Insurance portfolio risk aggregation and solvency capital computation with mathematical copula techniques

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Insurance Portfolio Risk Aggregation – a practitioner's view

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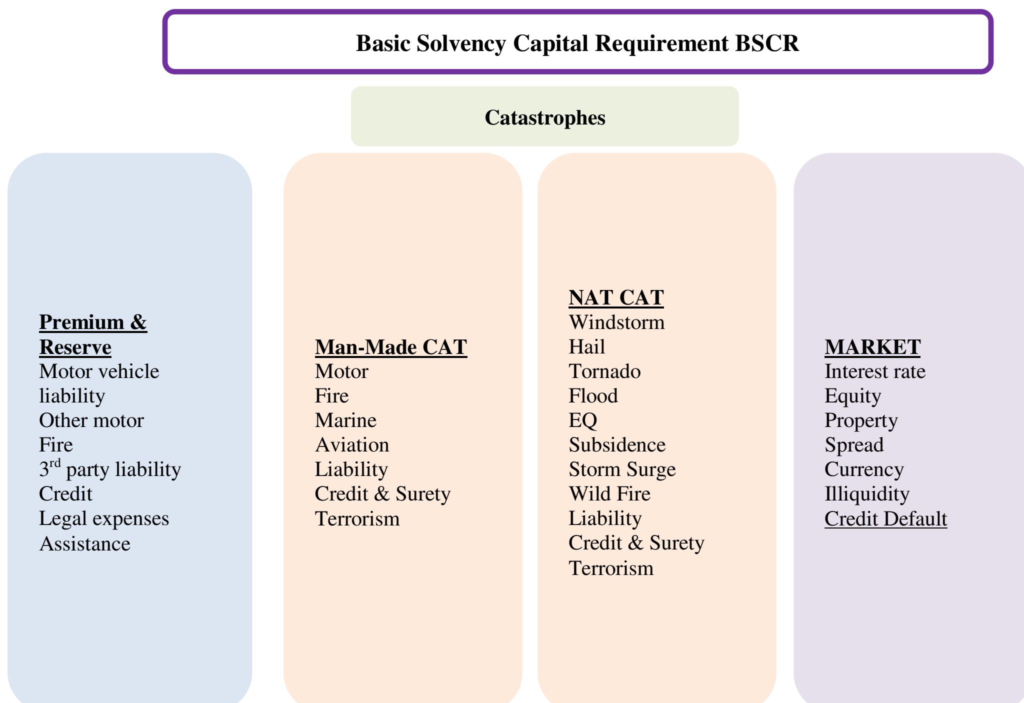
Illustration of a typical hierarchical tree aggregation structure in a non-life LOB or firm



1. Portfolio risk factor identification and mapping

	Risk Factor Type				
	Property / CAT	Casualty / CAT	Extreme Mortality	Market Price	Credit Default
Property LOB	125 individual risks policies	125 individual W.C. policies	none	holdings of 63 named variable price securities	holdings of 63 fixed income securities
Life LOB	none	none	2 blocks of 125 individual life insurance policies	holdings of 62 named variable price securities	holdings of 62 fixed income securities

Total Exposed Liabilities – risk exposure - for our portfolio:

physical liability	125,000,000 of *Property policy* holdings 125,000,000 of *Workers Compensation* policy holdings 250,000,000 of *Life Insurance* policy holdings
market & credit liability	125,000,000 of *variable price securities* holdings 125,000,000 of *fixed income securities* at 3.1% ARP

No holdings of re/insurance treaties – pure 'cash' insurance risk reserve

Treatment of one year forward Premium risk in NAT CAT P&C and Life

We assume fixed constant premium receivable for one year ahead. Instead of traditional P&L, example:

$$Propoerty\ CAT\ P\&L = \ \ MAX\{Loss_{ep=0.0001-0.9999} - Fixed\ Premium, 0\}$$

We have:

$Fixed\ Premium\ \rightarrow$
$Transaction\ and\ Operational\ Cost +$
$transfer\ to\ Variable\ Price\ Security\ Reserves$

We keep loss aggregation independent of premium levels and risk

Risk aggregation in Property and Casualty LOB

Modeled individual 125 risks, each with insured value of 1M USD, annual loss distributions with an AIR CAT model

risk #	E.V	2.00%	1.00%	0.90%	0.80%	0.70%	....	0.06%	0.05%	0.04%	0.03%	0.02%	0.01%
1	0.0374%	0.2497%	0.4592%	0.6730%	0.8862%	1.0967%	....	3.1842%	3.3921%	3.5973%	3.7998%	4.0063%	4.2124%
2	0.0371%	0.2481%	0.4563%	0.6687%	0.8807%	1.0898%	....	3.1644%	3.3710%	3.5749%	3.7762%	3.9815%	4.1864%
3	0.0372%	0.2488%	0.4576%	0.6706%	0.8831%	1.0929%	....	3.1731%	3.3802%	3.5847%	3.7865%	3.9924%	4.1978%
....	....	....	....	....	....	....	....	....	....	....	....	....	....
124	0.0371%	0.2480%	0.4561%	0.6684%	0.8802%	1.0893%	....	3.1628%	3.3693%	3.5732%	3.7743%	3.9795%	4.1843%
125	0.0376%	0.2509%	0.4615%	0.6763%	0.8906%	1.1021%	....	3.1997%	3.4086%	3.6147%	3.8182%	4.0257%	4.2328%

Semi - Simulated a **125X125** covariance matrix, with some input from stochastic losses, and computed a correlation matrix:

$$\sum \rho[i,j] = \frac{\sum COV[i:125, j:125]}{\sum \sigma(i:125)\sigma(j:125)}$$

Covariance / Correlation matrix estimation → suggested methodologies

(a) Stochastic modeled losses – equivalent to implied correlations – unstable – not suitable

(b) Historical events policy claims data – insufficient and sparse historical data – unreliable

(c') From modeled and historical hazard intensities and physical parameters – wind speed, EQ magnitudes, flood metrics – complex, lack of industry consensus, lack of professional expertise

Critique: Dependence Function → Linear Correlation – simplistic; does not fully allow to model tail dependencies of the marginal distributions
Suitable for elliptic copulas – Gaussian and Student-t - Extreme events i.e. dependence of marginal EVD's will be underestimated by linear correlation.

The Gaussian copula simulation algorithm

1. Model individual risk distributions with Normal PDF – invert probability mass to uniform quantiles / variates
2. Simulate Uniform *variates* - $[U_1, \ldots, U_N]$; convert them to Standard Normal *quantiles* $[Z_1, \ldots, Z_N]$; from $N(0,1)$
3. Apply Cholesky decomposition on the computed correlation matrix $\sum \rho[i,j] = LL'$
4. Apply vector $L\ to\ [Z_1, \ldots, Z_N]$ to create a *correlated standard normal space* $N_d\{0, \sum \rho[i,j]\}$

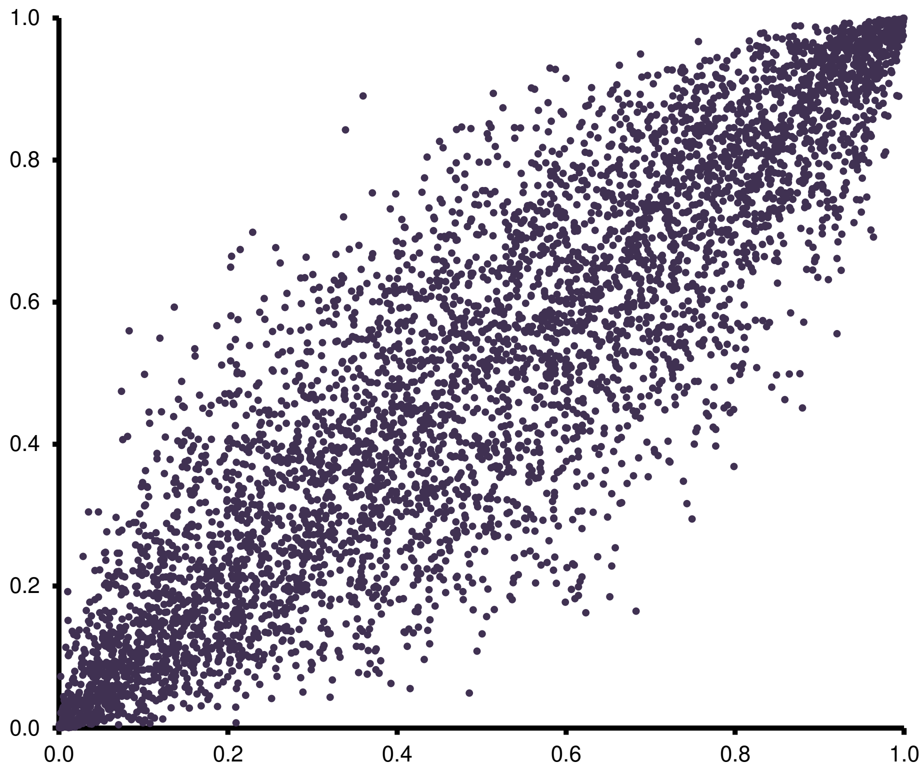
5. For every quintile of the space $N_d\{0, \sum \rho[i,j]\} \rightarrow [Z_1^\rho, \dots, Z_N^\rho]$ compute the standard normal cumulative density $\left[N\left(Z_1^\rho\right), \dots N\left(Z_N^\rho\right)\right] = [Q_1, \dots Q_N]$
6. Lastly to obtain tail metrics and quantiles from the empirical loss function we construct the empirical inverse distribution function $F = [f_1(Q_1), \dots, f_N(Q_N)]$, where $[f_1, \dots, f_N]$ are the empirical Gaussian loss distributions for the LOB: The risk metric becomes $F_\alpha = SCR_\alpha$

Computed property NAT CAT risk $SCR_{\alpha-PCR}$

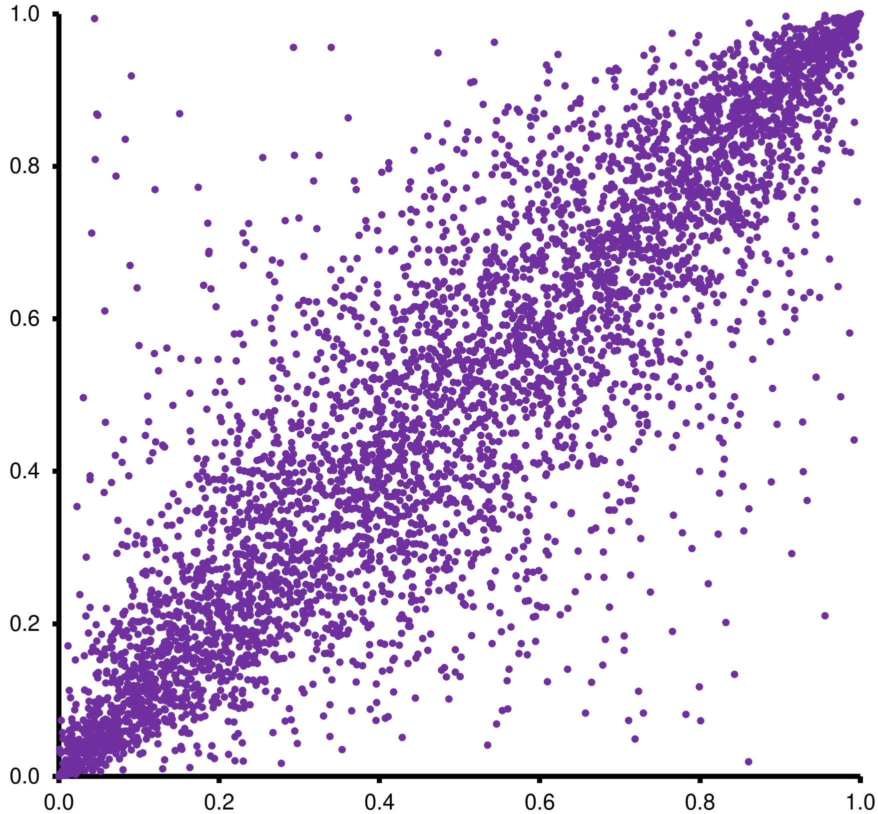
SCR	α = 99.5%	α = 97%	α = 95%
Gaussian	1,185,543	1,170,743	1,163,288
Student-t	1,185,467	1,176,791	1,171,502

Elliptical dependence structure of two randomly selected variables from the space $N_d\{0, \sum \rho[i,j]\} \rightarrow [Z_1^\rho, \dots, Z_N^\rho]$ with the standard normal cumulative densities $\left[N\left(Z_1^\rho\right), \dots N\left(Z_N^\rho\right)\right] = [Q_1, \dots Q_N]$

Gaussian copula



Student-t copula



Critique: very weak tail dependence for marginal functions with elliptical copulas. More research is needed in parameter estimation and fitting of EVT or Archimedean copulas – theoretically more suited for modeling extreme events.
Equivalent to multivariate distributions – assume that all marginal are of the same type

We model our 125, individual 1M USD workers compensations policies, with an AIR CAT model.

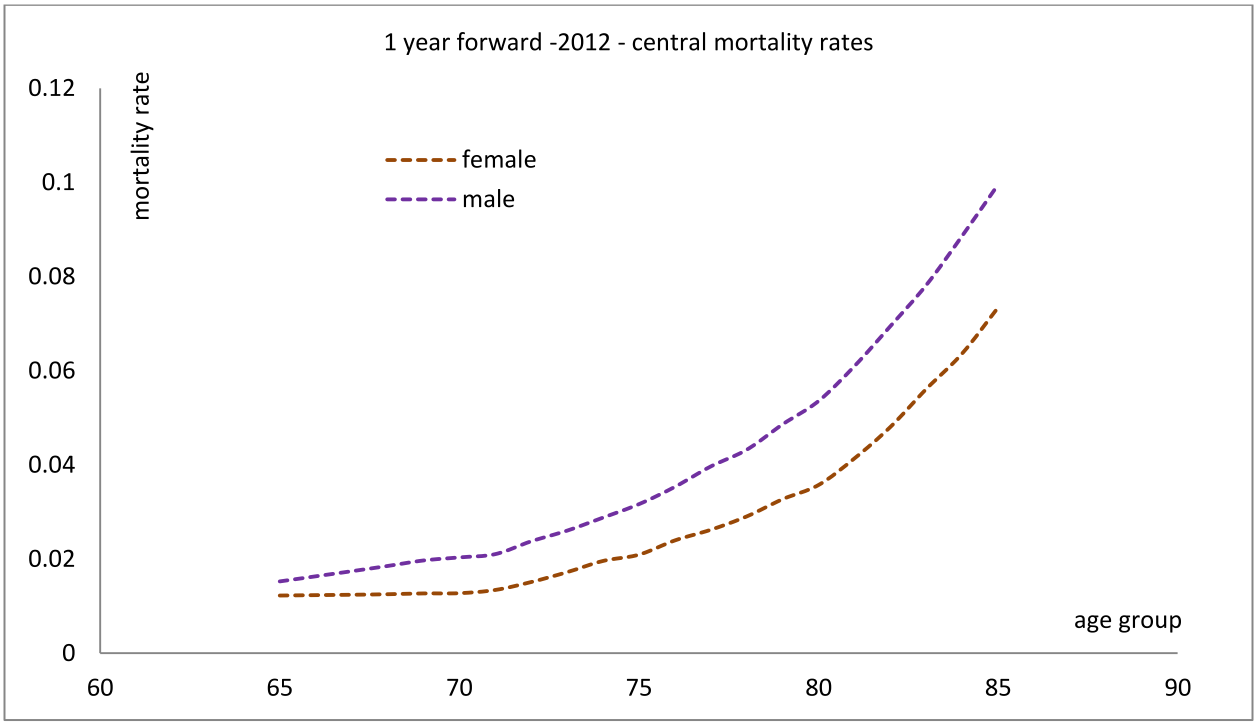
Computed $SCR_{\alpha-WCR}$

SCR	α = 99.5%	α = 97%	α = 95%
Gaussian	985,231	931,935	902,831
Student-t	989,832	932,897	901,886

Life LOB – life insurance excess mortality risk factor

Identify excess mortality to our 2 blocks – male and female – of 250 1M USD, life insurance policy holders as the single risk factor for this LOB. Our population exposure is in the *age band or cohorts: 65 - 85*

Obtain central mortality one year forward curves – projections, from an AIR baseline and excess mortality model. We define central mortality rate as: $\frac{number\ of\ deaths\ among\ policy\ holders}{total\ number\ of\ alive\ policy\ holders}$ for each age group in our policy holder population 65 – 85; forty 2*20 (male and female) central mortality rate curves in total for the whole portfolio.



Source: Mary H. Louie, Sr. Statistician, AIR Worldwide

As with P & C risk aggregation methodologies, there is very little consensus and research on how to practically estimate a covariance and correlation matrices for life risk factors such as excess mortality or longevity
For our purposes we assume a constant off- diagonal correlation factor

The correlation between two historical central mortality time series (1962 – 2010), with central mortality rates $\boldsymbol{M_i\ and\ M_j}$ and historical mean central mortality rates $\boldsymbol{\mu_i\ and\ \mu_j}$ for policy holdings respectively $\boldsymbol{i\ and\ j}$ in our *Life-risk LOB* is given by:

$$\boldsymbol{\rho_{i,j;M} = \frac{cov(M_i; M_j)}{\sigma_i \sigma_j} = \frac{\sum_{i,j}^{n}(M_i - \mu_i)(M_j - \mu_j)}{\sqrt{\sum_i^n (M_i - \mu_i)^2 \sum_j^n \left(M_j - \mu_j\right)^2}}}$$

The average implied pair-wise correlation for our life risk LOB becomes:

$$\boldsymbol{\rho_{average;LOB}} = \frac{\mathbf{1}}{\boldsymbol{n*(n-1)}} \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \sum_{\boldsymbol{i \neq j}}^{\boldsymbol{n}} \boldsymbol{\rho_{i,j;M}}$$
$$n = 250$$

Computed life - mortality risk $SCR_{\alpha-LLR}$

SCR α = 99.5%		α = 97%	α = 95%
Gaussian	1,147,447	1,231, 057	993,741
Student-t	2,009,309	1,147,447	1,023,791

Market Price risk

The P&C and Life LOB's share the same holding of 125 variable price securities (equities), which are subject to market price risk.

We estimate 90 days of historical price S_T log returns: $ln\left(\frac{S_t}{S_{t-1}}\right)$ and an average log return: $\mu_{S(t)} = \frac{1}{N}\sum_{N=1}^{90} ln\left(\frac{S_t}{S_{t-1}}\right)$

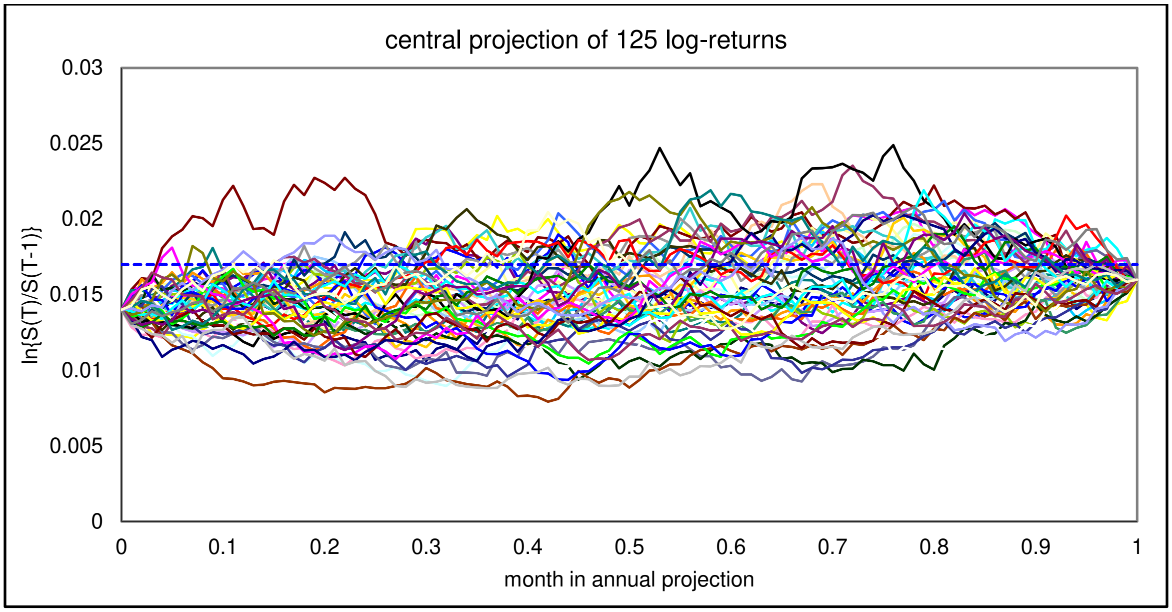
And a *Historical Volatility*: $\sigma_H = \sqrt{252} * \sqrt{\frac{1}{N}\sum_{i=1}^{90}\left(ln\left(\frac{S_t}{S_{t-1}}\right) - \mu_{S(t)}\right)^2}$

For practicality we assume the most simplistic returns one year forward simulation model:

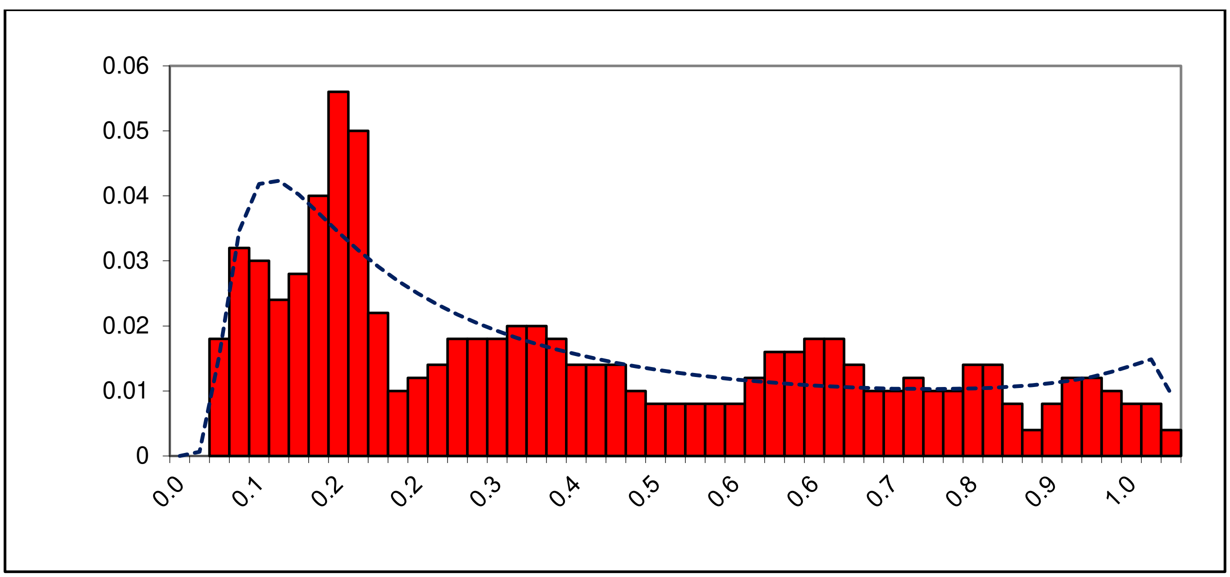
$$ln\left(\frac{S_t}{S_{t-1}}\right) = \theta dt + \sigma_H dW$$

Where returns drift - $\theta = 0$; σ_H – historical volatility; and - $W - Weiner\ process$: $W_t - W_s \rightarrow N \approx (0, t-s)$.

Central projections (averages) of MCBB simulation of $ln\left(\frac{S_t}{S_{t-1}}\right)$ of our 125 variable price securities



The probabilistic empirical distribution of simulated $ln\left(\frac{S_t}{S_{t-1}}\right)$



Correlation matrix: Given $\boldsymbol{r_i = ln\left(\frac{S_t}{S_{t-1}}\right) and\, r_j}$ and historical mean log-returns $\boldsymbol{R_i = \frac{1}{N}\sum_{N=1}^{90} ln\left(\frac{S_t}{S_{t-1}}\right) and\, R_j}$ the historical non-weighted pairwise correlation is

$$\rho_{i,j;R} = \frac{cov(r_i; r_j)}{\sigma_i \sigma_j} = \frac{\sum_{i,j}^{n}(r_i - R_i)(r_j - R_j)}{\sqrt{\sum_i^n (r_i - R_i)^2 \sum_j^n \left(r_j - R_j\right)^2}}$$

We build a positive definitive correlation matrix

$$\sum_{i=1}^{n} \sum_{i \neq j}^{n} \rho_{i,j;M}$$

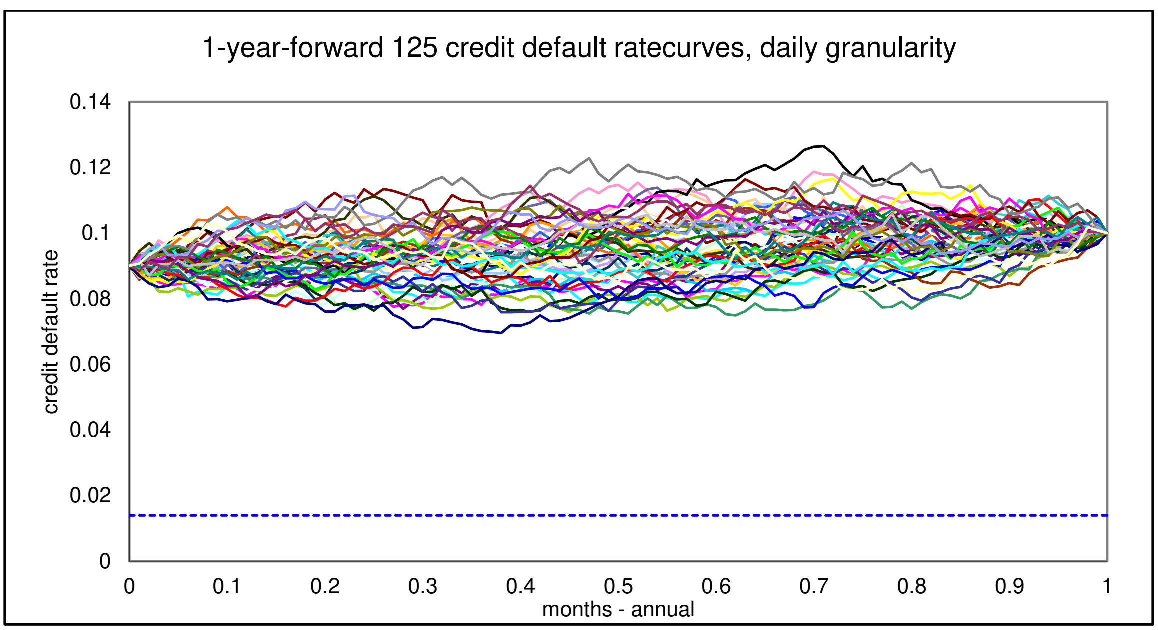
Computed market price risk $SCR_{\alpha - MPR}$

SCR α = 99.5%		α = 97%	α = 95%
Gaussian	153,139	94,310	68,223
Student-t	334,415	145,450	79,537

Credit Default risk

Credit default rates of sovereign bonds are collected and interpolated from market quotes of credit default swaps. The fundamental approach is to do a full macro – economic and credit risk research study - often times this is beyond the resources of even large firms.

We simulate 125 1-year-forward credit default curves from a single 5 year Sovereign CDS on UK 5 year Gilts.



In the absence of fundamental research on forward credit default correlations, we assume a constant quantity: $\rho_{i,j;CDR} = \mathbf{30}\%$

Computed credit default risk $SCR_{\alpha-CDR}$

SCR	α = 99.5%	α = 97%	α = 95%
Gaussian	7,658	5,197	4,183
Student-t	12,159	7,952	5,887

Portfolio risk factor $\boldsymbol{SCR_\alpha}$ view and aggregation methodology

For physical risk factors we are less conservative and use Gaussian SCR_α ; for market and credit risk factor we are more conservative and use Student –t SCR_α

	SCR by Risk Factor				
	Property CAT Risk	Workers Compensation CAT Risk	Life Longevity Risk	Market Price Risk	Credit default Risk
Property & Casualty LOB	1,185,543	985,231		334,415	12,159
Life LOB			1,147,447		

Do we know by fundamental research of a dependence structure between portfolio risk factors?
Are SII linear correlation factor overestimated?
Critique: Definition of correlation dependence factors and matrices – one of the delaying issues in adoption of SII

	Property CAT Risk	Workers Compensation CAT Risk	Life Longevity Risk	Market Price Risk	Credit default Risk
Property CAT Risk	1.00	0.25	0.00	0.00	0.00
Workers Compensation CAT Risk	0.25	1.00	0.10	0.00	0.00
Life Longevity Risk	0.00	0.10	1.00	0.00	0.00
Market Price Risk	0.00	0.00	0.00	1.00	0.10
Credit default Risk	0.00	0.00	0.00	0.10	1.00

Gaussian aggregation methodology
– With a portfolio correlation matrix: $\sum_{i,j=1}^{N} \rho_{i,j}^{P}$

$$SCR_{portfolio} = \sqrt{\sum_{i,j=1}^{N} \rho_{i,j}^{P} SCR_{\alpha,i} SCR_{\alpha,j}}$$

$$SCR_{portfolio} = 3{,}725{,}355$$

Conclusion → Consistency, Benchmarking, Stress Testing
SCR as a coherent risk measure - m:

Homogeneity –risk increases with size of positions, exposure, liability, etc.: $m[t(X)] = t * p(X)$

Monotonicity lower returns, P&L leads to higher risk: $X > Y \rightarrow m(X) > m(Y)$

Subadditivity – critical issue: - $m(X + Y) \leq m(X) + m(Y)$

Risk free condition: $m(X + a) = m(X) + a$

In practice only Expected Shortfall (ES) or Tail-SCR is a coherent risk measure

$$ES_{\propto} = T/SCR_{\alpha} = \frac{1}{1 - \alpha} \int_{1-\alpha}^{1} SCR(u)du$$

Benchmarks:
Comprehensive methodology for deriving dependence structures, beyond linear correlations, for physical insurance risk factors: NATCAT property, casualty, excess mortality, etc. from:

- historical events policy claims data – traditional but disputed and incomplete
- physical, modeled hazard event intensities – much further research is needed, underway at AIR

Evidence - empirical, analytical or otherwise - on preference in aggregation functions is insufficient
Both points are valid for market and credit risk factor types – moving beyond traditional techniques as: historical covariance, GBM simulation for log-returns, credit default rates derived from CDS
Studying and comparing aggregation and dependence functions beyond Gaussian and Student-t

Stress tests:
Historical stress tests – desirable but difficult to configure, (a) lack of data, (b) time consuming
Worst case – the perfect storm: what scenario will render the business insolvent, regardless of capital reserves?

- $credit\ default\ crisis$
- $credit\ default\ crisis + stock\ market\ shock$
- $credit\ default\ crisis + stock\ market\ shock + 200\ YRP\ NAT\ CAT\ event$
- $credit\ default\ crisis + stock\ market\ shock + 200\ YRP\ NAT\ CAT\ event + 200\ YRP\ pandemic\ event$

Appendix

Table 1: Individual component risk factor Solvency Capital Requirement as % of insured values by LOB

Component risk	Aggregation method	SCR confidence level = α		
		α = 99.5%	α = 97%	α = 95%
Property NAT CAT	Gaussian	0.1581%	0.1561%	0.1551%
	Student-t	0.1581%	0.1569%	0.1562%
Work Comp NAT CAT	Gaussian	0.1314%	0.1243%	0.1204%
	Student-t	0.1581%	0.1569%	0.1562%
Life - Excess Mort	Gaussian	0.1530%	0.1641%	0.1325%
	Student-t	0.2679%	0.1530%	0.1365%
Market price - equity	Gaussian	0.0204%	0.0126%	0.0091%
	Student-t	0.0446%	0.0194%	0.0106%
Credit default	Gaussian	0.0010%	0.0007%	0.0006%
	Student-t	0.0016%	0.0011%	0.0008%

Table 2: sensitivity analysis for Portfolio Solvency Capital Requirement as % of total portfolio insured value

Correlation	Aggregation method	Portfolio SCR confidence level = α		
		α = 99.5%	α = 97%	α = 95%
Recommended SII	Gaussian	0.4930%	0.4896%	0.4683%
	Student-t	0.5721%	0.5063%	0.4928%
Internal estimate	Gaussian	0.4844%	0.4817%	0.4606%
	Student-t	0.5629%	0.4974%	0.4838%
Non-weighted average	**Blend**	**0.5281%**	**0.4937%**	**0.4764%**