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Abstract

We employ the Lévy sections theorem in the analysis of selected dollar exchange rate time series. The theorem is an extension of the classical central limit theorem and offers an alternative to the most usual analysis of the sum variable. We find that the presence of fat tails can be related to the local volatility pattern of the series.

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1 Introduction.

In the benchmark econophysics study of Mantegna and Stanley [1] the self-similarity and fat tails observed in financial distributions were shown to be responsible for a variety of behaviors and, in particular, the ultraslow convergence to a Gaussian. They suggested a truncated Lévy flight [2] to explain the departures from the central limit theorem as well as the presence of scaling laws. A complementary approach can be built on the Lévy sections theorem [5]. Paul Lévy employed his notion of "sections" to outline a proof for a variant of the central limit theorem that considers the sums of correlated random variables [3]. Thus the theorem extends the central limit theorem to encompass dependent variables.

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In this work we show how the theorem's approach can be applied to time series [5] with the help of analytical techniques taken from the study of from complex systems. We analyze daily data from the dollar price of the British pound, Indian rupee and Chinese yuan, as well as an intraday, high frequency series of the Japanese yen-Deutschemark rate. The Lévy sections suggest an explanation based on volatilities for the stylized fact of elevated kurtosis. Larger than average kurtosis of the emerging market currency of China is explained by the duration of its exchange rate pegs.

Section 2 presents the Lévy sections theorem and its extension to the analysis of time series. Section 3 illustrates with data from exchange rate changes. Section 4 concludes.

2 The Lévy sections theorem.

Let X_n be a chain of random variables. The conditional probability of a given realization x_{n+1} of X_n is $P\left(x_{n+1}\left|x_1,\ldots,x_n\right.\right)$. This is the probability of x_{n+1} if the random variables $X_1,\ldots,X_n$ follow a particular chain walk $x_1,\ldots,x_n$. The conditional mean and variance of x_n are $\mu_n \equiv \left\langle X_{n+1}\right\rangle_{x_1,\ldots,x_n} = \int x_{n+1} P\left(x_{n+1}\left|x_1,\ldots,x_{n-1}\right.\right)$ and $m_n^2 = \left\langle X_{n+1}^2\right\rangle_{x_1,\ldots,x_n} - \left\langle X_{n+1}\right\rangle_{x_1,\ldots,x_n}^2$ respectively.

Consider the quantities $\lambda_n = \sum_{i=1}^{n} m_i^2$. Given a real positive t such that $\lambda_n \leq t \leq \lambda_{n+1}$, the chain walk $(x_1,\ldots,x_n)$ is said to belong to "section" t . Section is made up of all the chain walks obeying $\lambda_n \leq t \leq \lambda_{n+1}$. These chain walks can have different numbers of elements. The sum of the elements in a truncated sequence belonging to the section t is denoted by $S_t = x_1+\ldots+x_n$, and its variance is M_t^2. For the stochastic variable S_t the Lévy sections states that [3]:

Theorem 1 *For null conditional means $\mu_n = 0$ and random variables X_n satisfying the Lindeberg conditional condition (see [3] page 237), the probability distribution of $\frac{S_t}{\sqrt{t}}$ is such that*

$$\lim_{t\to\infty} P\left(\frac{S_t}{\sqrt{t}} < \eta\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\eta} \exp^{-\frac{x^2}{2}} dx$$

The Lévy sections theorem generalizes the central limit theorem in that it also considers chains of dependent random variables. It states that the sum S_t converges to a Gaussian even if the usual sum $S_n \equiv x_1+\ldots+x_n$ does not (*e.g.* due to the presence of correlations).Taking Lévy sections amounts basically to employing the inverse of the predictable quadratic variation as a random time change to transform a given process into a Gaussian one (see [5] and the references therein for further details).

To extend the theorem to time series we must first overcome some difficulties. One major challenge is to assess the "local" volatilities m_i^2 since it is impossible to get them from only one realization of the variable (the value of x_i taken from the data set). So we take the following steps. Let $(x_n)_{n=1,\ldots,N}$ be the elements of a time series, where N is series size. For a positive integer q we then define a new series as $(y_n)_{n=1,\ldots,N-2q}$, where the initial q and the last q terms of $(x_n)_{n=1,\ldots,N}$ are dropped. Assuming that this time series is a realization of a single process, we can calculate approximately the local mean (given a "size" q) through $\mu_n = \sum_{i=n}^{n+2q} \frac{x_i}{2q+1}\ m_n^2$. Then the local volatility is

$$m_n^2 = \frac{1}{2q+1} \sum_{i=n}^{n+2q} x_i^2 - \left(\frac{1}{2q+1} \sum_{i=n}^{n+2q} x_i \right)^2 ,\ n = 1, \ldots, N-2q \qquad (1)$$

The set of all the sums S_t is $y_i + y_{i+1} + \ldots + y_{n_i-1} + y_{n_i},\ i \in \{1, \ldots, N-2q\}$ such that the "section" t condition is fulfilled. The local volatility is a measure of the conditional variance of a given chain of random variables.

3 Illustrating with exchange rate returns

We take historical daily dollar denominated price variations of the British currency (8031 data points spanning from 4 January 1971 to 10 January 2003), the Chinese currency (5471 data points spanning from 2 January 1981 to 10 January 2003), and the Indian currency (7525 data points spanning from 2 January 1973 to 10 January 2003), as in [5]. We also take a high frequency series of the Japanese yen against the Deutschemark (158 973 data points covering the time period from 01 October 1992 to 30 September 1993, obtained from Olsen & Associates – Research Institute for Applied Economics).

Fig. 1 shows the yen-mark return rate's kurtosis for different values of q . To display the sections' kurtosis behavior we take 6735 sections (for $q = 2$) and let them vary by small steps Δt . The starting value of t is such that the section matches the original series. The step Δt is chosen in such a way to make the kurtosis curve smooth. Fig. 1 shows that the results are essentially the same for $q = 2, 5$ and 10. The kurtosis quickly decays to zero, in accordance with the Lévy sections theorem. We also present the behavior of the usual sum $S_n \equiv x_1 + \ldots + x_n$, which seems to slowly converge to the Gaussian, which is typical of a truncated Lévy flight [2]. The Lévy sections filter the effects on the local volatility so that the series present a near-Gaussian universal pattern.

To compare S_t and S_n, we considered the following (Fig. 1). Let us suppose a single realization of a random process, which gives a time series of N elements $X \equiv (x_i)_{i=1,\ldots,N}$. For this time series the sum variable $S_n = X_1 + \ldots X_n$ is

$$S_n \left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_{1+i}, \ldots \right)$$

We define a "variance time" as follows:

$$\tau_n = \frac{M_n^2}{\nu^2}$$

where $\nu^2 = \langle X^2 \rangle - \langle X \rangle^2$. For IID variables, $M_n^2 = n\nu^2 \Rightarrow \tau_n = n$. For Mandelbrot's fractional Brownian motion $M_n = \propto n^H \Rightarrow \tau_n = n^{2H}$, with H the Hurst exponent. From the definition of S_t we have:

$$S_n \left(\sum_{i=1}^{n_1} y_i, \sum_{i=1}^{n_2} y_{1+i}, \ldots \right)$$

Note that the number of terms in every sum belonging to the collection S_t depends on the particular chain walk, therefore $n_1 \neq n_2 \neq \ldots$ in general.

For the time series, the variance of S_t is given by

$$M_t^2 \equiv \left\langle S_t^2 \right\rangle - \langle S_t \rangle^2$$

and we can also define its variance time as $\tau_t = \frac{M_t^2}{\nu^2}$. We defined the variance time to compare the time evolution of S_n and S_t. Note that S_t is not indexed to "actual time" (as in the case of IID variables where $\tau_n = n$). Nevertheless, the variance time allows one to compare S_n and S_t. Clearly other scales can be imagined, and in the one suggested here the variance of both S_n and S_t is the same for every variance time. So we can assess the evolution of S_n and S_t by considering not actual time, but how their respective variances evolve. We assume that the time series is stationary when doing the above sum procedures. Though the stationarity assumption for a chain of random variables is not made in the Lévy sections theorem, our sum procedures to obtain S_t for an empirical time series make sense only if the series is stationary. So our sum procedure is to be blamed in the event of a possible failure of the extension of the Lévy sections theorem to time series. See [5] for further details.

Another interesting feature is how the presence of fat tails can be related to the local volatility pattern of the series. To see this, we compare the pound, rupee and yuan with a Gaussian random generator of "reduced" variables that are independent and identically distributed (IIDR), as defined in [6]. Essentially we consider $g_n, n = 1, \ldots, N - 2q$ numbers generated from a Gaussian distribution and multiply them by the local volatilities m_n (1) reckoned from the empirical time series. Then we find the sequence $z_n = m_n g_n$, where

$n = 1, \ldots, N-4$ (here we are using $q = 2$). The name $IIDR$ comes from the fact that, although z_n is not Gaussian, the reduced variable $\frac{z_n}{m_n}$ is. Note that if m_n is constant the distribution of $z_n = m_n g_n$ also collapses to a Gaussian.

The kurtosis of the IIRD variable for the pound, rupee and yuan are respectively $6.76, 118.9$ and 1547.7. They are in agreement with the kurtosis of the original series of daily changes, at least for pound and rupee. For the pound, rupee and yuan they are respectively $4.76, 118.3$ and 3486.1.

The effect of local volatilities is clear. Since g_n is Gaussian, the elevated kurtosis (bigger than a Gaussian's kurtosis, which equals 3) should be explained by the m_n.

Due to exchange rate pegs, the dispersion of data is low provided an exchange rate is fixed. In this way many observations fall out of the variance interval. For example, the extra high kurtosis of China can be explained by too many observations outside the variance interval (a symmetrical interval around the mean and two standard deviations wide, with respect to original data). This rationale is simpler than ones based on fat tails and autocorrelation.

Exchange rate time series are believed to be modeled by a Gaussian if government intervention is absent. This is because with free float the market tends to be efficient. Our results are consistent with the interpretation that foreign exchange intervention provokes departures from the Gaussian through a bias in the volatility evolution. So the greater the control is, the greater the kurtosis. This is so because the pegs tend to bring a series' dispersion closer to zero, thereby rendering many observations out of the distribution's variance interval. For further details, see [5].

4 Conclusions.

We employ the Lévy sections theorem [3,5] in the analysis of selected dollar exchange rate time series. The theorem is an extension of the classical central limit theorem and offers an alternative to the most usual analysis of the sum variable. We find that the presence of fat tails can be related to the local volatility pattern of the series. This occurs because in the sections, a time series converges to a near-Gaussian distribution regardless of the presence of correlations.

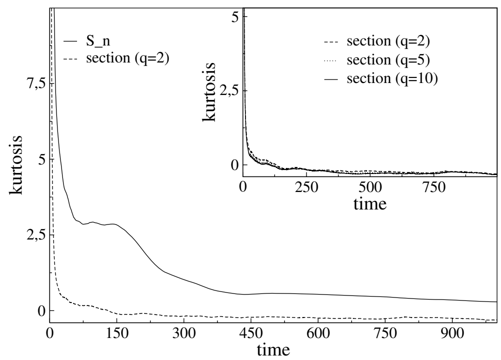

Fig. 1. Kurtosis behavior of the yen-deutschemark, for both S_n and S_t

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