Some remarks on restricted bargaining sets

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Abstract.
We analyze bargaining mechanisms for allocating resources in atomless economies. We provide results proving that it is not necessary to consider the formation of all coalitions in order to obtain the bargaining sets. This is shown under restrictions of different nature, triggering different equivalence results. In addition, several counterexamples state boundaries for the possibility of extending and generalizing our results.

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1 Introduction

The core of an economy is defined as the set of allocations that can not be blocked by any coalition. Nevertheless, it can be argued that the veto mechanism that defines the core is a single move, static, in the sense that it does not take into account any other consequence this move may have. For instance, one may ask whether this objection or veto is credible or, on the contrary, it is weak, not consistent enough so other agents in the economy may react to it and propose a re-blocking or counter-objection. The first outcome of this more dynamic conception of the veto mechanism was the work by Aumann and Maschler (1964) who introduced the concept of bargaining set, containing the core of a game. The main idea is to try to inject a sense of stability to the veto mechanism, and hence permitting the implementation of some allocations that otherwise would be formally blocked. In this sense, only objections that can not be counter-objected (the credible, the stable ones) should be allowed, and blocking would be more difficult. The bargaining set thus contains the allocations that can not be stably blocked: Any objection to them would result in a counter-objection.

The concept of bargaining set was adapted later to an atomless economy framework. Mas-Colell (1989), presents a modification of the definition of Aumann and Maschler's bargaining set, and an equivalence theorem where he shows, under conditions of generality similar to the Core-Walras equivalence theorem, that the bargaining set and the core coincide. On his side, Vind (1992) gives a definition of a bargaining set different from Mas-Colell's resulting in a bargaining set strictly larger than the core.

It has been widely argued in the existing literature that the veto mechanism is difficult to work freely and spontaneously. Moreover, in economies with many agents it is unlikely that all coalitions are easy to form. In this respect, the works by Schmeidler (1972), Grodal (1972) and Vind (1972) analyzing the core of atomless economies when restrictions on the formation of coalitions are applied are well known and assign a great motivation to apply the idea to the case of bargaining sets, where the veto mechanism is not only a single move and hence the the analysis of the coalitions that are required to be formed becomes even more interesting. In this spirit, Schjødt and Sloth (1994) analyze both Mas-Colell and Vind's bargaining sets when there are restrictions on the size of the coalitions involved in the objection and counter-objection process. It turns out that the Mas-Colell bargaining set becomes larger, whereas the corresponding

Vind bargaining set remains unaltered.

In this work we follow this train of thought and present a further study on both bargaining sets when restrictions on the size of the coalitions involved are applied. In this sense, our findings are twofold: Firstly, we have establish equivalence results for the full bargaining sets in terms of a restricted objection and counter-objection process, and secondly, we provide examples that state boundaries for the possibility of extending and generalizing our main equivalence results.

We have three main equivalence results. In the first one, we prove that the full bargaining set is equivalent to the one where not only limits to the size of the coalitions involved are set, but also the members are closely bound together as in Grodal's (1972) remark on the core. Vind's (1972) result leads us to prove a second result on restricted bargaining sets: It turns out that to counter-object an objection it is enough to consider the formation of coalitions of any fixed measure. Finally, our third equivalence result is based on the work by Hervés-Beloso and Moreno-García (2008), who provided a new characterization of competitive allocations, and hence of the core, by strengthening the veto power of the grand coalition, formed by all the agents in the economy. Following their approach, we prove that an objection has no full counterobjection if and only if the allocation involved in the objection is robustly efficient.

In order to make clear the different restrictions of coalitions formation we consider, a lemma is stated before each of our equivalence theorems. Although the lemmas can be obtained as consequences of these underlying core characterizations for atomless economies, we provide totally constructive proofs not only for the sake of completeness but also because the construction of the allocation which defines the objections or counterobjections for coalitions helps to a better understanding of the equivalence results.

We also provide examples that highlight the problems we face when trying to extend our results. Precisely, we show that our first two equivalence results do not hold if one considers Mas-Colell's (1989) definition of the bargaining set. Moreover, we also point out the impossibility of generalizing our results to an economy with a continuum of agents and infinite commodities without introducing new assumptions.

The rest of the work is presented as follows: In section 2, we set the model and the definitions of bargaining set and full bargaining set. Section 3 contains our main equivalence theorems, followed by the forementioned examples in section

4. Finally, an appendix is devoted to the proofs of the results.

2 The economy and bargaining sets

Let $\mathcal{E}$ be an exchange economy with a continuum of traders modeled by the finite measure space $(I, \mathcal{A}, \mu)$, where $I = [0, 1]$ is the set of agents, $\mathcal{A}$ is the Lebesgue σ-algebra on I and μ is the Lebesgue measure on $\mathcal{A}$.

There is a finite number ℓ of commodities to be traded and therefore $\mathbb{R}^\ell_+$ is the commodity space. Each agent t is characterized by her endowments $\omega(t) \in \mathbb{R}^\ell_+$ and a preference relation over the consumption set represented by the utility function $U_t : \mathbb{R}^\ell_+ \to \mathbb{R}_+$.

We assume $\int_I \omega(t)d\mu(t) \gg 0$ and the utility functions U_t are continuous and increasing. In addition, the mapping that associates to each agent her utility function is measurable (with respect the compact-open topology).

A coalition S is a measurable set of consumers such that $\mu(S) > 0$. An allocation $f : S \to \mathbb{R}^\ell_+$ is said to be attainable or feasible for the coalition S if $\displaystyle\int_S f(t)d\mu(t) \leq \int_S \omega(t)d\mu(t)$. The set of feasible allocations in the economy $\mathcal{E}$ is the set of allocations that are attainable for the big coalition I.

A coalition S blocks or improve upon an allocation f if there exists g which is an attainable allocation for S and $U_t(g(t)) > U_t(f(t))$ for every $t \in S$. The core of the economy $\mathcal{E}$, that we denote by $C(\mathcal{E})$, is the set of feasible allocations that cannot be improve upon by any coalition. That is, a feasible allocation is in the core if it has the property that there is no coalition which can redistribute its total endowments in such a way that every member becomes better off.

Thus, the core is defined taking into account the veto power from any coalition. However, one might argue that even though an assignment could be blocked (or objected) by a coalition, this would only occur if the blocking allocation cannot be "re-blocked" (or counterobjected), that is, no other coalition can propose another redistribution of resources which makes its member better off.

To capture this idea, the bargaining set notion introduced by Aumann and Maschler (1964) for cooperative games was adapted to atomless economies by Mas-Colell (1989) and Vind (1992) resulting in two different definitions of bargaining set. Mas-Colell's will be referred to as "bargaining set", whereas Vind's will be named “full bargaining set”[1]. To understand the similarities and differences between these concepts, let us now state both definitions.

Bargaining Set: The bargaining set introduced by Mas-Colell (1989) contains all the feasible allocations of the economy that, if objected, they could also be counter-objected.

Definition 2.1 *An objection to the allocation f in the economy $\mathcal{E}$ is a pair (S, y), where S is a coalition and y is an allocation defined on S such that:*

(i) $\int_S y(t)d\mu(t) \leq \int_S \omega(t)d\mu(t)$

(ii) $U_t(y(t)) \geq U_t(f(t))$ *for every* $t \in S$

(iii) $\mu\left(\{t \in S | U_t(y(t)) > U_t(f(t))\}\right) > 0.$

Definition 2.2 *A counter-objection to the objection (S, y) is another pair (T, z), where T is a coalition and z is an allocation defined on T such that:*

(i) $\int_T z d\mu(t) \leq \int_T \omega(t)d\mu(t)$

(ii) $U_t(z(t)) > U_t(y(t))$ *for every* $t \in T \cap S$

(iii) $U_t(z(t)) > U_t(f(t))$ *for every* $t \in T \setminus S.$

Note that $T \cap S$ may be even empty.

Full Bargaining Set: The full bargaining set introduced by Vind (1992) contains all the feasible allocations of the economy that, if full objected, they could also be full counter-objected. The definition of full objection and full counter-objection is as follows:

Definition 2.3 *A full objection to the allocation f in the economy $\mathcal{E}$ is a pair (S, y), where S is a coalition and y is a feasible allocation such that:*

(i) $\int_S y(t)d\mu(t) \leq \int_S \omega(t)d\mu(t)$

[1] We remark that the term “full” becomes natural once the definitions are precised

(ii) $U_t(y(t)) \geq U_t(f(t))$ *for every* $t \in S$

(iii) $\mu\left(\{t \in S | U_t(y(t)) > U_t(f(t))\}\right) > 0.$

Remark. *If* (S, y) *is a full objection to* f *then* $(S, y|S)$ *is an objection to f.*

Definition 2.4 *A full counter-objection to the full objection* (S, y) *is another pair* (T, z), *where* T *is a coalition and* z *is a feasible allocation such that:*

(i) $\int_T z d\mu(t) \leq \int_T \omega(t) d\mu(t)$

(ii) $U_t(z(t)) > U_t(y(t))$ *for every* $t \in T$

Remark. Note that in the full bargaining set, the allocations y and z are both attainable for the corresponding coalitions and feasible in $\mathcal{E}$, which implies that they specify a commodity bundle for every agent, whereas in the bargaining set they are only defined on S and T respectively.

Let $B(\mathcal{E})$ denote the bargaining set and $B^*(\mathcal{E})$ the full bargaining set for the economy $\mathcal{E}$.

3 Main results

Eight years after the publication of Aumann's (1964) core equivalence, Schmeidler (1972) showed that in an atomless economy, with finitely many commodities to be traded, any allocation that is not blocked by arbitrarily small coalitions is in the core[2]. In the same issue of Econometrica, Grodal (1972) showed that we can further restrict the set of coalitions in order to get the core. Precisely Grodals' result establishes that an allocation belongs to the core if and only if it cannot be blocked by a coalition which is the union of at most $\ell + 1$ groups, each of which has not only measure but also diameter arbitrarily small. Moreover, Vind (1972) showed a third consecutive remark on the core of an atomless economy which states that in order to block any non-competitive allocation it is enough to consider the veto power of arbitrarily large coalitions.

[2]Schmeidler's (1972) is stronger than what one reads in the statement of his theorem. More precisely, from the proof of Schmeidler's (1972) one deduces that if an allocation f is blocked by a coalition S via an allocation g, then, for any $\varepsilon > 0$, f can be blocked via the same allocation g by a coalition $S' \subset S$, with $\mu(S') \leq \varepsilon$.

On the other hand, Hervés-Beloso and Moreno-García (2008) provide a new characterization of competitive equilibrium based on the veto mechanism. They show that, in pure exchange economies with a continuum of non-atomic agents and a finite number of commodities, the competitive allocations, and hence the core, can be characterized by strengthening the veto power of the grand coalition, formed by all the agents in the economy.

These core characterizations for continuum economies allow us to obtain different equivalence results for the full bargaining sets in terms of a restricted objection and counter-objection process. In this section, each main result is preceded by a lemma stating the corresponding restriction on the formation of coalitions to either object or counter-object in the bargaining mechanism in exchange economies that we analyze. Although the lemmas can be obtained as consequences of the underlying core characterizations for atomless economies, we provide totally constructive proofs (see Appendix) not only for the sake of completeness but also because the construction of the allocation which defines the objections or counterobjection for coalitions helps to a better understanding of the equivalence results we obtain as well as the counterexamples stated in the next section that place limits to some generalizations or extensions of our main results.

One may argue that the lack of communication restricts the set of coalitions that can be formed to those that are unions of groups with small both measure and diameters. This restriction is the one considered in our first theorem that at the same time strengthen the main related results in Schjødt and Sloth (1994). For it, given $\varepsilon > 0$ let us consider the following set of coalitions

$$\mathcal{S}_\varepsilon = \left\{ S \subset I, \text{ such that } S = \cup_{i=1}^{\ell+1} S_i \text{ with } diam(S_i) \leq \varepsilon \text{ and } \mu(S_i) \leq \varepsilon \right\}$$

Let $B^*_\varepsilon(\mathcal{E})$ denote the full bargaining set where coalitions (for both objecting and counterobjecting) are restricted to those in $\mathcal{S}_\varepsilon$.

Lemma 3.1 *Let f be an allocation in an atomless economy $\mathcal{E}$ and (S, g) a full objection to f. Let (T, h) be a full counterobjection to (S, g). Then, for every $\varepsilon > 0$, there exists $H \in \mathcal{S}_\varepsilon$ that counterobjects (S, g) via the same h.*

Theorem 3.1 *For every ε, all the restricted full bargaining sets $B^*_\varepsilon(\mathcal{E})$ are the same and coincide with the full bargaining set $B^*(\mathcal{E})$*

The above equivalence result relies on the characterization of the core sated by Schmeidler (1972) and Grodal (1972) which exploit the veto power of arbitrarily small coalitions in atomless economies. Symmetrically, as we have already mentioned, large enough coalitions are able to eliminate any allocation that does not belong to the core. This Vind's (1972) result leads us to prove a further result on restricted bargaining sets. Precisely, we next lemma shows that to counter-object an objection it is enough to consider the formation of coalitions of any fixed measure.

Lemma 3.2 *Let f be an allocation in an atomless economy $\mathcal{E}$ and (S, g) a full objection to the allocation f. Let (T, h) be a full counterobjection to (S, g). Then, for all $\alpha \in (0, 1)$, there exists another full counterobjection (C, y) to (S, g) such that $\mu(C) = \alpha$.*

To state our second characterization in terms of a restricted bargaining mechanism, let α-$B^*_\varepsilon(\mathcal{E})$ be the restricted bargaining set which contains all the feasible allocations of the economy that, if full objected by a coalition in $\mathcal{S}_\varepsilon$, they could also be full counter-objected by a coalition in $C_\alpha = \{S \subset I,$ such that $\mu(S) = \alpha\}$.

Theorem 3.2 *α-$B^*_\varepsilon(\mathcal{E}) = B^*(\mathcal{E})$ for every $\alpha, \varepsilon \in (0, 1)$.*

In the previous coincidence results the blocking or objecting system is restricted to coalitions which are the union of at most $\ell + 1$ coalitions, each of which has measure and diameter less than ε. As Grodal (1972) pointed out that a coalition has measure and diameter less than a intuitively means that the coalition consists of relatively "few" agents, and that the agents in the coalition resemble one another in chosen characteristics. Thus, we may say that such a group of agents are stable and moreover their cost of join together and object an allocation is low. In this way, the equivalence results in Theorems 3.1 and 3.2 state that, if objecting coalitions are restricted to those whose formation cost is small and are stable, then the set of allocations arising from the bargaining mechanism is the same if, in addition, the counterobjecting process is restricted to either arbitrarily small coalitions or to the set of coalitions of any fixed measure.

In which follows we show that we can further restrict the coalitions that are allowed to form in order to counterobject and still have the coincidence with the full bargaining set. Precisely, we will only consider the re-objecting power of the grand coalition formed by all the agents, but exercised in a family of economies obtained by perturbing the agents' initial endowments. For this, given a an allocation f, a coalition A and a parameter $\alpha \in [0,1]$, the continuum economy $\mathcal{E}(A,f,\alpha)$ is the same as $\mathcal{E}$ except for the endowments of the agents in the coalition A that are the convex combination defined by α of f and ω, that is, the endowments in this perturbed economy $\mathcal{E}(A,f,\alpha)$ are the following:

$$\omega(A,f,\alpha)(t) = \begin{cases} \omega(t) \;\text{ if }\; t \in I \setminus A \\ \\ (1-\alpha)\omega(t) + \alpha f(t) \;\text{ if }\; t \in A \end{cases}$$

As in Hervés-Beloso and Moreno-García (2008), we say that a feasible allocation f is robustly efficient in the economy $\mathcal{E}$ if f is a non-dominated[3] allocation in every economy $\mathcal{E}(A,f,\alpha)$. Thus, if (S,g) is a full objection to an allocation, g is robustly efficient means that there is no full counterobjection (I,h) to (S,g) in any economy $\mathcal{E}(A,g,\alpha)$. In other words, the objection g is robustly efficient when the grand coalition I is not able to counter-object in any of the perturbed economies $\mathcal{E}(A,g,\alpha)$. Next lemma shows that in order to counter-object any objection it is enough to consider the formation of the grand coalition I whose veto power is exercised in a family of economies obtained by perturbing the agents' original endowments.

Lemma 3.3 *Let (S,g) be a full objection to the feasible allocation f. (S,g) has no full counterobjection if and only if g is robustly efficient.*

Theorem 3.3 *The full bargaining set $B^*(\mathcal{E})$ coincides with the set of allocations that are not full objected by any robustly efficient allocation.*

4 Counterexamples

It is easy to see that the bargaining set $B(\mathcal{E})$ and the full bargaining set $B^*(\mathcal{E})$ contain the core and hence the competitive allocations of the economy $\mathcal{E}$. The main result in Mas-Colell (1989) is that in atomless economies a converse is also true, namely, the bargaining set characterizes the core. In other words, if only

[3]An allocation h (feasible or not) is *dominated* (or blocked by the grand coalition) in an economy if there exists a feasible allocation g in such an economy such that every consumer t prefers $g(t)$ rather than $h(t)$.

objections which are not counterobjected are allowed, the bargaining process is equivalent to the veto mechanism, that is, the bargaining set and the core (or the set of competitive allocations) coincide. This is not the case for the full bargaining set which in general is larger that the core.

In the previous section, applying Grodal's (1972) core remark, we have extended and strengthened the results in Schjødt and Sloth (1994) by showing that in order to obtain the full bargaining set it is enough to consider the formation of arbitrarily small coalitions which are the union of a finite number of groups, each of which has also diameter arbitrarily small. In addition, using Vind's (1972) result, we have shown that the bargaining mechanism can be restricted to coalitions arbitrarily small for full objections and to the set of coalitions of any fixed measure for full counterobjections. For it, previous lemmas (Lemma 3.1 and 3.2) become essential. Next example shows that these lemmas do not hold for large coalitions and re-objections and then the restriction of coalition formation in the counterobjection process to those arbitrarily large does not guarantee the equivalence with the bargaining set.

Example 1. Let $\mathcal{E}$ be an economy with two commodities, x and y, and a continuum of agents represented by the unit real interval $I = [0,1]$, endowed with the Lebesgue measure μ. Every consumer $t \in I$ has the same preference relation, represented by the utility function $U(x,y) = \min\{x,y\}$, and the same endowments, given by $\omega(t) = (1,1)$ for every $t \in I$. Let f be the following feasible allocation in the atomless economy $\mathcal{E}$

$$f(t) = \begin{cases} (2,2) \text{ if } t \in [0,1/2) \\ (0,0) \text{ if } t \in [1/2,1] \end{cases}$$

The allocation f does not belong to the core. In fact, f is blocked by any coalition in $[1/2,1]$. In particular, (S,g), where $S = [1/2,3/4)$ and $g(t) = \omega(t) = (1,1)$ for every $t \in S$, is an objection to f. In addition, the coalition $A = [3/4,1]$ joint with the allocation $z(t) = \omega(t) = (1,1)$, for every $t \in A$, define a counterobjection to (S,g). However, there is no coalition with measure greater than 3/4 that is able to counterobject (S,g). To show this, let us assume that there exists $T \subset I = [0,1]$ with $\mu(T) > 3/4$ such that (T,h) is a counterobjection to (S,g). Then, $h(t) > (2,2)$ for every $t \in T_1 = T \cap [0,1/2)$, $h(t) > (1,1)$, for every $t \in T_2 = T \cap [1/2,3/4)$ and $h(t) > (0,0)$ for every $t \in T_3 = T \cap [3/4,1]$. Since $\mu(T) > 3/4$, we can ensure not only that T_1, T_2 and T_3 have positive measure but also $\mu(T_1) > \mu(T_3)$. Therefore $\int_T \omega(t) d\mu(t) = \mu(T)(1,1) < (2\mu(T_1) + \mu(T_2))(1,1) < \int_T h(t) d\mu(t)$, which is in contradiction with the fact that T can attain h.

A remark on large objecting coalitions. The previous example also points out that the restriction of the objecting mechanism in the bargaining process to large coalitions does not necessarily result in the full bargaining set. That is, in contrast to the core, the objecting power of large coalitions is not enough to characterize the bargaining scheme. To show this, note that $([1/2, 1], \omega)$ is a full objection to f which has no full counterobjection and hence f does not belong to the full bargaining set. Assume that in order to fully block an allocation coalitions are restricted to those with measure larger $\delta > 1/2$. Let (S, g) with $\mu(S) > \delta$ be a full objection to f. It remains to show that it has a full counterobjection. Note that both $S_1 = S \cap [0, 1/2)$ and $S_2 = S \cap [1/2, 1]$ has positive measure. Moreover $\mu(S_1) < \mu(S_2)$ and in addition, the set of consumers t in S_2 such that $U_t(g(t)) < 1$, denoted by H, has positive measure, actually its measure is greater than $\mu(S_1)$. This is due to the fact that g is a feasible allocation, which is attainable to S that blocks f via g. Then, we can conclude that (H, ω) is a fully counterobjection to (S, g). Therefore f belongs to the fully bargaining set where the coalitions to object are restricted to those with large enough measure, namely, greater that 1/2.

We stress that the proofs of our equivalence results rely on Liapunov's convexity theorem, which does not hold in an infinite dimensional framework. It should then not be surprising the example that follows, which shows that it is not possible to extend lemma 3.2 to economies with an infinite number of commodities unless we introduce additional assumptions on endowments and preferences. We will see that if a coalition blocks or counter-objects an allocation, it is not possible to find coalitions arbitrarily large that also block or counter-object such allocation. Then, the objecting power of arbitrarily large coalitions in atomless economies with an infinite dimensional commodity space is not sufficient in the bargaining mechanism.[4]

[4]We remark that extensions of these results to more general settings, where perfect competition is not guaranteed, require additional assumptions. For instance, Hervés-Beloso et al., (2000) have extended Schmeidler's, Grodal's and Vind's results to an infinite dimensional setting by requiring a kind of myopic behavior of the agents (see also Evren and Hüsseinov, 2008 and Pesce, 2010).

Example 2. Consider a pure exchange economy with a continuum of agents, represented by $I = [0,1]$. The commodity space is ℓ^∞, the space of bounded sequence.

For each $n \in \mathbb{N} \bigcup \{0\}$ let us consider a partition $\{I_n^i, i = 1, \ldots, 2^n\}$ of the unit real interval $I = [0,1]$ given by $I_n^i = \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)$ for $i = 1, \ldots, 2^n - 1$ and $I_n^{2^n} = \left[\frac{2^n-1}{2^n}, 1\right]$. Now, let A be a measurable subset of $I = [0,1]$ such that $0 < \mu(A \bigcap I_n^i) < \mu(I_n^i)$ for every n and every i and such that, for each n, $\mu(A \bigcap I_n^i) = a_n$ for every $i = 1, \ldots, 2^n$.[5]

The mapping $\omega : I \to \ell_+^\infty$ which associates to each agent $t \in I$ her initial endowment $\omega(t) = (\omega_j(t))_{j=1}^\infty$ is given by

$$\omega_j(t) = \begin{cases} 1 & \text{if } t \notin A \\ c(j) & \text{if } t \in A \bigcap I_n^i \text{ with } j = 2^n + i - 1 \\ 0 & \text{otherwise} \end{cases}$$

where $c(j) > 0$ verifies that $c(j) = c_n$ if $j = 2^n + i - 1$ for some $i \in \{1, \ldots, 2^n\}$. and c_n converges to zero when n goes to ∞. This last property implies that ω is bounded and Bochner integrable.

Preferences relations are given by the following utility functions:

$$\begin{aligned} U_t(x) &= \liminf_j \ x_j \quad \text{if } t \notin A, \quad \text{and} \\ U_t(x) &= \textstyle\sum_{j\geq 1} \alpha(j) \log(1 + x_j) \quad \text{if } t \in A \end{aligned}$$

where $\alpha(j) > 0$ verifies that $\alpha(j) = \alpha_n$ if $j = 2^n + i - 1$ for some $i \in \{1, \ldots, 2^n\}$, and $\sum_{j\geq 1} \alpha(j) < \infty$.

The coalition A blocks the initial allocation ω via the allocation x defined as follows:

$$\begin{aligned} x_2(t) &= \frac{c(2)\ \mu(A \bigcap I_1^1)}{\mu(A)}, \\ x_3(t) &= \frac{c(3)\ \mu(A \bigcap I_1^2)}{\mu(A)}; \quad \text{and} \\ x_j(t) &= \omega_j(t) \quad \text{if } j \neq 2, 3. \end{aligned}$$

[5]For example, we can take a non-negligible Cantor subset A_0^1 of $I_0^1 = I$; then we take again non-negligible Cantor subsets A_1^1 and A_1^2 of $I_1^1 \setminus A_0^1$ and $I_1^1 \setminus A_0^1$ respectively, with the property that $\mu(A_0^1 \bigcap I_1^1) + \mu(A_1^1) = \mu(A_0^1 \bigcap I_1^2) + \mu(A_1^2)$; and so on. In this way, $A = \bigcup_{n=0}^\infty \bigcup_{i=1}^{2^n} A_n^i$ and, for any n we have that $\mu(A \bigcap I_n^i) = \frac{\mu(A)}{2^n}$, for all $i = 1, \ldots, 2^n$.

To show this, note that by construction $\mu(A\bigcap I_1^1) = \mu(A\bigcap I_1^2) = \dfrac{\mu(A)}{2}$ and by definition $c(2) = c(3) = c_1$. Therefore, $x_2(t) = x_3(t) = \dfrac{c_1}{2}$.

On the other hand, since $\alpha(2) = \alpha(3) = \alpha_1$, we obtain that for every $t \in A$ the following inequality holds

$$\alpha(2)\ \log(1 + x_2(t)) + \alpha(3)\ \log(1 + x_3(t)) = \alpha_1\ \log\left(1 + \frac{c_1}{2}\right)^2 >$$

$$> \alpha_1\ \log(1 + c_1) = \alpha(2)\ \log(1 + \omega_2(t)) + \alpha(3)\ \log(1 + \omega_3(t))$$

Then, $U_t(x(t)) > U_t(\omega(t))$ for every agent t belonging to the coalition A.

Finally, note that the following equalities hold:

$$\int_A \omega_2(t)d\mu(t) = c(2)\mu(A\bigcap I_1^1) \quad \text{and}$$

$$\int_A \omega_3(t)d\mu(t) = c(3)\mu(A\bigcap I_1^2).$$

These equalities imply that $\int_A x(t)d\mu(t) = \int_A \omega(t)d\mu(t)$.

Therefore, the coalition A blocks ω via x. Furthermore, a similar argument shows that, for any natural number n, the coalition $A\bigcap\left[\frac{1}{2} - \frac{1}{2^n}, \frac{1}{2} + \frac{1}{2^n}\right]$ blocks ω. Then, we conclude that ω is blocked by arbitrarily small coalitions.

However, if S is a coalition such that $\mu(S\bigcap A^c) = \mu(S\bigcap(I \setminus A)) > 0$, then S can no longer block ω. To show this point it suffices to notice that $U_t(\omega(t)) = 1$ for every $t \notin A$, and $\int_S \omega_j(t)d\mu(t) = c(j)\mu(A\bigcap I_n^i) + \mu(S \setminus A)$, where n and i are such that $j = 2^n + i - 1$ and $1 \leq i \leq 2^n$.

Therefore, we conclude that ω is blocked by the coalition A and, given $\varepsilon < \mu(A)$, the initial allocation ω is also blocked by a coalition A_ε, with $\mu(A_\varepsilon) \leq \varepsilon$. However, if $\varepsilon > \mu(A)$, there is no coalition S with $\mu(S) \geq \varepsilon$ blocking ω.

Remark. We remark that the utility function $U(x) = \liminf_j\ x_j$ is not weak star continuous on ℓ_+^∞. To show it, let us consider the sequence x_n given by $(x_n)_j = 0$ if $j \leq n$ and $(x_n)_j = 1$ if $j > n$. Then, x_n converges to zero with respect to the Mackey topology[6] and hence with the weak star topology. However, note that $U(x_n) = 1$ for each n while $U(0) = 0$.

It is important to point out that weak star continuity of preferences leads to a precise condition of of the substitutability properties among commodities

[6]Recall that the Mackey topology coincides with the weak star topology on bounded subsets of ℓ^∞.

required for perfect competition and is related to economic thickness of markets (see, for instance Ostroy and Zame, 1994).

Appendix

Next we include the proofs of the main results stated in Section 3 which provide three characterizations of the bargaining mechanism in atomless economies by restricting the set of coalitions that can be formed to object or counterobject feasible allocations. As we have already remarked each equivalence result uses previous lemmas whose proofs are also collected in this Appendix.

Proof of Lemma 3.1. Since (T, h) is a full counterobjection to (S, g) we have that h is a feasible allocation, it is also attainable for T and every $t \in T$ prefers $h(t)$ rather than $g(t)$. By Liapunov's convexity theorem, for every $\varepsilon > 0$, there exists a coalition $H_\varepsilon \subset T$, such that h is attainable for H_ε and then (H_ε, h) is also a full counterobjection to (S, g) (see the proof in Schmeidler (1972) for further details).

As in Grodal (1972) let $(t_n, n \in \mathbb{N})$ be a dense subset in H_ε. Define $H_\varepsilon^1 = H_\varepsilon \cap B(t_1, \varepsilon/2)$ and $H_\varepsilon^n = H_\varepsilon \cap B(t_n, \varepsilon/2) \setminus \cup_{k=1}^{n-1} H_\varepsilon^k$, for $n > 1$. Let $h_n = \int_{H_\varepsilon^n} (h(t) - \omega(t))\, d\mu(t)$. Define $\mathcal{C}$ as the convex hull of the set $\Gamma = \{h_n | n \in N\}$, where $N = \{n \in \mathbb{N} | \mu(H_\varepsilon^n) > 0\}$. Now let χ be the smallest affine subspace containing $\mathcal{C}$. Note that $\sum_{n\in N} h_n = 0$ and 0 belongs to the interior of $\mathcal{C}$ relative to $\chi \subset \mathbb{R}^\ell$. By Caratheodory's theorem, there is $A \subset N$ with at most $\ell + 1$ elements such that we can write $0 = \sum_{j\in A} \alpha_j h_j$, with $\alpha_j \in [0, 1]$ and $\sum_{j\in A} \alpha_j = 1$. Liapunov's theorem allows us to guarantee that for each $j \in A$ there exists $H_j \subset H_\varepsilon^j$ such that $\mu(H_j) = \alpha_j \mu(H_\varepsilon^j)$ and $\int_{H_j} (h(t) - \omega(t))\, d\mu(t) = \alpha_j h_j$. Therefore, by construction the coalition $H = \cup_{j\in A} H_j$ belongs to $\mathcal{S}_\varepsilon$ and counterobjects (S, g) via h.

Q.E.D.

Proof of Theorem 3.1. Let us first prove that $B^*(\mathcal{E}) \subset B_\varepsilon^*(\mathcal{E})$. Indeed, assume that f is a feasible allocation such that $f \in B^*(\mathcal{E}) \setminus B_\varepsilon^*(\mathcal{E})$. Since $f \notin B_\varepsilon^*(\mathcal{E})$, there exists a full objection (S, g) to f, with $S \in \mathcal{S}_\varepsilon$, for which there is no full counter-objection (C, y) such that $C \in \mathcal{S}_\varepsilon$, and by Lemma 3.1, this is the same as saying that it does not exist any full counter-objection to (S, g). This means that f is full objected but not full counter-objected, so $f \notin B^*(\mathcal{E})$.

We will now prove that $B_\varepsilon^*(\mathcal{E}) \subset B^*(\mathcal{E})$. Let $f \in B_\varepsilon^*(\mathcal{E})$, and let (S, g) be a full objection to f. By Liapunov's and Caratheodory's theorems (see the proof of the Lemma 3.1) we can guarantee the existence of $K \subset S$ such that $K \in \mathcal{S}_\varepsilon$ and $\int_K g(t) d\mu(t) \leq \int_K \omega(t) d\mu(t)$, meaning that (K, g) is also a full objection to f. Since $f \in B^*_\varepsilon(\mathcal{E})$, (K, g) necessarily has a full counter-objection (C, y), with $C \in \mathcal{S}_\varepsilon$, which is straightforward also a full counter-obejection to (S, g), concluding that $f \in B^*(\mathcal{E})$.

Q.E.D.

Proof of Lemma 3.2. Since (T, h) is a full counterobjection to (S, g), the following holds:

(i) $\int_T h(t)d\mu(t) \leq \int_T \omega(t)d\mu(t)$.

(ii) $U_t(h(t)) > U_t(g(t))$ for almost all $t \in T$.

If $\alpha < \mu(T)$, consider the measure $\nu(A) = \big(\mu(A), \int_A (h(t) - \omega(t))d\mu(t)\big)$ restricted to measurable subsets of the coalition T. By Liapunov's convexity theorem we obtain that there exists $T_\alpha \subset T$, with $\mu(T_\alpha) = \alpha$, that blocks the allocation g via the same h.

Consider $\alpha > \mu(T)$. Let the measure $\eta(A) = \big(\mu(A), \int_A (h(t) - g(t))d\mu(t)\big)$ restricted to subsets of T. Applying Liapunov's convexity theorem we obtain that for any $\beta \in (0, 1)$ there exits $A \subset T$ such that $\mu(A) = \beta\mu(T)$ and $\int_A (h(t) - g(t))d\mu(t) = \beta \int_T (h(t) - g(t))d\mu(t)$. By continuity and measurability, there exist $\tilde{h}$ and $\delta > 0$ such that $\int_A \tilde{h}(t)d\mu(t) = \int_A h(t)d\mu(t) - \delta$ and $U_t(\tilde{h}(t)) > U_t(g(t))$ for every $t \in A$.

Let the allocation z defined as follows:

$$z(t) = \begin{cases} \tilde{h}(t) & \text{if } t \in A \\ \\ g(t) + \frac{\delta}{\mu(T \setminus A)} & \text{if } t \in T \setminus A \end{cases}$$

Note that $\int_T z(t)d\mu(t) = \int_T (\beta h(t) + (1 - \beta)g(t))\, d\mu(t)$ and $U_t(z(t)) > U_t(g(t))$ for every $t \in T$. As before, there exists $\gamma > 0$ and $\tilde{z}$ such that $\int_T \tilde{z}(t)d\mu(t) = \int_T z(t)d\mu(t) - \gamma$ and $U_t(\tilde{z}(t)) > U_t(g(t))$ for every $t \in T$. Applying Liapunov's theorem again to the above measure ν restricted to $I \setminus T$, we have that there exists $B \subset I \setminus T$ such that $\mu(B) = (1 - \beta)\mu(I \setminus T)$ and $\int_B (g(t) - \omega(t))\, d\mu(t) = (1 - \beta) \int_{I \setminus T} (g(t) - \omega(t))\, d\mu(t)$.

The coalition $C = T \cup B$ blocks g via de allocation y given by

$$y(t) = \begin{cases} \tilde{z}(t) & \text{if } t \in T \\ g(t) + \frac{\gamma}{\mu(B)} & \text{if } t \in B \\ \omega(t) & \text{otherwise} \end{cases}$$

By construction y is a feasible allocation and $\int_C y(t) d\mu(t) \leq \int_C \omega(t) d\mu(t)$. Taking $\beta = (1-\alpha)/(1-\mu(T))$, we conclude that $\mu(C) = \alpha$.

Q.E.D.

Proof of Theorem 3.2. We follow the same proof as in Theorem 3.1. To prove that $B^*(\mathcal{E}) \subset \alpha\text{-}B^*_\varepsilon(\mathcal{E})$, we apply Lemma 3.2 instead Lemma 3.1 and to show that $\alpha\text{-}B^*_\varepsilon(\mathcal{E}) \subset B^*(\mathcal{E})$ it is important to remark that if (S, g) is a full objection to f, then there exists $K \in \mathcal{S}_\varepsilon$ such that (K, g) is also a full objection to f. That is, both coalitions S and K uses the same feasible allocation g which is crucial in the proof.

Q.E.D.

Proof of Lemma 3.3. Let (S, g) be an objection to f. Since g has no full counterobjection, g is in the core. Let p a price system such that (p, g) is a competitive equilibrium for the economy $\mathcal{E}$. Suppose that there exist a coalition T and a number $\alpha \in (0, 1]$ such that g is dominated in the economy $\mathcal{E}(T, g, \alpha)$, that is, there exists an allocation h which is feasible in the perturbed economy $\mathcal{E}(T, g, \alpha)$ and $U_t(h(t)) > U_t(g(t))$ for almost all $t \in I$. Then, we have that $p \cdot h(t) > p \cdot \omega(t) \geq p \cdot g(t)$, for almost all agent $t \in I$.

Multiplying the above inequalities by $(1-\alpha)$ and by α, respectively, we obtain $p \cdot h(t) > p \cdot ((1-\alpha)\ \omega(t) + \alpha\ g(t))$ for almost all agent $t \in T$. Therefore, we have $\int_I p \cdot h(t)\ d\mu(t) > \int_{I \setminus T} p \cdot \omega(t) d\mu(t)\ +\ \int_T p \cdot ((1-\alpha)\omega(t) + \alpha g(t))\ d\mu(t) = \int_I p \cdot \omega(T, g, \alpha)(t)\ d\mu(t)$, which is a contradiction with the feasibility of h in the economy $\mathcal{E}(T, g, \alpha)$.

To show the converse, let g a non-dominated allocation for every economy $\mathcal{E}(A, g, \alpha)$. Assume that (S, g) has a full counter-objection, namely (T, z). Arguing as in the proof of Lemma 3.2, we can take z such that $\int_T z(t) d\mu(t) \leq \int_T \omega(t) d\mu(t) - \delta$, with $\delta > 0$ and moreover, given any $\alpha \in (0, 1)$, there exists an allocation $y : T \to \mathbb{R}^\ell_+$ such that $\int_T y(t) d\mu(t) = \int_T (\alpha z(t) + (1-\alpha) g(t)) d\mu(t)$ and $U_t(h(t)) > U_t(f(t))$ for every $t \in T$. Let us consider the allocation $h : I \to \mathbb{R}^\ell_+$

given by

$$h(t) = \begin{cases} y(t) & \text{if } t \in T \\ g(t) + \frac{\alpha}{\mu(I \setminus T)}\ \delta & \text{if } t \in I \setminus T \end{cases}$$

By construction, we can deduce $\int_I (h(t) - \omega(I \setminus T, g, \alpha)(t)) d\mu(t) = (1-\alpha) \int_I (g(t) - \omega(t)) d\mu(t) \leq 0$. Therefore, the grand coalition blocks g via h in the economy $\mathcal{E}(I \setminus T, g, \alpha)$, which is a contradiction with the fact that g is a non dominated allocation for every economy $\mathcal{E}(A, g, \alpha)$.

Q.E.D.

Proof of Theorem 3.3. It is enough to apply Lemma 3.3 and follow the same proof as in Theorem 3.1 and 3.2.

Q.E.D.

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