Covariances and relationships between price indices: Notes on a theorem of Ladislaus von Bortkiewicz on linear index functions

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The note examines a generalization of a theorem of Bortkiewicz which relates the difference between a Paasche and a Laspeyres price index to a covariance between price and quantity relatives. The generalized theorem is used to demonstrate a number of interesting special applications. It turns out that some known relationships between two index functions can be expressed more elegantly. In other cases where not much is known yet about how the two functions are related to one another, we could establish an interesting equation on the basis of this theorem. This demonstrates the remarkable flexibility and usefulness of the generalized Bortkiewicz - theorem.

1. Generalization of a theorem for additive indices of Ladislaus von Bortkiewicz

It is well known that Ladislaus von Bortkiewicz (1868 – 1931) found that the Paasche price index (P^P_{0t}) is related to the Laspeyres price index (P^L_{0t}) as follows

$$(1)\qquad \frac{P^P_{0t}}{P^L_{0t}} = 1 + \frac{\text{cov}}{P^L_{0t}Q^L_{0t}},$$

where Q^L_{0t} denotes the Laspeyres quantity index and cov is the (weighted) covariance between price and quantity relatives given by

$$(2)\qquad \text{cov} = \sum_{i=1}^{n}\left(\frac{p_{it}}{p_{i0}} - P^L_{0t}\right)\left(\frac{q_{it}}{q_{i0}} - Q^L_{0t}\right)w_{i0} = V_{0t} - P^L_{0t}Q^L_{0t} = Q^L_{0t}(P^P_{0t} - P^L_{0t}) = P^L_{0t}(Q^P_{0t} - Q^L_{0t}),$$

with base period expenditure weights $w_{i0} = p_{i0}q_{i0}/\sum p_{i0}q_{i0}$ of the n commodities (i = 1, …, n). As P^L_{0t} and Q^L_{0t} is the arithmetic mean of price and quantity relatives respectively the "centered" covariance $\frac{\text{cov}}{P^L_{0t}Q^L_{0t}}$ can also be written as follows

$$(3)\qquad \overline{\text{cov}} = \frac{\text{cov}}{P^L_{0t}Q^L_{0t}} = r_{pq}C_pC_q = \sum_i\left(\frac{p_{it}/p_{i0}}{P^L_{0t}} - 1\right)\left(\frac{q_{it}/q_{i0}}{Q^L_{0t}} - 1\right)w_{i0}.$$

Using the correlation coefficient r_{pq}, and the coefficients of variation C_p, C_q the theorem of Bortkiewicz can be written as

$$(4)\qquad \frac{P^P_{0t}}{P^L_{0t}} = \frac{V_{0t}}{P^L_{0t}Q^L_{0t}} = 1 + r_{pq}C_pC_q = 1 + \overline{\text{cov}}.$$

Interestingly this well known relationship between a Paasche and a Laspeyres price index turns out to be only a special case of a more general law of the ratio of two *additive* (*linear*) indices X_1 and X_0 respectively (see **fig. 1**).

An index function P($\mathbf{p}_0$,$\mathbf{q}_0$,$\mathbf{p}_t$,$\mathbf{q}_t$) is said to be linear when it can be expressed as a ratio of vector products as for example

$$P^L_{0t} = P^L(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t) = \frac{\sum p_t q_0}{\sum p_0 q_0} = \frac{\mathbf{p}_t'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_0}, \text{ and thus also as } P^L_{0t} = \sum \frac{p_{it}}{p_{i0}}\frac{p_{i0}q_{i0}}{\mathbf{p}_0'\mathbf{q}_0'}.$$

For example the function $P_{0t}^{\Lambda} = P^{\Lambda}(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t) = \prod(\frac{p_t}{p_0})^{\frac{p_0 q_0}{\Sigma p_0 q_0}}$ (which may be called the log -Laspeyres price index) is not a linear index.

Figure 1: Generalization of Bortkiewicz's theorem
(law of the ratio of two additive indices)
Taken from v. d. Lippe (2007), p. 196

first additive index X_0	second additive index X_t
$X_0 = \frac{\sum x_t y_0}{\sum x_0 y_0}$	$X_t = \frac{\sum x_t y_t}{\sum x_0 y_t}$

weighted (weights $w_0 = x_0 y_0 / \sum x_0 y_0$ throughout) arithmetic mean of the relatives

x_t/x_0 : $\bar{X} = X_0$	y_t/y_0 : $\bar{Y} = \frac{\sum y_t x_0}{\sum y_0 x_0}$ *)

variances (weights w_0) of the relatives

x_t/x_0 : $s_x^2 = \sum\left(\frac{x_t}{x_0} - \bar{X}\right)^2 w_0$	y_t/y_0 : $s_y^2 = \sum\left(\frac{y_t}{y_0} - \bar{Y}\right)^2 w_0$

the covariance is given by

(2a) $\text{cov} = s_{xy} = \sum\left(\frac{x_t}{x_0} - \bar{X}\right)\left(\frac{y_t}{y_0} - \bar{Y}\right) w_0 = \frac{\sum x_t y_t}{\sum x_0 y_0} - \bar{X}\cdot\bar{Y} = \bar{Y}(X_t - X_0)$

and the ratio of two additive indices

(1a) $\frac{X_t}{X_0} = 1 + r_{xy} C_x C_y = 1 + \frac{s_{xy}}{\bar{X}\cdot\bar{Y}}$ where $r_{xy} = \frac{s_{xy}}{s_x s_y}, C_x = \frac{s_x}{\bar{X}}$ and $C_y = \frac{s_y}{\bar{Y}}$ and

(1b) $\overline{\text{cov}}(x, y) = r_{xy} C_x C_y = \frac{s_{xy}}{\bar{X}\cdot\bar{Y}}$ is the centered covariance

*) The formula of $\bar{Y} = Y_0$ can be derived from $\bar{X} = X_0$ by interchanging x and y. In the same way we can derive Y_1 from X_1, so that $X_1/X_0 = Y_1/Y_0$

Now in view of fig. 1 we may substitute x- and y-vectors by prices and quantities as follows

01	$X_0 = \bar{X}$	X_1	x_0	x_t	y_0	y_t	w_{i0}	$\bar{Y}$
	P_{0t}^L	P_{0t}^P	p_{i0}	p_{it}	q_{i0}	q_{it}	$p_{i0}q_{i0}/\Sigma p_{i0}q_{i0}$	Q_{0t}^L

We then get according to fig. 1 for s_{xy} exactly the covariance cov as defined in (2) that is the covariance between price and quantity relatives weighted with base period expenditure shares $w_{i0} = p_{i0}q_{i0}/\Sigma p_{i0}q_{i0}$.

An alternative to (2) is (Siegel 1941a; 345, referring to Staehle for this result)

(2a) $$\text{cov}^* = \sum_{i=1}^{n}\left(\frac{p_{i0}}{p_{it}} - \frac{1}{P_{0t}^P}\right)\left(\frac{q_{i0}}{q_{it}} - \frac{1}{Q_{0t}^P}\right)\frac{p_{it}q_{it}}{\sum p_{it}q_{it}} = \frac{1}{Q_{0t}^P}\left(\frac{1}{P_{0t}^L} - \frac{1}{P_{0t}^P}\right)$$, so that

$$\frac{P_{0t}^P}{P_{0t}^L} = 1 + \frac{\text{cov}^*}{\left(P_{0t}^P\right)^{-1}\left(Q_{0t}^P\right)^{-1}}.$$

Another example is[1] a comparison between the Laspeyres and Walsh price index (the latter is defined as $P_{0t}^W = \frac{\sum p_t\sqrt{q_0 q_t}}{\sum p_0\sqrt{q_0 q_t}}$) where the elements x_0, x_t, y_0 and y_t may be defined as follows:

02	$X_0 = \overline{X}$	X_t	x_0	x_t	y_0	y_t	$\overline{Y}$
	P_{0t}^L	P_{0t}^W	p_0	p_t	q_0	$\sqrt{q_0 q_t}$	$\sum p_0\sqrt{q_0 q_t} / \sum p_0 q_0$

The relevant variances then are $\frac{x_t}{x_0} = \frac{p_t}{p_0}$ relative to the mean $\overline{X} = X_0 = P_{0t}^L$, and the variance of the $\frac{y_t}{y_0} = \sqrt{\frac{q_t}{q_0}}$ measured around $\overline{Y} = \sum\sqrt{\frac{q_t}{q_0}}\frac{p_0 q_0}{\sum p_0 q_0}$ so that the covariance then is given by $\text{cov} = \sum_{i=1}^{n}\left(\frac{p_t}{p_0} - P_{0t}^L\right)\left(\sqrt{\frac{q_t}{q_0}} - \overline{Y}\right)\frac{p_0 q_0}{\sum p_0 q_0} = \frac{\sum p_t\sqrt{q_0 q_t}}{\sum p_0 q_0} - \overline{X}\cdot\overline{Y}$ and

$$\overline{\text{cov}}(x, y) = \frac{\sum p_t\sqrt{q_0 q_t}}{\sum p_t q_0}\cdot\frac{\sum p_0\, q_0}{\sum p_0\sqrt{q_0 q_t}} = \frac{P_{0t}^W}{P_{0t}^L}$$

Thus the extent to which Walsh's index, P_{0t}^W is greater or smaller than Laspeyres' index, P_{0t}^L depends on the covariance between $\frac{p_t}{p_0}$ and $\sqrt{\frac{q_t}{q_0}}$. A consequence of this result is for example: if $\frac{p_t}{p_0}$ and $\frac{q_t}{q_0}$ are negatively correlated such that $P_{0t}^L > P_{0t}^P$ the same will be true for $\frac{p_t}{p_0}$ and $\sqrt{\frac{q_t}{q_0}}$ such that $P_{0t}^L > P_{0t}^W$. Thus not surprisingly we get: if $P_{0t}^L < P_{0t}^P$ then $P_{0t}^L < P_{0t}^W$ and if $P_{0t}^L > P_{0t}^P$ then also $P_{0t}^L > P_{0t}^W$.

2. Special cases of the general theorem

In order to find relationships between a weighted and an unweighted index number it is advisable to set one or two x or y variables equal to unity. It then turns out that the formulas given in fig. 1 are generally valid. For example upon setting $x_0 = y_0 = 1$ and thereby $w_0 = 1/n$ we get (as we do in general) $\overline{X} = X_0$ and $X_t = \frac{\frac{1}{n}\sum x_t y_t}{\overline{Y}}$ such that with $x_0 = y_0 = 1$ we end up with

(5) $$\text{cov} = s_{xy} = \sum\left(\frac{x_t}{x_0} - \overline{X}\right)\left(\frac{y_t}{y_0} - \overline{Y}\right)w_0 = \tfrac{1}{n}\sum\left(x_t - \overline{X}\right)\left(y_t - \overline{Y}\right)$$

[1] We henceforth leave out the subscript i to denote commodities over which the summation takes place. See also v. d. Lippe (2007), p. 195 for this particular example.

$$= \frac{\sum x_t y_t}{\sum x_0 y_0} - \overline{X} \cdot \overline{Y} = \overline{Y}\left(X_t - X_0\right)$$

Hence the "normal" formula for the (unweighted) covariance between x and y relatives is simply just a special case of Bortkiewicz's theorem. Using $\overline{X} = X_0$ we get

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Table 1: Some special variants of the generalized theorem of L. von Bortkiewicz on two additive indices

Taken from v. d. Lippe (2007), p. 196

$$\text{cov} = s_{xy} = \sum \left(\frac{x_t}{x_0} - \overline{X} \right)\left(\frac{y_t}{y_0} - \overline{Y} \right) w_0 = \overline{Y}\left(X_t - X_0\right)$$

Model	assumptions	$X_0 = \overline{X}$	X_t	$\overline{Y} = Y_0$	w_0
G	general	$\frac{\sum x_t y_0}{\sum x_0 y_0}$	$\frac{\sum x_t y_t}{\sum x_0 y_t}$	$\frac{\sum x_0 y_t}{\sum x_0 y_0}$	$\frac{x_0 y_0}{\sum x_0 y_0}$
A	$x_0 = 1$	$\frac{\sum x_t y_0}{\sum y_0}$	$\frac{\sum x_t y_t}{\sum y_t}$	$\frac{\sum y_t}{\sum y_0}$	$\frac{y_0}{\sum y_0}$
B	$x_t = 1$	$\frac{\sum y_0}{\sum x_0 y_0}$	$\frac{\sum y_t}{\sum x_0 y_t}$	$\frac{\sum x_0 y_t}{\sum x_0 y_0}$	$\frac{x_0 y_0}{\sum x_0 y_0}$
C	$y_0 = 1$	$\frac{\sum x_t}{\sum x_0}$	$\frac{\sum x_t y_t}{\sum x_0 y_t}$	$\frac{\sum x_0 y_t}{\sum x_0}$	$\frac{x_0}{\sum x_0}$
D	$y_t = 1$	$\frac{\sum x_t y_0}{\sum x_0 y_0}$	$\frac{\sum x_t}{\sum x_0}$	$\frac{\sum x_0}{\sum x_0 y_0}$	$\frac{x_0 y_0}{\sum x_0 y_0}$
E	$x_0 = y_0 = 1$	$\frac{\sum x_t}{n}$	$\frac{\sum x_t y_t}{\sum y_t}$	$\frac{\sum y_t}{n}$	$\frac{1}{n}$
F	$x_t = y_t = 1$	$\frac{\sum y_0}{\sum x_0 y_0}$	$\frac{n}{\sum x_0}$	$\frac{\sum x_0}{\sum x_0 y_0}$	$\frac{x_0 y_0}{\sum x_0 y_0}$

Strictly speaking the table is superfluous because all special cases (A through F) can easily be derived from G by setting certain x or y terms equal to unity. The table suggests that in many cases a choice among various models can be made when two indices are to be compared.

(5a) $\overline{\text{cov}} = \frac{X_t}{\overline{X}} - 1 = \frac{X_t}{X_0} - 1$.

3. Some examples

a) General theorem (model G)

In order to compare P_{0t}^{L} to the Marshall-Edgeworth index

(6) $P_{0t}^{ME} = \frac{\sum p_t \cdot \frac{1}{2}(q_0 + q_t)}{\sum p_0 \cdot \frac{1}{2}(q_0 + q_t)} = \frac{\sum p_t (q_0 + q_t)}{\sum p_0 (q_0 + q_t)}$

we proceed as indicated in row 3 of table 2. The index P_{0t}^{ME} can also be written as weighted arithmetic mean of P_{0t}^{L} and P_{0t}^{P}, viz. $P_{0t}^{ME} = \frac{1}{1+Q_{0t}^{L}} \cdot P_{0t}^{L} + \frac{Q_{0t}^{L}}{1+Q_{0t}^{L}} \cdot P_{0t}^{P}$ so that $\frac{P_{0t}^{ME}}{P_{0t}^{L}} = \frac{1+Q_{0t}^{P}}{1+Q_{0t}^{L}} = \frac{1+\lambda Q_{0t}^{L}}{1+Q_{0t}^{L}}$ where $\lambda = \frac{Q_{0t}^{P}}{Q_{0t}^{L}} = \frac{P_{0t}^{P}}{P_{0t}^{L}}$. Put otherwise $P_{0t}^{L} > P_{0t}^{P}$ (that is $\lambda < 1$) implies $P_{0t}^{L} > P_{0t}^{ME}$.

In the case of example 3 (row 3 of table 2) $\overline{Y} = \frac{\sum p_0(q_0+q_t)}{\sum p_0 q_0} = \sum \frac{q_0+q_t}{q_0} \frac{p_0 q_0}{\sum p_0 q_0} = 1 + Q_{0t}^{L}$ such that the relevant covariance is given by

$$\text{cov} = \sum \left(\frac{p_t}{p_0} - P_{0t}^{L}\right)\left(\frac{q_0+q_t}{q_0} - \overline{Y}\right)\frac{p_0 q_0}{\sum p_0 q_0} = \sum \left(\frac{p_t}{p_0} - P_{0t}^{L}\right)\left(\frac{q_t}{q_0} - Q_{0t}^{L}\right)\frac{p_0 q_0}{\sum p_0 q_0}.$$

This means that for comparing P_{0t}^{L} to P_{0t}^{ME} and P_{0t}^{L} to P_{0t}^{P} the result depends on the same covariance (as defined in eq. 2). It is again this covariance which also is involved in the comparison of P_{0t}^{L} (or P_{0t}^{P}) to Fisher's ideal index $P_{0t}^{F} = \sqrt{P_{0t}^{L} P_{0t}^{P}}$ because $\frac{P_{0t}^{F}}{P_{0t}^{L}} = \sqrt{\frac{P_{0t}^{P}}{P_{0t}^{L}}}$ and $\frac{P_{0t}^{F}}{P_{0t}^{P}} = 1/\sqrt{\frac{P_{0t}^{P}}{P_{0t}^{L}}}$.

Finally a simple function of this covariance is also in play when P_{0t}^{L} is compared to the following index[2]

(6a) $$P_{0t}^{DR*} = \tfrac{1}{2}(P_{0t}^{L} + P_{0t}^{P}).$$

As $\frac{P_{0t}^{DR*}}{P_{0t}^{L}} = \frac{1}{2}\left(1 + \frac{P_{0t}^{P}}{P_{0t}^{L}}\right)$ it follows: if $P_{0t}^{P} < P_{0t}^{L}$ then also $\frac{P_{0t}^{DR*}}{P_{0t}^{L}} <$ 1 and therefore $P_{0t}^{DR*} < P_{0t}^{L}$.

Table 2: Some examples for the general theorem

	$X_0 = \overline{X}$	X_t	x_0	x_t	y_0	y_t	$\overline{Y}$	w_0
3	P_{0t}^{L}	P_{0t}^{ME}	p_0	p_t	q_0	$q_0 + q_t$	see text above	$p_0 q_0/\Sigma p_0 q_0$
4	P_{0t}^{W}	P_{0t}^{ME}	p_0	p_t	$\sqrt{q_0 q_t}$	$(q_0 + q_t)/2$	see text below	see text below

The second example here (row 4) does not appear to be intuitively appealing because it may be difficult to find a meaningful interpretation for the "quantity relatives" $\frac{1}{2}(q_0 + q_t)/\sqrt{q_0 q_t}$ (which are ratios of an arithmetic and a geometric mean – over two periods – of quantities for each commodity $i = 1, \dots, n$ and therefore ≥ 1), nor appears $\overline{Y} = \frac{\frac{1}{2}\sum p_0 \cdot (q_0+q_t)}{\sum p_0 \sqrt{q_0 q_t}}$ to make much sense. However, the weights $w_0 = \frac{p_0 \cdot \sqrt{q_0 q_t}}{\sum p_0 \sqrt{q_0 q_t}}$ may clearly be viewed as expenditure shares for some fictitious (average) quantity.

b) x_0 or $x_t = 0$ (model A and B respectively)

As an alternative to example 1 we may compare P_{0t}^{L} to P_{0t}^{P} also as indicated in ex. 5 in the following table 3 where the critical covariance is

[2] It is another index of Drobisch in addition to the index P^{DR} which will be introduced shortly (eq. 7 below). Drobisch mentioned this index in Drobisch (1871), p. 425. It may be noted in passing that in this paper Drobisch was prepared to accept any kind of weighted arithmetic mean $\alpha P_{0t}^{L} + (1-\alpha)P_{0t}^{P}$, not only $\alpha = ½$. In the Anglo-American literature this index P^{DR*} is also known as index of Sidgwick – Bowley (Diewert (1997); p. 129).

$\text{cov} = \sum\left(\frac{p_t}{p_0} - P_{0t}^L\right)\left(\frac{1}{Q_{0t}^L}\frac{q_t}{q_0} - 1\right)\frac{p_0 q_0}{\sum p_0 q_0} = \frac{V_{0t}}{Q_{0t}^L} - P_{0t}^L = P_{0t}^P - P_{0t}^L.$

We get this also when we divide (2) by Q_{0t}^L.

Table 3: Some examples for the model A ($x_0 = 1$)

	$X_0 = \bar{X}$	X_t	x_0	x_t	y_0	y_t	$\bar{Y}$	w_0
5	P_{0t}^L	P_{0t}^P	1	p_t/p_0	$p_0q_0/\Sigma p_0q_0$	$p_0q_t/\Sigma p_0q_t$	1	$w_0 = y_0$
6a	$\bar{p}_t$	$\tilde{p}_t$	1	p_t	$1/n$	$q_t/\Sigma q_t$	1	$w_0 = y_0$
6b	$\bar{p}_0$	$\tilde{p}_0$	1	P_0	$1/n$	$q_0/\Sigma q_0$	1	$w_0 = y_0$

A bit more difficult appears at first glance, however, the comparison between Dutot's price index P_{0t}^D and the following index of Drobisch (example 6)

(7) $$P_{0t}^{DR} = \frac{\sum p_t q_t/\sum q_t}{\sum p_0 q_0/\sum q_0} = \frac{\tilde{p}_t}{\tilde{p}_0}.$$

By contrast to $P_{0t}^{DR*} = \frac{1}{2}(P_{0t}^L + P_{0t}^P)$ this index is much better known as an index suggested by Drobisch. However; unfortunately P_{0t}^{DR} is often called "unit value index". It is simply a ratio of two unit values $\tilde{p}_t$ and $\tilde{p}_0$.[3]

As a rule these two *quantity weighted* averages of prices are different from the unweighted averages $\bar{p}_t$ and $\bar{p}_0$ in $P_{0t}^D = \frac{1}{n}\sum p_t/\frac{1}{n}\sum p_0 = \bar{p}_t/\bar{p}_0$. Hence comparing P_{0t}^D to P_{0t}^{DR} boils down to comparing two kinds of average prices. This may be done in two steps: the first step (row 6a) results in the (numerator) covariance $c_n = \tilde{p}_t - \bar{p}_t$ and the second (row 6b) in the denominator covariance, which is $c_d = \tilde{p}_0 - \bar{p}_0$ so that we end up with

(7a) $$\frac{P_{0t}^{DR}}{P_{0t}^D} = \frac{1 + c_n/\bar{p}_t}{1 + c_d/\bar{p}_0}.$$

In a similar manner CSW 1980 derived a ratio with different covariances in numerator and denominator as an alternative to our eq. 8 (see below example 14).

c) y_0 or $y_t = 0$ (model C and D respectively)

We now make a comparison between P_{0t}^L and P_{0t}^{DR} using the fact that both indices are related to the value ratio (or value "index" $V_{0t} = \Sigma p_t q_t/\Sigma p_0 q_0$) as follows

- $P_{0t}^{DR} = \frac{V_{0t}}{Q_{0t}^D}$ where Q_{0t}^D is the quantity index of Dutot defined as $Q_{0t}^D = \frac{\Sigma q_t}{\Sigma q_0}$
- and P_{0t}^L can be written as $P_{0t}^L = V_{0t}/Q_{0t}^P$ so that

our ratio X_1/X_0 now is $\frac{P_{0t}^{DR}}{P_{0t}^L} = \frac{Q_{0t}^P}{Q_{0t}^D}$ so that a comparison between P_{0t}^L and P_{0t}^{DR} amounts to a comparison between Q_{0t}^P and Q_{0t}^D which is worked out as example 7.

We found in ex. 7 that $P_{0t}^{DR} = P_{0t}^L$ if $Q_{0t}^P = Q_{0t}^P$ or equivalently

(7b) $$Q_{0t}^P = \frac{\Sigma q_t p_t}{\Sigma q_0 p_t} = Q_{0t}^D = \frac{\Sigma q_t}{\Sigma q_0}$$

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[3] The problem is that unit values exist only for a group of homogeneous good. There is no "general" unit value over all goods, for the simple reason that for such a large aggregate die sum of quantities (Σq_t and (Σq_0) is not defined.

in which case the covariance vanishes. Given (7b) we see that in fact in the following definitional equation for $P_{0t}^{DR} = P_{0t}^{L}$ is true$\frac{\sum q_0 p_t}{\sum q_t p_t}$

$$P_{0t}^{DR} = \frac{\sum q_t p_t}{\sum q_0 p_0} : \frac{\sum q_t}{\sum q_0} = \frac{\sum q_t p_t}{\sum q_0 p_0} \frac{\sum q_0 p_t}{\sum q_t p_t} \qquad \text{using (7b)} \quad \frac{\sum q_0}{\sum q_t} = \frac{\sum q_0 p_t}{\sum q_t p_t} = \frac{\sum q_0 p_t}{\sum q_0 p_0} = P_{0t}^{L}.$$

Table 4: Some examples for the model C ($y_0 = 1$)

	$X_0 = \bar{X}$	X_t	x_0	x_t	y_0	y_t	$\bar{Y}$	w_0
7	Q_{0t}^{D}	Q_{0t}^{P}	q_0	q_t	1	p_t	$\frac{\sum p_t q_0}{\sum q_0} = \sum p_t \frac{q_0}{\sum q_0}$	$q_0/\Sigma q_0$
8	Q_{0t}^{D}	Q_{0t}^{L}	q_0	q_t	1	p_0	$\frac{\sum p_0 q_0}{\sum q_0} = \sum p_0 \frac{q_0}{\sum q_0} = \tilde{p}_0$	$q_0/\Sigma q_0$
9*	P_{0t}^{D}	P_{0t}^{L}	p_0	p_t	1	q_0	$\frac{\sum p_0 q_0}{\sum p_0} = \sum q_0 \frac{p_0}{\sum p_0} = \tilde{q}_0$	$p_0/\Sigma p_0$
10*	P_{0t}^{D}	P_{0t}^{P}	p_0	p_t	1	q_t	$\frac{\sum p_0 q_t}{\sum q_0} = \sum q_t \frac{p_0}{\sum p_0}$	$p_0/\Sigma p_0$

* see also examples 11 and 12 respectively

Note that the terms under $\bar{Y}$ can be viewed as weighted means of prices or quantities, referring either to t or to 0.

It may also be interesting to compare P_{0t}^{DR} to P_{0t}^{P} instead of P_{0t}^{L}. This means that we have to study the ratio $\frac{P_{0t}^{DR}}{P_{0t}^{P}} = \frac{Q_{0t}^{L}}{Q_{0t}^{D}}$ which is done in example 8.

The examples 9 and 10 may also be written in analogy to model D (see next table 5). This amounts to interchanging y_t and y_0 and as a consequence interchanging of X_0 and X_t. Also the weights w_0 and $\bar{Y}$ are affected when we move from model D to C.

Table 5: Some examples for the model D ($y_t = 1$)

	$X_0 = \bar{X}$	X_t	x_0	x_t	y_0	y_t	$\bar{Y}$	w_0
11	P_{0t}^{L}	P_{0t}^{D}	p_0	p_t	q_0	1	$\frac{\sum p_0}{\sum p_0 q_0}$	$p_0 q_0/\Sigma p_0 q_0$
12	P_{0t}^{P}	P_{0t}^{D}	p_0	p_t	q_t	1	$\frac{\sum p_0}{\sum p_0 q_t}$	$p_0 q_t/\Sigma p_0 q_t$

The terms under $\bar{Y}$ can be viewed as weighted means of reciprocal quantities, $1/q_0$ and $1/q_t$ respectively.

As to example 11 and 9 we find in CSW (1980), p. 19 the quite complicated formula (in our notation)

$$\text{(8)} \qquad \frac{P_{0t}^{L}}{P_{0t}^{D}} = \frac{1 + \overline{\text{cov}}(p_t, q_0)}{1 + \overline{\text{cov}}(p_0, q_0)},$$

using the unweighted covariances $\text{cov}(p_t, q_0) = \frac{1}{n}\Sigma(p_t - \bar{p}_t)(q_0 - \bar{q}_0)$ and $\text{cov}(p_0, q_0)$ defined analogously in which both averages, $\bar{p}$ and $\bar{q}$ are *unweighted* averages, while our less complicated formulas only needs *one*[4] covariance (between p_t/p_0 and . base period quantities q_0) *weighted*, however. The covariance in example 9 is

[4] with base period expenditure shares $p_0 q_0/\Sigma p_0 q_0$.

(8a) $\sum\left(\frac{p_t}{p_0}-P_{0t}^{D}\right)\left(q_0-\sum q_0\frac{p_0}{\sum p_0}\right)\frac{p_0}{\sum p_0}$ and in example 11

(8b) $\sum\left(\frac{p_t}{p_0}-P_{0t}^{L}\right)\left(\frac{1}{q_0}-\sum\frac{1}{q_0}\frac{p_0}{\sum p_0}\right)\frac{p_0q_0}{\sum p_0q_0}$.

This shows that there may well exist a number of different formulas for the relationship between the same two price indices. Again in the examples 10 and 12 the CSWD formula for comparing Paasche and Dutot

$$\text{(9)}\qquad \frac{P_{0t}^{P}}{P_{0t}^{D}}=\frac{1+\overline{\text{cov}}(p_t,q_t)}{1+\overline{\text{cov}}(p_0,q_t)}$$

based on two unweighted covariances (that is each product $(x-\bar{x})(y-\bar{y})$ is multiplied by 1/n),while our result is given by either

(9a) $\sum\left(\frac{p_t}{p_0}-P_{0t}^{D}\right)\left(q_t-\sum q_t\frac{p_0}{\sum p_0}\right)\frac{p_0}{\sum p_0}$ in example 10 or

(9b) $\sum\left(\frac{p_t}{p_0}-P_{0t}^{P}\right)\left(\frac{1}{q_t}-\sum\frac{1}{q_t}\frac{p_0}{\sum p_0}\right)\frac{p_0q_t}{\sum p_0q_t}$ in example 12

making use of one weighted covariance only. Note the striking resemblance between (9a) and (8a) on the one hand and (9b) and (8b) on the other.

We can also combine one of the formulas (8a) or (8b) to $\sqrt{\frac{P_{0t}^{L}}{P_{0t}^{D}}}$ with one of the formulas (9a) or (9b) for $\sqrt{\frac{P_{0t}^{P}}{P_{0t}^{D}}}$ in order to measure $\frac{P_{0t}^{F}}{P_{0t}^{D}}=\sqrt{\frac{P_{0t}^{L}}{P_{0t}^{D}}}\sqrt{\frac{P_{0t}^{P}}{P_{0t}^{D}}}$. For this task we find in CSW 1980; 31 a quite complicated expression using unweighted covariances only, viz.

$$\text{(9c)}\qquad \frac{P_{0t}^{F}}{P_{0t}^{D}}=\sqrt{\frac{1+\overline{\text{cov}}(p_t,q_0)}{1+\overline{\text{cov}}(p_0,q_0)}\cdot\frac{1+\overline{\text{cov}}(p_t,q_t)}{1+\overline{\text{cov}}(p_0,q_t)}}$$

with four $1+\overline{\text{cov}}$ terms involved, rather than only two. Note that the way how (9c) is composed of p and q terms bears some resemblance to $P_{0t}^{F}=\sqrt{\frac{\sum p_tq_0}{\sum p_0q_0}\frac{\sum p_tq_t}{\sum p_0q_t}}$.

c) $x_0 = y_0 = 1$, or $x_t = y_t = 0$ (model E and F respectively)

As an example (see row 13 in table 6 below) we compare the Dutot index $P_{0t}^{D}=\frac{\sum p_t}{\sum p_0}$ with the Carli index[5] given by

$$P_{0t}^{C}=\frac{1}{n}\sum\frac{p_{it}}{p_{i0}}.$$

For this reason we set $x_0 = y_0 = 1$, $x_t = p_t/p_0$ and $y_t = p_0/\Sigma p_0$. The result is shown in combination with some other comparisons in the following table 6.

Example 13 is particularly easy to understand. As usual $X_t = X_0$ holds when the covariance vanishes. The relevant covariance here is $\text{cov}=\frac{1}{n}\sum\left(\frac{p_t}{p_0}-P_{0t}^{C}\right)\left(\frac{p_0}{\sum p_0}-\frac{1}{n}\right)=$

[5] This index is also known as "Sauerbeck index". Laspeyres and some other authors in his days made extensively use of this formula (and also Sauerbeck's price statistics for British foreign trade). It was only in the 20th century that it became generally known that the formula originated from Giancarlo Carli.

$\frac{1}{n}(P_{0t}^{D} - P_{0t}^{C})$. When all ratios $\frac{p_{i0}}{\sum p_{i0}}$ are equal, viz. $\frac{p_{i0}}{\sum p_{i0}} = \frac{1}{n}$ then of course cov = 0 and $P_{0t}^{D} = \sum \frac{p_t}{p_0}\frac{p_0}{\sum p_0} = \sum \frac{p_t}{p_0}\frac{1}{n}$ reduces to P_{0t}^{C}.

Table 6: Some examples for the model E
(in all cases $w_0 = 1/n$)

	$X_0 = \bar{X}$	X_t	x_0	x_t	y_0	y_t	$\bar{Y}$
13*	P_{0t}^{C}	P_{0t}^{D}	1	p_t/p_0	1	$p_0/\Sigma p_0$	$1/n$
14*	P_{0t}^{C}	P_{0t}^{L}	1	p_t/p_0	1	$p_0q_0/\Sigma p_0q_0$	$1/n$
15	P_{0t}^{C}	P_{0t}^{P}	1	p_t/p_0	1	$p_0q_t/\Sigma p_0q_t$	$1/n$

* CSW (1980); p. 20 report the same formula

For CSW (1980), p. 27 there are good reasons to assume a negative correlation (between p_t/p_0 and $p_0/\Sigma p_0$) in the case of ex. 13, so for them $P_{0t}^{D} < P_{0t}^{C}$ should be fairly general the case.

In a similar vein in example 14 P_{0t}^{L} reduces to P_{0t}^{C} when the covariance cov $= \frac{1}{n}\sum\left(\frac{p_t}{p_0} - P_{0t}^{C}\right)\left(\frac{p_0q_0}{\sum p_0q_0} - \frac{1}{n}\right) = \frac{1}{n}(P_{0t}^{L} - P_{0t}^{C})$ vanishes, or put differently, when all base period expenditure shares are equal $(1/n)$[6] in which case of course also $Q_{0t}^{L} = Q_{0t}^{C}$.

Model E may also be used to find some relationships with the unweighted harmonic mean defined by $(P_{0t}^{H})^{-1} = \frac{1}{n}\sum\frac{p_{i0}}{p_{it}}$

Table 6 cont'd. ($w_0 = 1/n$)

	$X_0 = \bar{X}$	X_t	x_0	x_t	y_0	y_t	$\bar{Y}$
16	$1/P_{0t}^{H}$	$1/P_{0t}^{D}$	1	p_0/p_t	1	$p_t/\Sigma p_t$	$1/n = w_0$
17	$1/P_{0t}^{H}$	$1/P_{0t}^{L}$	1	p_0/p_t	1	p_tq_0	$\Sigma p_tq_0/n$
18	$1/P_{0t}^{H}$	$1/P_{0t}^{P}$	1	p_0/p_t	1	p_tq_t	$\Sigma p_tq_t/n$
19	P_{0t}^{C}	P_{0t}^{H}	1	p_t/p_0	1	p_0/p_t	$P_{t0}^{C} = 1/P_{0t}^{H}$

In 16 we get $X_1/X_0 = P_{0t}^{H}/P_{0t}^{D}$ and the covariance expressed in full is

$$\text{(10)} \qquad \text{cov} = \frac{1}{n}\sum\left(\frac{p_0}{p_t} - \frac{1}{P_{0t}^{H}}\right)\left(\frac{p_t}{\sum p_t} - \frac{1}{n}\right) = \frac{1}{n}\left(\frac{1}{P_{0t}^{D}} - \frac{1}{P_{0t}^{H}}\right) = \frac{1}{n}\left(\frac{P_{0t}^{H} - P_{0t}^{D}}{P_{0t}^{H}P_{0t}^{D}}\right)$$

thus cov < 0 entails $P_0^{H} < P_{0t}^{D}$. Alternatively with $x_t = p_t$ we get $\bar{Y} = \frac{\sum p_t}{n} = \bar{p}_t$ and therefore cov $= \frac{1}{n}\sum\left(\frac{p_0}{p_t} - \frac{1}{P_{0t}^{H}}\right)(p_t - \bar{p}_t) = \bar{p}_t\left(\frac{1}{P_{0t}^{D}} - \frac{1}{P_{0t}^{H}}\right)$.

It may also be interesting to compare Carli to the unweighted harmonic index which is done in example 19. From the general rule $\frac{X_t}{X_0} = \frac{X_t}{\bar{X}} = 1 + \frac{\text{cov}}{\bar{X}\,\bar{Y}}$ follows in this case

$\frac{P_{0t}^{H}}{P_{0t}^{C}} = P_{0t}^{H}P_{t0}^{H} = 1 + \frac{P_{0t}^{H} - P_{0t}^{C}}{P_{0t}^{C}} = \frac{P_{0t}^{H}}{P_{0t}^{C}} < 1.$

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[6] Already Drobisch was aware of this fact, when he criticized Laspeyres for his formula $P_{0t}^{L} = \sum p_tq_0/\sum p_0q_0$ (see Drobisch (1871); 423). As almost all other economists in these days Laspeyres used the formula P_{0t}^{C} not knowing that it was "invented" by Carli, and he developed his own formula (of which he never made much use) only in Laspeyres (1871), a paper Drobisch explicitly referred to.

This shows (in a quite simple manner) that both, Carli's index as well as the harmonic index fail the time reversal test (as $P_{0t}^{H}P_{t0}^{H} < 1$ and $P_{0t}^{C}P_{t0}^{C} > 1$).

Table 7 summarizes the 19 examples (indicating also the model used):

Table 7

	Carli	Dutot	Laspeyres	Paasche	Harmonic	Walsh	ME	Drobisch
Carli	-	13 E	14 E	15 E	19 E			
Dutot		-	9C / 11D	10C / 12D	16 E			6 A *
Lasp.			-	1 G / 5 A	17 E	2 G	3 G	7 C
Paasche				-	18 E			8 C
Harmon.					-			
Walsh						-	4 G	
ME							-	
Drobisch								-

* this is example 6a and 6b

It should not be too difficult to fill the gaps.

3. More functions of index formulas, e.g. the CSWD-index

We already examined some relations concerning Fisher's ideal index $P_{0t}^{F} = \sqrt{P_{0t}^{L}P_{0t}^{P}}$ that is the geometric mean of Laspeyres and Paasche and P_{0t}^{DR*}, the arithmetic mean of the same two indices. The following index

$$(11) \qquad P_{0t}^{CSWD} = \sqrt{P_{0t}^{C}P_{0t}^{H}} = \sqrt{\frac{\Sigma p_{it}/p_{io}}{\Sigma p_{io}/p_{it}}}$$

is known as index of Carruthers, Selwood, Ward and Dalen (or CSWD-index for short). Obviously $(P_{0t}^{H})^{-1} = \frac{1}{n}\Sigma\frac{p_{io}}{p_{it}} = P_{t0}^{C}$, or in Fisher's words P_{0t}^{H} is the "time antithesis" of P_{0t}^{C} and vice versa,[7] so

$$(12) \qquad \frac{P_{0t}^{CSWD}}{P_{0t}^{C}} = \sqrt{\frac{P_{0t}^{C}P_{0t}^{H}}{\left(P_{0t}^{C}\right)^{2}}} = \sqrt{\frac{P_{0t}^{H}}{P_{0t}^{C}}} = \left(\frac{P_{0t}^{CSWD}}{P_{0t}^{H}}\right)^{-1}$$

This means that example 19 enables us to compare a mixed index like P_{0t}^{CSWD} to one of its components, P_{0t}^{C} and P_{0t}^{H} respectively. The covariance in ex. 19 is given by

$\text{cov} = \frac{1}{n}\Sigma\left(\frac{p_t}{p_0} - P_{0t}^{C}\right)\left(\frac{p_0}{p_t} - P_{t0}^{C}\right) = 1 - P_{0t}^{C}P_{t0}^{C}$, and the centered covariance

$\overline{\text{cov}} = \frac{\text{cov}}{\bar{X}\,\bar{Y}} = \frac{1}{P_{0t}^{C}P_{t0}^{C}} - 1$ so that $\frac{P_{0t}^{CSWD}}{P_{0t}^{C}} = \sqrt{\frac{1}{P_{0t}^{C}P_{t0}^{C}}}$ and $\frac{P_{0t}^{CSWD}}{P_{0t}^{H}} = \sqrt{P_{0t}^{C}P_{t0}^{C}}$.

Finally it might be interesting to examine how P_{0t}^{CSWD} is related to P_{0t}^{D}. Using

[7] P_{0t}^{*} is the time antithesis of P_{0t} if $P_{0t}^{*} = (P_{t0})^{-1}$ (just like $P_{0t}^{H} = \left(P_{t0}^{C}\right)^{-1}$). A *geometric* mean of a pair of time antithetic indices as for example $P_{0t}^{CSWD} = \sqrt{P_{0t}^{C}P_{0t}^{H}}$ or $P_{0t}^{F} = \sqrt{P_{0t}^{L}P_{0t}^{P}}$ always satisfies the time reversal test

$$\frac{P_{0t}^{CSWD}}{P_{0t}^{D}} = \sqrt{\frac{P_{0t}^{C}}{P_{0t}^{D}}\frac{P_{0t}^{H}}{P_{0t}^{D}}} = \sqrt{f_1 f_2}.$$

Factor f_1 can be evaluated using ex. 13 and Factor f_2 with the help of ex. 16. Interchanging y_0 and y_1 in table 6 we get in the case of ex. 13 for f_1 the centered covariance (using $\bar{p}_0 = \sum p_0/n$ or $n\bar{p}_0 = \sum p_0$)

$$\overline{\text{cov}}_{(1)} = \sum\left(\frac{p_t/p_0}{P_{0t}^{D}} - 1\right)\left(\frac{\bar{p}_0}{p_0} - 1\right)\frac{p_0}{n\bar{p}_0},$$

or $\overline{\text{cov}}_{(1)} = \frac{P_{0t}^{C}}{P_{0t}^{D}} - 1 = f_1 - 1.$

We now consider factor $f_2 = P_{0t}^{H}/P_{0t}^{D}$ in a similar manner. For this purpose we are going back to ex. 16 where the relevant covariance is

$$\text{(9c)} \qquad \text{cov} = \frac{1}{n}\sum\left(\frac{p_0}{p_t} - P_{t0}^{C}\right)\left(\frac{p_t}{\sum p_t} - \frac{1}{n}\right) = \frac{1}{n}\left(\frac{P_{0t}^{H} - P_{0t}^{D}}{P_{0t}^{H}P_{0t}^{D}}\right),$$

from which we can easily derive

$$\overline{\text{cov}}_{(2)} = \frac{\text{cov}}{\frac{1}{P_{0t}^{H}}\cdot\frac{1}{n}} = \sum\left(\frac{p_0/p_t}{P_{t0}^{C}} - 1\right)\left(\frac{p_t}{\bar{p}_t} - 1\right)\frac{1}{n} = \frac{P_{0t}^{H}}{P_{0t}^{D}} - 1 = f_2 - 1$$

given the results for $\bar{X}$ and $\bar{Y}$ in ex. 16. We now can pull the strands together and conclude

$$\text{(13)} \qquad \frac{P_{0t}^{CSWD}}{P_{0t}^{D}} = \sqrt{\left(1 + \overline{\text{cov}}_{(1)}\right)\left(1 + \overline{\text{cov}}_{(2)}\right)}\ .$$

In order to compare P_{0t}^{CSWD} to Fisher's ideal index P_{0t}^{F} we again proceed in two steps, using ex. 14 for P_{0t}^{L}/P_{0t}^{C} and ex. 18 for P_{0t}^{P}/P_{0t}^{H} which results in

$$\text{(15)} \qquad \frac{1}{P_{0t}^{H}}\frac{P_{0t}^{F}}{P_{0t}^{CSWD}} = \sqrt{\frac{1 + \frac{P_{0t}^{L}}{n}\sum\left(\frac{p_t}{p_0} - P_{0t}^{L}\right)\left(\frac{p_0 q_0}{\sum p_0 q_0} - \frac{1}{n}\right)\frac{1}{n}}{1 + \frac{\overline{p_t q_t}}{P_{0t}^{H}}\sum\left(\frac{p_0}{p_t} - \frac{1}{P_{0t}^{H}}\right)\left(p_t q_t - \overline{p_t q_t}\right)\frac{1}{n}}} = \sqrt{\frac{1 + \overline{\text{cov}}(\frac{p_t}{p_0}, w_0)}{1 + \overline{\text{cov}}(\frac{p_0}{p_t}, w_t)}}$$

where $\overline{p_t q_t} = \frac{1}{n}\sum p_t q_t$ and $w_0 = \frac{p_0 q_0}{\sum p_0 q_0}$, $w_t = \frac{p_t q_t}{\sum p_t q_t}$, and this is precisely the same result which was derived by CSW (1980), p.31. who only made use of eq. (5) rather than the (generalized) Bortkiewicz theorem as exhibited in figure 1.

A final remark to P^{CSWD} (or $\sqrt{RH}$ in the notation of CSW)[8] may be in order: it is well known that the *geometric* mean of an index and its time antithesis will meet the time reversal test. This applies to $P_{0t}^{CSWD} = \sqrt{P_{0t}^{C}P_{0t}^{H}}$ or to $P_{0t}^{F} = \sqrt{P_{0t}^{L}P_{0t}^{P}}$, but of course it does not apply the *arithmetic* mean, that is to $\frac{1}{2}(P_{0t}^{C}+P_{0t}^{H})$ nor to $P_{0t}^{DR*} = \frac{1}{2}(P_{0t}^{L}+P_{0t}^{P})$.

4. Some additional remarks

Finally it appears useful to (once more) emphasize that firstly the relationship between any two index functions can possibly be expressed in a number of different (though after

[8] R stands for Carli's index

second thoughts equivalent) ways and secondly that the "message" of the somewhat abstract equations with covariances might not easily be grasped, and we therefore should give some thoughts to enhance understandability.

1. In Diewert and v. d. Lippe (2010) a number of bias formulas between two indices X_1 and X_0 were derived without reference to Bortkiewicz's theorem. We define

   bias $= [X_1/X_0] - 1 = \overline{\text{cov}}(x,y) = \frac{\text{cov}(x,y)}{\bar{X}\cdot\bar{Y}}$

   and found some biases between the Drobisch price index P_{0t}^{DR} and the price indices of Laspeyres (as in our example 7) and Paasche (ex. 8)[9]

   It may be useful, to introduce a simplified notation for the covariance[10]: in our ex. 7 $\text{cov}(q_t/q_0, p_t, q_0/\Sigma q_0)$ denotes our result $\sum\left(\frac{q_t}{q_0} - Q_{0t}^D\right)(p_t - \tilde{p}_t^*)\frac{q_0}{\sum q_0}$ (where $\tilde{p}_t^*$ is defined as $\tilde{p}_t^* = \frac{\sum p_t q_0}{\sum q_0}$. Now in Diewert and v. d. Lippe we find the following alternative covariances[11]

   $\text{cov}(p_t, q_t/\Sigma q_t - q_0/\Sigma q_0, 1/n)$

   $\text{cov}(p_t, (q_0/q_t)Q_{0t}^D, q_t/\Sigma q_t)$ and

   $\text{cov}(p_t, q_0/q_t - 1/Q_{0t}^D, q_t/\Sigma q_t)$.

   It may bewilder, but all four covariances boil down to the same relationship, and they all can be traced back to Bortkiewicz's theorem[12] (although they were developed without recourse to this formula). So we not only have a variety of formulas to describe basically the same thing, it may also be difficult to see how they are related to one another.

   This of course applies also to our ex. 8 where P_{0t}^{DR} is compared to P_{0t}^{P}

   $\text{cov}(q_t/q_0, p_0, q_0/\Sigma q_0) = \sum\left(\frac{q_t}{q_0} - Q_{0t}^D\right)(p_0 - \tilde{p}_0)\frac{q_0}{\sum q_0}$;

   this result in can also be expressed as[13]

   $\text{cov}(p_0, q_t/\Sigma q_t - q_0/\Sigma q_0, 1/n)$, or[14]

   $\text{cov}(p_0, \frac{q_t/\sum q_t}{q_0/\sum q_0} - 1, q_0/\Sigma q_0)$ and

   $\text{cov}(p_0, q_t/q_0, p_0, q_0/\Sigma q_0)$

   and they all can be identified as special cases of Bortkiewicz's formula and describe the same relationship, only in slightly different terms.

2. It is certainly a challenge to find good, intuitively appealing interpretations to such results and the underlying equations of the generalized theorem of von Bortkewicz

[9] We refrain from presenting here the corresponding bias- formulas between Drobisch and Laspeyres (according to our example 7)

[10] The rule should be cov(x-variable, y-variable, weights).

[11] These are equations 22, 25 and 29 in Diewert and v. d. Lippe (2010).

[12] I have shown this in v.d.Lippe (2010).

[13] Equations 13, 16 and 20 in Diewert and v. d. Lippe (2010).

[14] For this we gave the following verbal interpretation: "Thus the Drobisch index will have an upward bias relative to the Paasche index if products … whose quantity shares are growing … are associated with period 0 prices … which are above the arithmetic average of the period 0 prices" (p. 693). Note that with weights 1/n the mean of p_{i0} prices is $\bar{p}_t = \frac{1}{n}\sum p_{it}$ rather than $\tilde{p}_t = \sum p_{it} q_{it}/\sum q_{it}$.

which proved so widely applicable. Yet the results of such endeavours attained so far are not very promising. We present some ideas of the Hungarian statistician Pal Köves (1983), who in great detail dealt with Bortkiewicz's formulas (2) and (4), however, not with the generalization of the theorem. Köves introduced the ratio of two price indices X_1/X_0 which he called B in the honour of Bortkiewicz.[15] He made an attempt to interpret B – 1 (what we called "centered covariance") in terms of the elasticities and the slope of a regression of q_t/q_0 (dependent variable) on p_t/p_0 as regressor. It can easily be seen that for example

$$B = \frac{P_{0t}^{P}}{P_{0t}^{L}} = \frac{Q_{0t}^{P}}{Q_{0t}^{L}}\ ,\ \frac{P_{0t}^{F}}{P_{0t}^{L}} = \sqrt{B},\ \frac{P_{0t}^{DR*}}{P_{0t}^{L}} = \frac{1}{2}(1+B), \text{and}\ \frac{P_{0t}^{ME}}{P_{0t}^{L}} = \frac{q+B}{q+1} \quad (q = \frac{1}{Q_{0t}^{L}}).$$

Another concept, Köves introduced was the "factor quotient index" (Köves 1983; 93) which may be denoted by Φ. It turns out that Φ, defined as the ratio of a price indices and the corresponding quantity index is the same in the case of quite a few index functions: $\Phi = \frac{P_{0t}^{P}}{Q_{0t}^{P}} = \frac{P_{0t}^{L}}{Q_{0t}^{L}} = \frac{P_{0t}^{DR*}}{Q_{0t}^{DR*}}$ where $Q_{0t}^{DR*} = \frac{1}{2}(Q_{0t}^{L} + Q_{0t}^{P})$.

It seems doubtful, however, whether further proceeding along this kind of reasoning will really provide any new insights.

3. In Siegel 1941b we find a presentation of the difference between two linear indices $X_1 - X_0$ in the form of a determinant. Assume $X_0 = \sum x_i w_{i0}$ and $X_t = \sum x_i w_{it}$ where $x_i = p_{it}/p_{i0}$, $w_{i0} = p_{i0}q_{i0}/\Sigma p_{i0}q_{i0}$, and $w_{it} = p_{it}q_{it}/\Sigma p_{it}q_{it}$ then

$$\begin{bmatrix} w_{1t} & \cdots & w_{nt} \\ w_{10} & \cdots & w_{nt} \end{bmatrix} \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} = \begin{bmatrix} X_t & 1 \\ X_0 & 1 \end{bmatrix} = \mathbf{P}$$ and X_t - X_0 (which is $P_{0t}^{P} - P_{0t}^{L}$ with the x_i, w_{it}, and w_{i0} variable as defined above) is given as determinant $|\mathbf{P}|$. This may be interesting for some further generalizations of Bortkiewicz's theorem.

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15 Thus B is the bias of X_1 (relative to X_0) plus one.

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