A Note on Stability of Self-Consistent Equilibrium in an Asynchronous Model of Discrete-Choice with Social Interaction

Kaizoji, Taisei

Graduate School of Arts and Sciences, International Christian University

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Taisei Kaizoji[1]

Abstract

The aim of this paper is to demonstrate that dynamic paths in a model of discrete choice with social interactions, which have been developed by Brock and Durlauf (1999, 2001a, 2001b, 2006), converge some self-consistent equilibrium. To this aim, we propose an asynchronous model of discrete-choice with social interaction[2], in which the only individual selected cyclically is updated.

1. The model of discrete-choice with social interaction

We consider the model of discrete choice with social interaction proposed by Brock and Durlauf (1999, 2001a, 2001b, 2006). Each of I agents faces a binary choice. These choices are denoted by an indicator variable ω_i which has support $\{-1,1\}$. Agent i makes a choice in order to maximize a utility function,

$$U(\omega_i, \mu_i^e(\omega_{-i}), \varepsilon_i(\omega_i)) = u(\omega_i) + S(\omega_i, \mu_i^e(\omega_{-i})) + \varepsilon(\omega_i) \qquad (1)$$

where $\omega_{-i} = (\omega_1, ..., \omega_{i-1}, \omega_{i+1}, ..., \omega_I)$ denotes the vector of choices other than that of agent i and $\mu_i^e(\omega_{-i})$ denotes that individual's anticipations concerning the choices of other agents. The three terms which construct the utility function are named respectively the following: $u(\omega_i)$ is deterministic private utility, $S(\omega_i, \mu_i^e(\omega_{-i}))$ is deterministic social utility, and $\varepsilon(\omega_i)$ is random private utility. The deterministic social utility represents a conformity effect,

1 Taisei Kaizoji, International Christian University, 3-10-2 Osawa, Mitaka, Tokyo 181-8585 Japan. Email: kaizoji@icu.ac.jp

2 The asynchronous model is formulated in context of Neural Networks by Peterson, C. and Anderson, J. R., (1987) .

$$S\left(\omega_i, \mu_i^e\left(\omega_{-i}\right)\right) = -E_i \sum_{j \neq i} \frac{J_{i,j}}{2}\left(\omega_i - \omega_j\right)^2 . \qquad (2$$

The term $\frac{J_{i,j}}{2}$ represents the interaction weight which relates agent *i*'s choice to agent *j*'s choice expected by agent *i*, and is assumed to be equal to the cross-partial derivative of the social utility function,

$$J_{i,j} = \frac{\partial^2 S\left(\omega_i, \mu_i^e\left(\omega_{-i}\right)\right)}{\partial \omega_i \partial E_i\left(\omega_j\right)} \qquad (3)$$

which means that the function measures the strategic complementarity (and substitutability) between agent *i*'s choice and the expected choices of agent *j*.

It is assumed that the private deterministic utility function of agent *i* with a linear function,

$$u(\omega_i) = h\omega_i + k \qquad (4)$$

where h and k are chosen so that $h + k = u(1)$ and $-h + k = u(-1)$.

The random private utility $\varepsilon(\omega_i)$ is independent and extreme value distributed both within and across agents[3]. This means that the difference between the unobservable components is logistically distributed

$$\chi(\varepsilon(-1) - \varepsilon(1) \leq z) = \frac{1}{1 + \exp(-\beta z)}; \quad \beta \geq 0. \text{ d}, \qquad (5)$$

where $\chi(\cdot)$ denotes probability measures.

Under the above assumptions, agent *i*'s choice will obey the probability

$$\Pr(\omega_i) = \frac{\exp\left(\beta(u(\omega_i) + \sum_{i \neq j} J_{i,j}\omega_i E_i(\omega_j))\right)}{\sum_{\nu_i \in \{-1,1\}} \exp\left(\beta(u(\nu_i) + \sum_{i \neq j} J_{i,j}\omega_i E_i(\omega_j))\right)} \qquad (6).$$

The expected value of agent *i*'s choice, which is conditional on his anticipations concerning the

[3] See McFaden (1984).

behavior of others, can be written as

$$E(\omega_i) = \tanh(\beta h_i + \beta \sum_{j \neq i} J_{i,j} E_i(\omega_j)) \qquad (7)$$

To close the model, Brock and Durlauf (2000, 2001) considers a case of this model occurs when all of the agents possess rational expectations, i.e.

$$E_i(\omega_j) = E(\omega_j). \qquad (8)$$

Under the assumption of expectation formations, the subjective expectations can be replaced with their objective expecations, i.e.

$$E(\omega_i) = \tanh(\beta h_i + \beta \sum_{j \neq i} J_{i,j} E(\omega_j)) \quad (i = 1, 2, ..., I). \qquad (9)$$

These equations represent a continuous mapping of $C = (-1,1)^I \to C$. Therefore, it is clear from Brouwer's fixed point theorem that there is at least one fixed point solution, which implies Theorem 1. (see Brock and Durlauf (1999, 2001a, 2001b)).

Theorem 1. *Existence of self-consistent equilibria*

There exists at least one set of self-consistent equilibrium consistent with the binary choice model with interactions as specified by equation (9).

2. Dynamics

In order to consider the dynamic stability of the self-consistent equilibrium in the discrete-choice model from the point of view of dynamical system theory, we consider an asynchronous discrete-choice model in which the only individual selected cyclically is updated.

2.1. The synchronous discrete-choice model.

For simplicity of the description of the asynchronous discrete-choice model, let a mapping

$f : C = (-1,1)^n \to C$ be defined by

$$f_i(x) = \tanh\left(\beta \left(\sum_{j=1}^{n} J_{i,j} x_j(t) + h_i \right) \right) \quad (i = 1,...,n) \qquad (10)$$

where $x_i = E(\omega_i), \quad x = (x_1, x_2, ..., x_n)$, and, $f(x) =^t (f_1(x),..., f_n(x))$.

In order to consider the asynchronous discrete-choice model from the point of view of dynamical system theory, we define a mapping $f : C \to C$ as follows.

$$\tilde{f}_1(x) = f_1(x_1, x_2, ..., x_n)$$

$$\tilde{f}_2(x) = f_2(\tilde{f}_1(x), x_2, ..., x_n)$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$\tilde{f}_n(x) = f_n(\tilde{f}_1(x), \tilde{f}_2(x), ..., \tilde{f}_{n-1}(x), x_n)$$

and

$$\tilde{f}(x) =^t (\tilde{f}_1(x), ..., \tilde{f}_n(x)).$$

Then, the asynchronous discrete-choice model can be represented by

$$x(t+1) = \tilde{f}(x(t)) \quad (t = 0,1,2,...). \qquad (11)$$

2.2. Stability of self-consistent equilibrium

We assume that

$$J_{i,j} = J_{j,i}. \tag{12}$$

That is, the interactions which are driven by expectations of the behavior of others are symmetry, and no self-connection (i.e. $J_{i,i} = 0$).

Under these conditions, the following lemma 1 and Theorem 2 are demonstrated. The proofs are due to essentially Fleisher (1988) and Kurita and Funahashi (1996).

Lemma 1. *Assume that a matrix J of pairwise interactions between each individual choice and the expected choices of others are symmetric, that is,* $J_{i,j} = J_{j,i}$*, and no self-connection (i.e.* $J_{i,i} = 0$*). We define a Liapunov function,*

$$V(x) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} J_{i,j}x_i x_j - \sum_{i=1}^{\mathrm{n}} h_i x_i + \frac{1}{\beta}\sum_{i=1}^{n}\int_0^{x_i}\sigma^{-1}(x)dx \tag{13}$$

where $\sigma(x) = \tanh(x)$*. Then, the Liapunov function () of the synchronous discrete-choice model with the weight matrix* $J_{i,j}$ *decreases monotonically. If* $x' = \tilde{f}(x)$ *and* $x' \neq x$*, then* $V(x') < V(x)$*, when* $\tilde{f} : C \to C$ *is a mapping which defines the asynchronous discrete-choice model.*

Theorem 2 (Dynamic stability). *The state of the asynchronous discrete-choice model converges to some self-consistent equilibrium.*

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4. Appendix

The proof of Lemma 1 is given by Lemma 9 in Kurita and Funahashi (1996).

Proof of lemma 1 (Kurita and Funahashi (1996)). First, we prove the first assertion. We remember that

$$V(x) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} J_{i,j}x_i x_j - \sum_{i=1}^{n} h_i x_i + \frac{1}{\beta}\sum_{i=1}^{n}\int_0^{x_i}\sigma^{-1}(u)du.$$

We set

$$g(x_i) = \frac{1}{\beta}\int_0^{x_i}\sigma^{-1}(u)du.$$

Suppose that x_i changed to $x'_k = x_k + \Delta x_k$, the resulting change in V is given by

$$\Delta V = -\Delta x_k \left\{ \sum_{i=1}^{n} J_{k,j}x_j + h_k - \frac{g(x_k + \Delta x_k) - g(x_k)}{\Delta x_k} \right\}.$$

Applying the intermediate value theorem to the function $g(x)$, we get

$$\begin{aligned} \Delta V &= -\Delta x_k \left\{ \sum_{i=1}^{n} J_{k,j}x_j + h_k - g'_k(\xi) \right\} \\ &= -\frac{1}{\beta}\Delta x_k \left\{ \sigma^{-1}(x_k + \Delta x_k) - \sigma^{-1}(\xi) \right\}, \end{aligned}$$

where ξ is a point between x_k and $x_k + \Delta x_k$. Between both $g(x)$ and $\sigma^{-1}(x)$ are strictly increasing functions, if $\Delta x_k > 0$ then $x_k < \xi < x_k + \Delta x_k$ and this implies

$$\sigma^{-1}(\xi) < \sigma^{-1}(x_k + \Delta x_k)$$

and hence $\Delta V < 0$. A similar argument holds for the case $\Delta x_k < 0$. Of course $\Delta x_k = 0$ implies $\Delta V = 0$.

Second, we prove the last part of the lemma. From the first assertion, there is no periodic point in the asynchronous model. Hence if $x' = \tilde{f}(x)$ and $x' \neq x$, then $V(x') < V(x)$.

Q.E.D.

The proof of Theorem 2 is given by Theorem 5 in Kurita and Funahashi (1996).

Proof of Theorem 2 (Kurita and Funahashi (1996)).

We take any initial point $x \in C = (-1,1)^n$. We remark that $\bar{f}(C)$ is a relatively compact subset of C because $\sigma : R \to (-1,1)$ is a bounded function. We consider the ω-limit set of x:

$$\omega_{\bar{f}}(x) = \left\{ y \in C \,|\, \textit{there is a sequence } n_i \to \infty \textit{ such that } \bar{f}^{n_i}(x) \to y \right\}.$$

As $\bar{f}(C)$ is a relatively compact subset of $\omega_{\bar{h}}(x)$ is a nonempty relatively compact subset of C. Suppose that $\omega(x)$ contains at least two points p and q $(p \neq q)$. From the definition and the above lemma, we easily see that the Liapunov function $V(x)$ constant on $\omega_{\bar{f}}(x)$. Hence, for any $\varepsilon > 0$, there are two integers N_1 and N_2 such that $N_1 > N_2$,

$$\left| \bar{f}^{N_1}(x) - p \right| < \varepsilon \text{, and } \left| \bar{f}^{N_2}(x) - q \right| < \varepsilon .$$

Because $V(\bar{f}^{N_2}(x)) < V(\bar{f}^{N_1}(x))$, this implies $V(q) < V(p)$ from the above lemma and this is a contradiction. Therefore $\omega_{\bar{f}}(x)$ contains only one point. Q.E.D.

5. References

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