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Allocation Rules for Fixed and Flexible Networks: The Role of Players and their Links*

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Abstract

We propose an allocation rule that takes into account the importance of players and their links and characterizes it for a fixed network. Unlike previous rules, our characterization does not require component additivity. Next, we extend it to flexible networks à la Jackson (2005). Finally, we provide a comparison with other fixed (network Myerson and Position value) and flexible network (player and link based) allocation rules through a number of examples.

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1 Introduction

The study of networks under the framework of cooperative games revolves around two basic problems: how a network is formed and how to allocate the value it generates among its members. In this paper we propose an allocation rule that takes into account the importance of players as well as their links. Since a network describes the interaction structure between agents, our allocation rule covers both bilateral and multilateral interactions. We provide a characterization of this rule in terms of well-known axioms and compare it to other allocation rules in the literature.

The notion of a "Network game" in cooperative games is rather recent and dates back only to 2005 (Jackson,2005). The antecedents however are much older − in 1977, Myerson, incorporated graphs in a cooperative game and extended the notion of Shapley value. This value, later known as the Myerson value (Aumann and Myerson, 1988), is indeed the Shapley value applied to the modified TU game obtained from the original TU game and the exogenous (communication) network. Such a game was later termed as a communication game with the network structure describing the communication possibilities and value generation (see Jackson,2005). An alternative to Myerson value was the Position value proposed by Meessen (1988) in which each link in a communication game was viewed as a player and then applied the Shapley value to the corresponding link game. Since then many authors including Haeringer (1999), Borm et al. (1992) , Slikker (2005,2006), Slikker and Anne van den Nouweland (2001), Kamijo(2009), Kamijo and Kongo (2009) to name a few, have proposed the axiomatic characterizations of these values and their variants.

In their seminal paper, Jackson and Wolinsky (1995) argued that while communication games are a useful augmentation of cooperative games, they fail to be rich enough to capture most applications where network structures are important. They proposed another model where the value of a network can depend on exactly how agents are interconnected; the same pair of agents can generate different values if they are directly connected than when they are indirectly connected. Subsequently in his paper on allocation rules for network games Jackson (2005) further argued that in communication games, a group may generate the same productive value whether they are each connected to each other via a complete network or they are connected via a less complete network. To the extent that there are any costs associated with links, or benefits from shortened paths, it will generally be the case that the value generated in these two scenarios will differ. This led to the introduction of a new class of games which he termed as Network Games.

Two important allocation rules for network games, namely the Myerson and the Position value were proposed and characterized by Jackson and Wolinsky (1995) and Slikker (2006). These rules are indeed extensions of their counterparts in communication situations. In both these cases however, the network is taken as given or fixed. Jackson (2005) introduced the notion of flexible networks where the role of a player in all possible subnetworks of a given network is also taken into account. The underlying notion here is the idea that the network is not fixed and could ultimately evolve into one of the (better) subnetworks. He showed that while the Myerson value of a communication game also had useful characterizations in network games, it still inherits certain limitations and is based on some flawed properties as well. It can be seen that similar arguments also limit the usefulness of the Position value for network games. Jackson proposed another set of allocation rules for flexible networks among which, the Player Based Flexible Network allocation rule (PBFN) and the Link Based Flexible Network allocation rules (LBFN) are quite appealing. While flexible network allocation rules have many advantages, they may not always be feasible and may also involve tedious calculations. Hence in this paper we undertake the study of both fixed and flexible network allocation rules.

The Myerson value is designed with the players in mind by taking coalitions (that are restricted to the network) and computing the marginal contributions of the players in the subnetworks stemming from those coalitions. The Position value on the other hand emphasizes more on links by defining a player set based on these links, i.e., each link is replaced by a player. It essentially computes the marginal contributions of the links using the notion of Shapley value and splits them equally between the players in any link. Observe that these existing values have little account of simultaneous multilateral interactions. Even though, Jackson's flexible network values are able to account for both direct and indirect interactions between different players by allowing for flexibility in the network, yet such interactions are partially multilateral as the marginal contributions are computed over one link at a time and more importantly, they are exclusively either player based or link based. Thus it is useful to have an allocation rule that cares about both players and their links in a manner that incorporates the simultaneous multilateral interactions in a network.

In this paper, we have developed such an allocation rule for fixed network games and its extension to flexible networks. We provide an axiomatic characterization of our rule following standard axioms in cooperative game theory like linearity, dummy player axiom, anonymity, monotonicity and efficiency. In doing so, we assume that the effect of interactions among players through indirect links is intransitive i.e., if in a network, player 1 is linked to player 2 and player 2 is linked to player 3 while there is no link between 1 and 3, the interaction between 1 and 3 (information sharing, for example) takes place through their links with 2 and they will not be awarded extra for such indirect links. We obtain a Shapley like allocation rule which computes the weighted average of the marginal contributions of players in some sense. Our rule differs from the network Myerson value given by Jackson (2005), as the Myerson value is player oriented and no multilateral interaction is visible there. Our rule is also distinct from the Position value in the sense that in addition to allowing multilateral interactions, it also gives importance to the players by means of the values generated through their direct links, while the Position value allows only pairwise or bilateral interactions one at a time. By the same arguments, the extension of our rule to flexible networks differs from those of Jackson's PBFN and LBFN allocation rules. We focus on the role played by a player and her link set simultaneously while determining her marginal contributions. In our framework a player is important as in the existing player and link based rules, but the allocation is not determined by simply restricting the player's contributions over all coalitions under the given network or by adding one of its links at a time to all its subnetworks. Rather we examine her role in the presence of all her links together. Consequently, the allocation of a player depends on her networking capacity i.e., her ability to forming links and also on her ability to generate value under a particular game. A major drawback of both Myerson and the Position value is that they are characterized by a rather strong condition called "component efficiency" for which the associated game must necessarily be component additive. In the absence of a component additive game, the rules become arbitrary. Our rule does satisfy "component efficiency", however is not dependent on it and hence our characterization holds even if there are externalities across components.

The paper proceeds as follows. Section 2 provides the necessary mathematical preliminaries, includes a brief description of the existing allocation rules and their characterization results. Section 3 develops our notion of an interactive allocation rule and its characterization. In Sections 4 we use a number of examples to illustrate differences between the different fixed and flexible network allocation rules. Section 5 concludes.

2 Preliminaries

In this section, we present the definitions and results required for development of our model. To a large extent this section is based on Jackson (2005).

Players

Let $N = \{1, ..., n\}$ be a fixed set of players who are connected in some network relationship.

Networks

A network consists of a finite set of elements called nodes corresponding to players and a finite set of pairs of nodes called links which correspond to bilateral relationships between players. The network g is thus a list of unordered pairs of players $\{i, j\}$, where $\{i, j\} \in g$ indicates that i and j are linked in the network g. For simplicity, we write ij to represent the link $\{i, j\}$. The degree of a player in a network is the number of direct links it has in the network. Let g^N be the set of all subsets of N of size 2. $G = \{g | g \subseteq g^N\}$ denotes the set of all possible networks or graphs on N. The network obtained by adding another network g' to an existing network g is denoted by $g + g'$ and the network obtained by deleting subnetwork g' from an existing network g is denoted $g \setminus g'$. For $g \in G$, $L(g)$ denotes the set of all links in g and $l(g)$, the total number of such links.

Let $N(g)$ be the set of players who have at least one link in g. That is,

$$N(g) = \{i | \exists j \text{ such that } ij \in g\}$$

Let $n(g) = \#N(g)$ be the number of players involved in g. Let $L_i(g)$ be the set of links that player i is involved in, so that

$$L_i(g) = \{ij | \exists j \text{ s.t.} ij \in g\},$$

By $l_i(g)$ we denote the number of links in player i's link set. It follows that $l(g) = \frac{1}{2}\sum_i l_i(g)$.

Networks on subsets of players

Given any $S \subseteq N$, let g^S be the set of all subsets of S of size 2 i.e. the complete network formed by the players in S.

Let $g|_S$ denote the subnetwork of g formed by the players in S. Formally we have,

$$g|_S = \{ij \mid ij \in g \text{ and } i \in S, j \in S\}.$$

Components

A component of a network g, is a non-empty subnetwork $g' \subset g$, such that

(a) if $i \in N(g')$ and $j \in N(g')$ where $j \neq i$, then there exists a path in g' between i and j, and

(b) if $i \in N(g')$ and $ij \in g$, then $ij \in g'$.

Components of a network are the distinct connected subgraphs in it. The set of components of g is denoted by $C(g)$.

Value functions

A value function is a function $v : G \to \mathbb{R}$ such that $v(\emptyset) = 0$, where $\emptyset$ represents the empty network i.e. network without links. The set of all possible value functions is denoted by V.

The value function specifies the total worth that is generated by a given network structure. It may involve both costs and benefits whenever this information is available.

Component additive value function

A value function v is component additive if

$$v(g) = \sum_{g' \in C(g)} v(g'), \quad \text{for any} \quad g \in G.$$

Efficient networks

A network $g \in G$ is *efficient* relative to a value function v if $v(g) \geq v(g')$ for all $g' \in G$.

Monotonic game

A game v is monotonic if for every $g, g' \in G$, such that $g \subset g'$ we have $v(g) \leq v(g')$.
The following definition of a monotonic cover was given by Jackson (2005) for discussing properties of flexible networks.

Monotonic covers

Given a value function v, its *monotonic cover* $\hat{v}$ is defined by,

$$\hat{v}(g) = \max_{g' \subseteq g} v(g') \ \ \forall g \in G$$

Three special value functions

Let v_g denote the value function defined by

$$v_g(g') = \begin{cases} 1 & \text{if } g \subseteq g' \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

We call v_g a *basic* value function. The name signifies that any v can be written as a linear combination of basic value functions v_g's in a unique way. That is, any v can be represented as $v = \sum_g c_g v_g$ with some unique collection of scalars c_g.

Let us slightly change the above definition to obtain another value function $\hat{v}_g$ with respect to the network g as follows:

$$\hat{v}_g(g') = \begin{cases} 1 & \text{if } g \subsetneq g' \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

A third value function denoted by v_g^* is defined as follows:

$$v_g^*(g') = \begin{cases} 1 & \text{if } g = g' \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

All these value functions will be required at a later stage of our discussion.

Network games

A network game is a pair, (N, v), of a set of players and a value function. If N is fixed and no confusion arises about this, we denote the network game by only v. A network game v is *monotonic* if for $g, g' \in G$ with $g' \subset g$, we have $v(g') \leq v(g)$.

Allocation rules

An allocation rule is a function $Y : G \times V \to \mathbb{R}^n$ such that $Y_i(g, v)$ represents the allocation to player i with respect to v and g. An efficient allocation rule Y is such that $\sum_i Y_i(g, v) = v(g)$, $\forall v$ and g. It specifies how the value generated by the network is shared among the players. Note that in the literature on network game theory, an allocation rule is always efficient, however, we distinguish these two in order to present a more general notion of a solution concept.

Component efficiency: An allocation rule Y is component efficient if for any component additive v, $g \in G$, and $g' \in C(g)$,

$$\sum_{i \in N(g')} Y_i(g, v) = v(g').$$

Equal Bargaining power

An allocation rule satisfies equal bargaining power if for any component additive v , $g \in G$ and $i, j \in N$, it holds that,

$$Y_i(g, v) - Y_i(g - ij, v) = Y_j(g, v) - Y_j(g - ij, v).$$

Balanced link contributions

An allocation rule Y satisfies *Balanced link contributions* if for all g, v and N and all $i, j \in N$,

$$\sum_{g' \subseteq L_j(g)} [Y_i(g, v) - Y_i(g \setminus g', v)] = \sum_{g' \subseteq L_i(g)} [Y_j(g, v) - Y_j(g \setminus g', v)]$$

Balanced link contributions states that the total contribution of a player to the payoff of another player is the sum over all links of the first player of the payoff difference the second player experiences if such a link is broken.
Jackson points out that equality in bargaining power of two players makes sense when they are comparable in the network by some means. The same criticism applies to the axiom of balanced link contribution also. For instance, consider a network that is quite asymmetric with some players having many links and others a few links only. Then it is unlikely that these axioms will be satisfied in such a network.

Probabilistic Allocation rule

Fix a player i in a network g, and let $\left\{p_{\tilde{g}}^{g_i} | \tilde{g} \subseteq g \setminus g_i, g_i \subset g, \text{ with } i \in N(g_i)\right\}$ be a probability distribution. An allocation rule Y is a probabilistic allocation rule if for every v and for every $i \in N$,

$$Y_i(g, v) = \sum_{g' \subseteq g \setminus g_i} p_{\tilde{g}}^{g_i} \left\{v(\tilde{g} + g_i) - v(\tilde{g})\right\}.$$

Thus $Y_i(.,.)$ represents the expectation of player i's marginal contributions in every possible network $\tilde{g} \subseteq g \setminus g_i$.

The Myerson value

The Myerson value was extended to network games by Jackson and can be written as follows:

$$Y_i^{NMV}(g,v) = \sum_{S \subset N \setminus \{i\}} (v(g|_{S \cup i}) - v(g|_S)) \left(\frac{\#S!(n - \#S - 1)!}{n!} \right) \tag{2.4}$$

Following is a characterization of the Myerson value due to Jackson and Wolinsky (Jackson and Wolinsky, Theorem 4, pp 65) as restated by Jackson (2005).

Theorem 1. *(Jackson (2005), Theorem 1, pg 134) Y satisfies component efficiency and equal bargaining power if and only if* $Y(g,v) = Y_i^{NMV}(g,v)$*, for all* $g \in G$ *and any component additive* v*.*

The Position Value

The position value Y^{NPos} of a network game as extended by Slikker (2006), can be written as follows:

$$Y_i^{NPos}(g,v) = \sum_{\substack{j \neq i \\ j \in N(g)}} \left(\sum_{g' \subset g \setminus \{ij\}} \frac{1}{2} \{v(g' + ij) - v(g')\} \left\{ \frac{l(g')!(l(g) - l(g') - 1)!}{l(g)!} \right\} \right)$$

Thus the position value attributes to each player half of the value of each link she is involved in. Slikker (2006) gave a characterization of the Position value as follows:

Theorem 2. *(Slikker (2006), Theorem 3.1, pg 6) The Position value is the unique allocation rule for network games that satisfies component efficiency and balanced linked contributions.*

Note that both Myerson value and Position value (in communication situation as well as Networks) are Probabilistic allocation rules as the co-efficients in each expression form a probability distribution. Further, these rules are characterized by the requirement that the value function be component additive. In absence of this assumption the allocation rule becomes arbitrary. Moreover, they require assumptions which may not always hold.

Consequently, Jackson (2005) proposed a new approach to network games called flexible networks and provided some alternative rules. This approach takes the potential of a player into account- her allocation is based not only on what she contributes in the current network, but also in alternative networks related to this network. The rules give allocation to a player not only by looking at her current position in the given network but also by considering her roles in alternative networks. We begin with a few more definitions.

Flexible-network rules

An allocation rule Y is a flexible network rule if $Y(g, v) = Y(g^N, \hat{v})$ for all v and efficient g (relative to v).

Additivity

An allocation rule Y is additive if for any v and v', and scalars $a \geq 0$ and $b \geq 0$,

$$Y(g^N, av + bv') = aY(g^N, v) + bY(g^N, v').$$

A weaker version of additivity was also proposed by Jackson.

Weak additivity

An allocation rule Y is weakly additive if for any monotonic v and v', and scalars $a \geq 0$ and $b \geq 0$,

$$Y(g^N, av + bv') = aY(g^N, v) + bY(g^N, v'),$$

and if $av - bv'$ is monotonic, then

$$Y(g^N, av - bv') = aY(g^N, v) - bY(g^N, v').$$

The following is an important equity condition.

Equal treatment of vital players

An allocation rule Y satisfies equal treatment of vital players if v_g is a basic value function for some g, then,

$$Y_i(g,v) = \begin{cases} \dfrac{1}{n(g)}, & \text{if } i \in N(g) \\ 0, & \text{otherwise.} \end{cases}$$

The above conditions give rise to a player-based allocation rule given below.

The player-based flexible network (PBFN) allocation rule

Jackson's PBFN allocation rule for efficient networks is given by,

$$Y_i^{PBFN}(g,v) = \sum_{S \subset N \setminus \{i\}} (\hat{v}(g^{S \cup \{i\}}) - \hat{v}(g^S)) \left\{ \frac{\#S!(n - \#S - 1)!}{n!} \right\}. \tag{2.5}$$

Theorem 3. *(Jackson, 2005, Theorem 2, pp 141) An allocation rule satisfies equal treatment of vital players, weak additivity, and is a flexible network rule if and only if it is defined by (2.5) for all v and g that are efficient relative to v.*

In order to define PBFN for inefficient networks, we need the following condition.

Proportionality

An allocation rule Y is proportional if for each i and v, either $Y_i(g,v) = 0$ for all g, or for any g and g' such that $v(g') \neq 0$,

$$\frac{Y_i(g,v)}{Y_i(g',v)} = \frac{v(g)}{v(g')}$$

The PBFN allocation rule for inefficient networks is given by

$$Y_i^{PBFN}(g,v) = \frac{v(g)}{\hat{v}(g^N)} \sum_{S \subset N \setminus \{i\}} (\hat{v}(g^{S \cup \{i\}}) - \hat{v}(g^S)) \left\{ \frac{\#S!(n - \#S - 1)!}{n!} \right\}. \tag{2.6}$$

Jackson (2005) showed that for inefficient networks, an allocation rule that satisfies all the conditions of Theorem 3 along with proportional rule if and only if it is given by (2.6) for all v and g.

Another interesting allocation rule proposed by Jackson for flexible networks is the link-based allocation rule.

An allocation rule Y is *link-based* if there exists $\psi : V \times G \to \mathbb{R}^{n(n-1)/2}$ such that $\sum_{ij \in g^N} \frac{\psi_{ij}(g,v)}{2}$.

An allocation rule Y is said to satisfy *equal treatment of vital links* if v_g is a basic value function for some $g \neq \emptyset$, then,

$$Y_i(g, v_g) = \frac{l_i(g)}{2l(g)}.$$

The link-based flexible network (LBFN) allocation rule

Jackson's LBFN allocation rule for any networks is given by

$$Y_i^{LBFN}(g,v) = \frac{v(g)}{\hat{v}(g^N)} \sum_{j \neq i} \left[\sum_{g \subset g^N \setminus \{ij\}} \frac{1}{2} (\hat{v}(g+ij) - \hat{v}(g)) \left\{ \frac{l(g)!([n(n-1)/2] - l(g) - 1)!}{[n(n-1)/2]!} \right\} \right] \tag{2.7}$$

Theorem 4. *(Jackson (2005), Theorem 4, pp 143) An allocation rule satisfies equal treatment of vital links, weak additivity, and is a flexible network rule if and only if it agrees with Y^{LBFN} on efficient networks. It satisfies equal treatment of vital links, weak additivity, and is a flexible network and proportional rule if and only if it is Y^{LBFN}.*

Remark 1. (*Relationship between fixed and flexible network rules*)

The similarity between the LBFN rule and the network Position value lies, even though they arise from different model setups, in the fact that both allocate payoffs to the links instead of players and then divide it equally between the linked players. One can easily verify that when the value function is monotonic and g is the complete network, Y^{NPos} and Y^{NMV} coincides with Y^{LBFN} and Y^{PBFN} respectively.

3 An Interactive Allocation Rule

In this section, we develop a new allocation rule for fixed networks, emphasizing both the roles of the players and their links. We call such a rule: an *interactive allocation rule.*

Consider a network $g = \{ij, jk, ik\}$ with $L_i(g) = \{ij, ik\}$. Now, player i can interact with j and k separately one by one or simultaneously which may provide additional synergies. A simultaneous interaction (one to all) would result in group decisions while pairwise interactions (one to one) may focus on building up personal relationships and it is important to capture both types of interactions. This feature clearly distinguishes our interactive allocation rule from the rest of the existing rules.

We begin with the following definitions.

Definition 1. Let π be a permutation on N. For $v \in V$, the game πv is defined by, $\pi v(\pi g) = v(g)$, where πg is the network obtained after permuting the players in $g \in G$ under π.

Definition 2. A player i is dummy for the value function $v \in V$ with respect to the network g, if $\forall g' \subseteq g \setminus L_i(g)$, we have $v(g' + L_i(g)) = v(g') + v(L_i(g))$.

Note that a dummy player has no effect on g beyond its link set. We shall show that the following five axioms characterize our proposed allocation rule:

Axiom 1. *Linearity axiom (L)*: For $g \in G,\ v, v' \in V$, and $a, b \in \mathbb{R}$,

$$Y_i(g, av + bv') = aY_i(g, v) + bY_i(g, v'),\ \forall i \in N.$$

Axiom 2. *Dummy axiom (D)* : For a dummy player $i \in N$, $Y_i(g, v) = v(L_i(g))$, for every $v \in V$.

Axiom 3. *Anonymity axiom (A)*: For all $v \in V$, permutations π,

$$Y_i(g, v) = Y_{\pi i}(\pi g, \pi v),\ \ \forall\ i \in N.$$

Axiom 4. *Monotonicity axiom (M)*: If v is monotonic, then $Y_i(g, v) \geq 0$ for every $i \in N$.

Axiom 5. *Efficiency Axiom (E)*: For every $v \in V$, $\sum_{i \in N(g)} Y_i(g, v) = v(g)$.

Proposition 1. *If $Y(g, v)$ satisfies linearity axiom (L), then for every g and every v, there exists a family of real constants $\{a^g_{g'}\}_{g' \subseteq g}$ such that*

$$Y_i(g, v) = \sum_{g' \subseteq g} a^g_{g'} v(g') \tag{3.1}$$

Proof. Fix a network $g \in G$. Denote by G_g, the set of all subnetworks of g. Consider the class V_g of all network games defined on G_g. Clearly V_g is a $2^{l(g)}-1$ dimensional subspace of V. We say two games are equivalent in V, if they coincide in G_g. Then V_g partitions elements of V into equivalence classes so that there is a one to one correspondence between these equivalence classes and the members of V_g. Thus we can identify each equivalence class by a unique value function in V_g. With a little abuse of notation, we thus accept that every $v \in V$ can be identified with a $v^* \in V_g$. Since, every $v^* \in V_g$ has a unique representation

$$v^* = \sum_{g' \in G_g} v^*(g') v^*_{g'} \text{ where } g' \in G_g.$$

Therefore under this equivalence condition, every $v \in V$ can be represented in a unique way as $v = \sum_{g \in G_g} v(g) v^*_g$, and so the result follows by taking $a^g_{g'} = Y_i(g, v^*_{g'})$. □

Observation 1. Jackson (2005) gave an example to illustrate the insensitivity of Myerson value to alternative networks by taking two value functions v and v' defined by $v(\{12\}) = v(\{23\}) = v(\{12, 23\}) = 1$, and $v(g) = 0$ for any other network g, while $v'(g) = 1$, for every g. It is indeed interesting to see that under v player 2's involvement is needed to generate any value, while under g', no player is special. However, if we consider only fixed networks as the case may be where a flexible network is not possible, and if the two games coincide on that network, then their corresponding payoffs are essentially identical. This can be interpreted as a weaker version of consistency of fixed networks. This is what we have considered under the equivalence condition in the above proof of proposition 1.

Proposition 2. *If $Y(g, v)$ satisfies L and D, then for every $g \in G$, there exists a family of real constants $\{p^g_{g'}\}_{g' \subseteq g}$ such that*

$$Y_i(g, v) = \sum_{g' \subseteq g \backslash L_i(g)} p^g_{g'} \left\{ v(g' + L_i(g)) - v(g') \right\} \tag{3.2}$$

Proof. Consider $\{Y_i(g, v)\}_{i \in N}$ satisfying axioms L and D. By Proposition 1, there exists $\{a^g_{g'}\}_{g' \subseteq g}$ such that,

$$Y_i(g, v) = \sum_{g' \subseteq g} a^g_{g'} v(g') \tag{3.3}$$

We write,

$$Y_i(g,v) = \sum_{g' \subseteq g \setminus L_i(g)} a^g_{g'+L_i(g)} v(g'+L_i(g)) + \sum_{g' \subseteq g \setminus L_i(g)} a^g_{g'} v(g') \tag{3.4}$$

Assume now that i is a dummy player for v with respect to g. Then,

$$Y_i(g,v) = \sum_{g' \subseteq g \setminus L_i(g)} v(g') \left[a^g_{g'+L_i(g)} + a^g_{g'} \right] + a^g_{g'+L_i(g)} v(L_i(g)) \tag{3.5}$$

For an arbitrarily fixed $g' \subseteq g \setminus L_i(g)$, consider the game $v^*_{g'}$. Now i is a dummy player in $v^*_{g'}$ also. We have $Y_i(g, v^*_{g'}) = v^*_{g'}(L_i(g)) = 0$. Therefore, it follows from axiom D, that,

$$a^g_{g'+L_i(g)} + a^g_{g'} = 0.$$

Let $a^g_{g'+L_i(g)} = -a^g_{g'} = p^g_{g'}$. Consequently Equation (3.2) is obtained. □

Our next proposition is about the anonymity axiom A, which would imply that the real coefficients $p^g_{g'}$ are independent of choices of subnetworks of g as long as they have the same number of links.

Proposition 3. *Under axioms L, D and A and for every v, there exist real constants $p^g_{l(g')}$, where $l(g') = 0, ..., l(g) - l_i(g)$, such that,*

$$Y_i(g,v) = \sum_{g' \subseteq g \setminus L_i(g)} p^g_{l(g')} \left\{ v(g'+L_i(g)) - v(g') \right\} \tag{3.6}$$

Proof. By axioms L and D and propositions 1 and 2, we have,

$$Y_i(g,v) = \sum_{g' \subseteq g \setminus L_i(g)} p^g_{g'} \left\{ v(g'+L_i(g)) - v(g') \right\} \tag{3.7}$$

Let $g_1,\ g_2 \subseteq g \setminus L_i(g)$ such that $l(g_1) = l(g_2)$ and π be a permutation with $\pi g_1 = g_2$ leaving the rest invariant.

By axiom A (anonymity), we have,

$$Y_i(g, v^*_{g_1+L_i(g)}) = Y_{\pi i}(\pi g, \pi v^*_{g_1+L_i(g)}) = Y_i(g, v^*_{g_2+L_i(g)}).$$

It follows from equation (3.7) that,

$$p^g_{g_1} = p^g_{g_2}$$

This implies that there exist constants, $p^g_{l(g')}$'s, $l(g') = 0, 1, ..., l(g) \setminus l_i(g)$ depending on the size of $g' \subseteq g \setminus L_i(g)$ such that equation (3.6) holds. This completes the proof. □

The following result gives a characterizing property of the co-efficients $p^g_{l(g')}$ under axioms L,D and A.

Proposition 4. *Under axioms L,D and A , there exists a collection of constants*

$$\left\{p^g_{l(g')} : g' \subseteq g \setminus L_i(g)\right\}$$

satisfying

$$\sum_{g' \subseteq g \setminus L_i(g)} p^g_{l(g')} = 1$$

such that for all v,

$$Y_i(g,v) = \sum_{g' \subseteq g \setminus L_i(g)} p^g_{l(g')} \left\{v(g' + L_i(g)) - v(g')\right\} \tag{3.8}$$

Further under axiom M, each $p^g_{l(g')} \geq 0$.

Proof. Under axioms L,D and A, proposition 3 ensures the existence of expression (3.6). Consider the game $v_{L_i(g)}$. Clearly player i is dummy in it with respect to g. Thus,

$$\begin{aligned} 1 = v_{L_i(g)}(L_i(g)) = Y_i(g, v_{L_i(g)}) &= \sum_{g' \subseteq g \setminus L_i(g)} p^g_{l(g')} \left\{v_{L_i(g)}(g' + L_i(g)) - v_{L_i(g)}(g')\right\} \\ &= \sum_{g' \subseteq g \setminus L_i(g)} p^g_{l(g')}. \end{aligned}$$

Further, since the game $\hat{v}_{g'}$ (refer to equation 2.2) is monotonic for every $g' \subseteq g \setminus L_i(g)$; by axiom M, we have,

$$Y_i(g, \hat{v}_{g'}) = p^g_{l(g')} \geq 0.$$

This completes the proof. □

Thus, combining above results, we have the following.

Theorem 5. *If $Y(g,v)$ is an allocation rule that satisfies L,D,A and M, then it is a probabilistic allocation rule.*

We have so far obtained an axiomatic characterization of a probabilistic value for a network game. It provides different allocation rules depending on the different probability distributions one is considering. Thus to provide a precise model setup, we will consider here, a particular probability distribution under the following intuitive assumption:

Assumption 1 : *For any given network, all subnetworks having the same number of links are equally likely.*

Remark 2. Note that Assumption 1 is inherent in every Shapley like solution concept. The original Shapley hypothesis for cooperative games is that the probability of coalitions depend on size, with the total probability of each size being the same and is indifferent about the existence of all possible coalitions. In network literature, however it does not account for non existing links among players.

Proposition 5. *Let $Y_i(g, v)$ be a probabilistic value such that for all v and g,*

$$Y_i(g, v) = \sum_{g' \subseteq g \setminus L_i(g)} p^g_{l(g')} \left\{ v(g' + L_i(g)) - v(g') \right\}. \tag{3.9}$$

then under Assumption 1, it follows that,

$$p^g_{l(g')} = \frac{(l(g) - l_i(g) - l(g'))! l(g')!}{(l(g) - l_i(g) + 1)!}$$

Proof. Create a subnetwork g' of $g \setminus L_i(g)$. Draw a subnetwork at random consisting of all possible number of links $0, 1, 2, ..., l(g \setminus L_i(g)) = l(g) - l(L_i(g)) = l(g) - l_i(g)$, each number has probability

$\frac{1}{l(g) - l_i(g) + 1}$ to be drawn.

Probability of getting one such network g' of size $l(g')$ from $g \setminus L_i(g)$ is:

$$\frac{1}{{}^{l(g \setminus L_i(g))}C_{l(g')}}$$

If g' is formed, player i will be paid a proportion of $v(g' + L_i(g)) - v(g')$, the marginal contribution with respect to the direct links made by i in g with other players. Thus the total probability of drawing a subnetwork is given by: $\frac{1}{l(g) - l_i(g) + 1} \times \frac{1}{{}^{l(g \setminus L_i(g))}C_{l(g')}}$, giving us the required expression as:

$$p^g_{l(g')} = \frac{(l(g) - l_i(g) - l(g'))! l(g')!}{(l(g) - l_i(g) + 1)!}$$

□

Therefore, we obtain our interactive allocation rule by means of the following proposition:

Proposition 6. *Given Assumption 1, $Y : G \times V \to \mathbb{R}$ satisfies axioms L, D, A, and M if and only if for all i, it has the following with possibly a constant multiple,*

$$Y_i(g,v) = \begin{cases} \sum_{g' \subseteq g \setminus L_i(g)} \dfrac{\{l(g \setminus L_i(g)) - l(g')\}!\; l(g')!}{\{l(g \setminus L_i(g)) + 1\}!} \{v(g' + L_i(g)) - v(g')\}, & \text{if } i \in N(g) \\ 0, \quad \text{otherwise} \end{cases} \tag{3.10}$$

Proof. The proof follows immediately from Propositions $1, 2, 4, 5, 6$ and 7. □

Note that Y defined in (3.10) does not satisfy axiom E (Efficiency), however, an efficient interactive rule can be immediately obtained through simply multiplying it by the scaling factor $\frac{v(g)}{\sum_{i \in N(g)} Y_i(g,v)}$. Thus, we call Y a proportional Interactive allocation rule. An efficient Interactive allocation or simply an Interactive allocation rule denoted by Y^I is therefore deduced as follows:

$$Y_i^I(g,v) = \frac{v(g)}{\sum_{i \in N(g)} Y_i(g,v)} Y_i(g,v). \tag{3.11}$$

where, $Y_i(g,v)$ is given by equation (3.10). Thus, it follows that the Interactive allocation rule is unique up to a choice of the scaling factor.

Proposition 7. *The allocation rule Y^I given by equation (3.11) is component efficient.*

Proof. The proof is straight forward. □

Interactive Allocation Rule for Flexible Networks

Let us define the Interactive allocation rule for flexible network in two steps as follows:

(i) Let the proportional allocation rule for flexible network be defined as,

$$Y_i^{FN}(g,v) = \frac{v(g)}{\hat{v}(g^N)} \sum_{g' \subseteq g^N \setminus L_i(g^N)} \frac{\{l(g^N \setminus L_i(g^N)) - l(g')\}!\; l(g')!}{\{l(g^N \setminus L_i(g^N)) + 1\}!} \{\hat{v}(g' + L_i(g^N)) - \hat{v}(g')\} \tag{3.12}$$

Note that, under efficient g, $Y_i^{FN}(g,v) = Y_i^{FN}(g^N, \hat{v})$. Therefore, it follows from the definition of a flexible network rule that,

(ii) The Interactive flexible network allocation rule Y^{FNI} is given by,

$$Y^{FNI}(g,v) = \frac{v(g)}{\sum_{i \in N} Y_i^{FN}(g,v)} Y_i^{FN}(g,v) \tag{3.13}$$

A characterization of Y^{FNI} is possible exactly in the same way as the one for Y^I in fixed networks. We first obtain the proportional allocation rule for the fixed network game $(g^N, \hat{v})$ and multiply it with the factor $\frac{v(g)}{\hat{v}(g^N)}$ to get the corresponding proportional flexible network allocation rule. This factor does not disturb the characterization process since the proportional rule as its name signifies, can accommodate any such factor. Thus we have the following:

Proposition 8. *Given Assumption 1, $Y : G \times V \to \mathbb{R}$ satisfies axioms L, D, A, M and E and is a flexible network rule if and only if it is given by (3.13) in conjunction with (3.12).*

The next section deals with some examples to show the difference among our rule, the Myerson value, Position value and the PBFN and LBFN rules given by Jackson in a general network game.

4 Examples

We now provide some examples to show the relationship between the existing allocation rules we have considered in this paper and our proposed allocation rule.

Example 1. Let $N = \{1, 2, 3\}$ and $g = \{12, 23\}$. Let $v(\{12\} = v(\{23\} = v(\{12\}, \{23\}) = 1$ and $v(g) = 0$ for all other g. Then,

$$Y^{NMV}(g, v) = Y^{PBFN}(g, v) = \left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right)$$
$$Y^{NPos}(g, v) = Y^{LBFN}(g, v) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$$
$$Y^{I}(g, v) = Y^{FNI}(g, v) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$$

Here, Y^{NMV} identifies player 2 as the most important player and hence pays more to 2. Y^{NPos} takes into account the fact that the network with two links is no better than the ones with one link each. So it pays a little to player 2 and a bit more to the others. Y^I and Y^{FNI} also give player 2 less as multilateral interactions do not add anything.

Note that, if we change the value function slightly by assuming $v(\{12\}, \{23\}) = 2$ and keep everything else as it is, we find that all the above mentioned rules yield to the allocation $(\frac{1}{2}, 1, \frac{1}{2})$. Under this new v, the network with two links is more valuable than the same network under the previous v and hence Y^{NPos} and Y^{I} (similarly Y^{LBFN} and Y^{FNI}) do not penalize player 2, unlike the earlier case.

Example 2. Let $N = \{1,2,3\}$, $g = \{12, 23, 13\}$ and v be such that $v(g) = 4$, $v(12,13) = v(13,23) = v(12,23) = 6$ and $v(12) = v(23) = v(13) = 2$. Thus, we have, $\hat{v}(g) = \hat{v}(g^N) = 6$, $\hat{v}(12,13) = \hat{v}(13,23) = \hat{v}(12,23) = 6$, and $\hat{v}(12) = \hat{v}(23) = \hat{v}(13) = 2$. We have,

$$Y^{PBFN}(g,v) = Y^{LBFN}(g,v) = Y^{I}(g,v) = Y^{FNI}(g,v) = Y^{NMV}(g,v) = Y^{NPos}(g,v) = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right).$$

Note that in this case, the value function does not satisfy monotonicity. Since all subnetworks with same number of links generate the same value and the network under construction is the complete network, all values are identical. This is very similar to what is stated in Observation 1.

Example 3. $N = \{1,2,3\}$, $g = \{12, 23, 13\}$ and v be such that $v(g) = 4$, $v(12,13) = v(13,23) = 2$, $v(12,23) = 6$ and $v(12) = v(23) = 1$ and $v(13) = 0$. Thus, $\hat{v}(g) = \hat{v}(g^N) = 6$. We have,

$$Y^{NMV}(g,v) = \left(\frac{7}{6}, \frac{10}{6}, \frac{7}{6}\right).$$

$$Y^{NPos}(g,v) = \left(\frac{11}{12}, \frac{26}{12}, \frac{11}{12}\right).$$

$$Y^{I}(g,v) = (1,2,1).$$

$$Y^{PBFN}(g,v) = \left(\frac{22}{18}, \frac{28}{18}, \frac{22}{18}\right).$$

$$Y^{LBFN}(g,v) = \left(\frac{19}{18}, \frac{34}{18}, \frac{19}{18}\right).$$

$$Y^{FNI}(g,v) = \left(\frac{14}{13}, \frac{24}{13}, \frac{14}{13}\right).$$

Note that, here, link $\{13\}$ has spillovers for the other links in the network but it can not generate a value of its own. Accordingly Y^{NMV} and Y^{PBFN} evaluate the contributions of players 1 and 3 in the network whereas Y^{NPos} and Y^{LBFN} consider link $\{13\}$'s contribution as a whole. However, Y^{I} and Y^{FNI} emphasize both players as well as links, they try to realize the contributions of players 1 and 3 through their link $\{13\}$ and thus their payoffs under these rules lie between those given by the player based and the linked based rules.

Example 4. We finally apply our allocation rule to a more interesting network game with a larger players' set. Let $g = \{15, 12, 34\}$ and the value function assigns as $v(\{15\}) = v(\{12\}) = 1$, $v(\{34\}) = 0$, $v(\{15, 12, 34\}) = 6$, $v(\{15, 12\}) = 4$, $v(\{15, 34\}) = v(\{12, 34\}) = 2$. Note that there are two components in g with externalities across them, i.e., the value function is not component additive.

Then

$$Y^I(g, v) = (2.432, 1.135, 0.648, 0.648, 1.135)\,.$$

$$Y^{FNI}(g, v) = (2.332, 1.424, 0.41, 0.41, 1.424)$$

Here v is not component additive, however it is efficient and monotonic upto g. So the Myerson value and the PBFN coincide and we have,

$$Y^{NMV}(g, v) = Y^{PBFN} = (2.166, 1.417, 0.5, 0.5, 1.417)\,.$$

Similarly, The Position value and the LBFN being coincident are given by,

$$Y^{NPos} = Y^{LBFN} = (2.5, 1.25, 0.5, 0.5, 1.25)\,.$$

In this example, Y^{NPos} gives player 1 the highest because of the number of links she is involved in. Y^I gives her more than Y^{NMV} and less than Y^{NPos} because it takes both players and multilateral interactions into account simultaneously. For this same reason, our allocation rule in flexible networks also gives player 3 and 4 less and players 2 and 5 more than what is given by the other rules.

We conclude this section with an illustration of an important property of Y^I.

Property 1. *(Reward to a player with more links within a network): For players $i, j \in N(g)$, if $l_i(g) > l_j(g)$ then $Y_i(g, v) \geq Y_j(g, v)$.*

Proof. It is sufficient to show the result holds for a basic value function v_g as every value function is a unique linear combination of such basic value functions. Let us first compute the proportional allocation rule $Y(g, v)$:

$$\begin{aligned} Y_i(g, v_g) &= \sum_{g' \subseteq g \setminus L_i(g)} \frac{\{l(g \setminus L_i(g)) - l(g')\}\,!\; l(g')!}{\{l(g \setminus L_i(g) + 1\}!} \{v_g(g' + L_i(g)) - v_g(g')\} \\ &= \frac{\{l(g \setminus L_i(g)) - l(g \setminus L_i(g))\}! l(g \setminus L_i(g))!}{\{l(g \setminus L_i(g)) + 1\}!} \{v_g(g) - v_g(g \setminus L_i(g))\} \end{aligned}$$

Thus,

$$Y_i(g, v_g) = \frac{1}{l(g) - l_i(g) + 1} \qquad \text{so that}$$
$$Y_i^I(g, v_g) = \frac{1}{\sum_{i \in N(g)} Y_i(g, v_g) \times \{l(g) - l_i(g) + 1\}}.$$

The assertion follows. □

5 Conclusion

In this paper, we have obtained and characterized an allocation rule for fixed networks and subsequently extended it to flexible networks. These rules differ from the existing ones in the literature in the sense that they emphasize both players and their links. We characterize our allocation rule with a new set of axioms that have been used in cooperative games, but not in network games. Unlike previous rules our characterization does not require component additivity and thus allows for spillovers across components. Moreover our allocation rule incorporates multilateral interactions in both fixed and flexible networks. We believe that this research can lead to a more general characterization of network allocation rules.

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