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1 Introduction

The asymptotic behavior of expressions of the form $\sum_{t=1}^{n} f(r_n x_t)$ where x_t is an integrated process, r_n is a sequence of norming constants, and f is a measurable function has been the subject of a number of articles in recent years. We mention Borodin and Ibragimov (1995), Park and Phillips (1999), de Jong (2004), Jeganathan (2004), Pötscher (2004), de Jong and Whang (2005), Berkes and Horvath (2006), and Christopeit (2009) which study weak convergence results for such expressions under various conditions on x_t and the function f. Of course, these results also provide information on the order of magnitude of $\sum_{t=1}^{n} f(r_n x_t)$. However, to the best of our knowledge no result is available for the case where f is non-integrable with respect to Lebesgue-measure in a neighborhood of a given point, say $x = 0$. In this paper we are interested in bounds on the order of magnitude of $\sum_{t=1}^{n} |x_t|^{-\alpha}$ when $\alpha \geq 1$, a case where the implied function f is not integrable in any neighborhood of zero. More generally, we shall also obtain bounds on the order of magnitude for $\sum_{t=1}^{n} v_t |x_t|^{-\alpha}$ where v_t are random variables satisfying certain conditions. While the emphasis in this paper is on negative powers that are non-integrable in any neighborhood of zero (i.e., $\alpha \geq 1$), we also present results for $\alpha < 1$ whenever they are easily obtained. We make no effort to improve the results in case $\alpha < 1$, but we shall occasionally mention better results available in this case (or in subcases thereof) without attempting to be complete in the coverage of such (better) results specific to the case $\alpha < 1$. While my interest in the problem treated in the

*I would like to thank Kalidas Jana for inquiring about the order of magnitude of some of the quantities now treated in the paper. I am indebted to Robert de Jong for comments on an early draft that have led to an improvement in Theorem 1. I am grateful to Istvan Berkes, Hannes Leeb, David Preinerstorfer, Zhan Shi, the referees, and the editor Peter Phillips for helpful comments.

present paper is purely driven by mathematical curiosity, reciprocals and ratios of variables that may be integrated are not alien to economic models. Hence the results presented below are of potential interest for the econometric analysis of such models.

2 Results

Consider an integrated process

$$x_t = x_{t-1} + w_t$$

for integer $t \geq 1$, with the initial real-valued random variable x_0 being independent of the process $(w_t)_{t \geq 1}$ which is assumed to be given by

$$w_t = \sum_{j=0}^{\infty} \phi_j \varepsilon_{t-j}.$$

Here $(\varepsilon_i)_{i \in \mathbb{Z}}$ are independent and identically distributed real-valued random variables that have mean 0 and a finite variance, which – without loss of generality – is set equal to 1. The coefficients ϕ_j are assumed to satisfy $\sum_{j=0}^{\infty} |\phi_j| < \infty$ and $\sum_{j=0}^{\infty} \phi_j \neq 0$. Furthermore, ε_i is supposed to have a density q with respect to (w.r.t.) Lebesgue-measure. We note that under these assumptions x_t possesses a density w.r.t. Lebesgue-measure for every $t \geq 1$, and the same is true for w_t; cf. Section 3.1 in Pötscher (2004). Furthermore, the characteristic function ψ of ε_i is assumed to satisfy

$$\int_{-\infty}^{\infty} |\psi(s)|^{\nu} \, ds < \infty \tag{1}$$

for some $1 \leq \nu < \infty$. *These assumptions will be maintained throughout the paper.* They have been used in Pötscher (2004), while stricter versions occur, e.g., in Park and Phillips (1999), de Jong (2004), and de Jong and Whang (2005). A detailed discussion of the scope of condition (1) is given in Pötscher (2004), Section 3.1. In particular, we recall from Lemma 3.1 in Pötscher (2004) that under the maintained conditions of the present paper densities h_t of $t^{-1/2} x_t$ exist such that for a suitable integer $t_* \geq 1$

$$\sup_{t \geq t_*} \|h_t\|_{\infty} < \infty \tag{2}$$

is satisfied, where $\|\cdot\|_{\infty}$ denotes the supremum norm. In the following we set $\kappa = \sup_{t \geq t_*} \|h_t\|_{\infty}$.

2.1 Bounds on the Order of Magnitude of $\sum_{t=1}^{n} |x_t|^{-\alpha}$

We first consider the behavior of $\sum_{t=1}^{n} |x_t|^{-\alpha}$. Note that under our assumptions this quantity is almost surely well-defined and finite for every $\alpha \in \mathbb{R}$.[1] Recall

[1] In particular, how, and if, we assign a value in the extended real line to $|x_t|^{-\alpha}$ on the event $\{x_t = 0\}$ has no consequence for the results.

that we are mainly interested in the case $\alpha \geq 1$. While the next theorem provides an upper bound on the order of magnitude, lower bounds are discussed in Remarks 5 and 6 below.

Theorem 1

$$\sum_{t=1}^{n} |x_t|^{-\alpha} = \begin{cases} O_{\Pr}(n^{\alpha/2}) & \textit{if} \quad \alpha > 1 \\ O_{\Pr}(n^{1/2}\log n) & \textit{if} \quad \alpha = 1 \\ O_{\Pr}(n^{1-\alpha/2}) & \textit{if} \quad -2 \leq \alpha < 1. \end{cases}$$

Proof. Suppose first that $\alpha \geq 0$ holds. Since $\sum_{t=1}^{t_*-1} |x_t|^{-\alpha}$ is almost surely real-valued it suffices to prove the result for $\sum_{t=t_*}^{n} |x_t|^{-\alpha}$. For $0 < \delta < 1$ we have almost surely

$$\begin{aligned} \sum_{t=t_*}^{n} |x_t|^{-\alpha} &= \sum_{t=t_*}^{n} |x_t|^{-\alpha}\, \mathbf{1}\left(\left|t^{-1/2}x_t\right| > \delta/(nt)^{1/2}\right) \\ &\quad + \sum_{t=t_*}^{n} |x_t|^{-\alpha}\, \mathbf{1}\left(\left|t^{-1/2}x_t\right| \leq \delta/(nt)^{1/2}\right) \\ &= Q_n(\delta) + R_n(\delta) \end{aligned}$$

where t_* is as in (2) and $n \geq t_*$. First consider $R_n(\delta)$: Set

$$S_n(\delta) = \bigcup_{t=t_*}^{n} \left\{\left|t^{-1/2}x_t\right| \leq \delta/(nt)^{1/2}\right\}.$$

Observe that $\{R_n(\delta) > 0\} = S_n(\delta)$ up to null-sets and

$$\begin{aligned} \Pr\left(R_n(\delta) > 0\right) &= \Pr\left(S_n(\delta)\right) \leq \sum_{t=t_*}^{n} \Pr\left(\left|t^{-1/2}x_t\right| \leq \delta/(nt)^{1/2}\right) \\ &= \sum_{t=t_*}^{n} \int_{-\delta/(nt)^{1/2}}^{\delta/(nt)^{1/2}} h_t(z)dz \leq 2\kappa\delta n^{-1/2} \sum_{t=t_*}^{n} t^{-1/2} \\ &\leq 4\kappa\delta \end{aligned}$$

holds for all $n \geq t_*$ in view of (2) using the fact that $\sum_{t=t_*}^{n} t^{-1/2} \leq \sum_{t=1}^{n} t^{-1/2} \leq 2n^{1/2}$. Next we bound $Q_n(\delta)$: Observe that

$$EQ_n(\delta) = \sum_{t=t_*}^{n} t^{-\alpha/2} E\left(\left|t^{-1/2}x_t\right|^{-\alpha} \mathbf{1}(\left|t^{-1/2}x_t\right| > \delta/(nt)^{1/2})\right),$$

and that for $t \geq t_*$

$$
\begin{aligned}
& E\left(\left|t^{-1/2}x_t\right|^{-\alpha}\mathbf{1}(\left|t^{-1/2}x_t\right| > \delta/(nt)^{1/2})\right) \\
= \; & E\left(\left|t^{-1/2}x_t\right|^{-\alpha}\mathbf{1}(1 > \left|t^{-1/2}x_t\right| > \delta/(nt)^{1/2})\right) \\
& + E\left(\left|t^{-1/2}x_t\right|^{-\alpha}\mathbf{1}(\left|t^{-1/2}x_t\right| \geq 1)\right) \\
\leq \; & \int_{\delta/(nt)^{1/2}<|z|<1} |z|^{-\alpha} h_t(z)dz \; + \; 1 \; \leq \; 2\kappa \int_{\delta/(nt)^{1/2}}^{1} z^{-\alpha}dz \; + \; 1 \\
\leq \; & \begin{cases} 1 + 2\kappa(\alpha-1)^{-1}\delta^{1-\alpha}(nt)^{(\alpha-1)/2} & \text{if} \quad \alpha > 1 \\ 1 + 2\kappa\log\left(\delta^{-1}\right) + 2\kappa\log\left((nt)^{1/2}\right) & \text{if} \quad \alpha = 1 \\ 1 + 2\kappa(1-\alpha)^{-1} & \text{if} \quad 0 \leq \alpha < 1. \end{cases}
\end{aligned}
$$

Consequently, for $n \geq \max(t_*, 3)$ we have

$$
\begin{aligned}
E(Q_n(\delta)) \; & \leq \; \begin{cases} \left(1 + 2\kappa(\alpha-1)^{-1}\delta^{1-\alpha}\right) n^{(\alpha-1)/2} \sum_{t=t_*}^{n} t^{-1/2} & \text{if} \quad \alpha > 1 \\ \left(1 + 2\kappa + 2\kappa\log\left(\delta^{-1}\right)\right)(\log n) \sum_{t=t_*}^{n} t^{-1/2} & \text{if} \quad \alpha = 1 \\ \left(1 + 2\kappa(1-\alpha)^{-1}\right) \sum_{t=t_*}^{n} t^{-\alpha/2} & \text{if} \quad 0 \leq \alpha < 1. \end{cases} \\
& \leq \; \begin{cases} c(\alpha,\delta,\kappa)n^{\alpha/2} & \text{if} \quad \alpha > 1 \\ c(1,\delta,\kappa)n^{1/2}\log n & \text{if} \quad \alpha = 1 \\ c(\alpha,\delta,\kappa)n^{1-\alpha/2} & \text{if} \quad 0 \leq \alpha < 1. \end{cases}
\end{aligned}
$$

where $c(\alpha, \delta, \kappa)$ are positive finite constants.

Now, for arbitrary $\varepsilon > 0$ choose $\delta(\varepsilon)$ satisfying $0 < \delta(\varepsilon) < \min(1, \varepsilon/(8\kappa))$. Then choose $M = M(\varepsilon, \alpha, \kappa) > 0$ large enough to satisfy

$$M > 4\varepsilon^{-1}c(\alpha, \delta(\varepsilon), \kappa).$$

Then, with $d_n = n^{\alpha/2}$ in case $\alpha > 1$, $d_n = n^{1/2}\log n$ in case $\alpha = 1$, and $d_n = n^{1-\alpha/2}$ in case $0 \leq \alpha < 1$, we obtain using Markov's inequality

$$
\begin{aligned}
& \Pr\left(d_n^{-1}\sum_{t=t_*}^{n}|x_t|^{-\alpha} > M\right) \\
\leq \; & \Pr\left(d_n^{-1}Q_n(\delta(\varepsilon)) > M/2\right) \; + \; \Pr\left(d_n^{-1}R_n(\delta(\varepsilon)) > M/2\right) \\
\leq \; & 2d_n^{-1}EQ_n(\delta(\varepsilon))/M \; + \; \Pr\left(R_n(\delta(\varepsilon)) > 0\right) \; < \; \varepsilon
\end{aligned}
$$

for all $n \geq \max(t_*, 3)$. Since $\sum_{t=t_*}^{n}|x_t|^{-\alpha}$ is almost surely real-valued for all $n \geq t_*$, this completes the proof in case $\alpha \geq 0$.

Suppose next that $-2 \leq \alpha < 0$ holds. Observe first that

$$\sum_{t=1}^{n}|x_t|^{-\alpha} \leq \max\left(1, 2^{-\alpha-1}\right)\left(\sum_{t=1}^{n}|x_t - x_0|^{-\alpha} + n|x_0|^{-\alpha}\right). \tag{3}$$

By Lyapunov's inequality and noting that $E(x_t - x_0)^2$ is of the exact order t (since w_t is a linear process with absolutely summable coefficients satisfying $\sum_{j=0}^{\infty} \phi_j \neq 0$) we have

$$E \sum_{t=1}^{n} |x_t - x_0|^{-\alpha} \leq c \sum_{t=1}^{n} t^{-\alpha/2} = O(n^{1-\alpha/2})$$

for some finite constant c. But then an application of Markov's inequality gives $\sum_{t=1}^{n} |x_t - x_0|^{-\alpha} = O_{\Pr}(n^{1-\alpha/2})$. Together with (3) this establishes the claim. ■

Remark 2 (i) The proof of Theorem 1 in the previous version of this paper (dated January 2011) is incorrect. For a discussion of the errors and an alternative proof see the supplementary notes available on my webpage.

(ii) Remark 6 in the January 2011 version of this paper insinuated that there is a contradiction between Theorem 1 and results in de Jong and Whang (2005). However, the argument put forward in this remark is invalid as there is an elementary sign-mistake in the inequality presented in that remark. Hence, this remark is completely invalid and I owe apologies to de Jong and Whang.

Remark 3 (i) For values of α such that $x^{-\alpha}$ is well-defined for every x except possibly for $x = 0$, the quantity $\sum_{t=1}^{n} x_t^{-\alpha}$ is almost surely well-defined and real-valued. By the triangle inequality Theorem 1 applies also to $\sum_{t=1}^{n} x_t^{-\alpha}$.

(ii) Not surprisingly, the expectation of $\sum_{t=1}^{n} |x_t|^{-\alpha}$ will typically be infinite in the case $\alpha \geq 1$ (e.g., if the density of x_t is bounded from below in a neighborhood of zero as is the case if x_t is Gaussian). The expectation can, however, also be infinite in other cases (e.g., if $\alpha < -2$ and moments of x_t of order $-\alpha$ do not exist).

Remark 4 (i) It follows from Remark 5 below that the bound given for $-2 \leq \alpha < 0$ holds in fact for all $\alpha < 0$ provided the additional condition $\sum_{j=0}^{\infty} j^{1/2} |\phi_j| < \infty$ is satisfied. [The additional condition is perhaps unnecessary, but we do not make any effort to remove it as the focus in this paper is on the case $\alpha \geq 1$.]

(ii) If $Ex_0^2 < \infty$ holds, then $Ex_t^2 = E(x_t - x_0)^2 + Ex_0^2$ is of the order t and thus $E|x_t|^{-\alpha}$ is at most of the order $t^{-\alpha/2}$ for $-2 \leq \alpha < 0$ by Lyapunov's inequality. This shows that if $Ex_0^2 < \infty$ holds the proof of Theorem 1 for the case $-2 \leq \alpha < 0$ can be simplified.

Remark 5 Suppose the stronger summability condition $\sum_{j=0}^{\infty} j^{1/2} |\phi_j| < \infty$ is satisfied. Under this additional assumption more is known in case $-\infty < \alpha < 1$ than just the upper bound on the order of magnitude of $\sum_{t=1}^{n} |x_t|^{-\alpha}$ given by Theorem 1: If $-\infty < \alpha < 1$ then

$$n^{\alpha/2-1} \sum_{t=1}^{n} |x_t|^{-\alpha} \xrightarrow{d} |\sigma|^{-\alpha} \int_0^1 |W(s)|^{-\alpha} \, ds \tag{4}$$

for $n \to \infty$, with the limiting variable being positive with probability one; as a consequence, $n^{1-\alpha/2}$ is the exact order of magnitude in probability of

$\sum_{t=1}^{n}|x_t|^{-\alpha}$. Here W is standard Brownian motion and $\sigma=\sum_{j=0}^{\infty}\phi_j$, which is non-zero by assumption.[2] Relation (4) follows from the first claim in Corollary 3.3 in Pötscher (2004), applied to the function T given by $T(x)=|x|^{-\alpha}$ for $x\neq 0$ and $T(0)=0$, and from the observation that $n^{\alpha/2-1}\sum_{t=1}^{b}|x_t|^{-\alpha}\to 0$ as $n\to\infty$ for every fixed integer b. Note that T is locally integrable since $\alpha<1$ and that T satisfies $T(\lambda x)=|\lambda|^{-\alpha}T(x)$ for all $x\in\mathbb{R}$ and all $\lambda\neq 0$. Also note that the integral in (4) is almost surely well-defined and finite (independently of how one interprets $|W(s)|^{-\alpha}$ for $W(s)=0$ in case $\alpha>0$), cf. (2.4) and Remark 2.1 in Pötscher (2004). [In the case $\alpha\leq 0$, it is well-known that (4) holds even under much weaker conditions than used here, cf. Lemma A.1 in Pötscher (2004). Since the emphasis in this paper is on positive α, we make no attempt to spell out these sharper and well-known results for $\alpha\leq 0$.]

Remark 6 [3](i) We first provide a lower bound in case $\alpha=1$. Given the additional assumption $\sum_{j=0}^{\infty}j^{1/2}\left|\phi_j\right|<\infty$, a lower bound for the order of magnitude in probability of $\sum_{t=1}^{n}|x_t|^{-1}$ is given by $n^{1/2}$, in the sense that

$$\lim_{n\to\infty}\Pr\left(n^{-1/2}\sum_{t=1}^{n}|x_t|^{-1}>M\right)=1$$

holds for every real M, i.e., $n^{-1/2}\sum_{t=1}^{n}|x_t|^{-1}\to\infty$ in probability. To see this, let $T_{k,1}(x)=\min(k,|x|^{-1})$ for $k\in\mathbb{N}$ with the convention that $T_{k,1}(0)=k$. Then we have almost surely

$$n^{-1/2}\sum_{t=1}^{n}|x_t|^{-1}=n^{-1}\sum_{t=1}^{n}\left|n^{-1/2}x_t\right|^{-1}\geq n^{-1}\sum_{t=1}^{n}T_{k,1}(n^{-1/2}x_t)$$

for every $k\in\mathbb{N}$. Furthermore, $n^{-1}\sum_{t=1}^{n}T_{k,1}(n^{-1/2}x_t)$ converges in distribution to $\int_0^1 T_{k,1}(\sigma W(s))ds$ by Corollary 3.4 in Pötscher (2004).[4] Now, by Corollary 7.4 in Chung and Williams (1990) and the monotone convergence theorem we have almost surely

$$\int_0^1 T_{k,1}(\sigma W(s))ds=\int_{-\infty}^{\infty}T_{k,1}(\sigma x)L(1,x)dx\to|\sigma|^{-1}\int_{-\infty}^{\infty}|x|^{-1}L(1,x)dx=\infty$$

for $k\to\infty$, where L denotes standard Brownian local time. The last equality in the above display follows since $L(1,0)>0$ almost surely and $L(1,x)$ having almost surely continuous sample path together imply that there exists a neighborhood U of zero (that may depend on the realization of $L(1,\cdot)$) such that $\inf_{x\in U}L(1,x)>0$ holds almost surely. Note that the just established

[2] Clearly, σ^2 is nothing else than the so-called long-run variance.

[3] The lower bound results for $\alpha\geq 1$ given in this remark together with the lower bound results for the case $-\infty<\alpha<1$ implied by Remark 5 provide an improvement over Proposition 6.4 in Park and Phillips (1999) under weaker conditions.

[4] Since $T_{k,1}$ is continuous, this convergence in fact holds under weaker conditions on the process x_t then used here, cf. Lemma A.1 in Pötscher (2004).

lower bound (established under the stricter summability condition on ϕ_j imposed here) and the upper bound given by Theorem 1 agree up to a logarithmic term and in this sense are close to being sharp.

(ii) We next turn to the case $\alpha > 1$ and show that the upper bound $n^{\alpha/2}$ on the order of magnitude is also a lower bound in the sense that

$$\lim_{\varepsilon \to 0, \varepsilon > 0} \liminf_{n \to \infty} \Pr \left(n^{-\alpha/2} \sum_{t=1}^{n} |x_t|^{-\alpha} > \varepsilon \right) = 1 \tag{5}$$

holds: To this end let β_n be a sequence satisfying $\beta_n \to \infty$ and $n^{-1}\beta_n \to 0$ as $n \to \infty$. Then we have almost surely

$$\begin{aligned} \left(n^{-1}\beta_n\right)^{1-\alpha} n^{-\alpha/2} \sum_{t=1}^{n} |x_t|^{-\alpha} &= n^{-1} \sum_{t=1}^{n} \beta_n \left|\beta_n n^{-1/2} x_t\right|^{-\alpha} \\ &\geq n^{-1} \sum_{t=1}^{n} \beta_n T_{k,\alpha}(\beta_n n^{-1/2} x_t), \end{aligned}$$

where $T_{k,\alpha}(x) = \min(k, |x|^{-\alpha})$ for $k \in \mathbb{N}$ with the convention that $T_{k,\alpha}(0) = k$. Note that $T_{k,\alpha}$ is Lebesgue-integrable (since $\alpha > 1$) and bounded. The version of Theorem 3 in Jeganathan (2004) given as Proposition 15 in the Appendix below now shows that the right-hand side of the above display converges in distribution to

$$|\sigma|^{-1} \int_{-\infty}^{\infty} T_{k,\alpha}(x) dx L(1,0).$$

Since $L(1,0) > 0$ almost surely and $\int_{-\infty}^{\infty} T_{k,\alpha}(x)dx \to \infty$ for $k \to \infty$, it follows that

$$\lim_{n \to \infty} \Pr \left(\left(n^{-1}\beta_n\right)^{1-\alpha} n^{-\alpha/2} \sum_{t=1}^{n} |x_t|^{-\alpha} > M \right) = 1$$

holds for every real M, i.e., $\left(n^{-1}\beta_n\right)^{1-\alpha} n^{-\alpha/2} \sum_{t=1}^{n} |x_t|^{-\alpha} \to \infty$ in probability. Note that $\alpha > 1$ and that this result holds for *every* sequence β_n satisfying $\beta_n \to \infty$ and $n^{-1}\beta_n \to 0$. A fortiori it then holds for every sequence $\beta_n > 0$ satisfying $n^{-1}\beta_n \to 0$. Hence we have that $\eta_n n^{-\alpha/2} \sum_{t=1}^{n} |x_t|^{-\alpha} \to \infty$ in probability for *every* sequence $\eta_n \to \infty$. By Lemma 16 in the Appendix it follows that $n^{\alpha/2}$ is a lower bound in the sense of (5).

Remark 7 (i) All results above for $\sum_{t=1}^{n} |x_t|^{-\alpha}$ apply analogously to sums of the form $\sum_{t=a}^{n} |x_t|^{-\alpha}$ for any (fixed) integer $a > 1$. [This follows since $\sum_{t=1}^{a-1} |x_t|^{-\alpha}$ is almost surely finite]

(ii) In case $\alpha \leq 0$ all results given above for $\sum_{t=1}^{n} |x_t|^{-\alpha}$ carry over to $\sum_{t=0}^{n} |x_t|^{-\alpha}$. For $\alpha > 0$ this is again so, provided the distribution of x_0 does not assign positive mass to the point 0; otherwise, $\sum_{t=0}^{n} |x_t|^{-\alpha}$ is undefined on the event where $x_0 = 0$; if one chooses to define $|x_0|^{-\alpha} = \infty$ on this event, then the above results clearly do not apply (except for the lower bound given in Remark 6 which then holds a fortiori).

2.2 Bounds on the Order of Magnitude of $\sum_{t=1}^{n} v_t |x_t|^{-\alpha}$

We next illustrate how the above results can be used to derive upper bounds on the order of magnitude of $\sum_{t=1}^{n} v_t |x_t|^{-\alpha}$ where v_t for $t \geq 1$ are random variables defined on the same probability space as x_t. Note that this expression is almost surely well-defined and finite for every $\alpha \in \mathbb{R}$.[5] The leading case we have in mind is $v_t = w_{t+1}^k$ where $k \in \mathbb{N}$. Applying the Cauchy-Schwarz inequality gives almost surely

$$\left| \sum_{t=1}^{n} v_t |x_t|^{-\alpha} \right| \leq \left(\sum_{t=1}^{n} v_t^2 \right)^{1/2} \left(\sum_{t=1}^{n} |x_t|^{-2\alpha} \right)^{1/2}.$$

Hence, if $\sup_{t \geq 1} E v_t^2 < \infty$ (or more generally $\sum_{t=1}^{n} E v_t^2 = O(n)$) holds, we obtain from Theorem 1

$$\sum_{t=1}^{n} v_t |x_t|^{-\alpha} = \begin{cases} O_{\Pr}(n^{(\alpha+1)/2}) & \text{if} \quad \alpha > 1/2 \\ O_{\Pr}(n^{3/4} (\log n)^{1/2}) & \text{if} \quad \alpha = 1/2 \\ O_{\Pr}(n^{1-\alpha/2}) & \text{if} \quad -1 \leq \alpha < 1/2. \end{cases} \tag{6}$$

Under the additional assumption $\sum_{j=0}^{\infty} j^{1/2} |\phi_j| < \infty$ the bound $O_{\Pr}(n^{1-\alpha/2})$ in fact holds also for $\alpha < -1$, cf. Remark 5. Variations of the above bound can obviously be obtained by using Hölder's inequality.

Remark 8 In the case $\alpha = 0$ the problem reduces to determining the order of $\sum_{t=1}^{n} v_t$, a problem to which this paper has nothing to add to the literature. We only observe that in this case the above bound can clearly be improved to $O_{\Pr}(n^{1/2})$ whenever v_t satisfies a central limit theorem (as is, e.g., the case if $v_t = w_{t+1}$), or whenever $E \left(\sum_{t=1}^{n} v_t \right)^2 = O(n)$. The latter condition is, e.g., satisfied if v_t is mean-zero and weakly stationary with absolutely summable covariance function, or if v_t is a sequence of uncorrelated mean-zero random variables satisfying $\sup_{t \geq 1} E v_t^2 < \infty$. We do not further comment on such improvements as they are not related to the subject of the paper.

We next provide improvements on the bound (6) under appropriate assumptions on v_t. Note that the assumptions on v_t in the subsequent proposition are certainly satisfied if v_t is independent of x_t (or of $x_t - x_0$, respectively) for every $t \geq 1$ and the first absolute moment of v_t is bounded uniformly in t. In particular, these assumptions are satisfied for the important special case $v_t = w_{t+1}^k$ provided that $\phi_j = 0$ for all $j > 0$ (implying that $w_t = \varepsilon_t$) and that $E |\varepsilon_t|^k$ is finite.[6]

[5] In particular, how, and if, we assign a value in the extended real line to $v_t |x_t|^{-\alpha}$ on the event $\{x_t = 0\}$ has no consequence for the results.

[6] The condition that $\phi_j = 0$ for all $j > 0$ can of course be replaced by the more general condition $\phi_l \neq 0$ for some $l \geq 0$ and $\phi_j = 0$ for all $j \neq l$. This equally applies to the discussion immediately preceding Propositions 11 and 13.

Proposition 9 *Suppose that in addition to the maintained assumptions we have that* $\sup_{t\geq 1} E(|v_t|) < \infty$ *holds. Assume further that* $E(|v_t| \mid x_t) = E(|v_t|)$ *almost surely holds for all* $t \geq 1$ *if* $\alpha \geq 0$, *and that* $E(|v_t| \mid x_t - x_0) = E(|v_t|)$ *almost surely holds for all* $t \geq 1$ *if* $-2 \leq \alpha < 0$. *Then*

$$\sum_{t=1}^{n} |v_t| \, |x_t|^{-\alpha} = \begin{cases} O_{\Pr}(n^{\alpha/2}) & \text{if} \quad \alpha > 1 \\ O_{\Pr}(n^{1/2} \log n) & \text{if} \quad \alpha = 1 \\ O_{\Pr}(n^{1-\alpha/2}) & \text{if} \quad -2 \leq \alpha < 1. \end{cases}$$

A fortiori the same bound then holds for $\sum_{t=1}^{n} v_t \, |x_t|^{-\alpha}$.

Proof. Suppose $\alpha \geq 0$. For the same reasons as given in the proof of Theorem 1 it suffices to bound $\sum_{t=t^*}^{n} |v_t| \, |x_t|^{-\alpha}$. Define for $0 < \delta < 1$

$$Q_n'(\delta) = \sum_{t=t_*}^{n} |v_t| \, |x_t|^{-\alpha} \, \mathbf{1}\left(\left|t^{-1/2} x_t\right| > \delta/(nt)^{1/2}\right)$$

and

$$R_n'(\delta) = \sum_{t=t_*}^{n} |v_t| \, |x_t|^{-\alpha} \, \mathbf{1}\left(\left|t^{-1/2} x_t\right| \leq \delta/(nt)^{1/2}\right).$$

Observe that now the event $\{R_n'(\delta) > 0\}$ is contained in $S_n(\delta)$ up to null-sets where $S_n(\delta)$ has been defined in the proof of Theorem 1. Hence,

$$\Pr\left(R_n'(\delta) > 0\right) \leq 4\kappa\delta$$

as shown in the proof of Theorem 1. Furthermore, since $|v_t|$ is integrable and $|x_t|^{-\alpha} \, \mathbf{1}\left(\left|t^{-1/2} x_t\right| > \delta/(nt)^{1/2}\right)$ is a bounded x_t-measurable random variable, the law of iterated expectations and the assumptions on v_t imply that

$$EQ_n'(\delta) \leq \left(\sup_{t\geq 1} E(|v_t|)\right) \sum_{t=t_*}^{n} t^{-\alpha/2} E\left(\left|t^{-1/2} x_t\right|^{-\alpha} \mathbf{1}(\left|t^{-1/2} x_t\right| > \delta/(nt)^{1/2})\right)$$

holds. The remainder of the proof is then identical to the proof of Theorem 1. Next suppose $-2 \leq \alpha < 0$. Then

$$\sum_{t=1}^{n} |v_t| \, |x_t|^{-\alpha} \leq \max\left(1, 2^{-\alpha-1}\right) \left(\sum_{t=1}^{n} |v_t| \, |x_t - x_0|^{-\alpha} + |x_0|^{-\alpha} \sum_{t=1}^{n} |v_t|\right). \quad (7)$$

Observe that the second sum on the right-hand side of the above display is $O_{\Pr}(n)$ by an application of Markov's inequality (since $E\,|v_t|$ is uniformly bounded by assumption) and since $|x_0|^{-\alpha}$ is well-defined and real-valued. Furthermore, since $|v_t|$ is integrable and $|x_t - x_0|^{-\alpha}$ is a nonnegative real-valued random variable we may use the law of iterated expectations again (conditioning being on $x_t - x_0$) to obtain that the expectation of the first sum in (7) is bounded by

$$\left(\sup_{t\geq 1} E(|v_t|)\right) \sum_{t=1}^{n} E\left(|x_t - x_0|^{-\alpha}\right).$$

This bound is then further treated exactly as in the proof of Theorem 1. ■

Remark 10 If $Ex_0^2 < \infty$ is assumed, the condition $E(|v_t| \mid x_t - x_0) = E(|v_t|)$ almost surely can be replaced by $E(|v_t| \mid x_t) = E(|v_t|)$ almost surely also in case $-2 \leq \alpha < 0$. The proof then proceeds by directly bounding $E \sum_{t=1}^{n} |v_t| |x_t|^{-\alpha}$ by $\left(\sup_{t \geq 1} E(|v_t|)\right) \sum_{t=1}^{n} E\left(|x_t|^{-\alpha}\right)$; cf. Remark 4(ii).

We next turn to the case where v_t is a martingale difference sequence. The improvement over the bound (6) is obtained in this case by observing that the sequence $\sum_{t=1}^{n} v_t |x_t|^{-\alpha}$ is then a martingale transform and by combining Theorem 1 with results in Lai and Wei (1982). [Note that $\sum_{t=1}^{n} v_t |x_t|^{-\alpha}$ will typically not be a martingale as the first moment will in general not exist, cf. Remark 3(ii); hence, martingale central limit theorems are not applicable.] The assumptions in the subsequent proposition are in particular satisfied in the important special case where $v_t = w_{t+1}$ and $\phi_j = 0$ for all $j > 0$ (implying that $v_t = w_{t+1} = \varepsilon_{t+1}$) by choosing $\mathcal{F}_t$ as the σ-field generated by $x_{t+1}, \ldots, x_1$ for $t \geq 0$.

Proposition 11 *Suppose that in addition to the maintained assumptions we have that $(v_t)_{t \geq 1}$ is a martingale difference sequence with respect to a filtration $(\mathcal{F}_t)_{t \geq 0}$ such that $\sup_{t \geq 1} E\left(v_t^2 \mid \mathcal{F}_{t-1}\right) < \infty$ holds almost surely. Assume further that x_t is $\mathcal{F}_{t-1}$-measurable for every $t \geq 1$.*

(a) Then

$$\sum_{t=1}^{n} v_t |x_t|^{-\alpha} = \begin{cases} o_{\Pr}(n^{\alpha/2} (\log n)^{1/2+\tau}) & \text{if } \alpha > 1/2 \\ o_{\Pr}(n^{1/4} (\log n)^{1+\tau}) & \text{if } \alpha = 1/2 \\ o_{\Pr}(n^{(1-\alpha)/2} (\log n)^{1/2+\tau}) & \text{if } -1 \leq \alpha < 1/2 \end{cases}$$

holds for every $\tau > 0$. Under the additional assumption $\sum_{j=0}^{\infty} j^{1/2} \left|\phi_j\right| < \infty$ the bound given for the range $-1 \leq \alpha < 1/2$ continues to hold for the range $-\infty < \alpha < 1/2$.

(b)

$$\sum_{t=1}^{n} v_t^2 |x_t|^{-\alpha} = \begin{cases} o_{\Pr}\left(n^{\alpha/2+\tau}\right) & \text{if } \alpha \geq 1 \\ o_{\Pr}\left(n^{1-\alpha/2+\tau}\right) & \text{if } -2 \leq \alpha < 1 \end{cases}$$

holds for every $\tau > 0$. Under the additional assumption $\sum_{j=0}^{\infty} j^{1/2} \left|\phi_j\right| < \infty$ the bound given for the range $-2 \leq \alpha < 1$ continues to hold for the range $-\infty < \alpha < 1$.

Proof. Since $\sum_{s=1}^{t} w_s$ is a (nondegenerate) recurrent random walk under the assumptions of the proposition that is not of the lattice-type (as it has uncountably many possible values in the sense of Chung (2001, Section 8.3) by Lebesgue's differentiation theorem), it visits every interval infinitely often almost surely. From independence of x_0 and $(w_s)_{s \geq 1}$ we may conclude that almost surely $|x_t|$ falls into the interval $(1/2, 3/2)$ infinitely often. This shows that the sum $\sum_{t=1}^{n} |x_t|^{-\alpha}$ diverges almost surely for every value $\alpha \neq 0$, the divergence being trivial in case $\alpha = 0$. Now apply Lemma 2(iii) in Lai and Wei (1982) to conclude that

$$\sum_{t=1}^{n} v_t \left|x_t\right|^{-\alpha} = o\left(\left(\sum_{t=1}^{n} \left|x_t\right|^{-2\alpha}\right)^{1/2} \left(\log \sum_{t=1}^{n} \left|x_t\right|^{-2\alpha}\right)^{1/2+\theta}\right) \quad a.s.$$

and

$$\sum_{t=1}^{n} v_t^2 \left|x_t\right|^{-\alpha} = o\left(\left(\sum_{t=1}^{n} \left|x_t\right|^{-\alpha}\right)^{1+\theta}\right) \quad a.s.$$

for every $\theta > 0$. Apply Theorem 1 as well as Remark 5 (applied to 2α and α, respectively) to complete the proof. ∎

Remark 12 If $\sup_{t\geq 1} E\left(\left|v_t\right|^{\gamma} \mid \mathcal{F}_{t-1}\right) < \infty$ almost surely holds for some $\gamma > 2$, applying Corollary 2 in Lai and Wei (1982) yields the slightly better bound

$$\sum_{t=1}^{n} v_t \left|x_t\right|^{-\alpha} = \begin{cases} O_{\Pr}(n^{\alpha/2} (\log n)^{1/2}) & \text{if } \alpha > 1/2 \\ O_{\Pr}(n^{1/4} \log n) & \text{if } \alpha = 1/2 \\ O_{\Pr}(n^{(1-\alpha)/2} (\log n)^{1/2}) & \text{if } -1 \leq \alpha < 1/2, \end{cases}$$

where under the additional condition $\sum_{j=0}^{\infty} j^{1/2} \left|\phi_j\right| < \infty$ the bound for the range $-1 \leq \alpha < 1/2$ again continues to hold for $-\infty < \alpha < 1/2$.

In case the martingale difference sequence is square-integrable with a non-random conditional variance the bound in Part (a) of the above proposition can be somewhat improved. I owe this observation to a referee. Note that the subsequent proposition in particular covers the important special case $v_t = w_{t+1} = \varepsilon_{t+1}$ mentioned above.

Proposition 13 *Suppose that in addition to the maintained assumptions we have that $(v_t)_{t\geq 1}$ is a martingale difference sequence with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $E\left(v_t^2 \mid \mathcal{F}_{t-1}\right) = E\left(v_t^2\right)$ holds almost surely for all $t \geq 1$ and such that $\sup_{t\geq 1} E\left(v_t^2\right) < \infty$. Assume further that x_t is $\mathcal{F}_{t-1}$-measurable for every $t \geq 1$. For the case $-1 \leq \alpha < 0$ assume additionally $Ex_0^2 < \infty$. Then*

$$\sum_{t=1}^{n} v_t \left|x_t\right|^{-\alpha} = \begin{cases} O_{\Pr}(n^{\alpha/2}) & \text{if } \alpha > 1/2 \\ O_{\Pr}(n^{1/4} (\log n)^{1/2}) & \text{if } \alpha = 1/2 \\ O_{\Pr}(n^{(1-\alpha)/2}) & \text{if } -1 \leq \alpha < 1/2 \end{cases}$$

holds.

Proof. Assume $\alpha \geq 0$ first. For the same reasons as given in the proof of Theorem 1 it suffices to bound $\sum_{t=t^*}^{n} v_t \left|x_t\right|^{-\alpha}$. For $0 < \delta < 1$ write $\sum_{t=t^*}^{n} v_t \left|x_t\right|^{-\alpha}$ as $Q_n^*(\delta) + R_n^*(\delta)$ where

$$Q_n^*(\delta) = \sum_{t=t_*}^{n} v_t \left|x_t\right|^{-\alpha} \mathbf{1}\left(\left|t^{-1/2} x_t\right| > \delta/(nt)^{1/2}\right)$$

and

$$R_n^*(\delta) = \sum_{t=t_*}^{n} v_t \left|x_t\right|^{-\alpha} \mathbf{1}\left(\left|t^{-1/2} x_t\right| \leq \delta/(nt)^{1/2}\right).$$

Observe that $\{|R_n^*(\delta)| > 0\} \subseteq S_n(\delta)$ up to null-sets, and hence $\Pr\left(|R_n^*(\delta)| > 0\right) \leq 4\kappa\delta$ as shown in the proof of Theorem 1. Observe that the terms making up $Q_n^*(\delta)$ have a finite second moment since the factor multiplying v_t is bounded in view of $\alpha \geq 0$. By the martingale difference property of v_t, by the assumptions on its conditional variance, and since x_t is $\mathcal{F}_{t-1}$-measurable we obtain arguing similarly as in the proof of Theorem 1 and setting $c = \sup_{t\geq 1} E\left(v_t^2\right)$

$$\begin{aligned}
EQ_n^*(\delta)^2 &= \sum_{t=t_*}^{n} Ev_t^2 E\left(\left|x_t\right|^{-2\alpha} \mathbf{1}\left(\left|t^{-1/2} x_t\right| > \delta/(nt)^{1/2}\right)\right) \\
&\leq c\sum_{t=t_*}^{n} t^{-\alpha} E\left(\left|t^{-1/2} x_t\right|^{-2\alpha} \mathbf{1}\left(1 > \left|t^{-1/2} x_t\right| > \delta/(nt)^{1/2}\right)\right) \\
&\quad + c\sum_{t=t_*}^{n} t^{-\alpha} E\left(\left|t^{-1/2} x_t\right|^{-2\alpha} \mathbf{1}\left(\left|t^{-1/2} x_t\right| \geq 1\right)\right) \\
&\leq c\sum_{t=t_*}^{n} t^{-\alpha}\left(2\kappa\int_{\delta/(nt)^{1/2}}^{1} z^{-2\alpha} dz \ + \ 1\right).
\end{aligned}$$

This gives the bound

$$EQ_n^*(\delta)^2 = \begin{cases} O\left(n^{\alpha}\right) & \text{if } \alpha > 1/2 \\ O\left(n^{1/2}\log n\right) & \text{if } \ \alpha = 1/2 \\ O\left(n^{1-\alpha}\right) & \text{if } 0 \leq \alpha < 1/2. \end{cases}$$

An argument similar to the one in the proof of Theorem 1 then completes the proof in the case $\alpha \geq 0$. Next consider the case $-1 \leq \alpha < 0$. Since $Ex_0^2 < \infty$ is assumed, we have that $\left|x_t\right|^{-\alpha}$ is square-integrable for $-1 \leq \alpha < 0$. Since v_t is square-integrable by assumption, it follows that $v_t \left|x_t\right|^{-\alpha}$ is integrable and hence is a martingale difference sequence w.r.t. $(\mathcal{F}_t)_{t\geq 0}$. In fact, $v_t \left|x_t\right|^{-\alpha}$ is even square-integrable for $-1 \leq \alpha < 0$: since v_t^2 and $\left|x_t\right|^{-2\alpha}$ are nonnegative and integrable, the law of iterated expectations and the assumptions imply

$$E\left(v_t^2 \left|x_t\right|^{-2\alpha}\right) = E\left(\left|x_t\right|^{-2\alpha} E\left(v_t^2 \mid \mathcal{F}_{t-1}\right)\right) = E\left(\left|x_t\right|^{-2\alpha}\right) E\left(v_t^2\right) < \infty.$$

Now, $v_t \left|x_t\right|^{-\alpha}$ being a square-integrable martingale difference sequence implies that

$$E\left(\sum_{t=1}^{n} v_t \left|x_t\right|^{-\alpha}\right)^2 = \sum_{t=1}^{n} Ev_t^2 E\left(\left|x_t\right|^{-2\alpha}\right) \leq \sup_{t\geq 1} E\left(v_t^2\right) c_1 \sum_{t=1}^{n} t^{-\alpha} = O\left(n^{1-\alpha}\right)$$

where we use the fact that $E\left|x_t\right|^{-2\alpha} \leq c_1 t^{-\alpha}$ for a finite constant c_1 as shown in Remark 4(ii). An application of Markov's inequality then proves the result. ■

Remark 14 We note that the bounds in Propositions 9 and 13 are given only for $\alpha \geq -2$ or $\alpha \geq -1$, respectively. We have not invested effort into extending the validity of these bounds beyond this range. In the special case $v_t = w_{t+1}$ the bound for $\sum_{t=1}^{n} v_t |x_t|^{-\alpha}$ is again $O_{\Pr}(n^{(1-\alpha)/2})$ for $\alpha \leq -2$; this follows from Theorem 3.1 in Ibragimov and Phillips (2008) which establishes distributional convergence of $n^{(\alpha-1)/2} \sum_{t=1}^{n} w_{t+1} |x_t|^{-\alpha}$. This theorem makes assumptions on the process x_t that are stronger in some dimensions (e.g., higher moment assumptions) but are weaker in other respects (e.g., no assumption about existence of a density). However, for $\alpha > -2$ (which includes the case of negative powers of interest here) the results in Ibragimov and Phillips (2008) do not apply.

References

Berkes, I. & L. Horvath (2006): "Convergence of Integral Functionals of Stochastic Processes", Econometric Theory 22, 304–322.

Borodin, A.N. & I.A. Ibragimov (1995): "Limit Theorems for Functionals of Random Walks", Proceedings of the Steklov Institute of Mathematics 195(2).

Christopeit, N. (2009): "Weak Convergence of Nonlinear Transformations of Integrated Processes: The Multivariate Case", Econometric Theory 25, 1180–1207.

Chung, K.L. (2001): A Course in Probability Theory. 3rd ed. Academic Press.

Chung, K.L. & R.J. Williams (1990): Introduction to Stochastic Integration. 2nd ed. Birkhäuser.

de Jong, R.M. (2004): "Addendum to 'Asymptotics for Nonlinear Transformations of Integrated Time Series'", Econometric Theory 20, 627-635.

de Jong, R.M. & C. Wang (2005): "Further Results on the Asymptotics for Nonlinear Transformations of Integrated Time Series", Econometric Theory 21, 413-430.

Jeganathan, P. (2004): "Convergence of Functionals of Sums of R.V.s to Local Times of Fractional Stable Motions", Annals of Probability 32, 1771–1795.

Lai, T.L. & C.Z. Wei (1982): "Least Squares Estimates in Stochastic Regression Models With Applications to Identification and Control of Dynamic Systems", Annals of Statistics 10, 154-166.

Ibragimov, R. & P.C.B. Phillips (2008): "Regression Asymptotics Using Martingale Convergence Methods", Econometric Theory 24, 888-947.

Park, J.Y. & P.C.B. Phillips (1999): "Asymptotics for Nonlinear Transformations of Integrated Time Series", Econometric Theory 15, 269-298.

Pötscher, B.M. (2004): "Nonlinear Functionals and Convergence to Brownian Motion: Beyond the Continuous Mapping Theorem", Econometric Theory 20, 1-22.

A Appendix

We first present a variant of Theorem 3 in Jeganathan (2004). If $x_0 = 0$, the subsequent proposition follows immediately from Theorem 3 in Jeganathan (2004). As we show in the proof below, for general x_0 the proposition follows from that theorem combined with Remark 4 in Jeganathan (2004) plus a conditioning argument. We also note that the assumptions on x_t that we maintain here are stronger than necessary and the proposition could also be established under weaker conditions similar to the ones used in Jeganathan (2004). We do not discuss such a more general result here.

Proposition 15 *Suppose f is a Lebesgue-integrable real-valued function on $\mathbb{R}$ that is bounded. Then, under the maintained assumptions on x_t, it holds that*

$$n^{-1}\sum_{t=1}^{n}\beta_n f(n^{-1/2}\beta_n x_t) \xrightarrow{d} |\sigma|^{-1}\left(\int_{-\infty}^{\infty} f(y)dy\right) L(1,0) \tag{8}$$

for any sequence β_n satisfying $\beta_n \to \infty$ and $n^{-1}\beta_n \to 0$. (Recall $\sigma = \sum_{j=0}^{\infty}\phi_j \neq 0$.)

Proof. Without loss of generality we may assume that $\phi_0 \neq 0$ (otherwise shift the sequences ϕ_j and ε_i appropriately). Set $\gamma_n = n^{1/2}h(n)\phi_0^{-1}\sigma$ as in Jeganathan (2004) with positive $h(n)$, and note that $\gamma_n \neq 0$. From Proposition 1 in Jeganathan (2004) we obtain that $\gamma_n^{-1}S_n = \gamma_n^{-1}\sum_{t=1}^{n} w_t$ converges in distribution to $N(0,2)$. In view of the central limit theorem for linear processes and the fact that $h(n)$ is positive, we conclude that $h(n)$ converges to $2^{-1/2}|\phi_0|$. We also note that $\operatorname{sign}(\gamma_n) = \operatorname{sign}(\phi_0^{-1}\sigma)$ is independent of n. Observe that

$$n^{-1}\sum_{t=1}^{n}\beta_n f(n^{-1/2}\beta_n x_t) = \left(n^{1/2}/|\gamma_n|\right) n^{-1}\sum_{t=1}^{n}\bar{\beta}_n f^{*}(\gamma_n^{-1}\bar{\beta}_n x_t) \tag{9}$$

where $\bar{\beta}_n = n^{-1/2}|\gamma_n|\beta_n$ satisfies $\bar{\beta}_n \to \infty$ and $n^{-1}\bar{\beta}_n \to 0$ and where $f^{*}(y) = f(\operatorname{sign}(\phi_0^{-1}\sigma)y)$.

Assume first that $x_0 \equiv 0$. Then $x_t = S_t$ and since all assumptions in Theorem 3(i) (or (ii)) in Jeganathan (2004) are satisfied, we conclude from that theorem that the above expression converges weakly to $2^{1/2}|\sigma|^{-1}\left(\int_{-\infty}^{\infty} f^{*}(y)dy\right)\bar{L}(1,0)$ where $\bar{L}(1,0)$ is the local time as defined in Jeganathan (2004). Since $\int_{-\infty}^{\infty} f^{*}(y)dy = \int_{-\infty}^{\infty} f(y)dy$ and since $2^{1/2}\bar{L}(1,0)$ has the same distribution as $L(1,0)$ the result follows in case $x_0 \equiv 0$.

Next assume that $x_0 \equiv c$, a constant not necessarily equal to zero. By (9) it again suffices to show that $n^{-1}\sum_{t=1}^{n}\bar{\beta}_n f^{*}(\gamma_n^{-1}\bar{\beta}_n x_t) = n^{-1}\sum_{t=1}^{n} f_n^{*}(\gamma_n^{-1}S_t)$ converges to $\left(\int_{-\infty}^{\infty} f^{*}(y)dy\right)\bar{L}(1,0)$ weakly, where $f_n^{*}(y) = \bar{\beta}_n f^{*}(\bar{\beta}_n(y + c\gamma_n^{-1}))$. But, under the maintained assumptions on x_t, this follows from the extension of Theorem 3 discussed in Remark 4 in Jeganathan (2004) if we can verify the subsequent conditions for f_n^{*} (we may assume without loss of generality that $\bar{\beta}_n > 0$ for all n):

(i) By change of variables and the integrability assumption on f we have

$$\sup_n \int_{-\infty}^{\infty} |f_n^*(y)|\, dy = \sup_n \int_{-\infty}^{\infty} \left|\bar{\beta}_n f^*(\bar{\beta}_n(y + c\gamma_n^{-1}))\right| dy = \int_{-\infty}^{\infty} |f(y)|\, dy < \infty.$$

(ii) Correcting a typo in Jeganathan (2004), we have to show that

$$\limsup_n n^{-1} \int_{-\infty}^{\infty} |f_n^*(y)|^2\, dy = 0.$$

Note that the left-hand side can be written as

$$\limsup_n n^{-1} \int_{-\infty}^{\infty} \left|\bar{\beta}_n f^*(\bar{\beta}_n(y + c\gamma_n^{-1}))\right|^2 dy = \limsup_n n^{-1}\bar{\beta}_n \int_{-\infty}^{\infty} |f(y)|^2\, dy$$

by a change of variables and the definition of f^*. But this is zero since $n^{-1}\beta_n \to 0$ by assumption and since the integral is finite (f is quadratically integrable since it is integrable and bounded).

(iii) Again by a change of variables

$$\lim_{d\to\infty} \sup_n \int_{|y|\geq d} |f_n^*(y)|\, dy = \lim_{d\to\infty} \sup_n \int_{\left|\bar{\beta}_n^{-1} z - c\gamma_n^{-1}\right|\geq d} |f^*(z)|\, dz.$$

Since f is integrable, the limit for $d \to \infty$ is zero for each integral individually. Hence, it suffices to show that

$$\lim_{d\to\infty} \sup_{n\geq N} \int_{|y|\geq d} |f_n^*(y)|\, dy = \lim_{d\to\infty} \sup_{n\geq N} \int_{\left|\bar{\beta}_n^{-1} z - c\gamma_n^{-1}\right|\geq d} |f^*(z)|\, dz = 0$$

for a suitable N. Choose N such that $\bar{\beta}_n > 1$ and $\left|c\gamma_n^{-1}\right| \leq 1$ holds for $n \geq N$. Then we have for $d > 2$

$$\sup_{n\geq N} \int_{\left|\bar{\beta}_n^{-1} z - c\gamma_n^{-1}\right|\geq d} |f^*(z)|\, dz \leq \int_{|z|\geq d/2} |f^*(z)|\, dz = \int_{|z|\geq d/2} |f(z)|\, dz \qquad (10)$$

since

$$\left\{z : \left|\bar{\beta}_n^{-1} z - c\gamma_n^{-1}\right| \geq d\right\} \subseteq \{z : |z| \geq d/2\}$$

for $n \geq N$ and $d > 2$. The upper bound in (10) now converges to zero for $d \to \infty$ by integrability of f.

(iv) Define $F_n(y)$ as in Remark 4 in Jeganathan (2004). Then for $y \geq 0$ we obtain

$$F_n(y) = \int_0^y \bar{\beta}_n f^*(\bar{\beta}_n(u + c\gamma_n^{-1}))du = \int_{\bar{\beta}_n c\gamma_n^{-1}}^{\bar{\beta}_n(y+c\gamma_n^{-1})} f^*(z)dz,$$

whereas for $y < 0$ we obtain

$$F_n(y) = -\int_y^0 \bar{\beta}_n f^*(\bar{\beta}_n(u + c\gamma_n^{-1}))du = -\int_{\bar{\beta}_n(y+c\gamma_n^{-1})}^{\bar{\beta}_n c\gamma_n^{-1}} f^*(z)dz.$$

It follows that

$$F_n(y) \to F(y) = \begin{cases} \int_0^\infty f^*(z)dz & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -\int_{-\infty}^0 f^*(z)dz & \text{if } y < 0 \end{cases}$$

for every y. Observe that consequently $\int_{-\infty}^{\infty} \bar{L}(1,y)dF(y) = \left(\int_{-\infty}^{\infty} f^*(y)dy\right) \bar{L}(1,0)$.

(v) $\sup_{n,y} \bar{\beta}_n^{-1} |f_n^*(y)| = \sup_{n,y} \left|f^*(\bar{\beta}_n(y + c\gamma_n^{-1}))\right| = \sup_y |f^*(y)| = \sup_y |f(y)| < \infty$ since f is a bounded function.

This proves that (8) holds for arbitrary *nonrandom* starting values. If the starting value x_0 is random, we proceed as follows:

$$\begin{aligned} & \Pr\left(n^{-1}\sum_{t=1}^{n} \beta_n f(n^{-1/2}\beta_n x_t) \le u\right) \\ = & \int \Pr\left(n^{-1}\sum_{t=1}^{n} \beta_n f(n^{-1/2}\beta_n x_t) \le u \mid x_0 = c\right) dG(c) \\ = & \int \Pr\left(n^{-1}\sum_{t=1}^{n} \beta_n f(n^{-1/2}\beta_n (S_t + c) \le u\right) dG(c) \end{aligned}$$

where we have made use of independence of x_0 and $(S_1, \ldots, S_n)$ and where G denotes the distribution function of x_0. By what was shown above, we have that $\Pr\left(n^{-1}\sum_{t=1}^{n} \beta_n f(n^{-1/2}\beta_n (S_t + c) \le u\right)$ converges to the distribution function $\Pr\left(\left(\int_{-\infty}^{\infty} f(y)dy\right) L(1,0) \le u\right)$ for all continuity points of this distribution function. Since this distribution function does not depend on c, we can conclude from dominated convergence that

$$\Pr\left(n^{-1}\sum_{t=1}^{n} \beta_n f(n^{-1/2}\beta_n x_t) \le u\right) \to \Pr\left(\left(\int_{-\infty}^{\infty} f(y)dy\right) L(1,0) \le u\right)$$

for all continuity points. This completes the proof. ■

Lemma 16 *Suppose Y_n is a sequence of (real-valued or extended real-valued) nonnegative random variables. Then the following are equivalent:*

(i) $\eta_n Y_n \to \infty$ in probability as $n \to \infty$ for every *sequence η_n of real numbers satisfying $\eta_n \to \infty$.*

(ii) $\lim_{\varepsilon \to 0, \varepsilon > 0} \liminf_{n\to\infty} \Pr(Y_n > \varepsilon) = 1$.

(iii) $\liminf_{n\to\infty} \Pr(Y_n > \varepsilon_n) = 1$ *for* every *sequence of real numbers $\varepsilon_n > 0$ satisfying $\varepsilon_n \to 0$.*

Proof. We first show that (i) implies (iii): For given $\varepsilon_n > 0$ satisfying $\varepsilon_n \to 0$ define $\eta_n = \varepsilon_n^{-1}$. Clearly then $\eta_n \to \infty$ holds. From (i) we then have that $\Pr(\eta_n Y_n > 1) \to 1$ as $n \to \infty$. But this immediately translates into (iii).

Next we show that (iii) implies (i): Let $\eta_n \to \infty$ be a given sequence and let $0 < M < \infty$ be arbitrary. Define $\varepsilon_n = M/\eta_n$ which is well-defined and positive for sufficiently large n and satisfies $\varepsilon_n \to 0$. But then

$$1 = \liminf_{n\to\infty} \Pr(Y_n > \varepsilon_n) = \liminf_{n\to\infty} \Pr(\eta_n Y_n > M)$$

holds as a consequence of (iii). Since M was arbitrary, (i) follows.

That (ii) implies (iii) is obvious since for every $\varepsilon > 0$ we have $\Pr(Y_n > \varepsilon_n) \geq \Pr(Y_n > \varepsilon)$ for large n since $\varepsilon_n \to 0$.

We finally show that (iii) implies (ii): Suppose (ii) does not hold. Then

$$\lim_{\varepsilon\to 0, \varepsilon>0} \liminf_{n\to\infty} \Pr(Y_n > \varepsilon) < 1$$

must hold, noting that the outer limit exists due to monotonicity with respect to ε. In particular,

$$\lim_{k\to\infty} \liminf_{n\to\infty} \Pr(Y_n > 1/k) < 1$$

must hold. Hence we can find a strictly increasing sequence n_k of integers diverging to infinity and a constant $c < 1$ such that

$$\Pr(Y_{n_k} > 1/k) < c < 1$$

holds for every $k \geq k_0$ for some sufficiently large k_0. For $n \geq n_{k_0}$ define $\varepsilon_n = 1/k$ if $n_k \leq n < n_{k+1}$, and set $\varepsilon_n = 1$ for $n < n_{k_0}$. Then $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ for $n \to \infty$ holds. But

$$\liminf_{n\to\infty} \Pr(Y_n > \varepsilon_n) \leq \liminf_{k\to\infty} \Pr(Y_{n_k} > \varepsilon_{n_k}) \leq \liminf_{k\to\infty} \Pr(Y_{n_k} > 1/k) \leq c < 1,$$

showing that (iii) does not hold. ■