Pure self-confirming equilibrium

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PURE SELF-CONFIRMING EQUILIBRIUM

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Abstract. In a Self-Confirming Equilibrium (Fudenberg and Levine, 1993A) every player obtains partial information about other players' strategies and plays a best response to *some* conjecture which is consistent with his information. Two kinds of information structures are considered: In the first each player observes his own payoff while in the second the information is the distribution of players among the various actions. For each of these information structures we prove that *pure* Self-Confirming Equilibrium exists in some classes of games. Pure Nash equilibrium may fail to exist in these classes.

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1. Introduction

The game theory literature reluctantly accepted the idea that players will sometimes use the outcome of a random device in order to determine their action. Aumann (1987, page 15) writes that "Practically speaking, the idea that serious people would base important decisions on the flip of a coin is difficult to accept". But since in many games pure-strategy Nash equilibrium fails to exist, mixed strategies are still an inevitable part of the theory.

Recall that the reason for the need to randomize in equilibrium is to prevent the possibility that other players can gain by deviating. In other words, players use mixed strategies in order to 'hide' their action from other players and not in order to improve their own utility. Thus, the assumption that players can perfectly observe the strategies of their opponents underlies the necessity of randomization. The contribution of the current paper is to show that when this assumption is relaxed in a natural way the need to randomize in equilibrium is sometimes eliminated. This intuition is best summarized in Pearce's (1984, page 1034) paper on rationalizabilty:

> "The need for players to randomize in many Nash equilibria has long been considered somewhat puzzling. The incentive for randomization seems to be the need to "evade" one's opponents. But in the present context, opponents are not always able to figure out a player's strategic choice; such a player can hide without randomizing, camouflaged by the uncertainty of the other players."

If players only partially observe their opponents' strategies then one should define what one means by equilibrium. The notion of equilibrium we employ here is in the spirit of Fudenberg and Levine's (1993A) Self-Confirming Equilibrium (SCE). That is, equilibrium is viewed here as a steady state of a recurring interaction with no strategic links among the repetitions. Perhaps the context which makes this solution concept most plausible is the one in which a fixed game is being played repeatedly but by different players in different periods[1]. In this context, a steady state is a strategy tuple such that (i) every player's strategy is a best response to *some* belief about his opponents' strategies and (ii) the belief of every player is not contradicted by the information he obtains.

Unlike Fudenberg and Levine (1993A) our interest here is in normal form games. Every player has an exogenously given *information structure* which maps action profiles into signals. Since typically several action profiles induce the same signal, each player will face uncertainty as to the actual profile that is being played. A *consistent conjecture* for a player is any probability distribution over opponent's action profiles which induce the same signal as the one observed by this player. A

[1]This idea is the subject of Fudenberg and Levine's (1993B) paper where players leave the population after a finite number of rounds.

(pure) SCE with respect to a given information structure is any action profile in which the action of every player is a best response to some consistent conjecture.

We differentiate between two versions of the consistency requirement. A conjecture is *strongly consistent* if each action profile in its support induce the same signal as the one that the player actually observed. This is the standard notion in the literature. A weaker version of consistency, which we introduce here, only requires that the *expected* signal[2] according to the conjecture coincides with the observed signal. In accordance, an action profile is *Strong SCE (SSCE)* if player's actions are best responses to strongly consistent conjectures and *Weak SCE (WSCE)* if player's actions are best responses to weakly consistent conjectures.

Of course, we are not the first to use imperfect monitoring of other players actions as a way to achieve pure equilibrium[3]. The main difference between previous works and the current one is that we restrict attention to particular information structures and classes of games. Specifically, our results are limited to two kinds of information structures: In the first the signal to every player is his own payoff, while in the second the information available to every player is the distribution of his opponents among the various actions. The implications for the existence of pure SCE of each of these information structures are considered in several well-known classes of games.

The next section illustrates the main ideas of this paper by means of several examples. The general model is described in Section 3. Section 4 deals with the case where each player's information is his own payoff while in Section 5 the information consists of the number of players that chose each action. A discussion of related literature is deferred to Section 6. We conclude in Section 7 with some remarks.

2. Motivating examples

2.1. **Rock-Paper-Scissors.** The following zero-sum game is well known as Rock-Paper-Scissors.

	R	P	S
R	* 0	-1	1
P	1	0	-1
S	-1	1	0

Entries in the matrix are the payoffs paid by the column player (player 2) to the row player (player 1). The only Nash equilibrium is the strategy pair in which every player plays each strategy with equal probability of $\frac{1}{3}$. In particular, there is no pure equilibrium in this game.

[2]This notion of consistency is suitable only in the case where the set of signals has a linear structure so that expectations can be calculated (see Section 3).

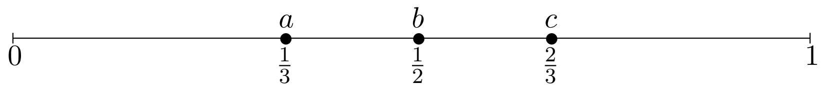
[3]As can be seen in the above quotation, this idea goes back at least to Pearce (1984).

In contrast to the Nash paradigm, assume that both players have only partial information about the strategy of their opponent. Specifically, assume that every player's signal is the payoff he obtains when playing his chosen action.

Now, consider the case where the action profile (R, R) is being played (the cell marked with an asterisk). Player 1's exclusive information is that when he plays R he gets a payoff of 0. Any conjecture of player 1 about the strategy of player 2 of the form $(1 - 2\alpha, \alpha, \alpha)$, $0 \leq \alpha \leq \frac{1}{2}$ is consistent with 1's signal in the weak sense that when 1 plays R and 2 plays such a strategy the average payoff to 1 is 0. In particular, it is possible that player 1's conjecture is that $\alpha = \frac{1}{3}$ which means that player 2 is playing his equilibrium strategy. A best response for player 1 in this case is playing R.

Similarly, player 2's belief might be that player 1 is playing each of his 3 actions with equal probabilities. A best response for player 2 in this case is to play R. Therefore, in the action profile (R, R) *every player is playing a best response to a conjecture which is consistent with his information.* Obviously, the same holds for the action profiles (P, P) and (S, S).

2.2. **Location game.** A unit measure of consumers is uniformly distributed in the $[0, 1]$ interval. Each of three sellers (the players of this game) should choose a location from the set $\{a, b, c\}$ (see picture). The payoff of a seller is equal to the measure of the set of consumers to whom he is the closest one (sellers split equally the payoff when located in the same place).



It is straightforward to verify that no pure equilibrium exists in this game. However, like in the previous example assume that every seller only knows his own payoff. We claim that this partial information makes any action profile in which all three sellers choose the same action a potentially steady state.

To see that this is indeed the case notice that there is a (completely mixed) Nash equilibrium in which every player plays each of the three actions with probability $\frac{1}{3}$. When this equilibrium profile is played the (expected) payoff to every player is $\frac{1}{3}$. In particular this means that if one of the players (say player 1) chooses any pure action while the other two players play the above equilibrium strategies then the payoff to player 1 will also be $\frac{1}{3}$.

Now, if all three players choose the same action (say a) then the payoff to each one of them is again $\frac{1}{3}$. Thus, when all three players are playing a player 1 might conjecture that players 2 and 3 are actually playing their equilibrium strategies. Such a conjecture is consistent in the sense that the actual payoff to player 1 (which is the information he has) is equal to the expected payoff according to 1's conjecture.

If this is the belief of player 1 then he is indifferent between the three actions. In particular a is a best response to his belief. The same is true for players 2 and 3 and for actions b and c.

2.3. **Majority rule game.** A society of three players should choose between 2 alternatives, a or b. Each of the 3 players can cast a vote for one of the alternatives or not to cast a vote at all. The winning alternative is chosen according to a majority rule. In case of a tie, the winner is decided by some predefined and commonly known tie-braking rule. Players' preferences are defined over the set of alternatives, so that the utility of every player depends only on the identity of the winner.

Fix some arbitrary preferences for the three players[4]. Existence of pure equilibria is not a question in this case since if all three players choose the same alternative then no player can change his payoff by deviating. Moreover, every player has a weakly dominant strategy which is to vote for his favorite alternative. However, there are also action profiles which are not equilibria. This is the case when a certain player is pivotal and can change the outcome so that his utility increases.

Assume, however, that players can only observe the winning alternative (or, equivalently, their own payoffs). In this case, *every* action profile is potentially stable since each player might think that the other two voted for the winner so that his ballot cannot make any difference.

2.4. **Hawaii versus Caribbeans.** Consider the case where each of three neighbors should choose whether to go to a vacation at Hawaii (H) or at the Caribbeans (C). Players are indifferent between the two locations but each of them would like to avoid his 'right' neighbor. That is, player 1 would like to avoid player 2, player 2 would like to avoid player 3 and player 3 would like to avoid player 1. Specifically, the payoffs to the players are given in the following matrices where 1 chooses row, 2 column and 3 matrix.

	H	C
H	$0,0,0$	* $1,1,0$
C	* $1,0,1$	* $0,1,1$

H

	H	C
H	* $0,1,1$	* $1,0,1$
C	* $1,1,0$	$0,0,0$

C

Clearly, there is no pure equilibrium in this game. But what happens if players are unable to observe the identity of their opponents in each of the locations? That is, if we assume that each player's information is only the *distribution* of players among the two locations, what action profiles can be stable?

[4]For simplicity we assume that no player is indifferent between the two alternatives.

The answer is that every action profile in which not all players are in the same location (the cells marked with asterisks) is potentially stable. This is so since the (unique) unfortunate player who couldn't avoid his 'right' neighbor might be thinking that he actually succeeded in doing so, and that he matched the choice of his 'left' neighbor. If this is his conjecture then he believes that he will only lose utility by deviating.

3. The model

3.1. **Games and information structures.**

A normal form game G is a tuple $G = (N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N})$. $N = \{1, 2, \ldots, n\}$ is the set of players. For each $i \in N$, A_i is a finite non-empty set of pure strategies (actions) of player i. Let $A = \times_{i\in N} A_i$ be the set of action profiles. For every $i \in N$, $u_i : A \to I\!R$ is the payoff function of player i.

The following notation will be used throughout the paper. If G is a game as above and $i \in N$ then A_{-i} denotes the set of action profiles of players other than i, that is $A_{-i} = \times_{j\in N\setminus\{i\}} A_j$. A typical element of A_{-i} will be denoted by a_{-i}. We will often write $(a_i; a_{-i})$ instead of a, where a is the action profile in which player i is playing $a_i \in A_i$ and other players are playing according to $a_{-i} \in A_{-i}$. If X is a finite set then $\Delta(X)$ is the set of probability measures over X. The set of mixed strategies of player i is $\Delta(A_i)$. As usual, every u_i is linearly extended to $\Delta(A)$. The extension will also be denoted by u_i.

To the standard description of a normal form game we now add a new ingredient. This is the *information structure* which determines the signal that every player observes as a function of the chosen profile of actions. Formally,

Definition 1. *Fix a game G.*
(i) An ***information structure for player*** **i** *is a pair $I_i = (S_i, s_i)$ where S_i is a set of* ***signals*** *and $s_i : A \to S_i$ is an arbitrary* ***signal function****.*
(ii) An ***information structure in*** **G** *is a list of information structures $I = \{I_i\}_{i\in N}$, one for every player in G.*

In some cases it is possible to compare the informativeness of two information structures in a game G. Intuitively, one information structure I is *more informative* (in a weak sense) than another information structure I' if every player can deduce from the signal he observes at I the signal that he would have seen had the structure been I'.

Definition 2. *Let $I = (S_i, s_i)_{i\in N}$ and $I' = (S'_i, s'_i)_{i\in N}$ be two information structures in G. I is* ***more informative*** *than I' if s'_i is s_i-measurable for every $i \in N$. Equivalently, I is more informative than I' if the partition of A induced by s_i is finer than the partition induced by s'_i, for every $i \in N$.*

3.2. **Weakly and strongly consistent conjectures.** The information available to player i about the actions profile is determined by the signal function s_i. Since typically many action profiles generate the same signal, player i will face uncertainty as to the actual profile being played. According to the standard paradigm, player i's preferences in this case can be described by his expected payoff, where the expectation is taken with respect to a certain probability distribution representing i's belief. We will refer to this belief as i's *conjecture*. Formally, a conjecture of player i is any probability distribution $\mu_{-i} \in \Delta(A_{-i})$.

Notice that Definition 1 does not explicitly require that a player is able to deduce his own action from the signal. Thus, *a priori* it may be that, while playing a_i, player i believes that he actually plays a_i' (or a mixture of a_i' and a_i''). But since we define a conjecture of i to be an element of $\Delta(A_{-i})$ (and not of $\Delta(A)$) such awkward beliefs are being ruled out. Thus, *it is an inherent property of our model that every player knows the action he chooses.*

Definition 3. *Let I be an information structure in a game G and let $a \in A$. A conjecture $\mu_{-i} \in \Delta(A_{-i})$ of player i is* ***strongly consistent at a*** *if $s_i(a) = s_i(a_i; a'_{-i})$ for every a'_{-i} in the support of μ_{-i}.*

The conjecture of i is strongly consistent at a iff it gives probability 1 to the set of action profiles a'_{-i} such that, when combined with a_i, generate the same signal as a. We will also use a weaker version of consistency. This weaker version can only be defined if additional assumptions on the structure of the signals set S_i are made. Namely, each S_i is assumed to be a subset of some linear space. This allows to compute the *expectation* of s_i with respect to some probability measure on A. Weak consistency is defined as follows.

Definition 4. *Let I be an information structure in a game G where, for every $i \in N$, S_i is a subset of some linear space. A conjecture $\mu_{-i} \in \Delta(A_{-i})$ of player i is* ***weakly consistent at a*** *if $\int_{A_{-i}} s_i(a_i; a'_{-i}) d\mu_{-i}(a'_{-i}) = s_i(a)$ for every $i \in N$.*

Thus, weak consistency at a only requires that the *expected* signal according to the conjecture μ_{-i} is equal to the signal of i at a. Since A_{-i} is finite the integral in the above definition is nothing but a finite convex combination of elements in S_i. Notice that every strongly consistent conjecture of some player i at a is also weakly consistent for i at a.

3.3. **Equilibrium.** We now define two versions of SCE. The difference between the two versions is only in the meaning of the consistency requirement. Since this paper only deals with pure SCE we do not define here SCE in mixed strategies. *Whenever we write SCE we mean pure SCE.*

Definition 5. *Let G be a game and I an information structure in G.*
(i) A profile of actions $a \in A$ is a ***Strong Self-Confirming Equilibrium (SSCE)***

if, for every $i \in N$, there is a strongly *consistent conjecture $\mu_{-i} \in \Delta(A_{-i})$ such that $u_i(a_i; \mu_{-i}) \geq u_i(a'_i; \mu_{-i})$ for every $a'_i \in A_i$.*
(ii) A profile of actions $a \in A$ is a ***Weak Self-Confirming Equilibrium (WSCE)*** *if, for every $i \in N$, there is a* weakly *consistent conjecture $\mu_{-i} \in \Delta(A_{-i})$ such that $u_i(a_i; \mu_{-i}) \geq u_i(a'_i; \mu_{-i})$ for every $a'_i \in A_i$.*

The following lemma lists several simple properties of SSCE and WSCE. Since these properties are straightforward consequences of the definitions the proofs are omitted.

Lemma 1. *Fix a game G.*

(1) *Every SSCE in G is also a WSCE in G (with respect to the same information structure).*
(2) *If I and I' are two information structures in G such that I is more informative than I' then every SSCE with respect to I is a SSCE with respect to I'.*
(3) *Every (pure) Nash equilibrium of G is a SSCE of G, no matter what is the information structure.*
(4) *If $a_i \in A_i$ is a strictly dominant action of player i then there is no WSCE (and by the first item of this lemma no SSCE) in which player i plays a_i.*

4. Observable payoffs

In the current section we restrict attention to the case where the signal to each player is (only) his own payoff[5]. Notice that in many economic situations this is the information available to agents. To give just one example, a firm may know it's own profits without knowing the production levels of other firms in the market.

The formal definition of the information structure is as follows. Fix a game G. For every $i \in N$, the set of signals for player i is $S_i = \mathbb{R}$. The signal function for i is defined by $s_i(a) = u_i(a)$. When this is the information available to each player we will say that G is a *game with observable payoffs*.

Every S_i has a natural linear structure so expectations of signal functions can be easily calculated. Notice that a weakly consistent conjecture for i at a in this case is any probability distribution $\mu_{-i} \in \Delta(A_{-i})$ with the property that $u_i(a_i; \mu_{-i}) = u_i(a)$. Therefore, a WSCE is an action profile a such that, for every player i, a_i is a best response to some μ_{-i} satisfying $u_i(a_i; \mu_{-i}) = u_i(a)$.

4.1. **Zero-sum games.** We start with a characterization of WSCE in two-person zero-sum games. Thus, we assume that $N = \{1, 2\}$ and $u_1(a_1, a_2) = -u_2(a_1, a_2)$ for every $a_1 \in A_1$ and $a_2 \in A_2$. The value of the game (for player 1) is denoted by[6] v.

[5]Lehrer (1992) considers a similar situation but in the context of a repeated game.
[6]Thus, if each player plays a maxmin strategy then player 2 pays v to player 1.

Proposition 1. *The pair $a = (a_1, a_2)$ is a WSCE of a zero-sum game G with observable payoffs iff the following two conditions hold:*
(i) $u_1(a) = v$; and
(ii) a_i is a best response to a maxmin *strategy of player $-i$, $i = 1, 2$.*

Proof. Assume first that (i) and (ii) hold. For $i = 1, 2$ let μ_{-i} be the maxmin strategy of player $-i$ implied by (ii). Since a_i is a best response to the maxmin strategy μ_{-i} it must be that $u_i(a_i, \mu_{-i}) = v$. Thus, by (i), μ_{-i} is a weakly consistent conjecture of player i at a. It follows from (ii) that a is a WSCE.

Conversely, assume that a is a WSCE. Then there are strategies $\mu_{-i},\ i = 1, 2$ such that for $i = 1, 2$

$$u_i(a_i; \mu_{-i}) = u_i(a_i; a_{-i}) \tag{1}$$

$$u_i(a_i; \mu_{-i}) \geq u_i(a'_i; \mu_{-i})\ \forall a'_i \in A_i \tag{2}$$

Combining (1) and (2) above, and not forgetting that $u_1 = -u_2$ we get that for every $(a'_1, a'_2) \in A_1 \times A_2$

$$u_1(a'_1, \mu_2) \leq u_1(a_1, \mu_2) = u_1(a_1, a_2) = u_1(\mu_1, a_2) \leq u_1(\mu_1, a'_2) \tag{3}$$

Since the inequality holds for every $a'_2 \in A_2$ it also holds for every mixed strategy of player 2. In particular, one can replace a'_2 with μ_2. It follows that μ_1 is a best response of player 1 to μ_2. Similarly, it is easy to see that μ_2 is a best response of player 2 to μ_1. Thus, the pair of conjectures (μ_1, μ_2) is an equilibrium of G. In particular, $u_1(\mu_1, \mu_2) = v$ and, for $i = 1, 2,\ \mu_i$ is a maxmin strategy for player i. Therefore, (i) and (ii) are consequences of (3) and (2) respectively. ■

The Rock-Paper-Scissors example of Section 2 shows that there may be WSCE which are not Nash equilibria of the game. Moreover, it is also possible that there will be SSCE which are not Nash. For instance, consider the following zero-sum game.

	L	R
T	* 0	-1
B	0	1

The numbers in the matrix are the payoffs to the row player (player 1). The action pair (T, L) (the cell marked with an asterisk) is not an equilibrium of the game since player 2's best response to T is R. However, it is a SSCE. Indeed, player 1 is playing a best response to the actual strategy of player 2, while player 2 is playing a best response to the strongly consistent conjecture (at (T, L)) that the action of player 1 is B.

4.2. **Constant-sum symmetric games.** The next class of games we consider is that of n-players constant-sum symmetric games. Thus, in this subsection we assume that all the players in the game have the same set of actions, denoted B. That is, $A = B^n$.

A game G is *constant-sum* if there is $c \in I\!R$ such that $\sum_{i \in N} u_i(a) = c$ for every $a \in A$. To define symmetry we first introduce the following notation. If $a \in A$ and $\pi : N \to N$ is a permutation of the players then $\pi(a) \in A$ is defined by $\pi(a)_i = a_{\pi(i)}$. A game G is *symmetric* if $u_i(a) = u_j(\pi(a))$ for every $i, j \in N$ (possibly $i = j$), every $a \in A$ and every permutation π satisfying $\pi(j) = i$.

Proposition 2. *Every symmetric constant-sum game with observable payoffs has a* symmetric *WSCE.*

Proof. It is well known that every symmetric game has a symmetric equilibrium, possibly in mixed strategies. Let $\mu \in \Delta(B)$ be one such equilibrium (that is, μ is a best response of every player to everybody else playing μ). Let $b \in B$ be in the support of μ, and denote $\underline{b} \in A$ the action profile in which all the players choose b. We show that $\underline{b}$ is a WSCE.

Fix $i \in N$. Since G is a symmetric and constant-sum it follows that $u_i(\underline{b}) = \frac{c}{n}$. Similarly, $u_i(\underline{\mu}) = \frac{c}{n}$ where $\underline{\mu}$ is the strategy profile in which everyone plays μ. Denote $(b; \underline{\mu}_{-i})$ the strategy profile in which player i plays b and any other player plays μ. Since $\underline{\mu}$ is an equilibrium with b in its support it must be that $u_i(b; \underline{\mu}_{-i}) = \frac{c}{n}$. This implies that $\underline{\mu}_{-i}$ is a weakly consistent conjecture for player i at $\underline{b}$ and that b is a best response to $\underline{\mu}_{-i}$. ■

Remark 1. *The location game of subsection 2.2 is an example of a constant-sum symmetric game. Obviously, the example can be significantly generalized while maintaining these two properties. Specifically, the* $[0, 1]$ *interval with the uniform measure may be replaced by any subset of an Euclidean space endowed with an arbitrary measure. The possible locations for the players can be any (finite) set of points.*

4.3. **Social choice games.** The games considered in this subsection arise when a society should choose among several possible outcomes. The action that each player chooses can be seen as the massage that this player sends to some mechanism. The mechanism specify the chosen outcome for each profile of massages. The characteristic property of these games is that the preferences of every player are defined on the set of outcomes and not on the set of action profiles. In other words, the payoff of every player depends on the action profile only through the prevailing outcome.

Formally, let O be a set of *outcomes*. We say that G is a *social choice game associated with* O if there is a *mechanism* $f : A \to O$ such that the utility of every player $i \in N$ satisfies $u_i(a) = u_i(a')$ whenever $f(a) = f(a')$. If a mechanism f satisfies this last property we say that f *is sufficient for* G.

Definition 6. *Let G be a social choice game associated with outcomes set O and let f be a sufficient mechanism for G.* ***Player*** $\mathbf{i} \in \mathbf{N}$ ***can prevent outcome*** $\mathbf{o} \in \mathbf{O}$ *if for every $a_{-i} \in A_{-i}$ there is $a_i \in A_i$ such that $f(a_i; a_{-i}) \neq o$.*

Proposition 3. *Let G be a social choice game with observable payoffs associated with outcomes set O and let f be a sufficient mechanism for G. If $o \in O$ is an outcome which no player can prevent then every $a \in f^{-1}(o)$ is a SSCE.*

Proof. By Lemma 1 (2) it is sufficient to prove the proposition when the signal to every player is the outcome (since every player can deduce his payoff from the outcome). Assume that $a \in f^{-1}(o)$ and that no player can prevent o. Fix $i \in N$. Since i can't prevent o there is $a'_{-i} \in A_{-i}$ such that $f(a'_i; a'_{-i}) = o$ for every $a'_i \in A_i$. In particular, $f(a_i; a'_{-i}) = o$ which means that a'_{-i} is a strongly consistent conjecture for i at a. Moreover, a_i is a best response to a'_{-i} since player i cannot change his payoff by deviating. ■

To illustrate the proposition consider the following example. There are 3 players and 2 actions (c and d) for each player. The set of outcomes is $O = \{o, o', o''\}$. The mechanism is given by the following matrices, where player 1 chooses row player 2 column and player 3 matrix.

	c	d
c	* o	o'
d	* o	* o

c

	c	d
c	* o	o''
d	o''	o'

d

Recall that the utility of the players depend only on the outcome. Player 1's preferences satisfy $u_1(o') > u_1(o) > u_1(o'')$, player's 2 preferences satisfy $u_2(o') > u_2(o'') > u_2(o)$ and 3's preferences satisfy $u_3(o'') > u_3(o) > u_3(o')$.

It is easy to check that no pure equilibrium exists in this game. However, if players only observe their own payoff (which is equivalent to the case where players observe only the outcome) then every profile that yields the outcome o (the cells marked with asterisks) is a SSCE. Indeed, every pair of players can guarantee that o will be the prevailing outcome. In other words, no player can prevent o. Proposition 3 implies that any profile $a \in f^{-1}(o)$ is a SSCE.

An important family of social choice games is the class of 'majority rule' games. In this class the action set of every player is equal to the outcome set[7]. The prevailing outcome is the one that was chosen by the largest number of players. If there is a tie the outcome is chosen according to some predefined and commonly known tie-breaking rule. An immediate consequence of Proposition 3 is the following.

[7]It is possible to extend the set of actions by allowing the players not to vote at all.

Corollary 1. *If G is a social choice game with observable payoffs and with at least 3 players such that the outcome is chosen according to a majority rule then* every *action profile is a SSCE.*

5. Distributional information

In many cases the information available to agents is of anonymous nature. That is, agents do not know the identity of the the players who chose each action but do know the distribution of the players among the various actions. For instance, drivers can hear on the radio the amount of traffic in each route but do not know the identity of the drivers (and, therefore, the probability of an accident); And a client entering a bank only sees the number of clients waiting in front of every clerk but do not know the kind of service that each client requires.

Let B be a finite set of actions. In this section we consider games G such that[8] $A_i = B$ for every $i \in N$. Thus, $A = B^n$. For $a \in A$, we denote by $dist(a)$ the distribution of players among the actions in B. That is, $dist(a) = (dist(a)_b)_{b \in B}$ where $dist(a)_b = \#\{i \in N \ : \ a_i = b\}$ is the number of players who choose the action b according to a. We say that G is a game with *distributional information* if $s_i(a) = dist(a)$ for every $i \in N$.

5.1. **Large continuous games.** The seminal work of Schmeidler (1973) demonstrates that in large anonymous games one should expect pure equilibria to exist. In Schmeidler's model the set of players is approximated by a non-atomic measure space. More recently, the results of Kalai (2004) imply that, with appropriate continuity condition on the payoff functions, equilibria of finite anonymous games 'self-purify' as the number of players grows to infinity[9].

In this subsection we use the results of Kalai to derive existence of approximate SSCE in large continuous games with distributional information. The idea behind this result is simple: On the one hand we drop the assumption of anonymous payoff functions, thus considering a more general class of games. On the other hand, the result we obtain is the existence of (approximate) SSCE and not of (approximate) Nash equilibrium. The anonymous nature of the information available to the players compensates for the lack of anonymity in the payoff functions.

For a given $\epsilon > 0$, we define ϵ-SSCE similarly to the definition of ϵ-Nash equilibrium. Namely, a profile of actions $a \in A$ is an *ϵ-SSCE* (with respect to a given information structure) if, for every $i \in N$, there is a strongly consistent conjecture $\mu_{-i} \in \Delta(A_{-i})$ such that $u_i(a_i; \mu_{-i}) \geq u_i(a'_i; \mu_{-i}) - \epsilon$ for every $a'_i \in A_i$.

[8]Our results can easily be extended to the case where the actions available to a player are some non-empty subset of B

[9]We do not wish to go into the exact meaning of this statement at the moment. It is only made to point out that in large continuous and anonymous games pure approximate equilibria exist. For details see Kalai (2004).

Let $\Gamma(B)$ be a family of normal form games such that $A_i = B$ for every player $i \in N$ in every game $G \in \Gamma(B)$. For the next definition we introduce the following notation. If a_{-i} and a'_{-i} are two profiles of actions of players other than $i \in N$ in some normal form game $G \in \Gamma(B)$ then $d(a_{-i}, a'_{-i}) = \#\{j \in N \setminus \{i\} \ : \ a_j \neq a'_j\}$ is the number of players which play differently in a_{-i} than in a'_{-i}.

Definition 7. *(i) The family* $\Gamma(B)$ *is* ***uniformly bounded*** *if there is* $M > 0$ *such that* $|u_i| \leq M$ *for every* $i \in N$ *and for every* $G \in \Gamma(B)$.
(ii) The family $\Gamma(B)$ ***exhibits a diminishing effect of a single player*** *if there is* $M > 0$ *such that* $|u_i(a_i; a_{-i}) - u_i(a_i; a'_{-i})| < \frac{M}{|N|}$ *for every* $G \in \Gamma(B)$*, every* $i \in N$*, every* $a_i \in B$ *and every* a_{-i}, a'_{-i} *satisfying* $d(a_{-i}, a'_{-i}) = 1$.

Proposition 4. *Let* $\Gamma(B)$ *be a family of normal form games which is uniformly bounded and exhibits a diminishing effect of a single player. Then for every* $\epsilon > 0$ *there is* $n_0 = n_0(\epsilon)$ *such that in any game* $G \in \Gamma(B)$ *with at least* n_0 *players and with distributional information there is an* $\epsilon - SSCE$.

Proof. The proof is divided into 3 steps.

Step 1: To any game $G \in \Gamma(B)$ we associate another game $\tilde{G} = (N, \{A_i\}_{i \in N}, \{\tilde{u}_i\}_{i \in N})$ with the same sets of players and actions. For every $i \in N$, the payoff function $\tilde{u}_i$ is defined by

(4) $\tilde{u}_i(a_i; a_{-i}) = \max\{u_i(a_i; \bar{a}_{-i}) \ : \ dist(a_i; \bar{a}_{-i}) = dist(a_i; a_{-i}), \ \bar{a}_{-i} \in A_{-i}\}$.

Thus, the payoff to i when he plays a_i and his opponents play a_{-i} is the maximal payoff he may get when he plays a_i and the distribution of his opponents' choices induce the same distribution as a_{-i}.

Define $\tilde{\Gamma}(B) = \{\tilde{G} \ : \ G \in \Gamma(B)\}$. It is easy to check that if $\Gamma(B)$ is uniformly bounded and exhibits a diminishing effect of a single player then so is $\tilde{\Gamma}(B)$.

Step 2: We claim that the family $\tilde{\Gamma}(B)$ satisfies the conditions of *semi-anonymity* and *uniform equicontinuity* of Kalai (2004, Definitions 2,3 in page 1637). Semi-anonymity (full anonymity actually) is a straightforward consequence of the definition of $\tilde{u}_i$. Uniform equicontinuity follows from the diminishing effect property when combined with anonymity. Indeed, for a given $\epsilon > 0$, let $\delta = \frac{\epsilon}{M|B|}$ where M is the constant in Definition 7 (ii). It should be shown that if a and a' are two action profiles satisfying $a_i = a'_i$ and $|dist(a)_b - dist(a')_b| < \delta|N|$ for every $b \in B$ then $|\tilde{u}_i(a) - \tilde{u}_i(a')| < \epsilon$.

Fix a player i and two action profiles a, a' as above. By permuting the players other than i one can find another action profile a'' satisfying $a_i = a''_i$, $dist(a'') = dist(a')$ and $d(a_{-i}, a''_{-i}) < \delta|N||B|$. By anonymity, $\tilde{u}_i(a') = \tilde{u}_i(a'')$. Repeated use of the diminishing effect property gives $|\tilde{u}_i(a) - \tilde{u}_i(a'')| < \frac{M}{|N|} \cdot \delta|N||B| = \epsilon$. it follows that $|\tilde{u}_i(a) - \tilde{u}_i(a')| < \epsilon$ as required.

Step 3: By Theorem 1 of Kalai (2004, see also subsection 3.2 in page 1642 there), for every $\epsilon > 0$ there is $n_0 = n_0(\epsilon)$ such that, if $\tilde{G} \in \tilde{\Gamma}(B)$ has at least n_0 players, then there is a pure ϵ-Nash equilibrium in $\tilde{G}$. Let a be a pure ϵ-Nash equilibrium in $\tilde{G}$. To complete the proof, we will show that a is an ϵ-SSCE in G.

Fix a player $i \in N$. Since a is an ϵ-Nash equilibrium in $\tilde{G}$ it follows that $\tilde{u}_i(a_i; a_{-i}) \geq \tilde{u}_i(a'_i; a_{-i}) - \epsilon$ for every $a'_i \in B$. Let $\bar{a}_{-i}$ be a maximizer of the righthand side of (4). Since always $\tilde{u}_i \geq u_i$ we get

$$u_i(a_i; \bar{a}_{-i}) = \tilde{u}_i(a_i; a_{-i}) \geq \tilde{u}_i(a'_i; a_{-i}) - \epsilon = \tilde{u}_i(a'_i; \bar{a}_{-i}) - \epsilon \geq u_i(a'_i; \bar{a}_{-i}) - \epsilon.$$

Finally, since $\bar{a}_{-i}$ is a strongly consistent conjecture for i at a, it follows that a is an ϵ-SSCE in G. ■

5.2. **Congestion games.** Congestion games were first introduced by Rosenthal (1973). Here, however, we use a slightly different model than that of Rosenthal which is a generalization of the model used by Milchtaich (1996). A game G will be called *congestion game* if it has the following two properties[10]:

(i) The payoff to a player depends only on the action he chooses and on the set of players who choose the same action as he did. That is, $u_i(a) = u_i(a')$ whenever $a_i = a'_i$ and $\{j \in N \setminus \{i\} \ : \ a_j = a_i\} = \{j \in N \setminus \{i\} \ : \ a'_j = a'_i\}$.

(ii) For every $i \in N$, every $a_i \in B$ and every a_{-i}, a'_{-i}, if $\{j \in N \setminus \{i\} \ : \ a_j = a_i\} \subseteq \{j \in N \setminus \{i\} \ : \ a'_j = a_i\}$ then $u_i(a_i; a_{-i}) \geq u_i(a_i; a'_{-i})$. In words, the utility of player i is decreasing (with respect to the inclusion relation) in the set of players that choose the same action as him.

Notice that we impose no symmetry assumption on the players nor on the actions. It was shown in Milchtaich (1996, Section 8) that games satisfying (i) and (ii) above need not have a pure Nash equilibrium[11]. Also, notice that the Hawaii versus Caribbeans game of subsection 2.4 is a congestion game with no pure equilibria. However, when players only have distributional information SSCE do exist, as stated in the following proposition.

Proposition 5. *Every congestion game with distributional information has a SSCE.*

Proof. Let G be a congestion game. Define the game $\tilde{G}$ (with the same sets of players and actions as G) as in the proof of Proposition 4. That is,

(5) $\tilde{u}_i(a_i; a_{-i}) = \max\{u_i(a_i; \bar{a}_{-i}) \ : \ dist(a_i; \bar{a}_{-i}) = dist(a_i; a_{-i}), \ \bar{a}_{-i} \in A_{-i}\}$.

Claim 1. *For every $i \in N$, every $a_i \in B$ and every $a_{-i}, a'_{-i} \in A_{-i}$, if $\#\{j \in N \setminus \{i\} \ : \ a_j = a_i\} = \#\{j \in N \setminus \{i\} \ : \ a'_j = a_i\}$ then $\tilde{u}_i(a_i; a_{-i}) = \tilde{u}_i(a_i; a'_{-i})$.*

[10] Recall that the setup of this section is that all players have the same action set B.

[11] The class of 'weighted congestion games' considered by Milchtaich (1996, Section 8) is narrower than the class of games considered here. Milchtaich shows that, even in this restricted class, pure equilibrium may fail to exist. This is in contrast to the class of games considered by Rosenthal (1973) in which every game has a pure equilibrim.

That is, the payoff to every player only depends on the action he chooses and on the number *of players playing the same action.*

Proof. Let $\bar{a}_{-i}$ satisfy $dist(a_i;\bar{a}_{-i}) = dist(a_i;a_{-i})$ and $\tilde{u}_i(a_i;a_{-i}) = u_i(a_i;\bar{a}_{-i})$. That is, $\bar{a}_{-i}$ is a maximizer of (5) above for $(a_i;a_{-i})$. Denote $D = \{j \in N \setminus \{i\} \ : \ a_j = a_i\}$, $E = \{j \in N \setminus \{i\} \ : \ a'_j = a_i\}$ and $F = \{j \in N \setminus \{i\} \ : \ \bar{a}_j = a_i\}$. By assumption $|D| = |E|$ and by the choice of $\bar{a}_{-i}$, $|D| = |F|$. It follows that there is $\bar{a}'_{-i}$ such that $dist(a_i;\bar{a}'_{-i}) = dist(a_i;a'_{-i})$ and $\{j \in N \setminus \{i\} \ : \ \bar{a}'_j = a_i\} = F$. By property (i) of congestion games and since always $u_i \leq \tilde{u}_i$ we get

$$\tilde{u}_i(a_i;a_{-i}) = u_i(a_i;\bar{a}_{-i}) = u_i(a_i;\bar{a}'_{-i}) \leq \tilde{u}_i(a_i;\bar{a}'_{-i}) = \tilde{u}_i(a_i;a'_{-i})$$

Due to symmetry we have the other inequality. This completes the proof of the claim. ■

Claim 2. *For every $i \in N$, every $a_i \in B$ and every $a_{-i}, a'_{-i} \in A_{-i}$, if $\#\{j \in N \setminus \{i\} \ : \ a_j = a_i\} \geq \#\{j \in N \setminus \{i\} \ : \ a'_j = a_i\}$ then $\tilde{u}_i(a_i;a_{-i}) \leq \tilde{u}_i(a_i;a'_{-i})$.*

Proof. Let $\bar{a}_{-i}$, D, E, and F be as in the previous proof. This time we have $|F| = |D| \geq |E|$. Thus, there is $\bar{a}'_{-i}$ such that $dist(a_i;\bar{a}'_{-i}) = dist(a_i;a'_{-i})$ and $\{j \in N \setminus \{i\} \ : \ \bar{a}'_j = a_i\} \subseteq F$. By property (ii) of congestion games we obtain

$$\tilde{u}_i(a_i;a_{-i}) = u_i(a_i;\bar{a}_{-i}) \leq u_i(a_i;\bar{a}'_{-i}) \leq \tilde{u}_i(a_i;\bar{a}'_{-i}) = \tilde{u}_i(a_i;a'_{-i}),$$

which proves the claim. ■

Claim 3. *The game $\tilde{G}$ has a pure Nash equilibrium.*

Proof. This follows from the two previous claims and from Theorem 2 in Milchtaich (1996). ■

To complete the proof of the proposition it is sufficient to show that every pure Nash equilibrium in $\tilde{G}$ is a SSCE in G. This can be done by repeating the argument in step 3 of the proof of Proposition 4 with $\epsilon = 0$. ■

6. Related literature

The notion of Self-Confirming Equilibrium originates in a paper of Fudenberg and Levine (1993A). Their solution concept is appropriate only for extensive form games since they assume that players observe the actions of their opponents (only) on the equilibrium path. In another paper, Fudenberg and Levine (1993B) show that SCE correspond to steady states of a particular learning process. SCE and learning in games with incomplete information are studied by Dekel et al. (2004).

A similar solution concept which is suitable for normal form games is Battigalli and Guaitoli's (1988) *Conjectural Equilibrium* (CE). In fact, SSCE which we used in this work is the same as CE[12]. Rubinstein and Wolinsky (1994) defined and analyzed the notion of *Rationalizable Conjectural Equilibrium* (RCE) which is a refinement of CE. In RCE players take into account the signal functions of the other players and the fact that all the participants are rational, which further restrict the set of consistent conjectures. Thus, RCE takes an intermediate position between Nash equilibrium and Rationalizability (Bernheim, 1984 and Pearce, 1984). The same idea is used by Dekel et al. (1999) to refine SCE. *Subjective Equilibrium* of Kalai and Lehrer (1993) is in the same spirit but in the context of a repeated game.

There were several attempts to break away from the idea that agents spin roulettes before making decisions. Notably, Harsanyi (1973) shows that adding small random perturbations to the payoff functions eliminates the need to randomize in the resulting incomplete information game. Aumann (1987) suggests that a mixed strategy should be seen as a 'plan of action' which specify the (pure) action to be taken after each possible *private* signal. In this way no player explicitly randomizes, and a mixed strategy of a certain player reflects the uncertainty of other players about his choice. A discussion of these two approaches can be found in Rubinstein (1991).

More recently, several papers refine SCE by requiring that players will play a best response to a particular consistent conjecture. In Lehrer's (2007) *Partially Specified Equilibrium* each player maximizes his payoff against the worst possible consistent conjecture. Jehiel (2005) introduced the notion of *Analogy-Based Expectation Equilibrium* in which players' information is the average strategy of their opponents in groups of nodes of a game tree. Each player's conjecture is that the strategy in each node is the same as the average strategy of the group containing this node.

7. Final remarks

7.1. **Independent conjectures.** A conjecture for player i is any element of $\Delta(A_{-i})$, not necessarily a product measure. That is, a player is allowed to believe that his opponents are using a correlated strategy. Restricting players to independent conjectures will sometimes shrink the set of SCE[13]. As a consequence, it is possible that no pure SCE will survive. However, all the results of this paper will not be affected by such restriction since the conjectures we use in the proofs are always independent.

7.2. **WSCE and stochastic signals.** One may argue that if a certain game is played repeatedly and some of the players are using mixed strategies then what a player will know is the *distribution* of signals and not just the expected signal. If

[12]We preferred the notion of self-confirming equilibrium over conjectural equilibrium since the former is much more known and frequently used in the literature. Also, we feel that the name 'self-confirming' catches the essence of this solution concept.

[13]In Azrieli (2007) it is essential that players think that their opponents play independently of each other.

this is the case then some weakly consistent conjectures (as defined in this paper) are no longer consistent with that player's information. For instance, in the Rock-Paper-Scissors example of subsection 2.1 the conjecture of player 1 that player 2 is playing $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is not consistent at (R, R) since it induces different distribution on 1's signals than what the actual strategy of player 2 induces.

A possible way out of this difficulty[14] is to assume that the signals themselves are stochastic and depend non-deterministically on the action profile. To illustrate this idea consider the observable payoffs case. The numbers in the payoff matrices can be thought of as the *expected* payoff to each player in every action profile. So in the Rock-Paper-Scissors example, when the action profile (R, R) is played, player 1's payoff is drawn from a certain distribution whose expectation is 0. If one assumes that the (random) payoff to player 1 in this case is 1 or -1 with equal probabilities then player 1 will not be able to distinguish between the true strategy of player 2 and player 2 playing $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

7.3. **Other signal functions and other classes of games.** Our results are restricted to two kinds of information structures and to several classes of non-cooperative games. It will be interesting to identify more classes of games which admit pure SSCE or WSCE with respect to these information structures or with respect to other natural information structures. It may also be of interest to study the case of asymmetric information, where different players have different kinds of information structures.

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