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Simwaka, Kisu

Reserve Bank of Malawi

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By

Kisu Simwaka

Reserve Bank of Malawi

Abstract

Fractional cointegration has attracted interest in time series econometrics in recent years (see among others, Dittmann 2004). According to Engle and Granger (1987), the concept of fractional cointegration was introduced to generalize the traditional cointegration to the long memory framework. Although cointegration tests have been developed for the traditional cointegration framework, these tests do not take into account fractional cointegration. This paper proposes a bootstrap procedure to test for time-varying fractional cointegration.

JEL Classification: C15, C22, C52

Keywords: Time-varying fractional cointegration, bootstrap procedure

1.0 Introduction

Fractional cointegration has attracted interest in time series econometrics in recent years (see among others, Dittmann 2004). Fractional cointegration analysis has emerged based on the view that cointegrating relationships between non-stationary economic variables may exist without observable processes necessarily being unit root $I(1)$ processes or cointegrating errors necessarily $I(0)$ processes.

Both fractional and standard cointegrations were originally defined at the same time in Engle and Granger (1987), but standard cointegration has attracted wide interest. In their standard approach, Engle and Granger (1987) and Johansen (1988) assumed that the cointegrating vector(s) do not change over time. However, when one takes into account such phenomenon as structural breaks and regime shifts, the assumption of fixed cointegrating vector(s) becomes quite restrictive. The fractional cointegration framework is more general since it allows the memory parameter to take fractional values and to be any positive real number.

Following Granger (1986), a set of $I(d)$ variables are said to be cointegrated, or $CI(d,b)$, if there exists a linear combination that is $CI(d-b)$ for $b>0$. To define fractional cointegration, let x_t by *n*-dimensional vector $I(1)$ process. Then x_t is fractionally cointegrated if there is an $a \in R^n$, $a \neq 0$, such that $a^{'} x_t \sim I(d)$ with $0<d<1$. In this case, d is called the *equilibrium long-memory parameter* and write $x_t \sim I(d)$. Compared to classical cointegration, where $d=0$, defining the cointegration rank is more difficult for fractionally cointegrated systems, because different cointegrating relationship need not have the same long-memory parameter.

Although cointegration tests have been developed for the traditional cointegration framework, these tests do not take into account fractional cointegration. The bootstrap has become a standard tool for econometric analysis. In general, the purpose of using the bootstrap methodology is two-fold: first, to find the distributions of statistics whose asymptotic distributions are unknown or dependent upon nuisance parameters, and second, to obtain refinements of the asymptotic distributions that are closer to the finite sample distributions of the statistics. It is well known that the bootstrap statistics have the same asymptotic distributions as the corresponding sample statistics for a very wide, if not all, class of models, and therefore, the unknown or nuisance parameter dependent limit distributions can be approximated by the bootstrap simulations. Furthermore, if properly implemented to pivotal statistics, the bootstrap simulations provide better approximations to the finite sample distributions of the statistics than their asymptotics (see Horowitz 2002).

The purpose of this paper is to propose a bootstrap procedure for testing for time-varying fractional cointegration. The rest of the paper is organized as follows. Section 1.1 examines the fractional cointegration framework while Section 1.2 introduces the time-varying cointegration framework. Section 1.3 presents the bootstrap procedure for testing for time-varying fractional cointegration.

1.1 Fractional cointegration

The fractional cointegration setup that we consider in this paper is based on an extension of the Johansen's (2008) Error Correction Mechanism (ECM) framework which is specified as follows:

$$\Delta X_t = \Pi X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \Phi D_t + \varepsilon_t \qquad (1.1)$$

where X_t is a vector of $I(1)$ series of order *k* x 1, D_t are deterministic terms, ε_t is a *k* x 1 vector of Gaussian errors with variance-covariance matrix Ω, and Π, $\Gamma_1,...,\Gamma_{k-1},\Phi$ are freely varying parameters. When the vector X_t is cointegrated, we have the reduced rank condition $\Pi = \alpha\beta^{'}$, where α and β are *N* x *r* constant parameter matrices, having rank *r*, representing the error correction and cointegrating coefficients, respectively.

Granger (1986) proposed the first generalization of the VECM model to the fractional case with the following form:

$$A*(L)\Delta^d X_t = (1-\Delta^b)\Delta^{d-b}\alpha\beta^{'} X_{t-1} + d(L)\varepsilon_t \qquad (1.2)$$

Where $A*(L)$ is a lag polynomial, X_t and ε_t are *N* x 1, ε_t~i.i.d(o,Σ); α and β are as defined in (1.1) above; and b and d are real values, with d representing order of fractional integration and $d-b$ representing order of co-fractional order. The process X_t is a fractional order of d and co-fractional order of, $d-b$. In other words, that is there exists β vectors for which $\beta' X_t$ is fractional of order $d-b$. L represents lag operator, and (Δ^d)

represents fractional difference parameter. Note that equation (1.2) has the conventional error correction representation when $d = 1$ and $d - b = 0$, i.e. $I(1)$ variables cointegrate to $I(0)$.

Dittman (2004) attempts to derive this model from a moving average form but, according to Johansen 2008, the results are not correctly proved. In this paper, we follow the formulation suggested by Johansen (2008):

$$\Delta^d X_t = \alpha\beta' \Delta^{d-b} L_b X_t + \sum_{i=1}^{k-1} \Gamma_i \Delta^d L_b^i X_t + \varepsilon_t \qquad (1.3)$$

This formulation implies the following changes from (1.2): $(1-\Delta^b)X_{t-1}$ is changed to $L_b X_t$; the lag polynomial $A^*(L)$ is changed to $A(L_b)$; i.e. the latter is lag polynomial in L_b (and not L_b). $L_b = 1-(1-L)^b$. The lag polynomial $d(L)$ is ignored.

When $d = 1$ and $d - b = 0$, i.e. $I(1)$ variables cointegrate to $I(0)$.

$$\Delta X = \alpha\beta' X_{t-1} + \sum_{i=1}^{k} \Gamma_i \Delta X_{t-i} + \varepsilon_t \qquad (1.4)$$

However, from (1.2) also note that the condition

$$(1-L)^{d-b} \beta' x_t \sim \mathrm{I}(0) \qquad (1.5)$$

is required so that the equation balances, having both sides $I(0)$. $d-b$ represents cointegrating rank. Setting $d = b = 1$ yields to the usual Johansen (1988, 1991) style VECM, but d and b can be real values with $d > 0$ and $0 < b \leq d$. In this model, all elements of x_t exhibit the sane order of integration, not necessarily unit, and similarly, the cointegrating residuals $\beta' x_t$ are all of order $d-b$. It should be noted that in fractional cointegration, the cointegrating residual is long memory and possibly even non-stationary, but has a lower order of integration than its constituent variables.

From equation (1.5), it follows that

$$\Delta(1-L)^{d-b} \beta' x_t = w_t \sim \mathrm{I}(0) \qquad (1.6)$$

Where $w_t = \psi(L)\varepsilon_t$,

1.2 Time-varying Fractional Cointegration Framework

In this model, we extend the Johansen (2008) Fractional VECM(p) framework to a time-varying framework as follows:

$$\Delta^d X_t = \Pi_t{}' \Delta^{d-b} L_b X_t + \sum_{i=1}^{k-1} \Gamma_i \Delta^d L_b^i X_t + \varepsilon_t \tag{1.7}$$

where $\Pi_t^{'} = \alpha\beta_t$, and β_t is time-varying cointegrating vector of coefficients. Our objective is to test the null hypothesis of time-invariant cointegration, $\Pi_t^{'} = \Pi^{'} = \alpha\beta$, where α and β are fixed *k* and *r* matrices with rank r, against the time varying parameter of the type

$$\Pi_t^{'} = \alpha(\beta_t^{'}), \tag{1.8}$$

Where β_t's are time varying *k* x *r* matrices, with constant rank *r,* and t represents time, where $t \geq 0$. In this case, α_t's are assumed to be fixed while β_t's are assumed to be time dependent.

Equation (1.7) is governed by the following assumptions:

Assumption 1. $\beta_t = \beta_{t/T}$, where each element of $\beta_t, t \in (0,1)$ is a function of time, t and twice-differentiable on $(0,1)$.

Assumption 2. X_t is an $\alpha-$mixing sequence with finite *8*-th moments

Assumption 3: u_t is a stationary martingale difference sequence with finite *4*-th moments, which is independent of X_t at all leads and lags

Assumption 1 is quite essential. It specifies that β is a deterministic function of time. It is interesting to note that it depends not only on the point in time t, but also on the sample size *T*. This is necessary as one needs the sample size that relates to that parameter to tend to infinity, for one to estimate consistently a particular parameter. This is achieved by allowing an increasing number of neighbouring observations in order to obtain more information about β at time *t*. In other words, we have to assume that as the sample size grows, the function β_t will extend to cover the whole period of the sample. This kind of setup has examples in the statistical literature. Assumptions 2 and 3 are standard mixing and moment conditions for the explanatory variables and the error term.

1.3 Testing for time-invariant fractional cointegration against time-varying fractional cointegration using bootstrap approach

We wish to test the hypothesis that $\beta_t = \beta \quad \forall t$ against the alternative hypothesis that β_t is non-constant and satisfies assumption 1. We start our analysis by looking at point-wise tests, i.e. tests that focus on particular time periods, and therefore consider a fixed i. Let us denote the estimate of β under the null as $\tilde{\beta}$. Depending on the assumptions made about u_t, standard methods can be used to estimate β under the null. For example, in the case where the disturbances are spherical and uncorrelated, from X_t OLS is an optimal estimator.

1.3.1 The Bootstrap approach

The bootstrap is a method for estimating the distribution of an estimator or test statistic by resampling one's data. It treats data as if they were the population for the purpose of evaluating the distribution of interest. What determines how reliably a bootstrap test performs is how well the bootstrap data generating processes (DGP) mimics the features of the true DGP that matter for the distribution of the test statistic.

There are various bootstrap methods used for re-sampling data. The first is the residual bootstrap, which assumes the residuals (error terms) of a regression are independent and identically distributed with common variance. It obtains estimated parameter and residuals from a given regression. Using rescaled residuals, the residual bootstrap data generating process generates a typical observation of the bootstrap sample. The bootstrap errors are said to be re-sampled. The second is the parametric bootstrap which is used when the distribution of the error term is known (i.e. normal distribution). The third one is the wild bootstrap and it is used if the error terms are not independently and identically distributed.

All of the bootstrap DGPs that have been discussed so far treat the error terms (or the data, in the case of the pairs bootstrap) as independent. When that is not the case, these methods are not appropriate. In particular, re-sampling (whether of residuals or data) breaks up whatever dependence there may be and is therefore unsuitable for use when there is dependence.

Several bootstrap DGPs for dependent data have been proposed.

(i) *Sieve bootstrap*

The sieve bootstrap method assumes that the error terms follow an unknown stationary process with homoscedastic innovations. It uses a finite autoregressive model (whose order is increasing with the sample size) to approximate this process and then re-samples from the approximated auto-regression. It obtains the residuals $\hat{u}_t$ and then estimates the AR(p) model

$$\hat{\varepsilon}_t = \sum_{i=1}^{p} \rho_i \hat{\varepsilon}_{t-i} + u_t \qquad (1.9)$$

After p has been chosen, and the preferred version of equation (1.13) estimated, the bootstrap error terms are generated recursively by the equation

$$\varepsilon_t^* = \sum_{i=1}^{p} \hat{\rho}_i \varepsilon_{t-i}^* + u_t^* \qquad (1.10)$$

where $\hat{\rho}_i$ are the estimated parameters, and the ε_t^* are re-sampled residuals. The method of the sieve bootstrap requires to fit the linear process (w_t) to a finite order VAR with the order increasing as the sample size grows. We may re-write (w_t) as a VAR

(ii) *Block bootstrap*

This involves dividing the series into b blocks and then re-sampling the blocks. One disadvantage of the method is that its performance can depend on the choice of b especially for a moderately small sample size.

(iii) *Sub-sampling bootstrap*

Sub-sampling bootstrap method is where b samples of the series are generated and the statistics of interest is calculated for each sub-series. The main difference between the subsampling and the Moving Block Bootstrap is that subsampling looks upon the blocks as "subseries", whereas the Moving Block use the blocks to construct a new pseudo-time series.

1.3.2 The Bootstrap test procedure for time-varying fractional cointegration

In this section, we introduce the bootstrap procedure for testing for time-varying fractional cointegration

Our objective is to test the null hypothesis of time-invariant cointegration, $\Pi_t^{'} = \Pi^{'} = \alpha\beta$, where α and β are fixed *k* and *r* matrices with rank r, against the time varying parameter of the type $\Pi_t^{'} = \alpha(\beta_t^{'})$, where β_t's are time varying *k* x *r* matrices, with constant rank *r,* and *t* represents time, where $t \geq 0$. The ρ lags of ΔX_{t-j} are added to account for serial correlation in the error terms, with ρ using AIC criteria.

From equation (1.6), we may write (w_t) as a VAR

$$\Phi(L)w_t = \varepsilon_t \tag{1.11}$$

It is therefore reasonable to approximate (w_t) as a finite order VAR

$$w_t = \Phi_1 w_{t-1} + ... + \Phi_q w_{t-q} + \varepsilon_{qt} \tag{1.12}$$

The order q of the approximated VAR is set to increase at a controlled rate of n, as we will specify below. In practice, it can be chosen by one of the commonly used order selection rules such as AIC and BIC.

Assumption 1

Let $q \to \infty$ and $q = o(n^{1/2})$ as $n \to \infty$

Below, is an outline of the bootstrap algorithm for the time-varying fractional cointegration:

(a) Fit an ARIMA model of order $p(T)$) and obtain estimated coefficients of the model and construct a set of residuals $(\hat{\varepsilon}_t)$

(b) Then fractionally difference the series according to estimates from (a) to estimate w_t in (1.6) and get the fitted values of $(\hat{\varepsilon}_t^*)$

(c) Apply the sieve estimation method to $(\hat{w}_t)$ to get the fitted values $(\hat{\varepsilon}_{qt})$ of $(\hat{\varepsilon}_{qt})$ i.e.

$$\hat{w}_t = \hat{\Phi}_1 \hat{w}_{t-1} + ... + \hat{\Phi} \hat{w}_{t-q} + \hat{\varepsilon}_{qt} \tag{1.13}$$

Obtain (ε_t^*) by re-sampling the centred fitted residuals

$$(\hat{\varepsilon}_{qt} - \tfrac{1}{n}\sum_{t=1}^{n}\hat{\varepsilon}_{qt})_{t=1}^{n}$$

(d) Specify dynamic model of differences. The objective is to estimate equation (1.7) to test the restriction of the null hypothesis $\beta_t = \beta$. The residuals from stage (b) are re-sampled with replacement and used to generate series according to (1.6) under H_0. Any suitable statistic to test for a cointegration relation can be computed from these. The values of these statistics in the observed data are located in the bootstrap distributions to yield an estimated *p*-value.

(e) Repeat steps (a) – (d), *B* times to obtain the empirical distribution of $\hat{\beta}^*$ and determine whether it is constant ($\hat{\beta}^*$) or time-varying $(\hat{\beta}_t^*)$

1.3.3 Bootstrap Asymptotics

The asymptotic theories of the estimators $\hat{\Pi}_n^*$ can be developed similarly as those for $\hat{\Pi}_n$. To develop their asymptotics, we develop the bootstrap invariance principle for (ε_t^*). We have

Lemma 1.1 Under 1.1

$$E^* | \varepsilon_t^* |^a = Op(1) \quad \text{as } n \to \infty$$

Generally, Lemma 1.1 allows us to regard the bootstrap samples (ε_t^*) as iid random variables with finite *a*-th moment, given a sample realization.

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