game traces out the whole process of formation and updating of conjectures starting with first order uninformative conjectures and keep updating by game theoretic reasoning to achieve convergence in conjectures, if there is any. Example 4 and 5 illustrate the BEIC solution of complete and incomplete information simultaneous games.

Example 4: Investment Entry Game

1\2	Enter (y)	Refrain (1-y)
Modern (w)	0, -2	7, 0
Antique (1-w)	4, 2	6, 0

There are three Nash equilibriums: (w=0, y=1), (w=1, y=0) and (w=1/2, y=1/5).

The reaction functions are: w=1 for y<1/5, $w \in [0,1]$ for y=1/5 and w=0 for y>1/5;

y=1 for w<1/2, $y \in [0,1]$ for w=1/2, y=0 for w>1/2.

The BEIC solution proceeds as follow:

Given the first order uninformative conjectures that $w = 0.5$, the second order conjecture is y=0.5 as the second player is indifferent between Enter and Refrain given that w=0.5. The third order conjecture is w=0 and the fourth order conjecture is y=1 and the fifth order conjecture is w=0 and the conjectures converge here. Starting with the first order conjecture that $y = 0.5$, the second order conjecture is $w = 0$ and the third and fourth order conjectures are y=1 and $w = 0$. The process converges here. The unique BEIC is (w=0, y=1).

The BEIC of a simultaneous game is very similar to the focal point. Both are convergence of conjectures (or predictions or expectations).[6] This similarity is obvious by looking at the best-response equivalent identical interest game of the entry-investment game analyzed above. A best-response equivalent game is a transformation of a game whereby the payoff matrix of the original game is transformed yet the reaction functions are preserved so that the strategic nature of the game is unchanged.[7] An identical interest game has the special feature that the payoffs of the players are exactly the same. In an identical interest game, there is therefore at least a natural focal point or Schelling point: the combination of strategies that yields the highest payoff.

The best response equivalent identical interest game of the investment-entry game is:

1\2	Enter (y)	Refrain (1-y)
Modern (w)	0, 0	5, 5
Antique (1-w)	8, 8	3, 3

Note that (0,1) is both the BRPE and focal point.

Since the BEIC is a point where conjectures converged, the BEIC approach therefore selects compelling stable equilibrium and eliminates unstable equilibrium. As mixed strategy equilibriums are in general unstable, they are generally eliminated in the selection of BEIC.[8]

The BEIC approach also has the merit that its solution agrees with that of selection of equilibrium by iterative elimination of (weakly) dominated strategies (if that could be done). The following game is an example.

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[6] Schelling (1960, p. 57): "focal point(s) for each person's expectation of what the other expects him to expect to be expected to do."

[7] Refer to Morris and Takashi (2004).

[8] Refer to Aumann (1985) for criticisms of mixed strategy equilibrium.

1\2	L (y)	R (1-y)
U (x)	0, 1	1, 1
D (1-x)	1, 1	1, 0

There are two Nash equilibriums: (D, L) and (U, R). However, only one of them (D, L) makes sense for (U, R) would be weeded out by the elimination of (weakly) dominated strategy.

The BEIC solution proceeds as follow: Given the first order uninformative conjecture that $x = 0.5$, the second order conjecture is $y = 1$ and the third order conjecture is $x = 0$ and the process converges here. Given the first order uninformative conjecture that $y = 0.5$, the second order conjecture is $x = 0$ and the third order conjecture is $y = 1$ and the process converges here. The unique BEIC is (x=0, y=1) and it agrees with the result from iterative elimination of (weakly) dominated strategies.

Example 5: Incomplete Information Investment Entry Game

The solution of incomplete information simultaneous game proceeds in likewise manner, as shown in the following example where firm 2 has two types, high cost type with probability 1/10 and low cost type with probability 9/10. There are multiple equilibriums by the Bayesian Nash equilibrium approach. The BEIC approach yields a unique equilibrium.

When facing the high cost firm 2, firm 1 has the following payoff matrix:

1\2	Enter (z)	Refrain (1-z)
Modern (w)	0, -5	7, 0
Antique (1-w)	4, 1	6, 0

Firm 1 when facing the low cost firm 2 has the following payoff matrix:

1\2	Enter (y)	Refrain (1-y)
Modern (w)	0, -2	7, 0
Antique (1-w)	4, 2	6, 0

The reaction functions are:

I. $w = 1$ if $1 > \frac{9}{2}y + \frac{1}{2}z, w \in [0,1]$ if $1 = \frac{9}{2}y + \frac{1}{2}z$ and $w = 0$ if $1 < \frac{9}{2}y + \frac{1}{2}z.$

II. $y = 1$ if $\frac{1}{2} > w, y \in [0,1]$ if $\frac{1}{2} = w$ and $y = 0$ if $\frac{1}{2} < w$

III. $z=1$ if $\frac{1}{6}>w, z\in[0,1]$ if $\frac{1}{6}=w$ and $z=0$ if $\frac{1}{6}<w$

Starting with w=0.5, the second order conjectures are y=0.5 and z=0. The third, fourth and fifth order conjectures are w=0, y=1 and z=1 and, w=0. Here the conjectures converge. Starting with y=0.5 and z=0.5, the second order conjecture is w=0. The third and fourth order conjectures are y=1, z=1 and w=0. Here the conjectures converge. The unique BEIC is w=0, y=1 and z=1.

5. Conclusions

Given its ability to narrow down the number of equilibrium normally to one, the BEIC approach is useful for solving games with multiple side incomplete information, multiple heterogeneous players and multiple decision variables. It would also be useful for analyzing games where expectations or predictions need to be endogenous, such as macroeconomics games involving rational expectation.

Finally, I highlight a few major differences between the BEIC approach and the current Nash Equilibrium based approach:

1. Consistency with other major solution concepts.
The equilibrium results of current Nash Equilibrium games theory sometimes contradict those derived by backward induction or iterative elimination of (weakly) dominated strategies. In contrast, the BEIC results are consistent with those of these two methods.

2. Use of reaction functions.
The current prevailing Nash Equilibrium games theory solves for equilibriums by constructing reaction functions and looks for their intersections. In contrast, the BEIC approach constructs reaction functions as well. It however uses first order uninformative conjectures and reaction functions to derive higher and higher orders of conjectures until a convergence of conjectures is achieved.

3. Definition of rationality.
The Nash Equilibrium based approach starts without defining rationality in the processing of information and forming of conjectures or prediction. It incorporates rationality in the processing of information and forming of predictions in an ad hoc manner latter through Perfect Bayesian Equilibrium and its many refinements. In

contrast, rationality in the processing of information and forming of predictions is the very foundation of the BEIC.

4. Equilibrium in Strategic Space versus Equilibrium in Subjective Probability Space

The Nash equilibrium approach defines equilibrium in the strategic/actions space. The incorporation of beliefs in incomplete information games does not change that basic feature. In contrast, the BEIC approach defines equilibrium in the subjective probability space with its use of convergence of conjectures. Of course, for conjectures to converge, they must also be consistent with the equilibrium they supported and so the BEIC's equilibrium in subjective probability space naturally incorporates equilibrium in strategic/action space as well.

5. Objective Versus Subjective Probability distribution function

The BEIC is based on the Bayesian view of subjective probability. This allows the tracing of updating of conjectures from first order uninformative conjectures to higher and higher order of conjectures and till convergence. The Nash equilibrium based approach largely sticks to the classical or frequentist view of probability.[9] In sequential games of incomplete information with pooling equilibriums, the use of off equilibrium beliefs is an exception that resort to subjective probability.

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