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11 June 2012

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June 2012

Abstract

This paper discusses the existence of a bubble in the pricing of an asset that pays positive dividends. I show that rational bubbles can exist in a growing economy. The existence of bubbles depends on the relative magnitudes of risk aversion to consumption and to wealth. Furthermore, I examine how an exogenous shock in technology might trigger bubbles.

Keywords: bubbles, the spirit of capitalism, growth

JEL Classification: E2, E44

*I am grateful to Danyang Xie, Pengfei Wang and Jenny Xu for helpful suggestions and comments.

1 Introduction

People use to believe that no bubble can exist in an infinite-horizon model with a finite number of rational individuals. However, the opinion is true only for the baseline case. In the baseline model, wealth merely provides consumption flows, which individuals only care about. No one likes to hold an asset, whose price is above its fundamental value, forever. Sooner or later, individuals will sell out the asset for the purpose of material rewards. It is the behavior that rules bubbles out.

Conditions that restrict the behavior, however, can help cause a bubble in an infinite-horizon model. The constraint on debt accumulation (or, no-Ponzi-game condition) is one of them, as argued by Kocherlakota (1992). Furthermore, Kocherlakota (2009) shows that the credit constraint naturally leads to a bubble in the pricing of the collateral. The reason is simple. An asset that works as collateral helps to relax credit constraints. Individuals facing credit constraints have incentives to hold the asset forever, regardless of a bubble in its pricing. These constraints make bubbles not to be ruled out from an infinite-horizon model. A bubble modeled by this way improves the efficiency of resource allocation. So it is good for the economy. But, the result is not consistent with our wisdom.

Another way to restrict individuals' behaviors that rule bubbles out is introducing the "spirit of capitalism". Based on the idea of Max Weber (1958), individuals, in an economy with the "spirit of capitalism", not only care about the material rewards provided by their wealth, but also enjoy holding the wealth itself. If individuals always enjoy the increase of their wealth but not just for the purpose of material rewards, they would like to hold an asset with a bubble in its pricing forever. Kamihigashi (2008, 2009) models the bubble in an infinite-horizon model by requiring the marginal benefit of holding wealth always to be positive. However, Zhou (2011) stresses the relative degree of the marginal benefit of holding wealth and the marginal utility of consumption. A bubble in the pricing of a zero-dividend asset might arise provided that the ratio of the marginal benefit of holding wealth to the marginal utility of consumption is positive at the end of the world. Zhou (2011) suggests that the bubble crowds out investment and retards economic growth.

This paper extends the discussion of Zhou (2011) to a bubble in the pricing of an asset with positive dividends. It shows that a rational equity bubble can arise in an infinite-horizon model with endogenous growth. In order to model the "spirit of capitalism", I follow the method of Heng-fu Zou (1991) to set the wealth term directly into the utility function. Dividends of the equity are the total profits of homogenous firms. I model the endogenous growth by a production function with positive externality. Given the standard preference function (1), I prove that the existence of an equity bubble depends on values of parameters that measure the degree of constant relative risk aversion (CRRA). When the value of the CRRA parameter to the consumption term is larger than that to the wealth term, a bubble possibly arises in the pricing of the equity. If the rate of time preference is larger than the real interest rate, an unstable bubbleless steady state also exists.

The reason why an equity bubble can exist is same as that given by Zhou (2011). The restriction on CRRA parameters helps to guarantee that an individual in a bubbly economy will still feel happy by holding one more unit of wealth itself, relative to from one more unit of consumption, at the end of the world. Therefore, the individual would like to hold an asset with a bubble in its pricing for enjoying the increase of his wealth itself, so that the bubble cannot be ruled out. The "spirit of capitalism" is important to cause a bubble. However, it might appear that there is a simple explanation why the equilibrium price of the equity could stay above its fundamental value. This explanation attributes the bubble term to flows of utility by holding the asset itself. This simple intuition is however incorrect because with another restriction on time preference, no bubble could exist, namely that the equilibrium price has to be equal to the fundamental value.

This paper also provides the dynamic analysis on both bubbly equilibrium and bubbleless equilibrium. Basing on the analysis, I use the phase diagrams to illustrate a scenario about the birth of a bubble. A sudden innovation in technology will force the initial bubbleless economy to jump onto the trajectory, which finally converges to the bubbly balanced growth path. It implies that the new higher technology would trigger bubbles, just like our conventional story.

The rest of paper is organized as follows. Section 2 describes the model. Section 3 focuses on the sufficient and necessary conditions for the existence of bubbles in an economy with endogenous growth. Section 4 provides a dynamic analysis by phase diagrams. Section 5 illustrates a scenario that a technology shock might trigger the birth of bubbles. Section 6 concludes.

2 The Model

Time is continuous. An infinite number of identical individuals, who live forever, are continuously and evenly distributed in the area of [0,1]. Every individual can rent his physical capital to firms, and receives a rental at the rate of r. The capital stock is denoted by k. Each individual is also able to invest in financial assets. For convenience, I suppose that there is only one kind of asset, the equity, in this economy. The total supply of this asset is normalized by 1. One unit of equity receives a dividend every period. The value of this dividend is equal to the total profit of firms, Π, which is endogenous.

Each individual wishes to maximize the sum of time discounted utility values

$$\int_0^\infty e^{-\rho t}\left(\frac{c^{1-\sigma}-1}{1-\sigma}+\eta\frac{a^{1-\gamma}-1}{1-\gamma}\right)dt, \quad \rho>0, \sigma\geq 1, \gamma\geq 1, \eta>0, \tag{1}$$

subject to below budget constraint.

$$\dot{a} = rk - c + (\dot{q}+\Pi)s, \tag{2}$$

where ρ is the rate of time preference, σ, γ, and η are all preference parameters. Here, s denotes the amount of the equity held by the representative individual, the price of this asset is given by q, c is the amount of consumption, and $a \equiv qs+k$ is the amount of wealth, which is equal to the sum of values of equity and physical capital. Following the methodology of Heng-fu Zou (1991), I set the wealth term directly into utility function to model "the spirit of capitalism".

The first order conditions are given by the Euler equation

$$-\sigma\frac{\dot{c}}{c} = \rho - \eta\frac{c^\sigma}{a^\gamma} - r, \tag{3}$$

and the non-arbitrage condition

$$\frac{\dot{q}+\Pi}{q} = r. \tag{4}$$

Below two transversality conditions should also be satisfied.

$$\lim_{t\to\infty} e^{-\rho t}\mu k = 0, \tag{5}$$

$$\lim_{t\to\infty} e^{-\rho t}\mu q s = 0. \tag{6}$$

The non-arbitrage condition (4) can be rewritten as

$$e^{\int_0^t -r_i di}(\dot{q} - rq) = -e^{\int_0^t -r_i di}\Pi = \frac{d(e^{\int_0^t -r_i di}q)}{t}.$$

From the second equality sign, we can obtain that

$$e^{\int_0^\infty -r_i di}q_\infty - e^{\int_0^t -r_i di}q = -\int_t^\infty e^{\int_0^t -r_i di}\Pi dt.$$

It is easy to find that

$$q = \int_t^\infty e^{\int_t^j -r_i di}\Pi_j dj + e^{\int_t^\infty -r_i di}q_\infty.$$

Here, the first term that I denote by q^f in the following, is the standard definition of fundamental value of equity in the literature, the second term is the equity bubble, which is denoted by q^b. That is,

$$q^f \equiv \int_t^\infty e^{\int_t^i -r_j dj}\Pi_i di, \tag{7}$$

and

$$q^b \equiv e^{\int_t^\infty -r_i di}q_\infty.$$

Taking derivative of t on both sides of the above two equations, respectively, we can obtain that

$$\dot{q}^f = -\Pi + rq^f, \tag{8}$$

and

$$\dot{q}^b = re^{\int_t^\infty -r_i di}q_\infty = rq^b. \tag{9}$$

This implies that the rational equity bubble always grows at the speed of real interest rate once it exists.

Since the fundamental value of the equity and the bubble term are both non-negative, the transversality conditions (6) can be rewritten as

$$\lim_{t\to\infty} e^{-\rho t}c^{-\sigma}q^f = 0, \tag{10}$$

$$\lim_{t\to\infty} e^{-\rho t}c^{-\sigma}q^b = 0. \tag{11}$$

Given the fact that

$$\Pi = f(k) - f'(k)k, \tag{12}$$

the non-arbitrage condition (4) can be rewritten as

$$\dot{q} = f'(k)(q+k) - \delta q - f(k).$$

At equilibrium, we know that

$$a = q + k.$$

Together with equation

$$\dot{k} = f(k) - \delta k - c,$$

we obtain that

$$\dot{a} = (f'(k) - \delta)a - c. \tag{13}$$

When the production function is of decreasing return to scale, the real economy will converge to some steady state. If a bubble exists, i.e., $q_0^b > 0$, by equation (9), it should eventually grow at the speed of r^*, which would be the constant rate of real interest at this steady state. Since this rate of real interest must be positive[1], wealth a will converge to infinity. Based on the specification of the utility function, it is easy to see that

$$\lim_{t\to\infty} \frac{c^\sigma}{a^\gamma} = 0.$$

Thus, the transversality condition (11) does not hold. This means that this kind of equity bubble cannot exist in a neo-classical growth economy. The reason can be briefly interpreted by the fact that the real economy, with the production function of decreasing returns to scale, cannot support the growth of a bubble. Therefore, in this paper we focus on the production function that could generate endogenous growth. For convenience, the production function is given by

$$f(k) \equiv Ak^\alpha \bar{k}^{1-\alpha}, 0 < \alpha < 1, \tag{14}$$

where A is the technology level, $\bar{k}$ is the average capital stock. This type of production also can guarantee positive dividends of firms' equities.

Under this special setup, the real interest rate, r, is given by

$$\alpha A k^{\alpha-1}\bar{k}^{1-\alpha} - \delta.$$

[1] When real interest rate is negative or zero, by equation (13), the wealth will finally converge to negative.

The profit, Π, is given by

$$(1-\alpha)Ak^{\alpha}\bar{k}^{1-\alpha}.$$

At equilibrium, individuals' capital stocks are all equivalent, i.e.,

$$k = \bar{k}$$

Thus, we have

$$r = \alpha A - \delta > 0,$$

and

$$\Pi = (1-\alpha)Ak.$$

Furthermore, this aggregate economy is determined by the following system of equations.

$$-\sigma\frac{\dot{c}}{c} = \rho - \eta\frac{c^{\sigma}}{a^{\gamma}} - (\alpha A - \delta) \tag{15}$$

$$\dot{a} = (\alpha A - \delta)a - c \tag{16}$$

$$\dot{k} = Ak - \delta k - c \tag{17}$$

$$\lim_{t\to\infty} e^{-\rho t}c^{-\sigma}a = 0 \tag{18}$$

3 The Existence of A Bubbly Economy

The following two propositions show that $\sigma > \gamma$ is the necessary and sufficient condition to guarantee a bubbly economy. The details of proofs are included in the appendix.

Proposition 1 *When $\sigma \leq \gamma$, a bubbly economy does not exist.*

The contrapositive of this proposition actually proves that $\sigma > \gamma$ is the necessary condition for the existence of bubbles. The sufficient condition is provided by the proposition below.

Proposition 2 *When $\sigma > \gamma$, a bubbly economy does exist.*

The first proposition just shows that under some parameter restriction no bubble could arise in an economy with the "spirit of capitalism". The result does help us to recognize that below intuition is not correct. People with the intuition simply assert that the value of bubble just comes from the direct utility flow by holding the asset. However, from the proposition, we do find that when $\sigma \leq \gamma$, the price of the equity is just equal to its fundamental value, which is from the discounted dividend flows. In the economy with the "spirit of capitalism", the direct utility of holding the asset is still positive. But, the bubble is gone. Obviously, above simple intuition cannot explain this result.

However, the direct utility of holding wealth is truly important to guarantee the existence of bubbles. But, it does not generate bubbles. It works by preventing the transversality conditions from ruling bubbles out. Zhou (2011) provides more technique details for how the direct utility of holding wealth works. Here, I just explain it intuitively. In this specified economy, when $\sigma > \gamma$, the marginal utility of holding wealth relative to the marginal utility of consumption, is always non-trivial, even when the time goes to infinity. Thus, even at the end of the world, people still have incentives to hold wealth but not for the purpose of consumption. Therefore, people are willing to hold an asset with a bubble in its pricing, even if the price of the asset is not supported by material reward flows.

Given the necessary and sufficient conditions for the existence of bubbles, it is natural to ask whether the bubbleless economy exists or not in the environment where bubbles might exist. The following proposition shows that only if the rate of time preference, ρ, is larger than the real interest rate, r, a bubbleless economy can exist.

Proposition 3 *In the case of $\sigma > \gamma$, when $\rho \leq \alpha A - \delta$, the bubbleless economy does not exist; when $\rho > \alpha A - \delta$, the steady state, where $c^* = [\frac{\rho - (\alpha A - \delta)}{\eta(\alpha A - \delta)^\gamma}]^{\frac{1}{\sigma - \gamma}}$, $a^* = [\frac{\rho - (\alpha A - \delta)}{\eta(\alpha A - \delta)^\sigma}]^{\frac{1}{\sigma - \gamma}}$ corresponds to the unique bubbleless steady state in space of c and a.*

4 Dynamic Analysis

This section discusses the dynamics of both the economy where $\sigma > \gamma$ and the economy where $\sigma < \gamma$.

The optimal behaviors of the representative agent can be described by the Euler equation (3) and the following equation

$$\dot{a} = ra - c,$$

which comes from the budget constraint (2) and equation (4). In this economy, the real interest rate, r, is always a positive constant, $\alpha A - \delta$. I suppose that

$\alpha A - \delta$ is less than the rate of time preference, ρ. As illustrated by Figure 1, in the space of c and a, the $\dot{a} = 0$ locus is a straight line through the origin, and the $\frac{\dot{c}}{c} = 0$ locus is convex when $\sigma < \gamma$, and concave when $\sigma > \gamma$. It is clear that a saddle path shown by the dash-dot line approaches a stable bubbleless steady state when $\sigma < \gamma$. When $\sigma > \gamma$, the bubbleless steady state is unstable and a trajectory shown by the bold solid line converges to the bubbly balanced growth path that is given by Proposition 2.

Using the similar method as that given by Zhou (2011), we also can obtain another two-dimensional dynamic system in the space of q and k, which consists of the following pair of differential equations.

$$\dot{q} = (\alpha A - \delta)q - (1 - \alpha)Ak, \tag{19}$$

and

$$\dot{k} = (A - \delta)k - c(k, q), \tag{20}$$

where

$$\frac{\partial c}{\partial k} > 0, \frac{\partial c}{\partial q} > 0.$$

Figure 2 gives the phase diagram for the case of $\sigma > \gamma$. With the increasing of capital stock, the $\dot{k} = 0$ locus should eventually be above the $\dot{q} = 0$ locus. From the indication of arrows which show the directions of motion of equity price and capital stock, we can easily find an unstable bubbleless steady state at the level of capital, $k^*_{non-bubble}$. To the right of the vertical dot-dash line of $k = k^*_{non-bubble}$, there is one trajectory that eventually converges to the bubbly balanced growth path. To the left of this vertical line, all trajectories converge to the origin and this has no economic meaning. Therefore, in this case, when the initial capital is just equal to $k^*_{non-bubble}$, the economy will always stay at the bubbleless steady state; when capital stock k is larger than $k^*_{non-bubble}$, equity price q must be along the trajectory approaching the bubbly balanced growth path and includes a bubble term. The economy along this trajectory must be a bubbly economy.

Figure 3 provides the phase diagram for the case $\sigma < \gamma$. In the figure, the $\dot{k} = 0$ locus will eventually be below the $\dot{q} = 0$ locus, with the increase in capital stock. Following the indication of arrows, we also easily find that a saddle path drawn by the dash-dot line converges to the stable bubbleless steady state, which was solved in the proof of Proposition 3. Claim 4 also proves that no bubble is on this saddle path. Thus, in this case, for any value of capital stock, k, equity price, q, must be along the saddle path and excludes any bubble term. The economy along this saddle path is a bubbleless economy.

5 The Birth of Bubbles

This section provides a scenario about how the equity bubbles might arise. The mechanism can be described by the phase diagrams given above.

In an environment where the preference parameter σ is larger than γ, suppose that the initial economy is just at the bubbleless steady state. For convenience, I set the rate of capital depreciation, δ, to be zero. As Figure 4 illustrates, an unexpected innovation of technology, or, a sudden news about the innovation, would shift the $\dot{k} = 0$ locus to the left. But, the locus of $\dot{q} = 0$ would not change. People cannot adjust their stock of capital immediately. However, the economy has to immediately jump on the new trajectory that approaches the new bubbly balanced growth path, which is described by the bold solid line. Therefore, equity price would increase. The increase of equity price is not only from higher expected dividends, but also because a bubble has been created.

6 Conclusion

This paper focuses on rational bubbles of equities with positive dividends in an economy with "the spirit of capitalism". I prove that the existence of a bubbly equilibrium depends on the degrees of constant relative risk aversion. Therefore, it would be of interest to empirically estimate the values of CRRA parameters both on consumption and on wealth. We leave this for future research.

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Appendix A: Necessary Claims and Lemmas

In this appendix, I give some claims and lemmas. These claims and lemmas are necessary to help us find the necessary and sufficient conditions that guarantee a bubbly economy. Their proofs are also provided.

For convenience, we need to make an equivalent transformation of the equations system given by equations of (15), (16), and (17). Under the definitions of

$$x \equiv \frac{\eta c^{\sigma}}{a^{\gamma}}, y \equiv \frac{c}{a}, z \equiv \frac{c}{k},$$

we find that

$$c = (\frac{x}{\eta y^{\gamma}})^{\frac{1}{\sigma-\gamma}}, a = (\frac{x}{\eta y^{\sigma}})^{\frac{1}{\sigma-\gamma}}, k = \frac{(\frac{x}{\eta y^{\gamma}})^{\frac{1}{\sigma-\gamma}}}{z},$$

and

$$\frac{\dot{x}}{x} = \sigma\frac{\dot{c}}{c} - \gamma\frac{\dot{a}}{a},$$

$$\frac{\dot{y}}{y} = \frac{\dot{c}}{c} - \frac{\dot{a}}{a},$$

$$\frac{\dot{z}}{z} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k}.$$

Thus, the Euler equation (15) and equation (16) can be rewritten as follows,

$$\frac{\dot{x}}{x} = x + \gamma y - (\gamma - 1)(\alpha A - \delta) - \rho, \tag{A.1}$$

$$\frac{\dot{y}}{y} = \frac{x}{\sigma} + y - \frac{(\sigma - 1)(\alpha A - \delta) + \rho}{\sigma}, \tag{A.2}$$

and equation (17) can be rewritten as

$$\frac{\dot{z}}{z} = \frac{x}{\sigma} + z + \frac{(\alpha A - \delta) - \rho}{\sigma} - (A - \delta). \tag{A.3}$$

In this new equation system, x, y, and z cannot converge to any balanced growth path, but do converge to some steady states. These steady states correspond to the balanced growth path or the steady state in the original equation system of (15), (17), and (16). Thus, the existence of bubble can be examined more easily by checking the property of the steady states in the space of $\{x, y, z\}$.

All of the steady states in the space of x and y, are listed below.

Steady State 1: $x^* = 0, y^* = 0$;
Steady State 2: $x^* = 0, y^* = \frac{(\sigma-1)(\alpha A-\delta)+\rho}{\sigma}$;
Steady State 3: $x^* = (\gamma - 1)(\alpha A - \delta) + \rho, y^* = 0$;
Steady State 4: $x^* = \rho - (\alpha A - \delta), y^* = \alpha A - \delta$.

It is easy to see that only steady state 4 corresponds to the steady state in the space of $\{c, a\}$, the other three steady states correspond to balanced growth paths. It is because that only at the steady state 4 the growth rate of consumption, $\frac{\dot{c}}{c}$, and that of wealth, $\frac{\dot{a}}{a}$, are both zero.

In the following analysis, it is necessary to discriminate which steady states correspond to the bubbleless economy and which correspond to the bubbly economy.

We begin our analysis at the only steady state in the space of $\{c, a\}$. Claim 4 contends that this steady state and trajectories to it, represent a bubbleless economy.

Claim 4 *No bubble exists at the steady state corresponding to the steady state 4 in the space of x and y; no bubble exists on the trajectory approaching this steady state.*

Proof. First, suppose that a bubble exists on trajectories to the steady state. By equation (9), the bubble grows at the rate of r. Since c converges to some positive constant, the capital level cannot be always decreasing. Or, the consumption level cannot be supported.

If $q_0^b > 0$, given that $a = q^f + q^b + k$, a will diverge. This contradicts the fact that the trajectory goes to the steady state.

If $q_0^b < 0$, to guarantee that a would converge to some positive constant, $q^f + k$ needs to always grow. Given the definition of q^f, k needs to always grow. Since $q = q^f + q^b = a - k$, asset price eventually becomes negative.

Therefore, no bubble exists on these trajectories. This is similar to proving that there is no bubble at the steady state. ∎

Before checking whether bubbles exist on balanced growth paths in the space of c and a, we need to know some basic properties of economies on the balanced growth path. These properties are described by the following two Lemmas.

Lemma 5 *At the balanced growth path, the growth rate of fundamental value of this equity, g_{q^f}, is equivalent to the growth rate of capital, g_k. The value of this growth rate is less than the real interest rate, $\alpha A - \delta$.*

Proof. At the balanced growth path, equation (7) can be rewritten as

$$q^f = (1-\alpha)Ak\int_t^{\infty} e^{[g_k-(\alpha A-\delta)](n-t)}dn.$$

When g_k is less than $\alpha A-\delta$, we obtain that $q^f = \frac{(1-\alpha)A}{\alpha A-\delta-g_k}k$. This implies that $g_{q^f} = g_k < \ \alpha A-\delta$.

However, when $g_k \geq \alpha A-\delta$, by equation (8), we obtain $g_{q^f} \leq \alpha A-\delta$.

Given the fact that $a = q^f + q^b + k$, if $g_k > r$, the growth rate of wealth, g_a, will eventually converge to g_k. From equation (13), we see that $g_a = r - \frac{c}{a}$. If g_a finally converges to g_k, which is larger than r, then the ratio of $\frac{c}{a}$ should eventually converge to a negative constant. This is impossible.

If $g_k = \alpha A-\delta$, then g_a will eventually converge to $\alpha A-\delta$ so that $g_a = g_k$. By equation (13), the ratio of $\frac{c}{a}$ finally converges to 0. This means that the growth rate of consumption, g_c, will be less than g_k at the balanced growth path. However, from equation (17), we can obtain that $\frac{\dot{k}}{k} = A-\delta-\frac{c}{k}$. Given $g_c < g_k$, the ratio of $\frac{c}{k}$ will finally converge to 0. Thus, g_k will converge to $A-\delta$, which is larger than $\alpha A-\delta$. This is contradictory to the previous assumption of $g_k = \alpha A-\delta$.

Therefore, $g_{q^f} = g_k < \alpha A-\delta$. ∎

This lemma is helpful in proving the next lemma, which can be used as a discriminant theorem on whether bubbles exist or not.

Lemma 6 *At the balanced growth path, when the growth rate of wealth,* $\frac{\dot{a}}{a}$ *is equal to the real interest rate,* $\alpha A-\delta$*, then bubbles exist, i.e.,* $q^b > 0$; *and, vice versa.*

Proof. When bubbles exist, that is, $q^b > 0$, from the fact that $a = q^f + q^b + k$, we obtain

$$\frac{\dot{a}}{a} = \frac{\frac{\dot{q}^f}{q^f}\frac{q^f}{q^b} + \frac{\dot{q}^b}{q^b} + \frac{\dot{k}}{k}\frac{k}{q^b}}{\frac{q^f}{q^b} + 1 + \frac{k}{q^b}}.$$

By Lemma 5 and equation (9), it can be easily obtained that the ratios of $\frac{q^f}{q^b}$ and $\frac{k}{q^b}$ will both converge to zero. Thus, at the balanced growth path, the growth rate of wealth, $\frac{\dot{a}}{a}$ will eventually converge to $\alpha A-\delta$.

When $\frac{\dot{a}}{a} = \alpha A-\delta$, and if no bubble exists, by Lemma 5, then $\frac{\dot{a}}{a} = g_k < \alpha A-\delta$. This is contradictory to $\frac{\dot{a}}{a} = \alpha A-\delta$. Thus, bubbles must exist. ∎

Using the above Lemma, we can easily judge whether balanced growth paths in the space of c and a, include bubbles or not. The following Claim presents this result.

Claim 7 *No bubble exists on the balanced growth path corresponding to the steady state 2 in space of x and y; no bubble exists on the trajectory approaching this balanced growth path. However, steady state 1 and steady state 3 correspond to the balanced growth paths in space of c and a, which include bubbles.*

Proof. At steady state 2, we know that $y^* > 0$, which means $\frac{\dot{a}}{a} < \alpha A - \delta$. By Lemma 6, we know that no bubble exists on this balanced growth path. In addition, it is also easy to prove that no bubble exists on the trajectory approaching this balanced growth path. If there is a bubble on this trajectory, then there must be a bubble on the balanced growth path. This means that $\frac{\dot{a}}{a} = \alpha A - \delta$ by Lemma 6. But, this is contradictory to $y^* > 0$.

However, at steady state 1 and steady state 3, it is clear that $y^* = 0$, which means $\frac{\dot{a}}{a} = \alpha A - \delta$. By Lemma 6, bubbles exist on the balanced growth paths in space of c and a, which correspond to steady state 1 and steady state 3. ■

Appendix B: Proofs of Propositions

This appendix provides the proofs of three propositions in section 3.

Proof of Proposition 1

Proof. Steady state 1 means that $\frac{\dot{a}}{a} = \alpha A - \delta$, and $\frac{\eta c^\sigma}{a^\gamma} = 0$. From Euler equation (3), we obtain that $-\sigma \frac{\dot{c}}{c} = \rho - (\alpha A - \delta)$, which means that $\frac{\dot{\mu}}{\mu} + \frac{\dot{a}}{a} = \rho$. Given the fact that $\mu a > 0$, the transversality condition is violated in this case. Therefore, the bubbly economy corresponding to this steady state and the trajectory approaching this steady state, does not exist.

Steady state 3 means that $\frac{\dot{a}}{a} = \alpha A - \delta$, $\frac{\dot{c}}{c} = \frac{\gamma}{\sigma}(\alpha A - \delta)$, $\frac{\eta c^\sigma}{a^\gamma} = (\gamma - 1)(\alpha A - \delta) + \rho > 0$, and $\frac{c}{a} = 0$. Given $\sigma \leq \gamma$, we know that $\frac{\dot{c}}{c} \geq \frac{\dot{a}}{a}$. Given the fact that bubbles exist and the capital stock and fundamental value of shares cannot be negative, the wealth a should be positive. Thus, unless $c = 0$, we cannot obtain $\frac{c}{a}$ equal to zero. However, $c = 0$, implies $\frac{\eta c^\sigma}{a^\gamma} = 0$, which is contradictory to the previous finding $\frac{\eta c^\sigma}{a^\gamma} > 0$. Thus, a bubbly economy corresponding to this steady state does not exist.

Therefore, no bubbly economy exists when $\sigma \leq \gamma$. ■

Proof of Proposition 2

Proof. There is still no bubble economy on the trajectory approaching steady state 1 in the space of x and y. The proof is same as the corresponding section in Proposition 1.

However, at steady state 3, we obtain that $\frac{\dot{a}}{a} = \alpha A - \delta$, $\frac{\dot{c}}{c} = \frac{\gamma}{\sigma}(\alpha A - \delta) < \frac{\dot{a}}{a}$, $\frac{c}{a} = 0$, and $\frac{\eta c^\sigma}{a^\gamma} = (\gamma - 1)(\alpha A - \delta) + \rho > 0$. It is easy to see that $\frac{\dot{\mu}}{\mu} + \frac{\dot{a}}{a} < \rho$, which ensures that the transversality condition holds. By equation (A.3), we obtain that z^* is 0, or, $A - \delta - \frac{\gamma(\alpha A - \delta)}{\sigma}$, which is a positive constant.

$z^* = 0$ means that $\frac{\dot{k}}{k} = A - \delta$, which implies that $\frac{\dot{a}}{a}$ eventually converges to $A - \delta$ unless $k = 0$. When $k \neq 0$, this is contradictory to $\frac{\dot{a}}{a} = \alpha A - \delta$. When $k = 0$, consumption has to be equal to zero. This contradicts $\frac{\eta c^\sigma}{a^\gamma} > 0$ given the fact that bubbles exist at this steady state. Thus, $z^* = 0$ cannot be an equilibrium.

When $z^* = A - \delta - \frac{\gamma(\alpha A - \delta)}{\sigma}$, we obtain $\frac{\dot{k}}{k} = \frac{\gamma(\alpha A - \delta)}{\sigma} = \frac{\dot{c}}{c} < \frac{\dot{a}}{a}$, $\frac{c}{k} = A - \delta - \frac{\gamma(\alpha A - \delta)}{\sigma} > 0$. This is the unique balanced growth path with bubbles in the space of c, a, and k.

In addition, it is also necessary to mention that the long-run growth rate of the bubbly economy, $\gamma r/\sigma$, is larger than the growth rate in the traditional endogenous growth model, $(r-\rho)/\sigma$. This is because that the existence of "the spirit of capitalism" stimulates the accumulation of capital, which in turn promotes the long-run growth. ■

Proof of Proposition 3

Proof. In the space of x and y, steady state 2 and 4 correspond to the possible non-bubble economies.

At the balanced growth path corresponding to steady state 2, we obtain $\frac{c}{a} = \frac{(\sigma-1)(\alpha A-\delta)+\rho}{\sigma} > 0$. This means that $\frac{\dot{c}}{c} = \frac{\dot{a}}{a}$. However, $x^* = 0$ means $\frac{\eta c^\sigma}{a^\gamma} = 0$. Given $\sigma > \gamma$, only $c = 0$ and $a \neq 0$ can guarantee above equality to hold. However, $c = 0$ and $a \neq 0$, imply that $\frac{c}{a} = 0$. This is contradictory to $\frac{c}{a} > 0$.

Steady state 4 corresponds to a non-bubble steady state in the space of c and a. When $\rho < \alpha A - \delta$, $x^* < 0$, it means that there are negative values for c or a; when $\rho = \alpha A - \delta$, steady state 4 coincides with steady state 2. Thus, in the case of $\rho \leq \alpha A - \delta$, the steady state corresponding to steady state 4 is not an economic equilibrium. When $\rho > \alpha A - \delta$, at steady state 4, we obtain $c^* = [\frac{\rho-(\alpha A-\delta)}{\eta(\alpha A-\delta)^\gamma}]^{\frac{1}{\sigma-\gamma}}$, $a^* = [\frac{\rho-(\alpha A-\delta)}{\eta(\alpha A-\delta)^\sigma}]^{\frac{1}{\sigma-\gamma}}$. By equation (17), we obtain $k^* = \frac{c^*}{A-\delta}$. From the definition of q^f given by equation (7), we obtain $q^{f*} = \frac{(1-\alpha)Ak^*}{\alpha A-\delta}$. We also find that $q^{f*} + k^*$ is exactly equal to a^*. Thus, this is the unique bubbleless steady state. ■

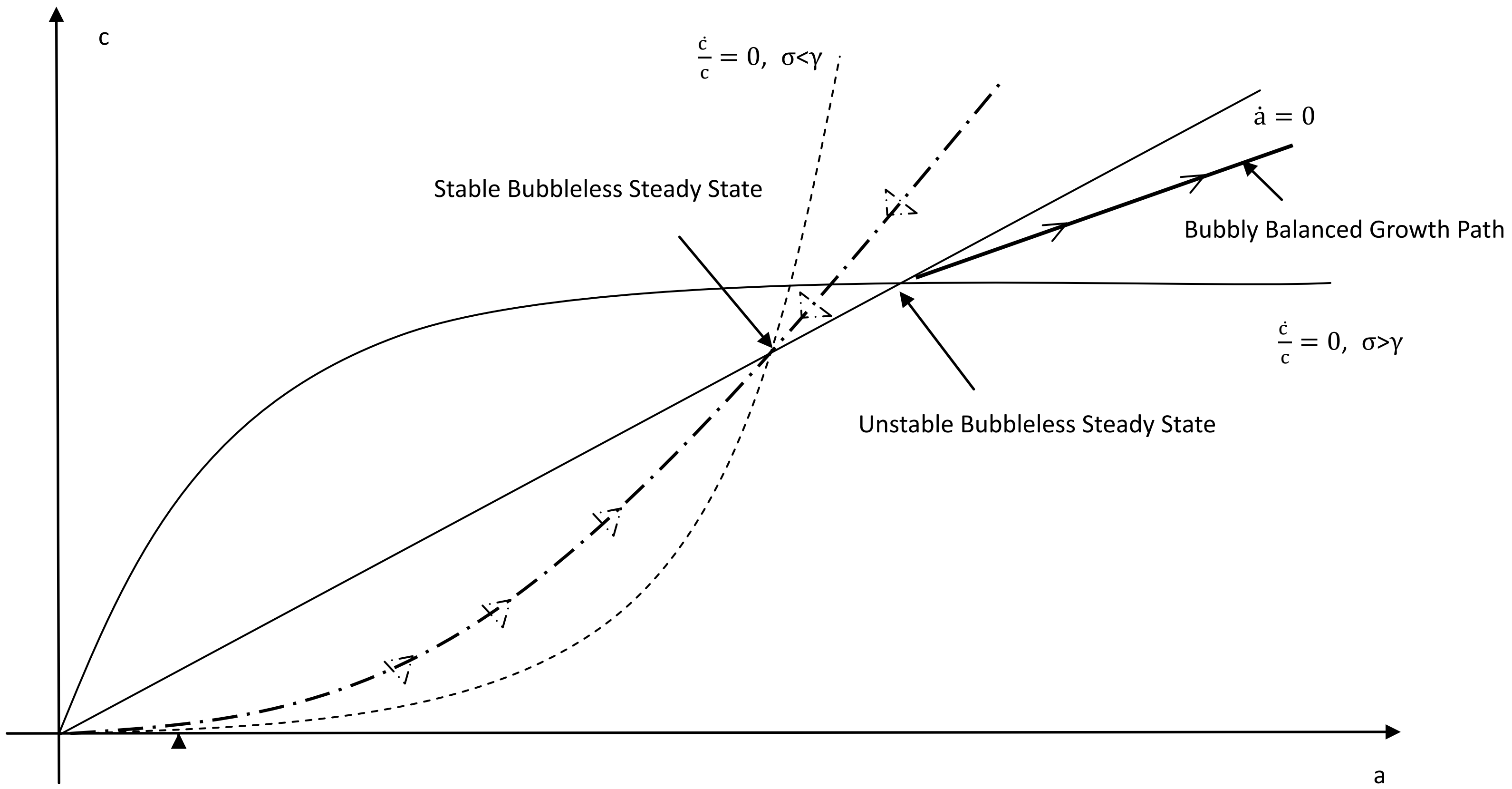

Figure 1: Dynamic Analysis

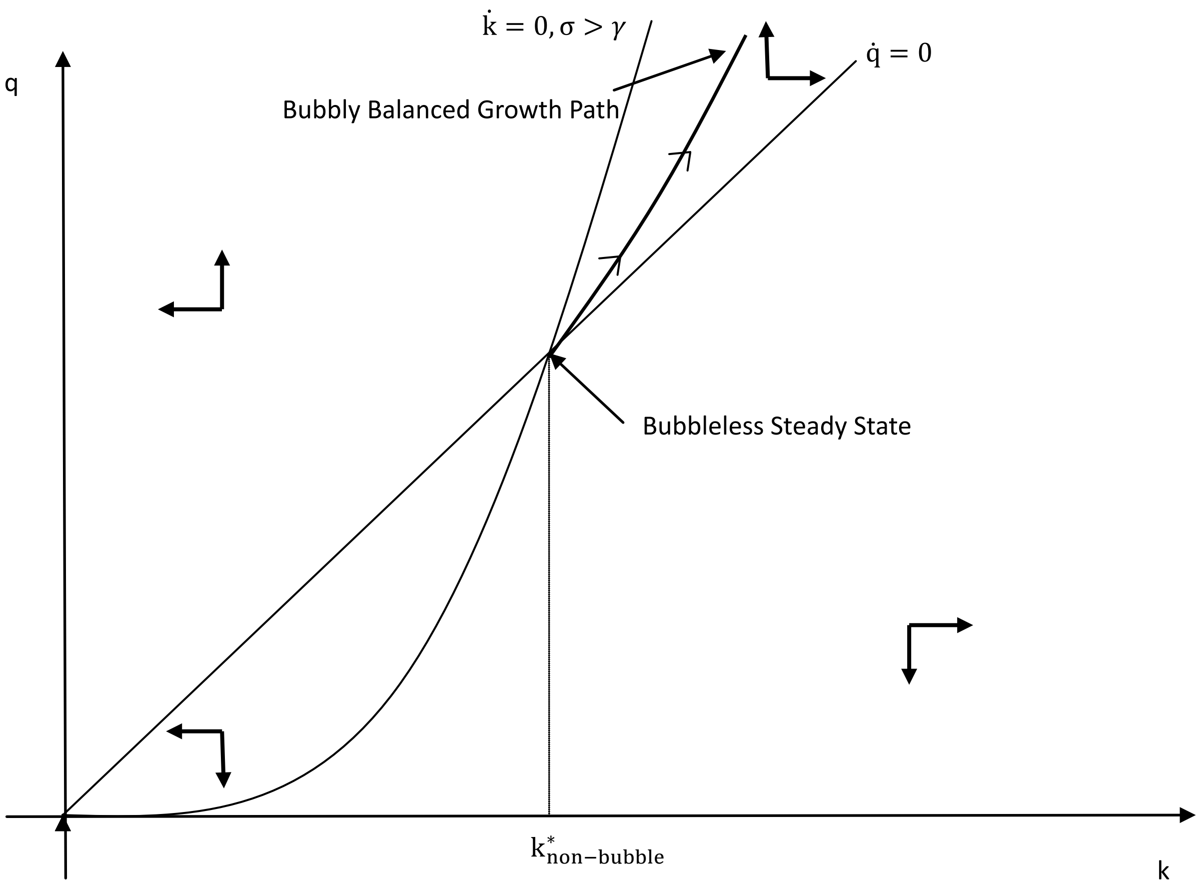

Figure 2: Bubbly Economy When $\sigma > \gamma$

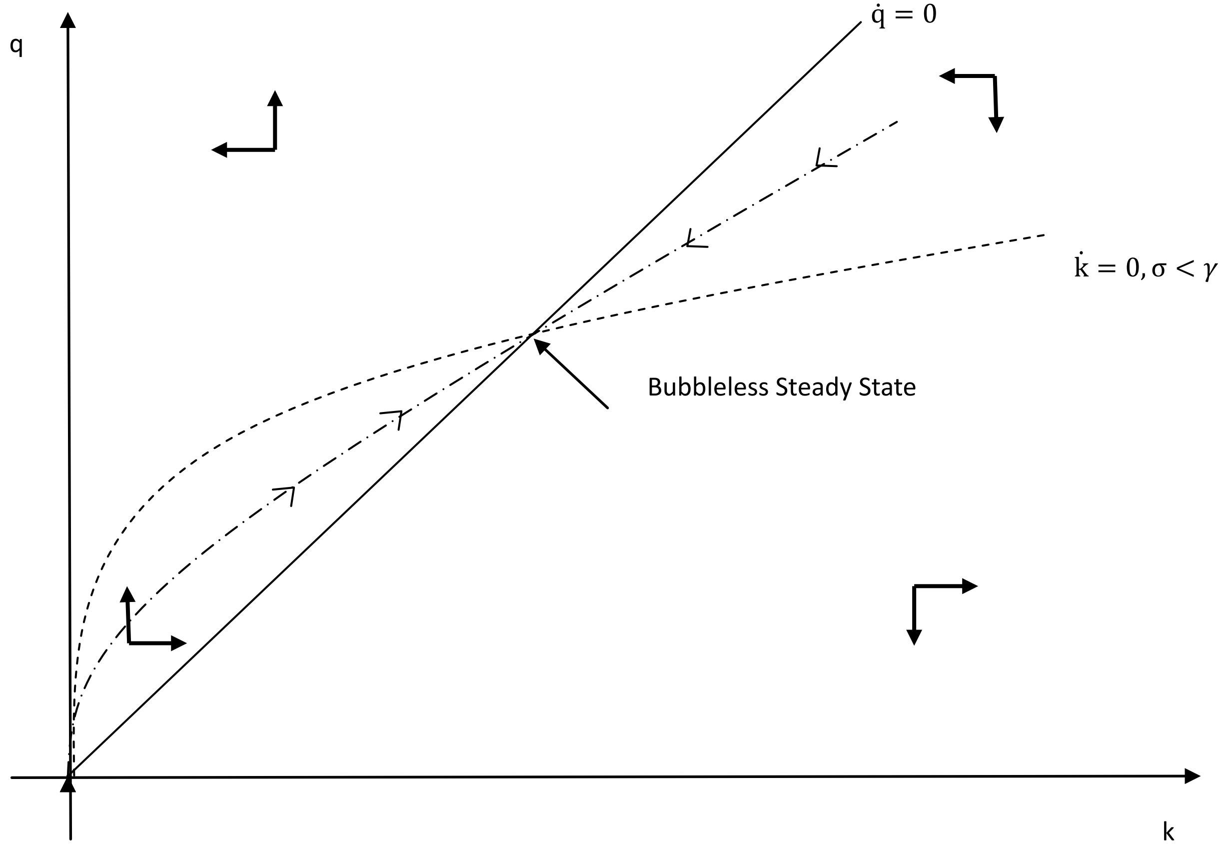

Figure 3: Bubbleless Economy When $\sigma < \gamma$

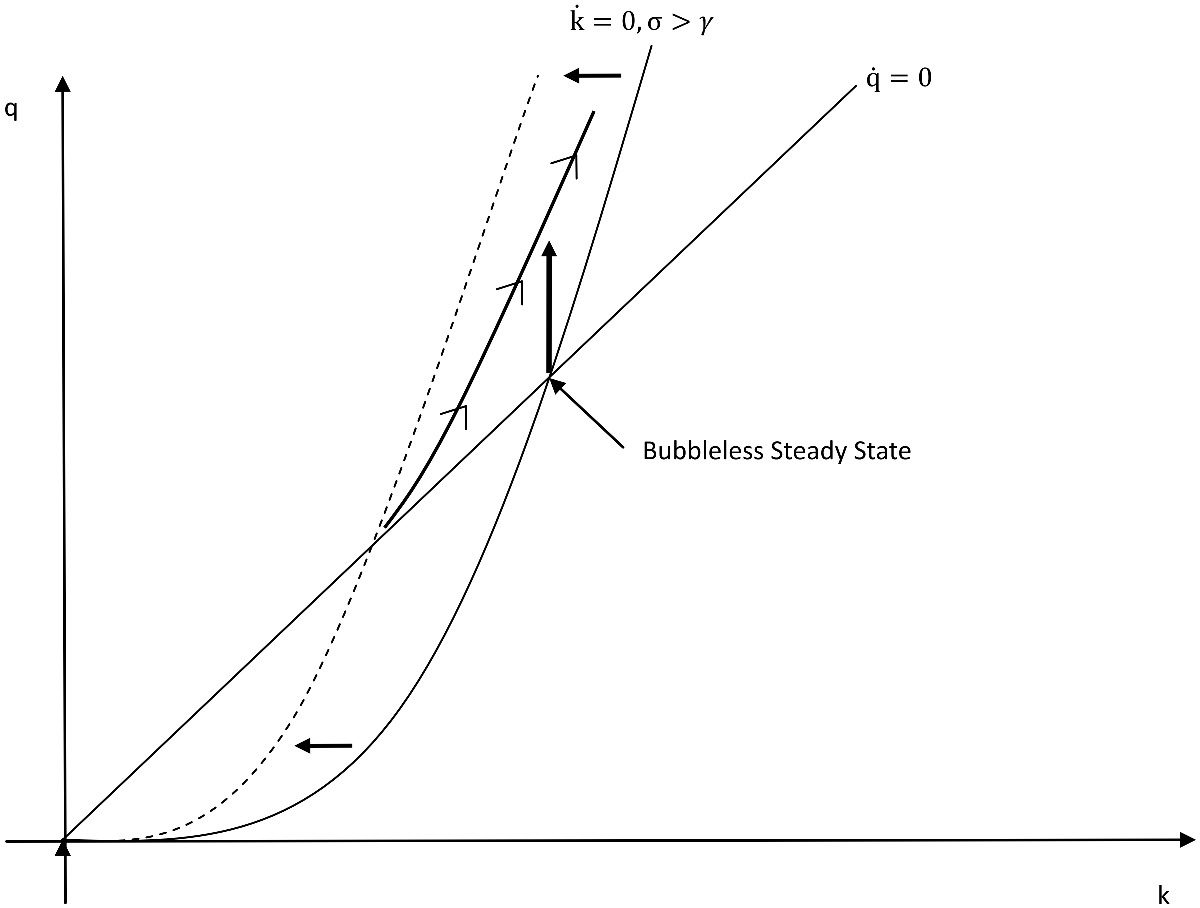

Figure 4: Birth of Bubbles