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Abstract

This paper conducts a comparative evaluation of the predictive performance of various Value at Risk (VaR) models such as GARCH-normal, GARCH-t, EGARCH, TGARCH models, variance-covariance method, historical simulation and filtred Historical Simulation, EVT and conditional EVT methods. Special emphasis is paid on two methodologies related to the Extreme Value Theory (EVT): The Peaks over Threshold (POT) and the Block Maxima (BM). Both estimation techniques are based on limits results for the excess distribution over high thresholds and block maxima, respectively. We apply both unconditional and conditional EVT models to management of extreme market risks in stock markets. They are applied on daily returns of the Tunisian stock exchange (BVMT) and CAC 40 indexes with the intension to compare the performance of various estimation methods on markets with different capitalization and trading practices. The sample extends over the period July 29, 1994 to December 30, 2005. We use a rolling windows of approximately four years (n= 1000 days). The sub-period from July, 1998 for BVMT (from August 4, 1998 for CAC 40) has been reserved for backtesting purposes. The results we report demonstrate that conditional POT-EVT method produces the most accurate forecasts of extreme losses both for standard and more extreme VaR quantiles. The conditional block maxima EVT method is less accurate.

Keywords : Financial Risk management, Value-at-Risk, Extreme Value Theory, Conditional EVT, Backtesting

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1. Introduction

Over the last seventeen years, risk management gained great importance due to increase in the volatility of financial markets and a desire of less volatile financial markets and less fragile financial system. Value-at-Risk models have been implemented throughout the financial industry and by non-financial corporations as well. VaR has became the key and standard measure that financial analysts use to quantify risk. It is defined as the maximum potential loss in value of an asset or a portfolio with a given probability over a certain horizon. It measures the potential loss on a portfolio that would result if relatively large adverse price movement were to occur. It is a number that indicates how much a financial institution or an investor can lose with a given probability over a given time horizon. The VaR's great popularity originates from the aggregation of several components of risk at firm and market into a single number.

The Basel Commitee on banking supervision (1996) at the bank for international settlements imposes to financial institutions such as banks and investment firms to meet capital requirement based on VaR estimates. It is crucially interesting to provide accurate estimates. If risk is not properly estimated, these can lead to a sub-optimal allocation.

VaR works on multiple levels, from the position-specific micro level to the portfolio-based macro level. It has become a common language for communication about aggregate risk taking, both within and outside an organisation.

A key element to VaR calculation is the distribution function we choose for the price change of an asset or portfolio. To calculate VaR, we can choose from three main methods: parametric, historical simulation and Monte Carlo simulation. Each method has some strengths and some weaknesses, and together offer a more comprehensive perspective of risk. The parametric method estimates VaR with equation that specifies parameters such as volatility and correlation, it is accurate for traditional assets and linear derivatives but less accurate for non linear derivatives and for skewed distributions. The Historical Simulation estimates VaR by reliving history; takes actual historical rates and revalues positions for each change in the market. Monte Carlo simulation method estimates VaR by simulating random scenarios and revaluing positions in the portfolio. The last two methods are appropriate for all types of instruments, linear and non linear and are mechanically identical in that they both revalue instruments, given changes in market rates. The difference lies in how they generate market scenarios. HS method takes actual past market movements as scenarios while MC method generates random hypothetical scenarios.

In this paper, we perform an evaluation of the predictive performance of the most popular and conventional VaR models such as GARCH-normal, GARCH-t, EGARCH, TGARCH models, variance-covariance method, historical simulation and filtred Historical Simulation, Unconditional EVT and conditional EVT as described by Mc Neil and Frey's (2000). The two authors calculate conditional VaR measures by filtering return series with a GARCH model and then apply threshold-based EVT tools to the independently identically distributed (iid) residuals. We extend this approach to the Block maxima method and create a conditional VaR forecasts based on the block maxima method. The performance of conditional block maxima EVT method is evaluated and compared with the conditional POT-EVT approach and with the others conventional methods. The models are backtested for their out-of-sample predictive ability by using Christoffersen's (1998) likelihood ratio tests for coverage probability. The data set used throughout this paper consists of daily returns on two indexes: the Tunisian stock exchange (BVMT) index and the CAC 40 index. The sample period is July 29, 1994 to

December 30, 2005. We use a rolling windows of approximately four years (n= 1000 days). The sub-period from July, 1998 for BVMT index (and from August 4, 1998 for CAC 40) has been reserved for backtesting purposes.

The present paper is organized as follows. The EVT, conditional EVT and conventional VaR estimation methods are introduced in section 2. Section 3 describes the evaluation framework for VaR estimation. Empirical results and predictive performance evaluation of several models are presented in section 4. We conclude afterwards.

2. VaR models

VaR has become a standard for measuring and assessing risk. It is defined as a quantile of the distribution of returns (or losses) of asset or portfolio in question. It is defined also as the predicted worst-case loss at a specific confidence level over a certain period of time. Some practitioners prefer to make in consideration the negative of this quantile, so that higher values of VaR correspond to higher level of risk.

Formally, Let $r_t = \log(p_t / p_{t-1})$ be the returns at time t where p_t is the price of an asset (or portfolio) at time t. We denote the (1-p)% quantile estimate at time t for a one-period-ahead return as VaR (p), so that

$$\Pr(r_t < VaR_t(p)) = p$$

The VaR's popularity originates from the aggregation of several components of risk at firm and market into a single number.

More formally, VaR is calculated based on the following equation

$$VaR_t = F^{-1}(p)\sigma_t$$

given that $F^{-1}(p)$ is the corresponding quantile of the assumed distribution and σ_t is the forecast of the conditional standard deviation at time t-1.

2.1 Variance-covariance method

The variance-covariance method is one of the simplest approach among various models used to estimate the VaR. let as assume that returns can be written as: $r_t = \mu_t + \varepsilon_t$ where ε_t has a distribution function F with zero mean and variance σ_t^2. The VaR can be calculated as

$$VaR_t^q = \hat{\mu}_t + F^{-1}(q)\hat{\sigma}_t \qquad (1)$$

where $F^{-1}(q)$ is the qth quantile value of an unknown distribution function F. We can estimate μ_t and σ_t^2 by the sample mean and the sample variance by

$$\hat{\mu}_t = \frac{1}{n}\sum_{i=1}^{n} r_t \qquad \hat{\sigma}^2_t = \frac{1}{n-1}\sum_{i=1}^{n}(r_t - \hat{\mu}_t)^2$$

For high quantile of a fat-tailed empirical distribution, the variance-covariance method underestimates risk since the normality assumption for financial series is usually rejected. In addition, this method is not appropriate for asymmetric distributions. Therefore, in order to estimate an accurate VaR number, researcher must make conjectures about the underlying distribution and about conditional variance innovation.

2.2 Historical simulation

An approach to VaR modelling is to estimate the quantile nonparametrically. A Conventional way is to use the historical simulation. This method is powerful because of its simplicity and its relative lack of theoretical baggage. It assume that the distribution of return will remain the same in the past and in the future and hence historical returns will be used in the forecast of Value-at-risk.

$$VaR_p = Quantile\left\{\left\{y_t\right\}_{t=1}^{n}, 100p\right\} \tag{2}$$

This method need not to make distributional assumptions (although parameter fitting may be performed on the resulting distribution). It accommodates non-normal distributions and therefore it accounts for fat tails and non-zero skewness. HS provides a full distribution of potential portfolio values, not just a specific percentile. The key assumption of this method is that the series under consideration is IID. For more turmoil periods, it can turn out to be a very bad measure of risk since risk can change significantly. VaR estimate using this simple approach is extremely sensitive to the choice of the sample length n. If n is too large, the most recent returns, that probably can describe better the future, have the same weight with the earliest observations. If n is too small, then a few or an insufficient extreme events will be observed and possibility to incorporate tail risk became more difficult.

2.3 Filtred Historical simulation

This method consists to combine volatility models (parametric) and historical simulation method (non-parametric). Such combination might lessen the problematic use of the traditional approaches, since it can accommodate the volatility clustering, the observed ”fat” tails and the skewness of the empirical distribution.

By using the quantiles of the standardized residuals and the conditional standard deviation forecast from a volatility model, the VaR number is calculated as:

$$VaR_p = \hat{\mu}_t + Quantile\left\{\left\{\varepsilon_t\right\}_{t=1}^{n}, 100p\right\}\sigma_{t+1} \tag{3}$$

In our empirical investigation, we assume that the volatility estimates and the corresponding quantiles are being generated via a GARCH(p,q) process.

2.4 GARCH models[2]

We assume that the return series is decomposed into two parts, the predictive and unpredictable component, $r_t = \mu_t + \varepsilon_t$ where μ_t is the conditional mean and ε_t is the unpredictable part or innovation process. The conditional mean return can be expressed as a s-th order autoregressive process, AR(s):

$$\mu_t = \phi_0 + \sum_{i=1}^{s} \phi r_{t-i} \tag{4}$$

The unpredictable component ε_t can be expressed as an ARCH process as follow:

$$\varepsilon_t = z_t \sigma_t$$

[2] For more detail on the use of volatility univariate and multivariate GARCH to meaure and evaluate risk, see Andersen, T.G., T. Bollerslev, P.F. Christoffersen and F.X. Diebold (2005)

where z_t is a sequence of independently and identically distributed random variables with zero mean and unit variance. The conditional variance on information at time t-1 of innovations ε_t is σ_t^2.

2.4.1 GARCH model:

Engle (1982) introduced the ARCH (p) model and expressed the conditional variance as a linear function of the past p squared innovations

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2$$

The conditional variance will be positive, if $\alpha_0 > 0$ and $\alpha_i \geq 0$ for i=1,2,…,p. Bollerslev (1986) proposed a generalization of the ARCH model, the GARCH(p,q) model. Generalized AutoRegressive Conditional Heteroskedasticity model permits to express the conditional variance as a linear function of lagged squared error terms and lagged conditional variance terms.

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{ji=1}^{q} \beta_j \sigma_{t-j}^2 \tag{5}$$

where $\alpha_0 > 0$ and $\alpha_i \geq 0$ for i=1,2,…,p. If $\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_j < 1$, the process ε_t is covariance stationary and its unconditional variance is equal to

$$\sigma_t^2 = \frac{\alpha_0}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{i=1}^{q} \beta_j}$$

The GARCH (p,q) model is successfully captures several characteristics of financial time series, such as thick tailed returns and volatility clustering.

2.4.2 EGARCH model:

In the GARCH model, the signs of residuals or shocks have no effects on conditional volatility only squared residuals enter in the conditional variance equation. However, a stylized fact of financial volatility is that bad news (negative shocks) tends to have a larger impact on volatility than good news (positive shocks). Bad news tends to drive down the stock price, thus increasing the leverage of the stock and the stock will be more volatile (Black , 1976) .
Nelson (1991) proposed the following exponential GARCH (EGARCH) model to allow for leverage effects:

$$In(\sigma_t^2) = \alpha_0 + \sum_{i=1}^{p} \alpha_i \frac{|\varepsilon_{t-i}| + \gamma_i \varepsilon_{t-i}}{\sigma_{t-i}} + \sum_{j=1}^{q} \beta_j In(\sigma_{t-j}^2) \tag{6}$$

In contrast to the GARCH model, no restrictions need to imposed on the model estimation, since the logarithmic transformation ensures that the forecasts of the variance are non-negative. Note that when ε_{t-i} is positive or there is a good news, the total effect of ε_{t-i} is $(1+\gamma_i)|\varepsilon_{t-i}|$; in contrast, when ε_{t-i} is negative or there are bad news, the total effect of ε_{t-i} is $(1-\gamma_i)|\varepsilon_{t-i}|$.

2.4.3 TGARCH model:

Another GARCH model that is capable of modelling leverage effects is the threshold GARCH (TGARCH) model or also known as the GJR model (Glosten, Jagannathan, and Runkle, 1993) which has the following form:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \gamma_i S_{t-i} \varepsilon_{t-i}^2 + \sum_{ji=1}^{q} \beta_j \sigma_{t-j}^2 \tag{7}$$

where

$$S_{t-i} = \begin{cases} 1 & if \quad \varepsilon_{t-i} < 0 \\ 0 & if \quad \varepsilon_{t-i} \geq 0 \end{cases}$$

That is, depending on whether innovation ε_{t-i} is above or below the threshold value of 0, ε_{t-i}^2 has different effects on the conditional variance: if innovation is negative, the total effects are $(\alpha_i + \gamma_i)\varepsilon_{t-i}^2$; when innovation is positive the total effect are given by $\alpha_i \varepsilon_{t-i}^2$.

Engle (1982) assumed that standardized residual z_t is normally distributed.

$$D(\varepsilon_t) = (2\pi)^{(-1/2)} \exp(-\frac{\varepsilon_t^2}{2})$$

However, given the well known fat tails in financial time series, it may be more desirable to use a distribution which has fatter tails than the normal distribution. Bollerslev (1987) proposed to use the standardized symetric t-distribution with $\upsilon > 2$ degrees of freedom with a density given by

$$D(\varepsilon_t, v) = \frac{\Gamma((v+1)/2)}{\Gamma(v/2)\sqrt{\pi(v-2)}} (1 + \frac{\varepsilon_t^2}{v-2})^{-\frac{v+1}{2}}$$

where $\Gamma(.)$ is the gamma function.

The one-step –ahead conditional variance forecast $\hat{\sigma}_t^2$, for the GARCH (p,q) model is given by

$$\hat{\sigma}_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{ji=1}^{q} \beta_j \hat{\sigma}_{t-j}^2 \tag{8}$$

For the EGARCH(p,q) model, one –step-ahead conditional variance forecast is given by

$$In(\hat{\sigma}_t^2) = \alpha_0 + \sum_{i=1}^{p} \alpha_i \frac{|\varepsilon_{t-i}| + \gamma_i \varepsilon_{t-i}}{\sigma_{t-i}} + \sum_{j=1}^{q} \beta_j In(\hat{\sigma}_{t-j}^2) \tag{9}$$

In the case of TGARCH (p,q) model, we forecast conditional variance using the following expression

$$\hat{\sigma}_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \gamma_i S_{t-i} \varepsilon_{t-i}^2 + \sum_{ji=1}^{q} \beta_j \hat{\sigma}_{t-j}^2 \tag{10}$$

In our empirical study, we will evaluate the predictive performance of GARCH-N model in which the error term is assumed normally distributed, GARCH-t model in which we assume that the error term follows a student-t distribution with v degrees of freedom, TGARCH and EGARCH models not in an econometric laboratory but in a risk management environment[3].

[3] For more information on volatility forecasting in financial markets, see Poon and Granger (2003).

2.5 Extreme Value Theory

Extreme Value Theory is a classical topic in probability theory. EVT is a powerful and yet fairly robust framework in which to study the tail behaviour of a distribution. It can be conveniently thought as a complement to the cetral limit theory: while the latter deals with fluctuations of cumultative sums, the formers deals with fluctuations of sample maxima. The main result is due to Fisher and Tippet (1928), who specify the form of the limit distribution appropriately normalised maxima.

There have been a number of extreme value studies in the finance literature in recent years. De Haan, Jansen, Koedijk and de Vries (1994) study the quantile estimation using extreme value theory. Mc Neil (1998) study the estimation of the tails of loss severity distributions and the estimation of the quantile risk measures for financial data using extreme value theory. Embrechts et al. (1998) overview the extreme value theory as a risk management tool. Muller et al. (1998) and Pictet et al. (1998) study the probability of exceedences and compare them with GARCH models for the foreign exchange rates. Mc Neil (1999) provides an extensive overview of the extreme value theory for risk managers. One year after, Mc Neil and Frey develop a new approach in two steps that permits to estimate the tail- related risk measures for heteroskedastic financial time series, a such method is a combination between GARCH models and EVT method. Some applications of EVT to finance and insurance can be found in Embrechts, Klueppeelberg and Mikosch (1997) and Reiss and Thomas (1997).

In the following , we present two approaches to study extreme events. The first one is a direct modelling of the distribution of minimum or maximum realizations. The other one is modelling the exceedances of a particular threshold. In addition, we present Mc Neil et al (2000) approach called conditional EVT and used to estimate tail-related risk measures in the case of heteroskedastic financial time series.

2.5.1 The block maxima method

Let X_1, X_2, X_3,…, X_n be a sequence of independently and identically distributed random variables with a common distribution function CDF $F(x) = \Pr(X_t \leq x)$ which has mean (location parameter) μ and variance (scale parameter) σ^2. Throughout this work, a loss is treated as a positive number and extreme events occur when losses take values in the right tail of the loss distribution F. Under the block maxima method, the data are divided into k blocks with n observations in each block corresponding to n trading intervals. Let the sample maxima of X_n denote the worst-case loss in a sample of n losses. From the iid assumption, the CDF of M_n is given by

$$\Pr(M_n \leq x) = \Pr(\max(X_1,\ X_2,...,X_n) \leq x) = \prod_{i=1}^{n} \Pr(X_i \leq x) = \left[F_X(x)\right]^n \tag{11}$$

F^n is assumed to be unknown and the empirical distribution function is often a very poor estimator of $F^n(x)$. Fisher and Tippet (1928) have shown that for X_t that are independent and drawn from the same non-degenerate distribution function H such that

$$\lim_{n\to\infty} \Pr\left\{\frac{M_n - d_n}{c_n} \leq y\right\} = \lim_{n\to\infty}\left\{F^n \leq c_n y + d_n\right\} = H(y) \tag{12}$$

Then H belongs to one of the three standard extreme value distributions, regardless of the original distribution of the observed data.

Fréchet: $$\phi_{\varepsilon}(x) = \begin{cases} 0, & x \le 0 \\ \exp(-x^{-\alpha}), & x > 0, \quad \alpha > 0 \end{cases}$$

Weibull: $$\psi_{\alpha}(x) = \begin{cases} \exp[-(-x^{-\alpha})], & x \le 0, \quad \alpha < 0 \\ 1, & x > 0 \end{cases}$$

Gumbel : $$\Lambda(x) = \exp(-\exp(-x)), \qquad x \in \Re$$

Fréchet and Weibull distributions attain the shape of a Gumbel distribution when the tail index parameter α goes to ∞ and $+\infty$, respectively. By taking the reparameterization $\xi = \frac{1}{\alpha}$, due to Jenkin (1955) and Von Mises (1936), these three extreme value distributions can be represented in an unified model with a single parameter

$$H_{\xi}(x) = \begin{cases} \exp\left\{-(1+\xi x)^{-1/\xi}\right\}, & if \quad \xi \neq 0, \quad 1+\xi x > 0 \\ \exp\left\{-\exp(-x)\right\}, & if \quad \xi = 0 \end{cases} \tag{13}$$

The parameter $\xi = \frac{1}{\alpha}$ is a shape parameter and determines the tail behaviour of H_{ξ}. The parameter α is called the tail index if ξ>0. This representation is known as the generalized extreme value distribution (GEV). The tail behaviour of the distribution F of the underlying data determines the shape parameter ξ of the GEV distribution. If the tail of F declines exponentially, then H_{ξ} is of the Gumbel type and ξ=0. Normal, log-normal, exponential and gamma distributions are thin tailed distributions and are in the domain of attraction of the Gumbel type. For these distributions, all moments usually exist. Distributions in the domain of attraction of the Fréchet type (ξ>0) include fat tailed distributions like the Stundent-t, Cauchy, Pareto, and mixture distributions. For these distributions, not all moments are finite. If the tail of F is finite then H_{ξ} is of the weibull type and ξ<0. Distributions in the domain of attraction of the weibull type include distributions with bounded support such as uniform and beta distributions. All moments exist for these distributions.

The Fisher-Tippet Theorem is the analog of the Central Limit Theorem for extreme values. Whereas the Central Limit Theorem applies to normalized sums of random variables, the Fisher- Tipett Theorem applies to standardized maxima of random variables.

The GEV distribution characterizes the limiting distribution of the standardized maxima. It turns out that the GEV distribution is invariant to location and scale transformation

$$H_\xi(z) = H_\xi\left(\frac{x-\mu}{\sigma}\right) = H_{\xi,\mu,\sigma}(x)$$

For n large, the Fisher- Tippet theorem may then interpreted as follows

$$\Pr(Z_n < z) = \Pr\left(\frac{M_n - \mu_n}{\sigma_n} < z\right) \approx H_\xi(z)$$

Letting $x = \sigma_n z + \mu_n$ then

$$\Pr(M_n < x) \approx H_{\xi,\mu,\sigma}\left(\frac{x-\mu_n}{\sigma_n}\right) = H_{\xi,\mu_n,\sigma_n}(x) \tag{14}$$

This result is used in practice to make inferences about the maximum loss M_n.

The parameters of the GEV distribution $(\xi,\ \mu_n\ and\ \sigma_n)$ are estimated by fitting the GEV distribution to the data. The likelihood function for the parameters of the GEV distribution is constructed from the sample of block maxima $\left\{M_1,\ M_2,\ ...M_k\right\}$. We assume that each block is of size n sufficiently large so that the Fisher-Tppet Theorem holds.

$$MaxL_H(\xi,\ \mu,\ \sigma) = Max\sum_i In(h_{\xi,\ \mu,\ \sigma}(x_i))$$

where

$$h_{\xi,\ \mu,\ \sigma}(x_i) = \frac{1}{\sigma}\left(1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right)^{(-1/\xi)-1} \times \exp\left(-\left(1+\xi\left(\frac{x_i-\mu}{\sigma}\right)^{-1/\xi}\right)\right)$$

is the density function of the GEV distribution if $\xi \neq 0$ and $1+\xi\left(\frac{x_i-\mu}{\sigma}\right) > 0$.

and

$$L_H(\xi,\ \mu,\ \sigma) = -kIn(\sigma) - (1+1/\xi)\sum_{i=1}^{k} In\left\{1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right\} - \sum_{i=1}^{k}\left\{1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right\}^{-1/\xi} \tag{15}$$

is the log function assuming iid observations from a GEV distribution whith $\xi \neq 0$.

For the case where $\xi = 0,$ the likelihood is given by

$$L_H(\mu,\ \sigma) = -kIn(\sigma) - \sum_{i=1}^{k}\left(\frac{x_i-\mu}{\sigma}\right) - \sum_{i=1}^{k}\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)\right\} \tag{16}$$

By inverting equation ( ), we can go from asymptotic GEV distribution of maxima to the distribution of the observations themselves and we can get an expression for (unconditional) VAR_p quantiles associated with a given probability p

$$VaR_p = R_{n,k} = H^{-1}\left(1-\frac{1}{k}\right) = \hat{\mu} + \frac{\hat{\sigma}}{\hat{\xi}}\left\{-1+\left(-In\left(1-\frac{1}{k}\right)\right)^{-\hat{\xi}}\right\} \tag{17}$$

where ξ, μ *and* σ have been substituted by their maximum likelihood estimates. If observations X_i are independent then

$$\left(1-\frac{1}{k}\right) = \Pr(X_1 \le R_{n,k},..., X_n \le R_{n,k}) = F^n(R_{n,k})$$

In the case of iid series, The (1-1/k) quantile, $R_{n,k}$, for the distribution of maxima M_n corresponds to the $\left(1-(1/k)\right)^{1/n}$ quantile of the marginal distribution of X_i. Suppose for example that we consider our model for annual (260 days) maxima. Then, the return that we expect to be exceeded once every 30 years, the 30 year return level corresponds to the $\left(1-1/30\right)^{(1/260)} = 0{,}99987$ quantile.

We have only considered the case of stationary and independently distributed random variables. In the case of non-iid variables which are supposed to hold in most financial markets, we can fit a slightly modified GEV distribution to stationary series that show that show the kind of clustering behaviour.

For iid series, we can easily calculate the distribution of the sample maxima from a distribution F of sample observations

$$F_M = [F_X]^n$$

As this case is unrealistic for financial time series, we extend the asymptotic proprieties of maxima derived for an iid variable to the non-i.i.d case. Let (X_n) be a stationary variable with marginal distribution F and ($\tilde{X}_n$) an associated independent process which have the same marginal distribution F and let $\tilde{M}_n = \max(\tilde{X}_1,....,\tilde{X}_n)$. We define a new parameter $\theta \in [0,1]$ called extremal index such that

$$F_M = P(M_n \le R_{n,k}) \approx P^{\theta}(\tilde{M}_n \le R_{n,k}) = F^{n\theta}(R_{n,k}) = [F_X]^{n\theta}$$

This means that the maximum of n observations from the non-i.i.d series have behaviour like the maximum of $n\theta$ observations from the associated iid variables. θ can be interpreted as the reciprocal of the mean cluster size and nθ as counting the number of pseudo-independent clusters in n observations. The asymptotic distribution of maxima for non-iid series is in fact a GEV distribution that converge in probability to $H^{\theta}(x)$. The GEV distributions for iid and non-iid series have the same tail index, because raising H(x) to the power θ only affects scale and location parameters. As Mc Neil (1998) points out, the extremal index θ can be interpreted as a reciprocal of the mean cluster size and nθ as counting the number of pseudo-independent clusters in n observations. The extremal index can be estimated asymptotically as

$$\hat{\theta} = n^{-1} \frac{In(1-K_u/k)}{In(1-N_u/nk)} \tag{18}$$

where N_u is the number of observations that exceed a certain high threshold, K_u is the number of blocks in which this threshold is exceeded, and k and n, which should be large, are the number of blocks and length of these blocks respectively.

2.5.2 The POT method

The distribution function F_u is called the conditional excess distribution function (cedf) an is defined as the conditional probability:

$$F_u(y) = P(X-u \le y / X > u), \;\; 0 \le y \le x_F - u \tag{19}$$

where X is a random variable, u is a given threshold, y=x-u is the excess over u and x_F is the right endpoint of F.

$$G_{\xi,\alpha}(y) = \begin{cases} \left[1-(1+\dfrac{\xi}{\alpha}y)\right]^{\frac{-1}{\xi}} & if \quad \xi \ne 0 \\ 1-e^{\frac{-y}{\alpha}} & if \quad \xi = 0 \end{cases} \tag{20}$$

for $0 \le y \le x_F - u$. ξ is the tail index.

$$F(x) = (1-F(u))F_u(y) + F(u) \tag{21}$$

The function $F(u)$ can be estimated non parametrically by $\dfrac{n-N_u}{n}$ where n is the total number of observations and N_u represents the number of exceedences over the threshold u[4]. After replacing $F_u(y)$ by $G_{\xi,\alpha}(y)$, we get the following estimate for $F(x)$:

$$\hat{F}(x) = \frac{N}{n_u}(1-(1+\frac{\hat{\xi}}{\hat{\sigma}}(x-u))^{-\xi}) + (1-\frac{N_u}{n}) = 1-\frac{N_u}{n}(1+\frac{\hat{\xi}}{\hat{\sigma}}(x-u))^{-\hat{\xi}} \tag{22}$$

By inverting this expression, we get an expression for (unconditional) VAR_p quantiles associated with a given probability p:

$$VAR_p = u + \frac{\hat{\sigma}}{\hat{\xi}}((\frac{n}{N_u}p)^{-\hat{\xi}} - 1) \tag{23}$$

4 The and Mean excess function (MEF) and hill plot two tools that are used to threshold determination. For a detailed discussion and several examples of the hill-plot, see Embrechts et al. (1997).

2.5.3 The Conditional EVT approach

To obtain value-at-risk estimates, we follow MC Neil and Frey's (2000) two-step estimation procedure called conditional EVT[5]:

Step 1: Fit a GARCH-type model to the return data by quasi-maximum likelihood. That is, maximize the log-likelihood function assuming normal innovations.

Step 2: Consider the standardized residuals computing in step 1 to be realizations of a strict white noise process and use extreme value theory (EVT) to model the tail of innovations using EVT and estimate the quantiles of innovations for $q \geq 0.95$.

We assume that the dynamic of log-negative returns can be modelled by

$$r_t = \mu_t + \sigma_t Z_t \tag{24}$$

where $\mu_t = \phi_0 + \sum_{i=1}^{s} \phi r_{t-i}$, ϕ_i are parameters, r_{t-i} are lagged returns and Z_t are iid innovations with zero mean and unit variance and marginal distribution $F_Z(z)$. We assume that the conditional variance σ_t^2 of the mean-adjusted series $\varepsilon_t = r_t - \mu_t$ follows a GARCH (p,q) process:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{ji=1}^{q} \beta_j \sigma_{t-j}^2 \tag{25}$$

The conditional mean is given by $\mu_t = \phi_0 + \sum_{i=1}^{s} \phi r_{t-i}$, and the likelihood function of a sample of m iid observations for a GARCH model with normal innovations is given by

$$L(\theta) = -\frac{m}{2}\log(2\prod) - \frac{1}{2}\sum_{t=2}^{m}\log(\sigma_t) - \frac{1}{2}\sum_{t=2}^{m}\frac{(r_t - \mu_t)^2}{\sigma_t} \tag{26}$$

After maximizing, we can obtain parameter estimates and compute standardized residuals to check the adequacy of the GARCH modelling and to use in stage 2 of the method. They are calculated as

$$(z_{t-m+1}, z_{t-m+2}, \ldots, z_t) = \left(\frac{r_{t-m+1} - \hat{\mu}_{t-m+1}}{\hat{\sigma}_{t-m+1}}, \frac{r_{t-m+2} - \hat{\mu}_{t-m+2}}{\hat{\sigma}_{t-m+2}}, \ldots \frac{r_t - \hat{\mu}_t}{\hat{\sigma}_1} \right) \tag{27}$$

The one-step forecast for the conditional variance in t+1 is given by

$$\hat{\sigma}_{t+1}^2 = \hat{\alpha}_0 + \sum_{i=1}^{p} \hat{\alpha}_i \varepsilon_{t-i+1}^2 + \sum_{ji=1}^{q} \hat{\beta}_j \hat{\sigma}_{t-j+1}^2 \tag{28}$$

where $\hat{\varepsilon}_t = r_t - \hat{\mu}_t$. The one-step-ahead VaR forecasts is given by

$$VaR_p = \hat{\mu} + \hat{\sigma}_{t+1} VaR(Z)_p \tag{29}$$

[5] McNeil, A., & Frey, R.(2000). Estimation of tail-related risk measures for heteroscedastic financial times series: An extreme value approach. Journal of Empirical Finance 7, 271– 300.

where $VaR(Z)_p$ is given by equation 23 (or by equation 17) applied to negative standardized residuals to obtain VaR forecasts with conditional EVT-POT method (conditional block maxima method)[6].

3. Statistical evaluation tests:

Our objective is to evaluate the adequacy of the realized VaR forecasts in a risk management environment. It is well know that there are many sources of error in VaR figures: Sampling errors, data problems, inappropriate specification, model error, ect. All these factors will cause our estimate often to be biased. Various methods and tests have been suggested for evaluating VaR model accuracy. In this paper, statistical adequacy will be tested based on Kupiec's and Christoffersen's backtesting measures.

3.1 Unconditional coverage:

Let I_{t+1} be a sequence of VaR violations that can be described as:

$$I_{t+1} = \begin{cases} 1 & if \quad y_{t+1} < VaR_{t+1/t} \\ 0 & if \quad y_{t+1} \geq VaR_{t+1/t} \end{cases}$$

and therefore $N = \sum_{t=1}^{T} I_t$ be the number of days over a T period that the portfolio loss was greater than the VaR forecast.

The failure number follows a binomial distribution and consequently the appropriate likelihood ratio statistic, under the null hypothesis that the exception frequency equals to the excpected one (N/T=p) is:

$$LR_{uc} = 2In[(1-\frac{N}{T})^{T-N}(\frac{N}{T})N] - 2In[(1-p)^{T-N}p^{N}] \sim \chi^2(1) \qquad (30)$$

This test can reject a model that has generated too many or too few VaR violations. As stated by Kupiec, this can reject a model for both high and low failures but its power is generally poor especially for high confidence levels, it can not indicate an inadequate model, even if the difference between the observed and the expected failure is significant.

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3.2 Conditional coverage:

A more complete test was made by Christoffersen (1998), which jointly examines the conjecture that the total number of failures is statistically equal to the expected one and the VaR violations are independent. The main advantage of this test that it takes account of any conditionality in forecasts: if volatilities are low in some period and high in others, the forecast should respond to this clustering from distribution event. Under the null huypothesis that an expectation occurring is independent on what happened the day before and the expected proportion of violations is equal to p, the appropriate likelihood ratio is given by

$$LR_{cc} = -2In[(1-p)^{T-N}(p)^{N}] + 2In[(1-\pi_{01})^{n00}\,\pi_{01}^{n_{01}}(1-\pi_{11})^{n10}\,\pi_{11}^{n_{11}}] \sim \chi^2(2) \qquad (31)$$

where n_{ij} is the number of observations with value i followed by j, for i, j =0,1 and $\pi_{ij} = \frac{n_{ij}}{\sum_j n_{ij}}$

[6] For the conditional block maxima EVT method, estimate of the extremal index is not necessary because the GARCH filtred series are expected to be iid or close to iid

Table 1

Descriptive statistics on stock returns

Index	Sample	Mean	stdev	Min	Max	JB	Q(12)	Q(24)	$Q^2(12)$	$Q^2(24)$	LM
BVMT	*In sample*	0.019	0.431	3.93	2	12083.8	495.3	593.6	295.2	327.3	78.5
	All sample	0.039	0.692	3.93	3.62	1572.5	570.23	603.0	2140.0	2434.4	776.3
CAC40	*In sample*	0.067	1.131	4.37	6.1	67.7	18	34.59	69.9	84.5	50.5
	All sample	0.028	1.393	7.678	7	813.0	26.3	47.4	1739.1	2876.9	521.9

Mean, standard deviation, min and max are in percent. JB is the Jarque-Bera test for normality. Q(.) are the Ljung-Box tests for returns and for squared returns. LM refers to the Engle (1982) Lagrange Multiplier test for the presence of ARCH effect at lag 12.

$$\pi_{ij} = \frac{n_{ij}}{\sum_j n_{ij}}$$

are the corresponding probabilities. i,j=1denotes that an expectation has been made, while i,j=0 indicates the opposite. If the sequence of values is independent, then the probabilities to observe or not a VaR violation in the next period must be equal, which can be written more formally as $\pi_{01} = \pi_{11} = p$. This test can reject a VaR model that generates either too many or too few clustered violations, but it needs several observations to became more accurate.

4. Empirical Results

1 .4 Data and descriptive statistics

The data set is the daily closings of Tunis stock Exchange (BVMT) index and CAC 40 for the same period from July 29, 1994 to December 30, 2005. There are observations 2868 in data set for BVMT index and 2890 for CAC 40 index. The period July 24, 1998 to December 30, 2005 has been reserved for backtesting the predictive performance of alternative models for BVMT index. For CAC 40 index, the period reserved for backtesting cover Aout 4, 1998 to December 30, 2005. The daily returns are defined by $r_t = \log(p_t / p_{t-1})$ where p_t is the price of an asset (or portfolio) at time t. In fig.1, the level of BVMT index and the corresponding daily returns are presented. The sample histogram of negative BVMT returns (returns multiplied with -1) is presented in Fig.2.

All computations shown hereafter were carried out with finmetrics module of S-Plus 6.1.

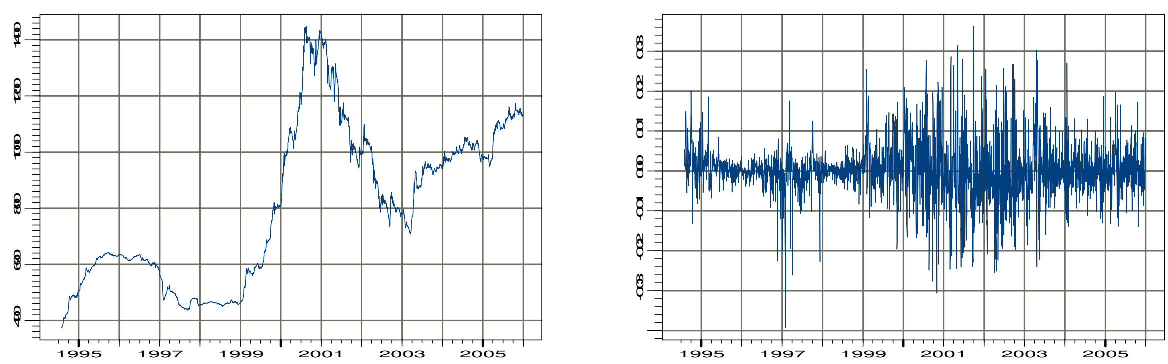


Fig. 1 Evolution of BVMT daily index and daily return (period: from July 29,1994 to December 30, 2005)

Table 2

Parameters estimates of AR-GARCH models for THE two index, as well as statistics on the standardized residuals.

	BVMT	CAC 40	BVMT	CAC 40
	Normal		Student's t	
$\phi_0 \times 10^4$	0.409	-6.966*	-0.957	-6.942*
ϕ_1	0.337*	-0.004	0.23 *	-0.001
ϕ_2	0.155*		0.148 *	
$\alpha_0 \times 10^7$	0.485*	6.341	1.066 *	8.967
α_1	0.333*	0.033 *	0.39 *	0.03 *
α_2	-0.261*		-0.321*	
β_1	0.927*	0.963 *	0.928 *	0.963*
t			3.66*	16.91*
Q(6)	3.68	9.18	12.77	12.43
Q(12)	12.14	17.61	22.52	17.56
$Q^2(6)$	0.85	8.93	0.49	10.04
$Q^2(12)$	9.74	10.17	13.35	11.17
LM (12)	9.83	10.04	13.54	10.95

* Significance at 95% level. Q(.) are the Ljung-Box tests. LM is the Lagrange multiplier test.

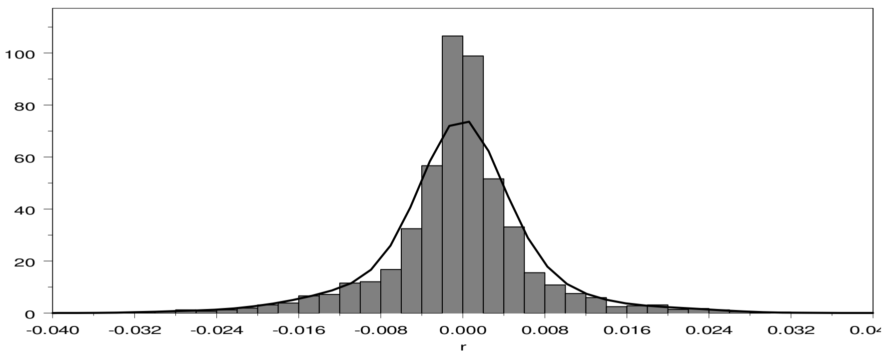


Fig. 2: Histogram of daily negative BVMT returns (losses)

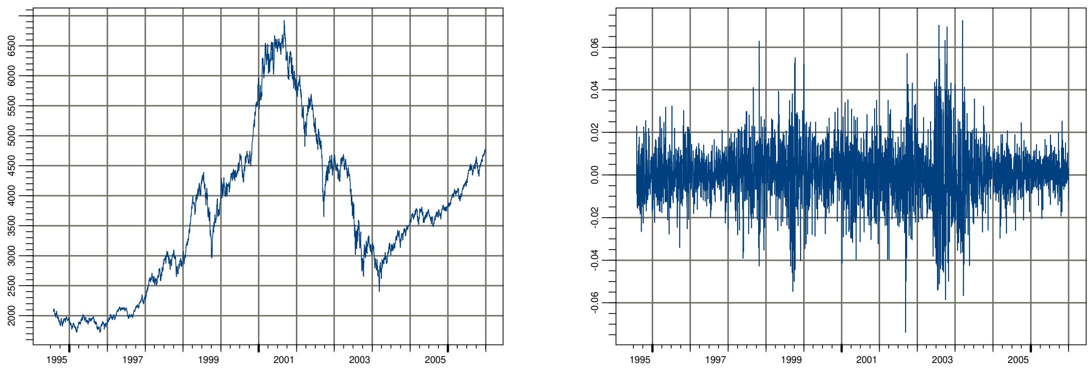


Fig. 3 : Evolution of CAC 40 daily index and daily return (period: from July 29, 1994 to December 30, 2005)

The descriptive statistics for daily returns of each index are presented in table 1. These statistics include the mean, standard deviation, median, maximum, minimum, Jarque-Bera statistics and Ljung-Box tests for raw and squared returns. The Jarque-Bera statistic indicate that daily returns for the two markets are not normally distributed. On the basis of Ljung-Box Q statistic and for raw returns series, the hypothesis that all correlation coefficients up to twelve and up to twenty four are jointly zero is rejected for the two markets. Therefore, we can conclude that two return series present some linear dependence in returns. In addition, the statistically significant serial correlations in squared returns imply that there is non linear dependence in return series. This indicates volatility clustering and a GARCH type modelling should be considered in VaR estimations.

4.2 In-sample evidence

The first step was to fit the model in Eqs. 24 and 25 to each return series. To identify the most adequate AR-GARCH model for each time series, we employ the Akaike criterion (AIC). For BVMT return series, we choose the AR(2)- GARCH(2,1) model. For CAC 40 returns, as in previous studies, we choose the AR(1)-GARCH(1,1) model. Parameter estimates for the models selected were obtained by the method of quasi-maximum likelihood and the log-likelihood function of the data was constructed by assuming that innovations are conditionally distributed as Gaussian. For the AR-GARCH models with normal distributed and with t-distributed errors, maximum likelihood estimates as well as some statistics on the standardized residuals are presented in table 2.

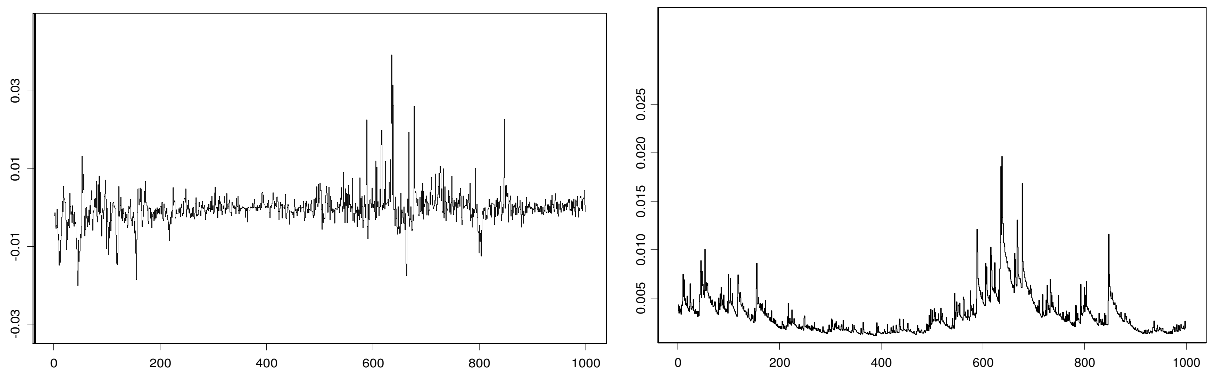


Fig. 4: 1000 day excerpt from series of negative log returns on BVMT index; plot in the right shows estimate of the conditional standard deviation derived from AR(2)-GARCH(2,1) model

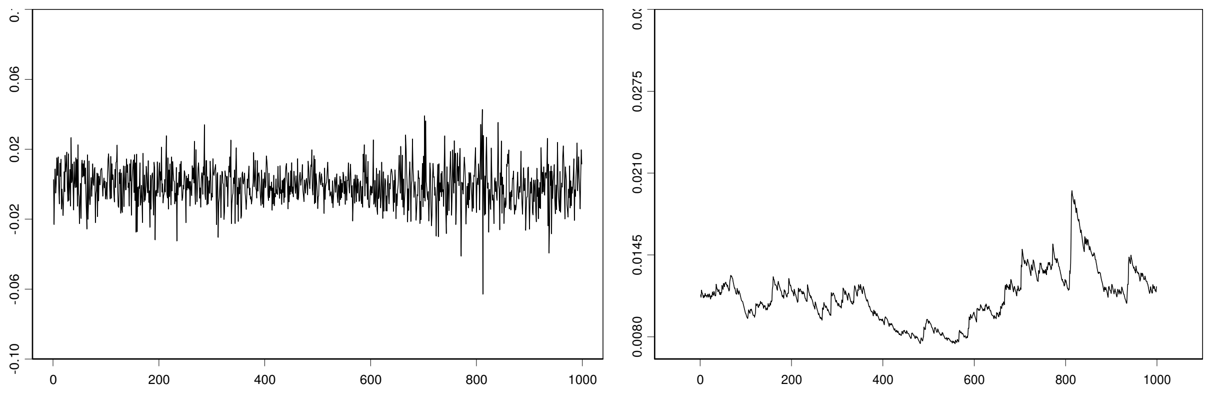


Fig. 5: 1000 day excerpt from series of negative log returns on CAC 40 index; plot in the right shows Estimate of the conditional standard deviation derived from AR (1)-GARCH (1, 1) model

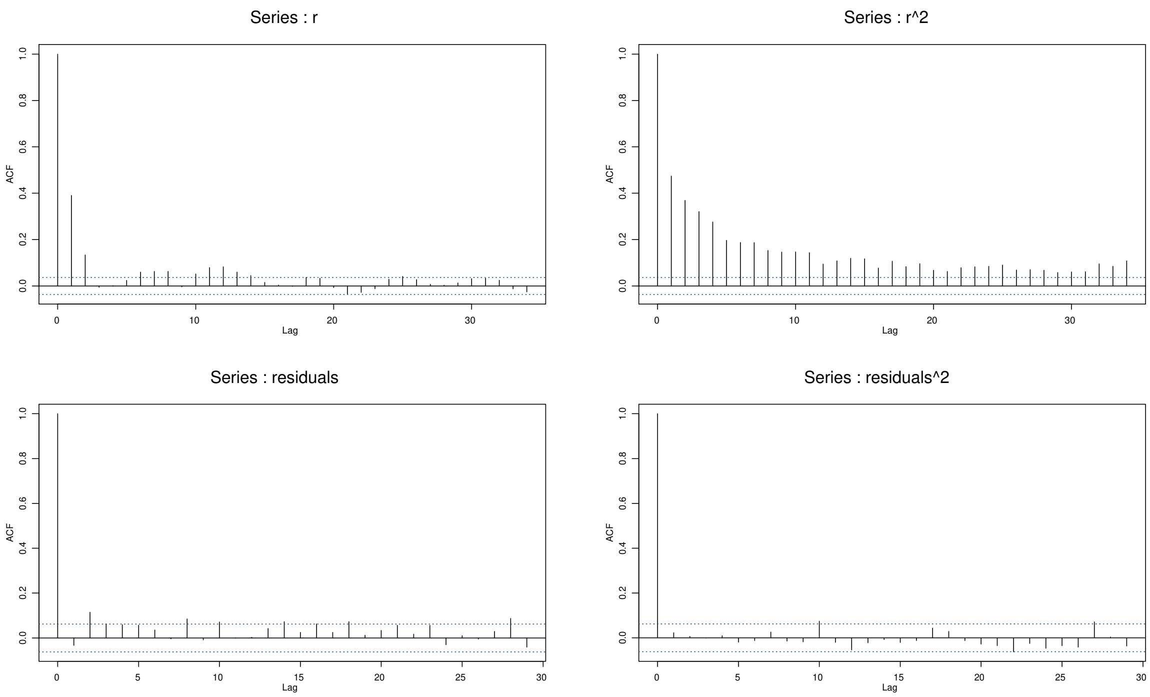


Fig. 6: Correlograms for the raw data (BVMT) and their squared values as well as for the residuals and squared residuals.

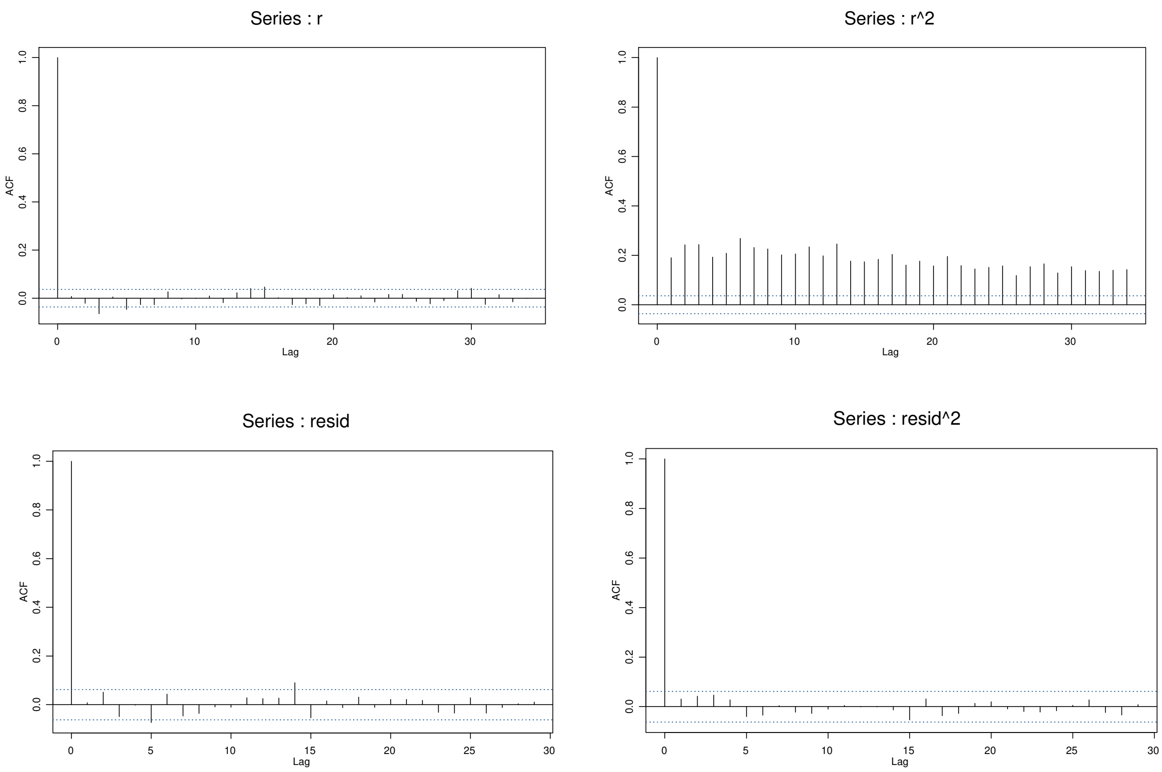


Fig. 7: Correlograms for the raw data (CAC 40) and their squared values as well as for the residuals and squared residuals.

Specification tests carried out after estimation failed to detect serial correlation and missing ARCH effects, suggesting that the selected functional form is adequate to the data.

In Fig. 4 and Fig.5 we show an arbitrary thousand day excerpt from our dataset; the estimated of the conditional standard deviation derived from GARCH fit is shown in the right.

In Fig.6 and 7, we plot correlograms for the raw data and their squared values as well as for the residuals and squared residuals. While the raw data are clearly not iid, this assumption may be tenable for residuals.

The mean excess plots for the BVMT and CAC 40 data are illustrated in Fig. 8[7].

A simple graphical technique infers the tail behaviour of observed losses is to create a qq-plot using the exponential distribution as a reference distribution. If the excesses over thresholds are from a thin- tailed distribution, then the GPD is exponential with $\xi = 0$ and the qq-plot should be linear. If the qq-plot is non-linear this indicate either bounded tails ($\xi < 0$) or fat-tailed behaviour ($\xi > 0$). Fig.9 shows qq- plots with exponential references distribution for the BVMT negatives returns and the CAC 40 negatives returns over the threshold u. There is a slight departure from linearity for the negative CAC 40 returns and a large departure from linearity for the negative CAC 40 index.

A simple graphical technique infers the tail behaviour of observed losses is to create a qq-plot using the exponential distribution as a reference distribution. If the excesses over thresholds are from a thin- tailed distribution, then the GPD is exponential with $\xi = 0$ and the qq-plot should be linear. If the qq-plot is non-linear this indicate either bounded tails ($\xi < 0$) or fat-tailed behaviour ($\xi > 0$). Fig.9 shows qq- plots with exponential references distribution for the BVMT negatives returns and the CAC 40 negatives returns over the threshold u. There is a slight departure from linearity for the negative CAC 40 returns and a large departure from linearity for the negative CAC 40 index.

4.3 Out-sample evidence

In order to compare the accuracy of EVT for VaR calculation with other alternatives, we backest each method on each return series by the following steps. Let $r_1, r_2, r_3, \ldots, r_m$ be a historical return series. The condition quantile is computed on t days in the set $T = \{n,...m-1\}$ using window of n days each time. Unless otherwise stated, we leave the last four years of the sample for prediction (we choose n=1000 days). In a long backtest it is less feasible to examine the fitted model carefully every day and to choose a new value of the constant k, which defines the number of exceedences above the threshold u, for the tail estimator each time. For this reason and as suggested by Mc Neil and Frey (2000), the constant k is set so that the 90th percentile of the innovation distribution is estimated by historical simulation.

On each day t, we fit a new AR(s)-GARCH(p,q) model and determine a new GPD to losses, wich are computed from the standardized residuals series. Such procedure, as mentioned earlier, is called conditional EVT.

The VaR estimates, in- sample and on December 30, 1995, for all the methods implemented and all significance level, are presented in table 3 and 4. This evaluation is based on one-step ahead forecast that have produced from a series of rolling samples with a size equal to 1000 observations. In the same tables, we calculate mean of VaR forecasts for the out-of-sample period.

[7] The mean excess function is the sum of the excess over the threshold u divided by the number of data points which exceeds the threshold u. It is an estimate of MEF that describes excpected overshoot of u once an exceedance occurs.

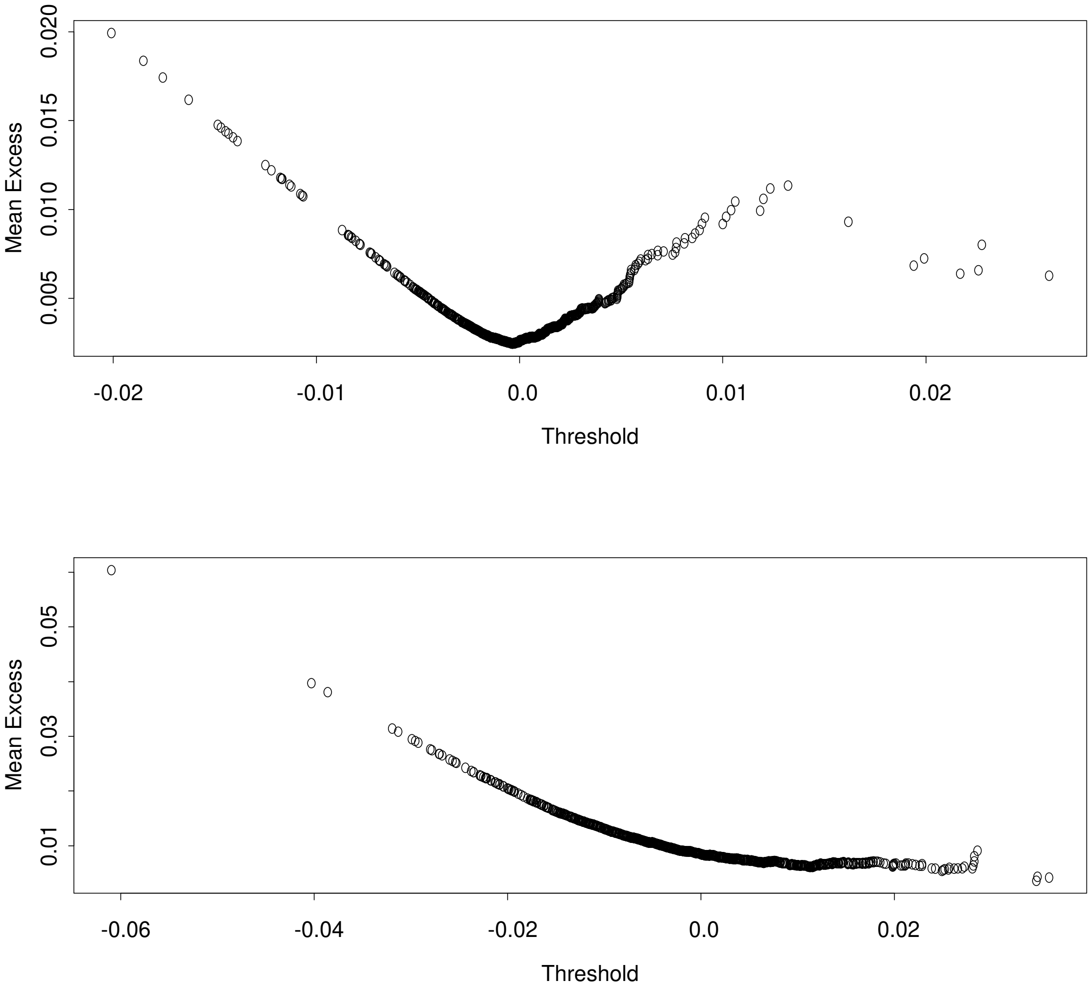


Fig. 8: Sample mean excess function for the BVMT and CAC 40 index (In sample period)

The relative out-of-sample performance for each model in term of violation ratio for the left tail (losses) at the window size of 1000 observations is calculated and presented in table 5 for BVMT index and in table 6 for the CAC 40 index. The number in parentheses are the ranking between ten competing models for each quantile. The violation ratio is defined as the number of times where the realized return is greater than estimated return (number of violation) divided by the total number of forecasts. An accurate and correct model is obtained when the expected violation ratio is equal to α . At qth quantile, the model predictions are expected to underpredict the realized return α = (1-q) percent of the time. A high violation ratio at q^{th} quantile greater than α implies that the model excessively underepredicts the realized return. In the case of a violation ratio less than α, there is excessive overprediction of the realized return by the underlying model. For instance, at the 0.95^{th} quantile, the realized BVMT return is 4.711% of the time greater than what the conditional POT-EVT model predicts implies that the model excessively overestimates risk. In contrary, the filtred historical simulation excessively underpredicts realized return as the violation ratio is greater than 5%.

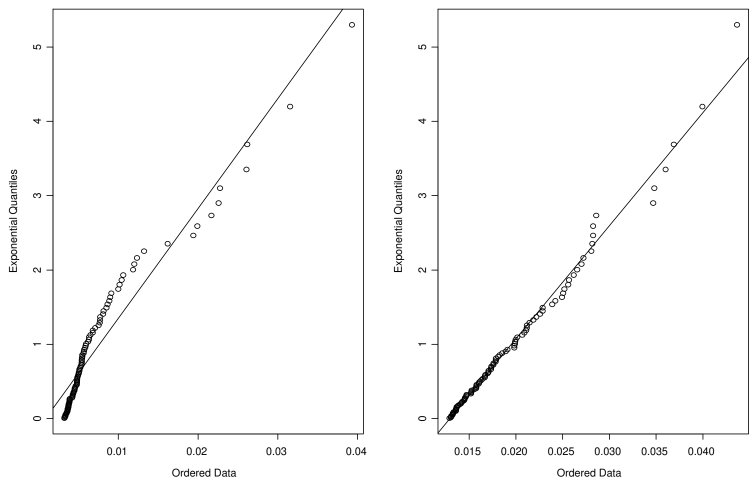


Fig. 9 : QQ plots with exponential references distribution for the BVMT negatives returns and the CAC 40 negatives returns over the threshold u

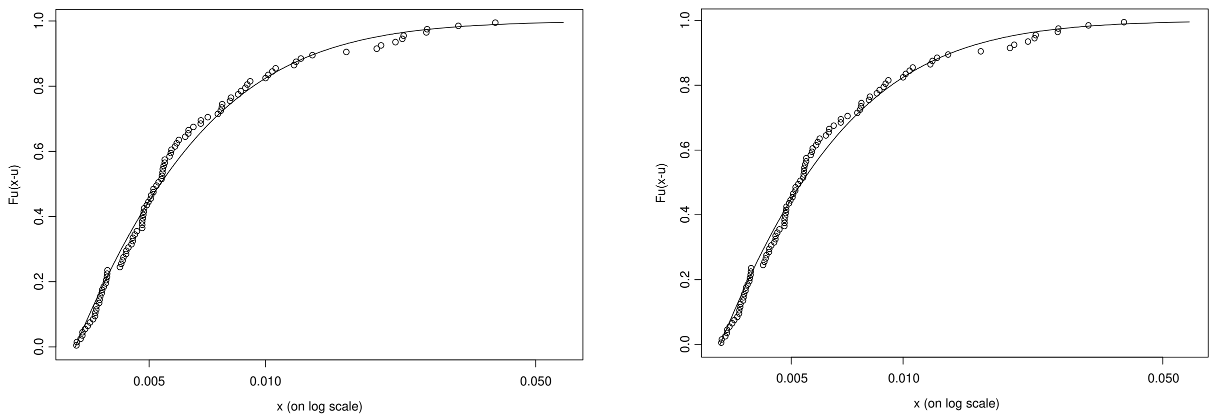


Fig.10: Fit of the estimated Generalized Pareto function for the BVMT (in the left) and CAC 40 (in the right)

A violation ratio excessively greater than the expected ratio implies that the model signals less capital allocation and the portfolio risk is not properly hedged. In this case, the model will increase the risk exposure by underestimating it. A excessively lower violation ratio implies that the model signals a capital allocation more than necessary. In this case, the portfolio holder allocates more to liquidity and registers an interest rate loss[8].

For BVMT index, the Var-cov, unconditional EVT, HS and conditional block maxima EVT methods are again the worst models for quantiles lower than the 0.99^{th} quantile. AR-GARCH models and filtred historical simulation provide the most performance for these quantiles. Historical simulation provides the best results for quantiles higher than 0.98^{th} except the 0.999^{th}

[8] Gençay, Selçuk, Ulugulyagci (2003), High volatility, thick tails and extreme value theory, Insurance: Mathematics and Economics.33 337–356.

quantile where Var-cov, filtred historical simulation and GARCH(t) perform best. This is the only quantile in which var-cov method not significantly overestimates nor underestimates the risk. Three VaR estimation methods give violation ratio that is statistically not overestimate nor underestimate risk at 95% level and for all quantiles: GARCH(t), EGARCH and conditional POT- EVT method.

At the 0.97th and 0.98th quantiles, TGARCH model performs the best with a violation ratio of 3.105% and 2.088% respectively. It is followed by normal GARCH model and filtred historical simulation. At 0.997th quantile, both historical simulation and GARCH(t) provide the best violation ratio of 0.214% which amounts to 0.086% over-rejection. Conditional and unconditional POT EVT methods rank third with a ratio of 0.161% (0.139% over-rejection). The worst ratio is given by Var-cov, EGARCH and GARCH (N) models.

The GARCH(t) model provide the best performance at 0.95th , 0.997th , 0.999th quantiles, it ranks second at 0.995th and fourth at the other remainder quantiles. Both conditional EVT methods overestimate realized returns at all quantiles while the unconditional EVT underestimates risk at all quantile except at 0.997th and 0.999th quantiles. We can conclude that conditional POT- EVT method should be placed at the middle of the performance ranking between ten competing models while both conditional block EVT and unconditional EVT should be placed at the bottom.

For CAC 40 index, the conditional POT-EVT method provides the best violation ratio for all quantiles except at 0.95th and at 0.99th quantile where it is placed at the second rank. The second best model is filtred Historical Simulation which provide also an excellent performance essentially at 0.97th , 0.99th and 0.995th quantiles. At 0.95th quantile, Historical Simulation provides the best performance but its performance deteriorates at higher quantiles. It is followed by the conditional POT-EVT method and filtred historical simulation that ranks third. At 0.999th quantile, conditional EVT methods provide the best results. The performance of conditional block maxima at 0.98th, 0.995th and 0.997th quantiles is not bad but deteriorates at lower quantiles less than 0.995th except the 0.98th quantile.

Three VaR estimation methods give violation ratio that is statistically not overestimate nor underestimate risk at 95% level and for all quantiles: GARCH(t), Filtred Historical Simulation and conditional POT- EVT method. The unconditional EVT, var- cov and all GARCH models underestimate risk at all quantiles.

In table 7, we present the Likelihood ratio tests statistics for the conditional LR_{cc} for the ten methods implemented and at eight differents significance levels. Our goal is to checks whether the probability of an exception occurring in one day is independent on events occurred in the day before. We reconfirm for both indices the previous results in tables 5 and 6 where Var-Cov and unconditional POT-EVT methods are not appropriate risk management techniques, as for the majority of cases, LR_{cc} statistics are significant (p-value<5%). Conditional POT- EVT and filtred Historical Simulation methods are the best performers along with GARCH-t method. The GARCH models for the BVMT index have also recorded a similar success. Conditional Block Maxima method for the CAC 40 index produce acceptable VaR forecasts at high confidence.

While Historical Simulation method gives VaR violation ratios that are not significant most of time and for both indices and provides sometimes and at some quantiles the best results in the base of conditional coverage criterion, it offers LR_{cc} statistics that are significant. Specifically, if VaR violation occurs, the probability to observe an exceedence the next day is high. Hence, we observe clustered violations, as this method does not update the VaR number quickly when market volatility increases.

Table 3
VaR(%) estimates-in absolute values- for the BVMT index on July 24,1998 and on December 30, 2005 and their mean for the out-sample period.

Model	**VaR (%)**	**95%**	**96%**	**97%**	**98%**	**99%**	**99.5%**	**99.7%**	**99.9%**
Var-Cov	*In sample*	0.666	0.722	0.796	0.901	1.088	1.287	1.449	1.815
	30/12/2005	1.145	1.241	1.366	1.544	1.860	2.197	2.462	3.092
	Mean	1.145	1.241	1.367	1.548	1.869	2.210	2.479	3.117
HS	*In sample*	0.523	0.569	0.682	0.887	1.328	2.258	2.607	3.155
	30/12/2005	1.161	1.276	1.373	1.597	1.892	2 .209	2.235	2.508
	Mean	1.154	1.273	1.429	1.649	1.999	2.367	2.517	2.853
Filtred HS	*In sample*	0.448	0.488	0.557	0.634	0.717	0.811	1.012	1.044
	30/12/2005	0.656	0.700	0.772	0.867	0.964	1.084	1.112	1.195
	Mean	1.108	1.197	1.327	1.520	1.841	2.286	2.531	2.731
GARCH (N)	*In sample*	0.434	0.446	0.483	0.518	0.574	0 .626	0.661	0.731
	30/12/2005	0.726	0.773	0.831	0.909	1.031	1.143	1.221	1.374
	Mean	1.155	1.231	1.324	1.448	1.643	1.822	1.945	2.191
GARCH (t)	*In sample*	0.397	0.428	0.470	0.533	0.654	0.796	0.916	1.233
	30/12/2005	1.135	0.751	0.822	0.923	1.841	2.177	1.425	3.087
	Mean	1.135	1.229	1.352	1.528	1.842	2.177	2.443	3.087
TGARCH	*In sample*	0.441	0.460	0.488	0.524	0.584	0.637	0.669	0.745
	30/12/2005	0.736	0.785	0.844	0.922	1.047	1.160	1.238	1.394
	Mean	1.181	1.232	1.325	1.449	1.677	1.859	1.945	2.234
Unc EVT POT	*In sample*	0.534	0.617	0.736	0.930	1.350	1.914	2.450	4.086
	30/12/2005	1.070	1.202	1.369	1.601	1.983	2.349	2.601	3.140
	Mean	0.914	1.036	1.194	1.419	1.816	2.234	2.562	3.353
Cond EVT POT	*In sample*	0.460	0.500	0.552	0.624	0.746	0.855	0.933	1.078
	30/12/2005	0.660	0 .701	0.755	0.831	0.963	1.101	1.205	1.438
	Mean	1.124	1.222	1.349	1.530	1.845	2.170	2.418	2.974
EGARCH	*In sample*	0.519	0.545	0.578	0.622	0.692	0.755	0.799	0.886
	30/12/2005	0.688	0.734	0 .790	0.865	0.983	1.091	1.165	1.313
	Mean	1.130	1 .204	1.296	1.412	1.607	1.783	1.903	2.143
Cond EVT Block	*In sample*	0.528	0.567	0.618	0.689	0.808	0.926	1.011	1.192
	30/12/2005	0.733	0.779	0.836	0.917	1.048	1.174	1.263	1.444
	Mean	1.256	1.357	1.489	1.678	2.005	2.344	2.604	3.194

Table 4
VaR(%) estimates-in absolute values- for the CAC 40 index on Aout 4,1998 and on December 30, 2005 and their mean for the out-sample period.

Model	**VaR (%)**	**95%**	**96%**	**97%**	**98%**	**99%**	**99.5%**	**99.7%**	**99.9%**
Var-Cov	*In sample*	1.727	1.876	2.068	2.345	2.835	3.356	3.766	4.741
	30/12/2005	2.340	2 .535	2.718	3.149	3.791	4.475	5.012	6.289
	Mean	2.393	2.593	2.853	3.225	3.885	4.587	5.139	6.453
HS	*In sample*	1.733	1.892	2.109	2.425	2.807	3.469	3.601	3.996
	30/12/2005	2.454	2.692	3.018	3.694	4.356	5.251	5.548	5.835
	Mean	2.458	2.698	3.005	3.492	4.149	4.836	5.062	5.576
Filtred HS	*In sample*	1.872	2.060	2.226	2.587	3.051	3.281	3.539	4.084
	30/12/2005	0.878	0.936	1.023	1.105	1.415	1.567	1.639	2.116
	Mean	2.353	2.526	2.691	2.983	3.386	3.748	3.958	4.693
GARCH (N)	*In sample*	1.903	2.030	2.187	2.396	2.724	3.025	3.239	3.644
	30/12/2005	0.837	0.896	0.970	1.067	1.220	1.360	1.456	1.649
	Mean	2.235	2.382	2.562	2.803	3.182	3.528	3.767	4.243
GARCH (t)	*In sample*	1.874	2.011	2.182	2.416	2.801	3.172	3.440	4.009
	30/12/2005	0.822	0.887	0.969	1.083	1.271	1.455	1.590	1.881
	Mean	2 .212	2.371	2.570	2.842	3.289	3.719	4.031	4.692
TGARCH	*In sample*	2.115	2.253	2.422	2.646	3.000	3.324	3.548	3.992
	30/12/2005	0.767	0.819	0.884	0.969	1.104	1.227	1.312	1.482
	Mean	2.243	2.388	2.566	2.802	3.175	3.527	3.752	4.221
Unc EVT POT	*In sample*	1.747	1.895	2.083	2.348	2.796	3.240	3.563	4.252
	30/12/2005	2.247	2.464	2.746	3.144	3.829	4.521	5.035	6.150
	Mean	2.192	2.401	2.669	3.045	3.687	4.326	4.795	5.800
Cond EVT POT	*In sample*	1.941	2.093	2.285	2.544	2.962	3.349	3.617	4.144
	30/12/2005	0.884	0.955	1.045	1.171	1.382	1.586	1.733	2.038
	Mean	2.343	2.507	2.714	2.997	3.465	3.915	4.240	4.927
EGARCH	*In sample*	2.229	2.373	2.551	2.788	3.160	3.502	3.737	4.205
	30/12/2005	0.813	0.867	0.933	1.021	1.159	1.285	1.372	1.546
	Mean	2.205	2.346	2.521	2.753	3.118	3.452	3.682	4.141
Cond EVT Block	*In sample*	2.137	2.265	2.431	2.668	3.081	3.506	3.830	4.557
	30/12/2005	0.962	1.047	1.153	1.295	1.521	1.726	1.866	2 .141
	Mean	2.584	2.741	2.943	3.222	3.695	4.170	4.526	5.331

Table 5

VaR violation ratios for the left tail (losses) of daily BVMT returns (in %)

Model	95%	96%	97%	98%	99%	99.5%	99.7%	99.9%
VAR-COV	6.317 (7)	5.407(7)	4.550 (9)	3.212 (9)	2.195 (10)	1.124 (10)	0.589 (10)	0.107 (1)
HS	7.013 (9)	5.514 (8)	4.229 (8)	3.051 (8)	0.963 (1)	0.428 (1)	0.214 (1)	0 (6)
Filtred HS	5.139 (3)	3.854 (1)	2.730 (4)	1.820 (3)	0.696 (6)	0.214 (7)	0.107 (6)	0.107 (1)
GARCH(N)	4.604 (5)	3.479 (5)	2.891 (2)	2.088 (1)	1.231 (3)	0.696 (5)	0.535 (8)	0.321 (10)
GARCH(t)	5.032 (1)	3.640 (4)	2.730 (4)	1.713 (4)	0.749 (4)	0.375 (2)	0.214 (1)	0.107 (1)
TGARCH	4.336 (6)	3.694 (3)	3.105 (1)	2.088 (1)	1.124 (2)	0.856 (9)	0.482 (5)	0.268 (4)
Unc EVT-POT	8.940 (10)	7.441 (10)	5.728 (10)	3.908 (10)	2.034 (9)	0.642 (3)	0.161 (3)	0 (6)
Cond EVT POT	4.711 (4)	3.426 (6)	2.623 (6)	1.659 (6)	0.749 (4)	0.268 (6)	0.161 (3)	0 (6)
EGARCH	4.925 (2)	4.336 (2)	3.266 (3)	2.355 (5)	1.392 (7)	0.642 (3)	0.535 (8)	0.268 (4)
Cond EVT block	2.998(8)	2.409(9)	1.927 (7)	1.445 (7)	0.428 (8)	0.161 (8)	0.107 (6)	0 (6)

The numbers in parentheses are the ranking between ten competing models for each quantile. Shaded number indicate statistically significant overestimation or underestimation of risk at 95% level.

Table 6

VaR violation ratios for the left tail (losses) of daily CAC 40 returns (in %)

Model	95%	96%	97%	98%	99%	99.5%	99.7%	99.9%
VAR-COV	5.873 (8)	4.550 (6)	4.074 (9)	3.175 (8)	1.905 (9)	1.005 (8)	0.635 (7)	0.159 (3)
HS	4.974 (1)	4.180 (3)	3.598 (4)	2.592 (5)	1.587 (5)	0.794 (5)	0.582 (6)	0.317 (9)
Filtred HS	4.815 (3)	3.915 (2)	2.963 (1)	2.222 (2)	1.005 (1)	0.476 (1)	0.423 (4)	0.159 (3)
GARCH(N)	5.503 (6)	4.709 (7)	3.598 (4)	2.646 (7)	1.693 (6)	0.794 (5)	0.529 (5)	0.265 (8)
GARCH(t)	5.714 (7)	4.762 (8)	3.704 (6)	2.592 (5)	1.481 (3)	0.582 (3)	0.370 (2)	0.212 (5)
TGARCH	5.291 (4)	4.339 (4)	3.386 (3)	2.487 (4)	1.693 (6)	0.952 (7)	0.741 (8)	0.212 (5)
Unc-EVT	6.825 (10)	5.661 (10)	4.339 (10)	3.439 (10)	2.063 (10)	1.356 (10)	0.794 (9)	0.212 (5)
Cond-EVT	4.868 (2)	4.021 (1)	2.963 (1)	2.063 (1)	0.794 (2)	0.476 (1)	0.265 (1)	0.106 (1)
EGARCH	5.344 (5)	4.497 (5)	3.809 (8)	3.228 (9)	1.799 (8)	1.005 (8)	0.846 (10)	0.370(10)
Cond EVT block	3.757 (9)	2.910 (9)	2.222 (7)	1.640 (3)	0.529 (4)	0.317 (4)	0.217 (3)	0.106 (1)

The numbers in parentheses are the ranking between ten competing models for each quantile. Shaded number indicate statistically significant overestimation or underestimation of risk at 95% level.

Table 7

Likelihood ratio tests statistics for the conditional LR_{cc}

	Index	95%	96%	97%	98%	99%	99.5%	99.7%	99.9%
VAR-COV	BVMT	122.989	95.601	103.478	70.1886	73.608	27.375	14.651	0.0156
	CAC 40	12.308	9.053	15.052	20.337	24.733	13.644	15.247	0.566
HS	BVMT	129.179	120.375	80.062	65.591	12.444	0.281	0.533	3.738
	CAC 40	11.475	7.787	11.086	9.996	12.060	10.813	4.070	5.695
Filtred HS	BVMT	4.762	0.701	0.666	1.616	1.952	3.932	3.100	0.016
	CAC 40	1 .785	1.556	0.152	0.514	0.406	0.117	0.927	0.566
GARCH (N)	BVMT	2.854	1.481	3.353	1.779	1.538	1.480	2.919	5 .748
	CAC 40	1.415	4.266	2 .402	4.053	7.949	3.033	2.815	3.540
GARCH (t)	BVMT	1.172	0.828	3.402	1.975	1.523	0.706	0.533	0.016
	CAC 40	2.199	4.479	5.077	3.562	4.498	0.383	0.351	1.801
TGARCH	BVMT	4.901	0.624	0.155	1.779	0.780	4.223	1.838	3.619
	CAC 40	1.041	2.247	1.336	4.399	7.980	6.501	8.905	1.801
Unc EVT-POT	BVMT	166.655	167.513	141.484	101.734	15.532	10.546	1.475	3.738
	CAC 40	20.769	21 .095	19.236	26.518	27.4	23.517	18.608	1.801
Cond EVT-POT	BVMT	5.257	2.038	3.646	2.252	1.523	2.473	1.475	3.738
	CAC 40	1.589	1 .281	0.152	0.137	1.131	0.022	0.114	0.013
EGARCH	BVMT	4.009	5.356	1.147	3.312	3.345	0.866	2.919	3.619
	CAC 40	1.865	3.717	9.282	16.088	15.157	9.272	14.975	8.184
Cond EVT block	BVMT	19.478	16.532	9.871	4.05	7.919	5.893	3.096	3.735
	CAC 40	7.452	6.613	4.362	1.747	5.228	1.499	0.572	0.013

Shaded numbers indicate significance at 95% level. LR_{cc} is $\chi^2(2)$ distributed.

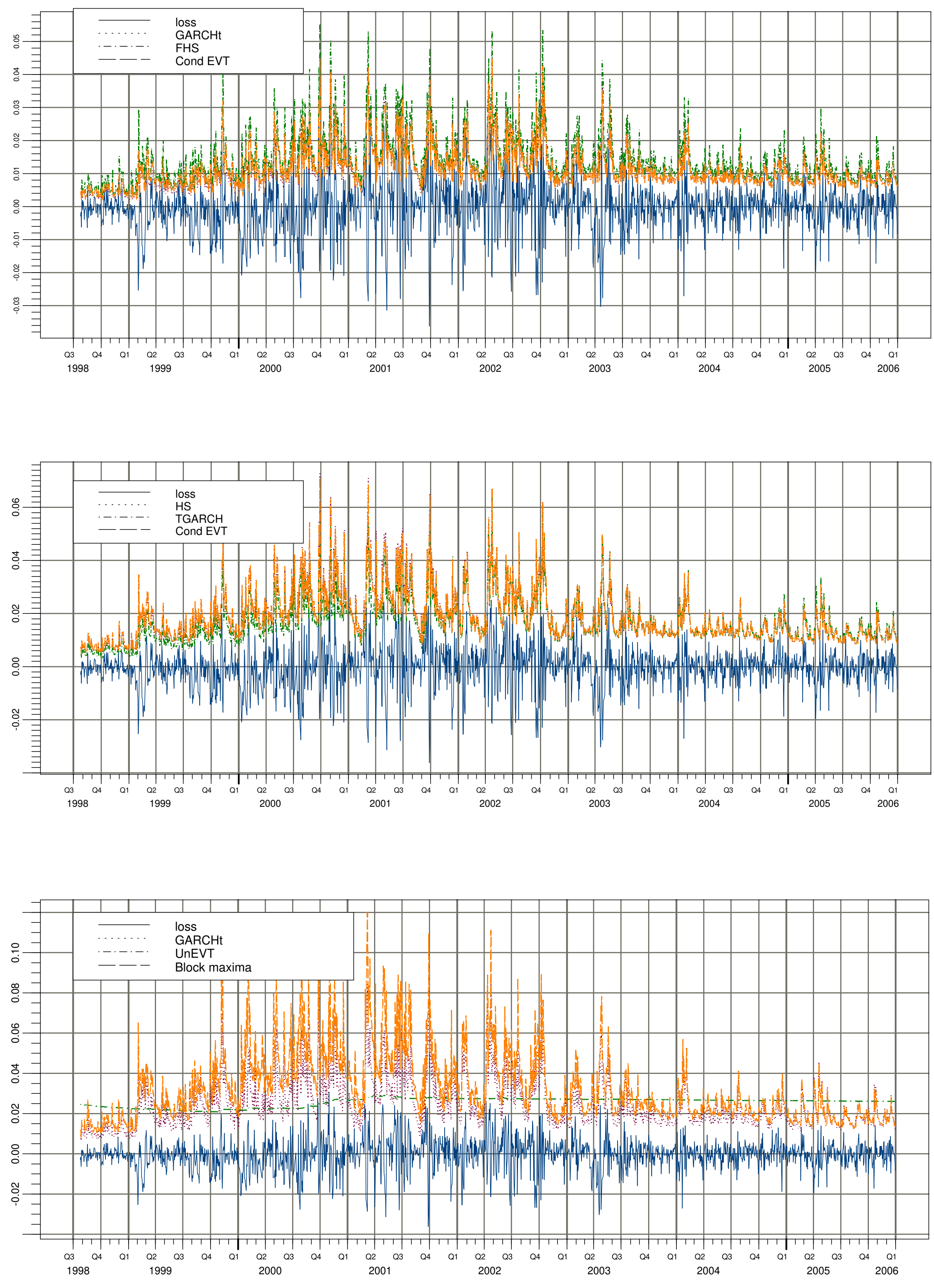
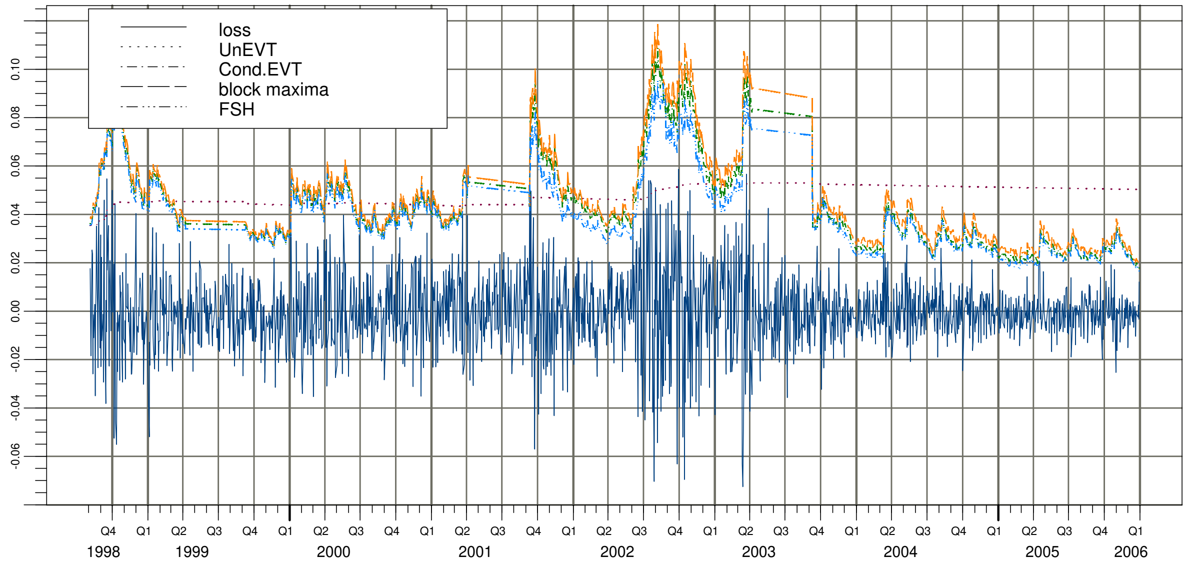


Fig. 11. Top: Daily BVMT negatives returns and VaR (95%) estimates. Middle: Daily BVMT negatives returns and VaR (99%). Bottom: Daily BVMT negatives returns and VaR (99.9%)



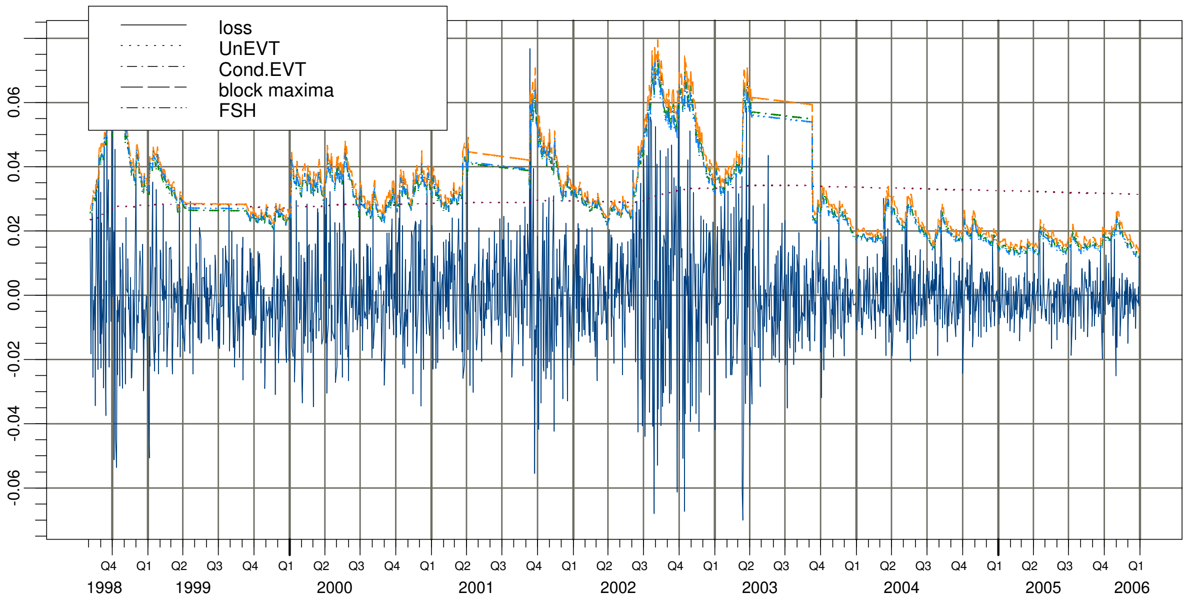


Fig. 12: Top: DailyCAC 40 negatives returns and VaR (99.7%) estimates.

Bottom: DailyCAC 40 negatives returns and VaR (98%) estimates

Fig. 9 offers a visual presentation of BVMT negative return and the estimated VaR with some of the more performing models and for three confidence level (95%, 99%, 99,9%). In Fig. 10, we plot the negative returns of the CAC 40 together with VaR forecasts (for 98% and 99,7% confidence level). We observe that unconditional models produce VaR forecasts that react to changing market conditions slowly. In contrast, the reaction of conditional models to changing market volatility is much quicker. Unconditional extreme value estimates are generally higher and are considerably less volatile than the GARCH models and two conditional EVT methods. The rolling samples do not generate substantial change of the data set of extreme observations and as a result the unconditional VaR estimates are almost time independent. Unconditional EVT models are more suitable for long run forecasts of the extreme losses rather than being a day-to-day tool to measure the market risk.

For GARCH models and conditional EVT methods, variances are forecasted by an exponential model with declining weights on past observations and therefore are crucially dependent on the last few observations that is added in the sample. Conditional VaR forecasts increase with increasing volatility but also decrease with decreasing volatility indicate that conditional VaR estimates correspond more closely to the actual returns than the unconditional VaR estimates.

5. Conclusion

The purpose of this paper has been to attempt a comparative study of the predictive ability of VaR estimates from various estimation techniques. The main emphasis has been given to the Extreme Value methodology and to evaluate how well EVT- models perform in modelling the tails of distributions and in estimating and forecasting VaR measures.

Two different stock indexes, the BVMT and the CAC 40, have been investigated, and some differences between the indexes have been pointed at. Empirical results show that Var-Cov and unconditional POT-EVT methods are not appropriate risk management techniques for majority cases. Conditional POT- EVT and filtred Historical Simulation methods are the best performers along with GARCH-t method. The GARCH models for the BVMT index have also recorded a similar success. The conditional Block Maxima method for the CAC 40 index and at high confidence level produce acceptable VaR forecasts. GARCH models and conditional EVT offer high volatile quantile forecasts, while Historical simulation and unconditional EVT methods provide stable quantile forecasts. These two methods do not update the VaR number quickly when market volatility increases: when VaR violation occurs this day,the probability to observe an exceedence the next day is high. Hence, we observe clustered violations;

There are possible directions for future research. Methods presented and studied above are well-suited for providing forecasts of portfolio level risk measures such as the aggregate VaR. However they are less well-suited for providing input into the active portfolio and risk management process. A multivariate approach should be adopted to have a complete picture of the risk and to know the optimal portfolio weights to minimize portfolio variance. Multivariate models provide a forecast for the entire covariance matrix and are also better suited for calculating sensitivity risk measures. We can compute VaR variation when we add additional shares to my portfolio. Variety of multivariate volatility models can be used such as symmetric and asymmetric MGARCH and DCC models, Flexible multivariate GARCH introduced by Ledoit, Santa-Clara and Wolf (2003). Multivariate Extreme Value Theory offers also a tool for exploring cross-asset tail dependencies, which are not captured by standard correlation measures. Modelling the dependence structure of multivariate financial data using copulas is an approach recently rediscovered by a number of authors. The copula function provides a complete description of the association and the co-dependence proprieties of random variables at each point of a distribution.

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