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Abstract

A general equilibrium model has been constructed in a stochastic endogenous growth economy driven by an Itô-Lévy diffusion process. The minimum time to "economic maturity" for an underdeveloped economy has been computed both in the preference manifold of *the modified Ramsey fashion* and in that of *the modified Radner fashion* with its support, i.e., fiscal policies and savings strategy, endogenously determined. Furthermore, the effects of different information structures to the endogenous time have been thoroughly investigated, and local sensitivity analyses of optimal consumption per capita with respect to the initial level of capital stock per capita have been smoothly incorporated into the current macroeconomic model.

Keywords: Stochastic endogenous growth; Minimum time to "economic maturity"; Optimal taxation policies; Endogenous savings rate; Preference manifold; Information structure; Local sensitivity analyses; Optimal stopping time; Lévy diffusion.

JEL classification: C61; C63; D82; D91; E62; H21; O11.

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1. INTRODUCTION

For any underdeveloped economy, like China, both the government and the people are motivated to choose appropriate fiscal policies and optimal investment strategies, respectively, to make the economy reach its maturity level[2] as quickly as possible. The state of "economic maturity" can be, in the category of macroeconomics, translated into the well-known von Neumann equilibrium (see, Neumann, 1945-1946; Kemeny et al, 1956; Howe, 1960; Yano, 1998), "turnpike"[3] (e.g., Hicks, 1961; Radner, 1961; Morishima, 1961; McKenzie, 1963a, 1963b; Atsumi, 1965; Cass, 1966; and Gale, 1967), the Golden Age or modified Golden Age (e.g., Champernowne, 1962; Pearce, 1962; Phelps, 1961, 1962, 1965; Samuelson, 1965). And in turn, provided the existence of the von Neumann path or the "turnpike" of the economy, the problem facing us, including the government and the representative agent, is to choose appropriate fiscal policies and savings strategy, respectively, to effectively support the convergence of the economical system, thereby implying the economy will spend almost all time staying at least in the neighborhood of the von Neumann equilibrium or the "turnpike" (see, Cass, 1966; Yano, 1984b; McKenzie (1998) and references therein), which indeed represents the maximal and sustainable terminal path level (e.g., Kurz, 1965; McKenzie, 1976) of the corresponding economy in the present model.

And the current paper is devoted to confirm the existence of the unique von Neumann path or the well-known "turnpike" of an aggregate endogenous growth economy equipped with AK production technology (e.g., Barro, 1990; Rebelo, 1991; Turnovsky, 2000; Aghion, 2004), in the background of a general equilibrium framework. Nonetheless, the major goal of this paper is to *explicitly compute* the minimum time needed to reach the "economic maturity" for an underdeveloped economy and in an uncertainty environment. Moreover, it's easy to notice that our

[2] Undoubtedly, it should reflect not only high speed of economic growth but also high quality of economic development. More about this topic of growth and development, one can refer to Solow (2003).
[3] Related preferences see, Inada, 1964; Morishima, 1965; Nikaidô, 1964; Tsukui, 1966, 1967; McKenzie, 1982; Winter, 1967; Coles, 1985; Yano, 1984a, 1985; Bewley, 1982; Gantz, 1980; Drandakis, 1966; Araujo and Scheinkman, 1977; and Joshi, 1997; Dai, 2012.

paper is a natural extension of the seminal and interesting paper of Kurz (1965)[4], where optimal paths of capital accumulation under the minimum time objective are thoroughly investigated. It is, nevertheless, worth emphasizing that our results are based upon the general equilibrium framework and the minimum time is endogenously determined provided the welfare of the representative agent is maximized[5].

The advantage of the method used here is that the endogenous time[6] or the minimum time to "economic maturity" can be *explicitly computed*[7] in some conditions, e.g., when the preference or the criterion of *the modified Radner fashion* (1961) is employed. Noting that the minimum time is endogenously determined, even applying economic intuitions, by the optimal savings strategy of the representative agent and the optimal taxation policies of the government, which are thoroughly explored under different information distributions or information structures, thereby implying that the endogenous time can be completely characterized and comparatively studied in different information structures, which obviously throws new insights into our understanding of the minimum time needed to reach "economic maturity" for an underdeveloped economy.

The current paper proceeds as follows. Section 2 introduces the general model and the basic idea behind the macroeconomic model. Section 3 computes the endogenous time in preference manifold one. Section 4 computes the endogenous time in preference manifold two. Section 5 analyzes the effects of different information structures to the corresponding endogenous time. Section 6 gives the local sensitivity analyses of the optimal consumption strategy, which supports the existence of the endogenous time in section 3, with respect to the initial level of capital stock per capita. There is a brief concluding section. All proofs, unless otherwise noted in the text, appear in the Appendix.

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4 It is regarded as a continuation of Srinivasan's work (1962) in a certain sense.
5 In other words, pursue of speed of economic growth is based upon the quality of economic development.
6 In the current paper, we will take no difference between "the endogenous time" and "the minimum time to 'economic maturity'".
7 That is, a simple formula is supplied for the first time. And also, it is easy to see that the maximal terminal path level of capital stock per capita is utility-optimal and simultaneously determined with the endogenous time in the present model.

2. THE GENERAL MODEL

2.1. Two Types of Preference Manifolds

In order to determine the minimum time needed to reach the so-called von Neumann path or "economic maturity" for an underdeveloped economy, the following two kinds of criterions are naturally and indeed comparatively investigated.

The first one has been widely employed to prove the well-known turnpike theorems, and noting that it is pioneered by Radner (1961), we call it *the Radner fashion*. However, it is worth noting that the discount factor is naturally incorporated into the criterion while it is excluded in the seminal paper of Radner, that is, we employ *the modified Radner fashion* in the current paper. Formally, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, similar to Dai (2012), the corresponding problem can be written as,

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho\tau} u\left(c(\tau)\right) \mathbf{1}_{\{\tau<\infty\}}\right],$$

where $\mathcal{T}$ denotes admissible stopping times, $0<\rho<1$ denotes subjective discount factor, c denotes consumption per capita, $u:\mathbb{R}_+ \to \mathbb{R}$ is a strictly concave instantaneous utility function, and $\mathbf{1}_{\{\tau<\infty\}}$ represents the indicator function of set $\{\omega \in \Omega; \tau(\omega)<\infty\}$.

The second one has been widely used in studying aggregate economic growth and optimal fiscal policies. And the idea is certainly due to Ramsey (1928), who studied endogenous saving with this kind of criterion. As a consequence, we call it *the Ramsey fashion*. As usual, and to meet the regular requirements, only finite time horizon, endogenously determined, and discounted sum are discussed in the current paper, i.e., only *the modified Ramsey fashion* is considered. Formally, based on the same stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$, the corresponding problem is expressed as,

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\int_0^{\tau} e^{-\rho t} u\left(c(t)\right) dt + e^{-\rho\tau} u\left(y(\tau)\right) \mathbf{1}_{\{\tau<\infty\}}\right],$$

where y denotes national income per capita and other notations are the same as in *the modified Radner fashion*. And it is worth emphasizing that *the modified Ramsey fashion* internally requires perfect foresight of the representative agent.

REMARK. It is easy to see from our specification that there is a natural one to one correspondence between the optimal stopping time and the minimum time needed to "economic maturity" for any underdeveloped economy. Accordingly, this equivalence reflects the fact that the above two kinds of preference manifolds, i.e., *the modified Radner fashion* and *the modified Ramsey fashion*, imply different standards characterizing the corresponding state of "economic maturity". Notice that *the modified Radner fashion* reflects some psychological effects that would be called as "the peak preference" [8] or its natural correspondence "the Ratchet effect" in traditional consumption theory. Consequently, we may claim that *the modified Radner fashion* is much stronger than the *the modified Ramsey fashion* in certain sense. In other words, *the modified Radner fashion* requires much higher level of standard about "economic maturity". Therefore, the "turnpike" of *the modified Radner fashion* should be located above that of *the modified Ramsey fashion* for any given economy.

2.2. Computation Algorithm of the Endogenous Time

As usual, the environment consists of the firm, the representative agent and the government. And the firm is, without loss of any generality, assumed to be competitive. There are alternative goals for the government, that is, government is either motivated to choose taxation policies so as to maximize the welfare of the representative agent or directly to minimize the time to "economic maturity". For the representative agent, she will first determines the minimum time to "economic maturity" given the taxation policies of the government, then to choose optimal savings strategy based upon the objective of discounted sum of future instantaneous utility in finite time horizon provided the taxation policies of the government. That is

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[8] That is, the representative agent pursues the highest level of utility or welfare of any single period. And it is just the highest level of the welfare that represents the corresponding state of "economic maturity" in the current model.

to say, the order of action is like this: the government moves first to choose optimal taxation policies, then the representative agent determines the minimum time to "economic maturity" or the time horizon based upon the optimal taxation policies, and finally, the representative agent chooses optimal savings strategy conditional on the optimal taxation policies and the endogenous time horizon representing the process leading to "economic maturity".

Therefore, based on the backward induction rationality principle in computing sub-game perfect Nash equilibrium in dynamic game theories, we introduce the following computation algorithm of the current model,

STEP 1: *The representative agent chooses optimal savings strategy given the taxation policies and the finite time horizon of the program.*

STEP 2: *Based on the results of Step* 1*, the representative agent will determine the minimum time to reach "economic maturity" with the criterions introduced in section* 2.1.

STEP 3a: *If the goal of the government is to choose taxation policies so as to maximize the welfare of the representative agent, thus based upon the results of Step* 1 *and Step* 2*, the optimal tax rates are derived.*

STEP 3b: *If the goal of the government is to choose taxation policies in order to directly minimize the time to "economic maturity" derived in Step* 2*, then the corresponding optimal taxation policies are endogenously determined and hence the endogenous time is completely characterized with these optimal tax rates.*

STEP 4: *The step is necessary only when Step* 3a *is chosen. Substituting the optimal tax rates into the endogenous time derived in Step* 2*, and so the minimum time to "economic maturity" is finally and completely determined.*

3. PREFERENCE MANIFOLD ONE

3.1. Firm

In the current paper, we introduce the following Cobb-Douglas type production function[9],

$$Y(t) = G_p(t)^{\alpha} K(t)^{1-\alpha}, \quad 0 < \alpha < 1 \tag{1}$$

where K denotes the capital stock and G_p represents the flow of services from government spending[10] on the economy's infrastructure. Particularly, suppose that these services are not subject to congestion so that G_p is a pure public good. Further to put $G_p = g_p Y$ [11], that is government will claim a fraction, g_p, of aggregate output Y, for expenditure on infrastructure. And, in particular, to make things easier and without loss of any generality, g_p will be assumed to be exogenously given [12] with $0 < g_p < 1$ throughout the paper, then the production function in (1) can be rewritten as,

$$Y(t) = g_p^{\alpha/1-\alpha} K(t), \text{ or } \quad y(t) = g_p^{\alpha/1-\alpha} k(t) \tag{1'}$$

which reveals that the Cobb-Douglas type function given in (1) rather exhibits AK production technology, which indeed enssures ongoing economic growth. Therefore, equilibrium wage rate is equal to zero and equilibrium return to capital reads as follows,

$$r_k = g_p^{\alpha/1-\alpha}, \tag{2}$$

where the depreciation rate is assumed to be zero for the sake of simplicity.

3.2. Representative Agent

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9 For simplicity's sake, endogenous labor supply has been excluded in the present paper. However, it is easy to show that endogenous labor supply can be naturally incorporated into the current model, thereby inducing a much more complicated model.

10 Gong and Zou (2002) set up a theoretical model linking the growth rate of the economy to the growth rate and volatility of different government expenditures. On a theoretical basis, they found that volatility in government spending can be positively or negatively associated with economic growth depending on the intertemporal elasticity in consumption. And it follows from our specification of the government spending that the volatility is endogenously determined by the unbalanced macro-economy as a whole. That is to say, the volatility of government spending is not exogenously given but internally and closely linked to the whole economic body. And we argue from the specification that government in reality is indeed deeply involved with the whole economy and therefore it itself will unavoidably be affected by the macroeconomic activities. One may certainly exogenously add volatility to the government spending, which however will be strongly disagreed by the theory of real business cycle (see, Kydland and Prescott, 1982; Long and Plosser, 1983).

11 This specification follows from Turnovsky (2000).

12 This in some extent follows from Kydland and Prescott (1977)'s analyses that policymakers should follow rules rather than have discretion.

It is assumed that the economy consists of $L(t)$ identical individuals at time t, each of whom possesses perfect foresight. Suppose that $\{B(t)\}_{0\le t\le T}$ is a standard Brownian motion defined on the following filtered probability space $(\Omega^{(B)}, \mathcal{F}^{(B)}, \{\mathcal{F}_t^{(B)}\}_{0\le t\le T}, \mathbb{P}^{(B)})$ with $\{\mathcal{F}_t^{(B)}\}_{0\le t\le T}$ the $\mathbb{P}^{(B)}$ – augmented filtration generated by $\{B(t)\}_{0\le t\le T}$ with $\mathcal{F}^{(B)} = \mathcal{F}_T^{(B)}$. Furthermore, we assume that a Poisson random measure $N(dt, dz)$ associated with a Lévy process is defined on the stochastic basis $(\Omega^{(\widetilde{N})}, \mathcal{F}^{(\widetilde{N})}, \{\mathcal{F}_t^{(\widetilde{N})}\}_{0\le t\le T}, \mathbb{P}^{(\widetilde{N})})$. And we denote by $\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ the compensated Poisson random measure associated with a Lévy process $\eta(t) \triangleq \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds, dz)$ with jump measure $N(dt, dz)$ and Lévy measure $\nu(O) = \mathbb{E}[N([0,1], O)]$ for $O \in \mathfrak{B}(\mathbb{R}_0)$, i.e., O is a Borel set with its closure $\overline{O} \subset \mathbb{R}_0$, where $\mathbb{R}_0 \triangleq \mathbb{R} - \{0\}$. In what follows, our reference stochastic basis will be $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ with $\Omega = \Omega^{(B)} \times \Omega^{(\widetilde{N})}$, $\mathcal{F} = \mathcal{F}^{(B)} \otimes \mathcal{F}^{(\widetilde{N})}$, $\mathcal{F}_t = \mathcal{F}_t^{(B)} \otimes \mathcal{F}_t^{(\widetilde{N})}$ and $\mathbb{P} = \mathbb{P}^{(B)} \otimes \mathbb{P}^{(\widetilde{N})}$, and also the underlying probability measure space is assumed to satisfy the so-called "usual conditions"[13]. Based on the above constructions and assumptions, we now define[14],

$$dL(t) = L(t^-)\left[ndt + \sigma dB(t) + \int_{\mathbb{R}_0} \gamma z \overline{N}(dt, dz)\right], \tag{3}$$

where n denotes the natural growth rate of population, $\sigma \in \mathbb{R}_0$ is an exogenously given constant, $\gamma z > -1$ a.s. $-\nu$, $B(0) = 0$ a.s. $-\mathbb{P}$ and,

$$\overline{N}(dt, dz) \triangleq \begin{cases} N(dt, dz) - \nu(dz)dt \triangleq \widetilde{N}(dt, dz), & |z| < Z \\ N(dt, dz), & |z| \ge Z \end{cases} \tag{4}$$

for some $Z \in [0, \infty]$. As usual, we define the following law of motion of capital

[13] That is, the probability space is complete and the filtration satisfies right continuity.

[14] It is in line with Merton (1975) that the uncertainty comes from the growth of population. Itô-Lévy process has been widely applied in finance, e.g., Yan et al (2000). And here we apply Lévy diffusion to macroeconomics, which would be regarded as reasonable via noting the properties of both Lévy diffusions and macroeconomic phenomenon.

accumulation,

$$\dot{K}(t) = \mu[(1-\tau_k)r_k K(t) - (1+\tau_c)C(t)]$$

$$= \mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k) - (1+\tau_c)(1-g_p-r_s)]K(t), \tag{5}$$

where $\mu \in \mathbb{R}_0$ is some exogenously given parameter, r_k denotes the equilibrium return to capital given in (2), τ_k denotes tax rate on capital income, τ_c represents consumption tax rate, C denotes aggregate consumption level and r_s denotes the savings rate. Hence, combining (3) with (5) and by applying Itô formula for Itô-Lévy process, we get,

$$dk(t) = \left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k) - (1+\tau_c)(1-g_p-r_s)] - n + \sigma^2\right\} k(t^-)dt$$

$$+\int_{|z|<Z} \tfrac{(\gamma z)^2}{1+\gamma z}\nu(dz)k(t^-)dt - \sigma k(t^-)dB(t) - k(t^-)\int_{\mathbb{R}_0} \tfrac{\gamma z}{1+\gamma z}\overline{N}(dt,dz), \tag{6}$$

Without loss of any generality, we put $Z = \infty$, then by (4), (6) becomes,

$$dk(t) = \left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k) - (1+\tau_c)(1-g_p-r_s)] - n + \sigma^2 + b\right\} k(t^-)dt$$

$$-\sigma k(t^-)dB(t) - k(t^-)\int_{\mathbb{R}_0} \tfrac{\gamma z}{1+\gamma z}\widetilde{N}(dt,dz), \tag{7}$$

where,

$$b \triangleq \int_{\mathbb{R}} \tfrac{(\gamma z)^2}{1+\gamma z}\nu(dz), \tag{8}$$

Suppose that the representative agent performs log preferences and the intertemporal objective function is specifically given as,

$$U = \mathbb{E}\left[\int_0^{\hat{\tau}} e^{-\rho(s+t)} \ln c(t)dt + U^{\hat{\tau}}\right]$$

$$= \mathbb{E}\left[\int_0^{\hat{\tau}} e^{-\rho(s+t)} \ln\left((1-g_p-r_s)g_p{}^{\alpha/1-\alpha}k(t)\right)dt + U^{\hat{\tau}}\right]. \tag{9}$$

where $\mathbb{E}$ denotes expectation operator with respect to probability measure $\mathbb{P}$, ρ is the subjective discount factor, $\forall 0 \le s < \hat{\tau}$ and $\hat{\tau}$ is an $\mathcal{F}_t$ − optimal stopping time, which with the term $U^{\hat{\tau}}$ are simultaneously determined by the following optimal stopping problem of *the modified Radner fashion*,

$$\hat{g}(\tau, k(\tau)) \triangleq \sup_{\tau\in\mathcal{T}} \mathbb{E}^{(s,k)}\left[e^{-\rho(s+\tau)} \ln\left(g_p{}^{\alpha/1-\alpha}k(\tau)\right)\mathbf{1}_{\{\tau<\infty\}}\right]$$

$$= \sup_{\tau \in \mathcal{T}} \mathbb{E}^{(s,k)} \left[e^{-\rho(s+\tau)} \ln(y(\tau)) \mathbf{1}_{\{\tau<\infty\}} \right]$$

$$= \mathbb{E}^{(s,k)} \left[e^{-\rho(s+\hat{\tau})} \ln\left(g_p{}^{\alpha/1-\alpha} k(\hat{\tau}) \right) \mathbf{1}_{\{\hat{\tau}<\infty\}} \right]. \tag{10}$$

subject to the stochastic differential equation (SDE) in (7), $\mathbb{E}^{(s,k)}$ denotes expectation operator based on initial condition $(s,k) \triangleq (s,k(0))$, $\mathbf{1}_{\{\tau<\infty\}}$ is an indicator function of set $\{\omega \in \Omega; \tau(\omega) < \infty\}$, and $\mathcal{T} \triangleq \{\mathcal{F}-\text{stopping times}\}$.

Now it follows from Step 1 introduced in section 2.2 that we are to consider the following stochastic optimal control problem facing the representative agent,

$$\max_{0<r_s<1} \mathbb{E}\left[\int_0^{\hat{\tau}} e^{-\rho(s+t)} \ln\left((1-g_p-r_s) g_p{}^{\alpha/1-\alpha} k(t) \right) dt + U^{\hat{\tau}} \right]. \tag{9'}$$

s.t.

$$dk(t) = \left\{ \mu g_p{}^{\alpha/1-\alpha} [(1-\tau_k)-(1+\tau_c)(1-g_p-r_s)] - n + \sigma^2 + b \right\} k(t^-) dt$$

$$-\sigma k(t^-) dB(t) - k(t^-) \int_{\mathbb{R}_0} \tfrac{\gamma z}{1+\gamma z} \widetilde{N}(dt,dz), \tag{7'}$$

where $\hat{\tau}$ and $U^{\hat{\tau}}$ are taken as exogenously given up to present. We prove that there exists a continuously differential function $V(t,k(t))$, satisfying the following stochastic Bellman partial differential equation (PDE),

$$-V_t(t,k(t)) - \tfrac{1}{2}\sigma^2 k^2(t) V_{kk}(t,k(t))$$

$$-\int_{\mathbb{R}_0} \left[V\left(t, k(t) - \tfrac{\gamma z}{1+\gamma z} k(t)\right) - V(t,k(t)) + \tfrac{\gamma z}{1+\gamma z} k(t) V_k(t,k(t)) \right] \nu(dz)$$

$$= \max_{0<r_s<1} \left\{ \begin{aligned} &\exp(-\rho(s+t)) \ln[(1-g_p-r_s) g_p{}^{\alpha/1-\alpha} k(t)] + V_k(t,k(t)) k(t) \\ &\times \left\{ \mu g_p{}^{\alpha/1-\alpha} [(1-\tau_k)-(1+\tau_c)(1-g_p-r_s)] - n + \sigma^2 + b \right\} \end{aligned} \right\}. \tag{11}$$

with the boundary condition,

$$V(\hat{\tau}, k(\hat{\tau})) = U^{\hat{\tau}}, \tag{12}$$

Thus, we get,

LEMMA 1. *Conditional on the above constructions and assumptions, and up to the present step, we obtain the optimal savings rate as follows,*

$$\hat{r}_s = 1 - g_p - \frac{\rho}{\mu g_p{}^{\alpha/1-\alpha}(1+\tau_c)},$$

Moreover, the value function $V(t,k(t))$ *satisfies the following boundary condition,*

$$V(\hat{\tau},k(\hat{\tau}))=\exp(-\rho(s+\hat{\tau}))[C_1+\rho^{-1}\ln k(\hat{\tau})]=U^{\hat{\tau}}.$$

where,

$$C_1\triangleq\rho^{-1}\left\{\begin{matrix}\ln\frac{\rho}{\mu(1+\tau_c)}+\rho^{-1}[\mu g_p{}^{\alpha/1-\alpha}(1-\tau_k)-n+\sigma^2+b]\\ -1-\frac{1}{2}\sigma^2\rho^{-1}+\rho^{-1}\int_{\mathbb{R}_0}\left(\ln\frac{1}{1+\gamma z}+\frac{\gamma z}{1+\gamma z}\right)\nu(dz)\end{matrix}\right\}.$$

Proof. See Appendix A. ▌

REMARK. Lemma 1 represents a conclusion of Step 1 introduced in section 2.2. That is, provided the taxation policies of the government and the finite time horizon of the program, the optimal savings rate is derived. And the boundary condition shown in Lemma 1 will be useful in computing the exact form of the endogenous time as is shown in the sequel.

Now, by applying Step 2 of the computation algorithm in section 2.2, we are in the position to calculate the term $U^{\hat{\tau}}$ and the optimal stopping time $\hat{\tau}$, given in (9), in a stochastic diffusion process. Firstly, via applying Lemma 1, (7) can be rewritten as,

$$dk(t)=\left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]-n+\sigma^2+b\right\}k(t^-)dt$$
$$-\sigma k(t^-)dB(t)-k(t^-)\int_{\mathbb{R}_0}\tfrac{\gamma z}{1+\gamma z}\widetilde{N}(dt,dz), \qquad (7'')$$

Let $Y(t)\triangleq(s+t,k(t))^\top$, $Y(0)\triangleq(s,k)^\top$, then the generator of $Y(t)$ reads as follows,

$$\mathcal{A}\phi(s,k)=\tfrac{\partial\phi}{\partial s}+\left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]-n+\sigma^2+b\right\}k\tfrac{\partial\phi}{\partial k}$$
$$+\tfrac{1}{2}\sigma^2k^2\tfrac{\partial^2\phi}{\partial k^2}+\int_{\mathbb{R}_0}[\phi(s,k-\tfrac{\gamma z}{1+\gamma z}k)-\phi(s,k)+\tfrac{\gamma z}{1+\gamma z}k\tfrac{\partial\phi}{\partial k}]\nu(dz), \qquad (13)$$

for $\forall\phi\in C^2(\mathbb{R}^2)$. If we try a function ϕ of the form,

$\phi(s,k)=e^{-\rho s}k^\beta$, for some constant $\beta\in\mathbb{R}$

We obtain,

$$\mathcal{A}\phi(s,k)=e^{-\rho s}k^\beta\left(-\rho+\beta\left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]-n+\sigma^2+b\right\}\right)$$
$$+\tfrac{1}{2}e^{-\rho s}k^\beta\sigma^2\beta(\beta-1)+e^{-\rho s}k^\beta\int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^\beta-1+\tfrac{\gamma z\beta}{1+\gamma z}]\nu(dz)$$
$$=e^{-\rho s}k^\beta h(\beta),$$

in which,

$$h(\beta) \triangleq -\rho + \beta\left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)] - n + \sigma^2 + b\right\}$$
$$+\tfrac{1}{2}\sigma^2\beta(\beta-1) + \int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^\beta - 1 + \tfrac{\gamma z\beta}{1+\gamma z}]\nu(dz)\,. \tag{14}$$

Notice that,

$$h(0) = -\rho < 0 \quad \text{and} \quad \lim_{|\beta|\to\infty} h(\beta) = \infty\,.$$

Therefore, there exists $\beta > 0$ such that $h(\beta) = 0$ and with this value of β, we put

$$\phi(s,k) = \begin{cases} e^{-\rho s}Ck^\beta, & (s,k)\in D \\ e^{-\rho s}\ln(g_p{}^{\alpha/1-\alpha}k), & (s,k)\notin D \end{cases} \tag{15}$$

for some constant $C > 0$ and the continuation region D, to be determined. Thus, if we define

$$g(s,k) \triangleq e^{-\rho s}\ln(g_p{}^{\alpha/1-\alpha}k)\,.$$

We have, by (13),

$$\mathcal{A}g(s,k) = e^{-\rho s}\left\{-\rho\ln(g_p{}^{\alpha/1-\alpha}k) + \mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]\right.$$
$$\left. - n + \tfrac{1}{2}\sigma^2 + b + d\right\} \tag{16}$$
$$> 0$$
$$\Leftrightarrow k < g_p{}^{-\alpha/1-\alpha}\exp\left\{\tfrac{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)] - n + \frac{1}{2}\sigma^2 + b + d}{\rho}\right\}$$

where

$$d \triangleq \int_{\mathbb{R}_0}\left(\ln\tfrac{1}{1+\gamma z} + \tfrac{\gamma z}{1+\gamma z}\right)\nu(dz)\,, \tag{17}$$

Hence, we can define,

$$U = \left\{(s,k); k < g_p{}^{-\alpha/1-\alpha}\exp\left(\tfrac{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)] - n + \frac{1}{2}\sigma^2 + b + d}{\rho}\right)\right\}, \tag{18}$$

Thus, it is natural to guess that the continuation region D has the form,

$$D = \{(s,k); 0 < k < \hat{k}\}\,. \tag{19}$$

for some $\hat{k}$ such that $U \subseteq D$, i.e.,

$$\hat{k} \geq g_p{}^{-\alpha/1-\alpha}\exp\left(\tfrac{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)] - n + \frac{1}{2}\sigma^2 + b + d}{\rho}\right), \tag{20}$$

Thus, (15) can be rewritten as follows,

$$\phi(s,k)=\begin{cases} e^{-\rho s}Ck^{\beta}, & 0<k<\hat{k} \\ e^{-\rho s}\ln(g_p{}^{\alpha/1-\alpha}k), & k\geq\hat{k} \end{cases} \tag{21}$$

where $\hat{k}>0$ and C remain to be determined. Moreover, continuity and differentiability of ϕ at $k=\hat{k}$ give,

$$C(\hat{k})^{\beta}=\ln(g_p{}^{\alpha/1-\alpha}\hat{k})$$

$$C\beta(\hat{k})^{\beta-1}=(\hat{k})^{-1}$$

Combining the above equations reveals that,

$$\tfrac{C(\hat{k})^{\beta}}{C\beta(\hat{k})^{\beta}}=\ln(g_p{}^{\alpha/1-\alpha}\hat{k})$$

$$\Leftrightarrow \hat{k}=g_p{}^{-\alpha/1-\alpha}\exp(\tfrac{1}{\beta}) \tag{22}$$

And

$$C=\tfrac{1}{\beta}(\hat{k})^{-\beta}=\tfrac{1}{\beta}[g_p{}^{-\alpha/1-\alpha}\exp(\tfrac{1}{\beta})]^{-\beta}. \tag{23}$$

To summarize, we have,

LEMMA 2. *Under the above assumptions and constructions, if* $\sigma<0$, $-1<\gamma z<0$ *a.s.* $-\nu$, $\int_{\mathbb{R}_0}[(\frac{1}{1+\gamma z})^{\beta}-1]^2\nu(dz)<\infty$ *and*

$$n-\tfrac{1}{2}\sigma^2-\int_{\mathbb{R}}\gamma z\nu(dz)$$

$$<\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]$$

$$\leq\min\left\{\rho+n-\sigma^2-b,\rho+n-\tfrac{3}{2}\sigma^2-\tfrac{1}{2}\int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^2-1]\nu(dz)-\int_{\mathbb{R}}\gamma z\nu(dz)\right\},$$

And,

$$\Big|2\beta\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]+2\beta\int_{\mathbb{R}}\gamma z\nu(dz)$$

$$-2\beta n+(\beta+2\beta^2)\sigma^2+\int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^{2\beta}-1]\nu(dz)\Big|<\infty,$$

where b *is defined in* (8). *Then we obtain the optimal* $\mathcal{F}_t-$ *stopping time,* $\hat{\tau}\triangleq\inf\{t\geq 0;k(t)=\hat{k}\}$. *In other words,*

$$\hat{g}(s,k)=e^{-\rho s}\tfrac{1}{\beta}(\hat{k})^{-\beta}k^{\beta}=U^{\hat{\tau}},$$

which is a supermeanvalued majorant of $g(s,k)$ *with* $\hat{k}$ *given by* (22) *and* β *is a solution of* $h(\beta)=0$ *in* (14).

Proof. See Appendix B. ▌

REMARK. Obviously, Lemma 2 can be regarded as a conclusion of Step 2 introduced in section 2.2. And $\hat{k}$ given in (22) would be seen as the maximal and sustainable terminal path level of capital stock per capita that is criterion-of-*the-modified-Radner-fashion* optimal. Noting that $\hat{k}$ is endogenously determined in the current paper while the maximal terminal path level is usually exogenously specified in existing literatures[15], for instance, the interesting paper of Kurz (1965). Consequently, we argue that the advantage of the theory of optimal stopping time employed here is that it is available for us to make the minimum time to "economic maturity" and the utility-optimal and sustainable terminal path level of capital stock per capita simultaneously and endogenously determined.

3.3. Government

It is assumed that the government continues to tie expenditure levels to aggregate output as before, i.e., $G_p = g_p Y$ with $0 < g_p < 1$, thus, in the absence of debt, tax revenues and government expenditures must satisfy the following balanced budget constraint,

$$\tau_k r_k K(t) + \tau_c c(t) L(t) = g_p Y(t)\,, \tag{24}$$

Using (1'), (2) and Lemma 1, (24) can be rewritten as,

$$\tau_k + \tau_c (1 - g_p - \hat{r}_s) = g_p\,. \tag{25}$$

Now, following from Step 3a shown in section 2.2, we consider the following case,

CASE 1. *The goal of the government is to maximize the welfare of the representative agent.*

Substituting (25) into (7'') gives,

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[15] See, Cass, 1966; and McKenzie, 1976.

$$dk(t)=k(t^-)\Big[(\mu g_p{}^{\alpha/1-\alpha}\hat{r}_s-n+\sigma^2+b)dt-\sigma dB(t)-\int_{\mathbb{R}_0}\tfrac{\gamma z}{1+\gamma z}\widetilde{N}(dt,dz)\Big], \qquad (26)$$

And hence the stochastic optimal control problem facing the government can be expressed as follows,

$$\max_{\substack{0<\tau_c<1\\0<\tau_k<1}}\mathbb{E}\left[\int_0^{\hat{\tau}}e^{-\rho(s+t)}\ln\Big((1-g_p-\hat{r}_s)g_p{}^{\alpha/1-\alpha}k(t)\Big)dt+U^{\hat{\tau}}\right]. \qquad (9'')$$

s.t.

$$dk(t)=k(t^-)\Big[(\mu g_p{}^{\alpha/1-\alpha}\hat{r}_s-n+\sigma^2+b)dt-\sigma dB(t)-\int_{\mathbb{R}_0}\tfrac{\gamma z}{1+\gamma z}\widetilde{N}(dt,dz)\Big].$$

Accordingly, the corresponding stochastic Bellman partial differential equation (PDE) amounts to,

$$\begin{aligned}&-W_t(t,k(t))-\tfrac{1}{2}\sigma^2k^2(t)W_{kk}(t,k(t))\\&-\int_{\mathbb{R}_0}\Big[W\Big(t,k(t)-\tfrac{\gamma z}{1+\gamma z}k(t)\Big)-W(t,k(t))+\tfrac{\gamma z}{1+\gamma z}k(t)W_k(t,k(t))\Big]\nu(dz)\\&=\max_{\substack{0<\tau_c<1\\0<\tau_k<1}}\begin{Bmatrix}\exp(-\rho(s+t))\ln[(1-g_p-\hat{r}_s)g_p{}^{\alpha/1-\alpha}k(t)]\\+W_k(t,k(t))k(t)(\mu g_p{}^{\alpha/1-\alpha}\hat{r}_s-n+\sigma^2+b)\end{Bmatrix}.\end{aligned} \qquad (27)$$

with the following boundary condition,

$$W(\hat{\tau},k(\hat{\tau}))=U^{\hat{\tau}}, \qquad (28)$$

where $W(t,k(t))$ denotes the value function. To solve the above dynamic optimal control problem, the following lemma is derived,

LEMMA 3. *Provided the balanced budget constraint given in* (25) *and the optimal control problem expressed in* (9''), *then the optimal capital income tax rate is equal to* $\tau_k^*=g_p$ *while optimal consumption tax rate is zero. Moreover, we have,*

$$W(\hat{\tau},k(\hat{\tau}))=\exp(-\rho(s+\hat{\tau}))(C_3+\rho^{-1}\ln\hat{k})=U^{\hat{\tau}},$$

where $\hat{\tau}$ *and* $U^{\hat{\tau}}$ *are defined in Lemma* 2, $\hat{k}$ *is given in* (22) *and,*

$$C_3\triangleq\rho^{-1}\begin{Bmatrix}\ln\frac{\rho}{\mu}+\rho^{-1}[\mu g_p{}^{\alpha/1-\alpha}(1-g_p)-n\\+\sigma^2+b]-1-\frac{1}{2}\sigma^2\rho^{-1}+\rho^{-1}d\end{Bmatrix},$$

where b *and* d *are given in* (8) *and* (17), *respectively.*

Proof. See Appendix C. ▌

REMARK. Lemma 3 would be regarded as a conclusion of Step 3a of the computation algorithm introduced in section 2.2, and the boundary condition given in Lemma 3 will play a crucial role in determining the exact form of the endogenous time in the sequel. Moreover, it is worthwhile mentioning that Lemma 3 provides us with a case against the well-known argument that capital income should not be taxed (Chamley, 1986; Judd, 2002) and even that the optimal income tax rate should be negative (Judd, 1997). Not only that, the optimal capital income tax rate is equal to an exogenously given constant which is known and controlled by the government, and which therefore implies a simple rule of taxation for the government. And it is from this character that we claim that our model is in accord with Kydland and Prescott (1977).

Hence, by combining Lemma 3 with Lemma 1, we have,

$$\hat{r}_s = 1 - g_p - \tfrac{\rho}{\mu g_p{}^{\alpha/1-\alpha}}, \tag{29}$$

And substituting (29) and the results in Lemma 3 into (14) produce,

$$\begin{aligned} h(\beta) \triangleq & -\rho + \beta[\mu g_p{}^{\alpha/1-\alpha}(1-g_p) - \rho - n + \sigma^2 + b] \\ & + \tfrac{1}{2}\sigma^2\beta(\beta-1) + \int_{\mathbb{R}_0} [(\tfrac{1}{1+\gamma z})^{\beta} - 1 + \tfrac{\gamma z \beta}{1+\gamma z}]\nu(dz), \end{aligned} \tag{14'}$$

Now, by Lemma 2, we have,

$$U^{\hat{\tau}} = e^{-\rho s} \tfrac{1}{\beta} (\hat{k})^{-\beta} k^{\beta}, \tag{30}$$

where $k = k(0) > 0$, $\hat{k}$ is given in (22), and β is a solution of equation $h(\beta) = 0$ in (14'). Combining (30) with Lemma 3 shows that,

$$\begin{aligned} W(\hat{\tau}, k(\hat{\tau})) &= \exp(-\rho(s+\hat{\tau}))(C_3 + \rho^{-1}\ln \hat{k}) \\ &= \exp(-\rho s)\tfrac{1}{\beta}(\hat{k})^{-\beta} k^{\beta} \\ &= U^{\hat{\tau}}. \end{aligned}$$

which implies that,

$$\hat{\tau} = \rho^{-1} \ln[\beta(\tfrac{\hat{k}}{k})^{\beta}(C_3 + \rho^{-1}\ln \hat{k})], \tag{31}$$

To summarize, we have the following theorem,

THEOREM 1. *Based on Lemma* 1 *to Lemma* 3*, and suppose the goal of the government is to maximize the welfare of the representative agent, we have,*

$$\hat{\tau} = \rho^{-1} \ln[\beta(\tfrac{\hat{k}}{k})^{\beta}(C_3 + \rho^{-1} \ln \hat{k})],$$

where $k = k(0) > 0$, $\hat{k}$ *is given in* (22)*,* β *is a solution of* $h(\beta) = 0$ *in* (14')*, and* C_3 *is given in Lemma* 3.

REMARK. It is by Theorem 1 that we confirm that the minimum time needed to "economic maturity" is endogenously determined and explicitly represented. And, in particular, the endogenous time depends on the following relevant parameters: the subject discount factor, the initial level of capital stock per capita, the utility-optimal and sustainable terminal path level of capital stock per capita, the natural growth rate of population, the exogenous level of government spending and also the volatility of the macro-economy. And one may, if motivated, develop more thorough comparative static analyses of the endogenous time with respect to the above relevant parameters.

Noting that Theorem 1 is a conclusion of Step 4 of the computation algorithm in section 2.2, we now consider the following case corresponding to Step 3b of the computation algorithm.

CASE 2. *The goal of the government is to minimize the optimal stopping time of the representative agent.*

Now by Lemma 2, we have,

$$U^{\hat{\tau}} = e^{-\rho s} \tfrac{1}{\beta} (\hat{k})^{-\beta} k^{\beta}, \tag{32}$$

where $k = k(0) > 0$, $\hat{k}$ is given in (22), and β is a solution of equation $h(\beta) = 0$ in (14). Combining (32) with Lemma 1 and Lemma 2 shows that,

$$\begin{aligned} V(\hat{\tau}, k(\hat{\tau})) &= \exp(-\rho(s+\hat{\tau}))[C_1 + \rho^{-1} \ln k(\hat{\tau})] \\ &= \exp(-\rho(s+\hat{\tau}))(C_1 + \rho^{-1} \ln \hat{k}) \\ &= \exp(-\rho s) \tfrac{1}{\beta} (\hat{k})^{-\beta} k^{\beta} \\ &= U^{\hat{\tau}}. \end{aligned}$$

which implies that,

$$\hat{\tau} = \rho^{-1} \ln[\beta(\tfrac{\hat{k}}{k})^{\beta}(C_1 + \rho^{-1} \ln \hat{k})]. \quad (33)$$

where C_1 is given in Lemma 1. Thus, the problem facing the government can be expressed as,

PROBLEM 1. *The government is motivated to choose taxation policies so as to minimize the stopping time defined in* (33).

REMARK. Problem 1 is actually a nonlinear optimization problem and here we don't try to solve it due to its complication. Moreover, it is worth emphasizing that the stopping time given in (33) may be fundamentally different from that given in Theorem 1. It is easy to notice that different goals of the government usually lead to different fiscal policies, thereby resulting different short-run and direct economic consequences and even different speeds and paths of economic development. And it is especially worth noting that there is a conjecture or possibility that the minimum time needed to "economic maturity" when the goal of the government is to minimize the endogenous time may be much *longer* than that when the goal of the government is to maximize the welfare of the representative agent. And here we provide one reasonable explanation that the incentive or motivation of investment of the representative agent may be terribly distorted when the goal of the government is not to maximize the welfare of the representative agent but to directly minimize the time needed to "economic maturity", thereby implying the micro-foundation of economic development is also distorted and hence retarding the speed of economic development. That is, there may exist a *trade-off* for the government, i.e., the speed of long-term economic development on the one hand and the short-term welfare of the representative agent on the other hand. Therefore, the lesson for us is that for the government of an underdeveloped economy, choosing an appropriate development strategy and hence appropriate fiscal policies are of crucial importance in affecting and even determining the long-term speed and path of the *convergence* of the corresponding economical system, and thus the long-term equilibrium level of the economy and welfare level of the representative agent.

4. PREFERENCE MANIFOLD TWO

In this section, our goal is to introduce a new type of preference manifold of the representative agent different from that in section 3. The firm will employ the same kind of production technology as is shown in section 3.1, so we begin our analyses from the representative agent.

4.1. Representative Agent

Our analyses will proceed according to the computation algorithm introduced in section 2.2, that is, the representative agent will choose an optimal savings rate and then the optimal stopping time. Different from (9), we introduce the following objective function of *the modified Ramsey fashion*,

$$U = \mathbb{E}\left[\int_0^{\tau^*} e^{-\rho(s+t)} \ln\left((1-g_p-r_s)y(t)\right)dt + e^{-\rho(s+\tau^*)} \ln y(\tau^*)\right]. \tag{34}$$

where $0 \le s < \tau^*$ and τ^* is an $\mathcal{F}_t -$optimal stopping time, which is determined by the following optimal stopping problem,

$$\begin{aligned}
&g^*(\tau,k(\tau)) \\
&\triangleq \sup_{\tau\in\mathcal{T}} \mathbb{E}^{(s,k)}\left[\int_0^{\tau} e^{-\rho(s+t)} \ln\left((1-g_p-r_s)g_p{}^{\alpha/1-\alpha}k(t)\right)dt + e^{-\rho(s+\tau)} \ln\left(g_p{}^{\alpha/1-\alpha}k(\tau)\right)\mathbf{1}_{\{\tau<\infty\}}\right] \\
&= \sup_{\tau\in\mathcal{T}} \mathbb{E}^{(s,k)}\left[\int_0^{\tau} e^{-\rho(s+t)} \ln\left((1-g_p-r_s)y(t)\right)dt + e^{-\rho(s+\tau)} \ln\left(y(\tau)\right)\mathbf{1}_{\{\tau<\infty\}}\right] \\
&= \mathbb{E}^{(s,k)}\left[\int_0^{\tau^*} e^{-\rho(s+t)} \ln\left((1-g_p-r_s)g_p{}^{\alpha/1-\alpha}k(t)\right)dt + e^{-\rho(s+\tau^*)} \ln\left(g_p{}^{\alpha/1-\alpha}k(\tau^*)\right)\mathbf{1}_{\{\tau^*<\infty\}}\right]
\end{aligned} \tag{35}$$

subject to the SDE defined in (7), and one may easily tell the difference between (35) and (10). Next, similar to (9'), we consider the optimal control problem as follows,

$$\max_{0<r_s<1} \mathbb{E}\left[\int_0^{\tau^*} e^{-\rho(s+t)} \ln\left((1-g_p-r_s)y(t)\right)dt + e^{-\rho(s+\tau^*)} \ln y(\tau^*)\right], \tag{36}$$

s.t.

$$dk(t)=\left\{\mu g_p^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s)]-n+\sigma^2+b\right\}k(t^-)dt$$

$$-\sigma k(t^-)dB(t)\ -k(t^-)\int_{\mathbb{R}_0}\tfrac{\gamma z}{1+\gamma z}\widetilde{N}(dt,dz)\,. \qquad (37)$$

where τ^* is taken as exogenously given up to present step. To solve the above dynamic optimization problem and employ $V(t,k(t))$ as the corresponding value function, then we get,

LEMMA 4. *Provided the above constructions and assumptions, the following optimal savings rate is derived,*

$$r_s^*=1-g_p-\frac{\rho}{\mu g_p^{\alpha/1-\alpha}(1+\tau_c)},$$

And the value function $V(t,k(t))$ *satisfies the following boundary condition,*

$$V(\tau^*,k(\tau^*))=e^{-\rho(s+\tau^*)}[C_1+\rho^{-1}\ln k(\tau^*)]=e^{-\rho(s+\tau^*)}\ln[g_p^{\alpha/1-\alpha}k(\tau^*)].$$

where C_1 *is defined in Lemma* 1.

REMARK. Lemma 4 is a natural correspondence to Lemma 1.

Noting that the proof of Lemma 4 is the same as that of Lemma 1, so we take it omitted. In what follows, we will determine the optimal stopping time τ^*. After applying Lemma 4, the optimal path of capital accumulation can be expressed as follows,

$$dk(t)=\left\{\mu g_p^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\sigma^2+b\right\}k(t^-)dt$$

$$-\sigma k(t^-)dB(t)\ -k(t^-)\int_{\mathbb{R}_0}\tfrac{\gamma z}{1+\gamma z}\widetilde{N}(dt,dz)\,, \qquad (37')$$

Let $Y(t)\triangleq(s+t,k(t))^\top$, $Y(0)\triangleq(s,k)^\top$, then the generator of $Y(t)$ reads as follows,

$$\mathcal{A}\phi(s,k)=\tfrac{\partial\phi}{\partial s}+\left\{\mu g_p^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\sigma^2+b\right\}k\tfrac{\partial\phi}{\partial k}$$

$$+\tfrac{1}{2}\sigma^2k^2\tfrac{\partial^2\phi}{\partial k^2}+\int_{\mathbb{R}_0}[\phi(s,k-\tfrac{\gamma z}{1+\gamma z}k)-\phi(s,k)+\tfrac{\gamma z}{1+\gamma z}k\tfrac{\partial\phi}{\partial k}]\nu(dz)\,, \qquad (38)$$

for $\forall\phi\in C^2(\mathbb{R}^2)$. If we try a function ϕ of the form,

$\phi(s,k)=e^{-\rho s}\varphi(k)$, for $\varphi\in C^2(\mathbb{R})$

Then we have,

$$\mathcal{A}\phi(s,k) = e^{-\rho s}\left(-\rho\varphi(k) + \left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\sigma^2+b\right\}k\varphi'(k)\right)$$

$$+e^{-\rho s}\left\{\tfrac{1}{2}\sigma^2k^2\varphi''(k)+\int_{\mathbb{R}_0}[\varphi(\tfrac{1}{1+\gamma z}k)-\varphi(k)+\tfrac{\gamma z}{1+\gamma z}k\varphi'(k)]\nu(dz)\right\}$$

$$\triangleq e^{-\rho s}\mathcal{A}_0\varphi(k). \tag{39}$$

Define $g(k)\triangleq \ln(g_p{}^{\alpha/1-\alpha}k)$, $f(k)\triangleq \ln[(1-g_p-r_s^*)g_p{}^{\alpha/1-\alpha}k]$, by (35) and (39), we see that,

$$\mathcal{A}_0 g(k)+f(k)>0$$

$$\Leftrightarrow \rho\ln(g_p{}^{\alpha/1-\alpha}k)-\ln[(1-g_p-r_s^*)g_p{}^{\alpha/1-\alpha}k]<$$

$$\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\tfrac{1}{2}\sigma^2+b$$

$$+\int_{\mathbb{R}_0}[\ln(\tfrac{1}{1+\gamma z})+\tfrac{\gamma z}{1+\gamma z}]\nu(dz)$$

$$\Leftrightarrow k<g_p{}^{-\alpha/1-\alpha}(1-g_p-r_s^*)^{1/(\rho-1)}\exp\left\{\frac{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\frac{1}{2}\sigma^2+b+d}{\rho-1}\right\}.$$

where b and d are defined in (8) and (17), respectively. Hence,

$$U=\left\{(s,k);k<g_p{}^{-\alpha/1-\alpha}(1-g_p-r_s^*)^{1/(\rho-1)}\exp\left(\frac{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\frac{1}{2}\sigma^2+b+d}{\rho-1}\right)\right\} \tag{40}$$

In view of $U\subseteq D$ it is natural to guess that the continuation region D has the form,

$$D=\{(s,k);0<k<k^*\}. \tag{41}$$

for some k^* satisfying,

$$k^*\geq g_p{}^{-\alpha/1-\alpha}(1-g_p-r_s^*)^{1/(\rho-1)}\exp\left(\frac{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\frac{1}{2}\sigma^2+b+d}{\rho-1}\right), \tag{42}$$

Now, in D we try to solve the equation,

$$\mathcal{A}_0\varphi(k)+f(k)=0. \tag{43}$$

The homogenous equation $\mathcal{A}_0\varphi_0(k)$ has a solution $\varphi_0(k)=k^r$ if and only if,

$$h(r)\triangleq -\rho+r\left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\sigma^2+b\right\}$$

$$+\tfrac{1}{2}\sigma^2r(r-1)+\int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^r-1+\tfrac{\gamma zr}{1+\gamma z}]\nu(dz)=0, \tag{44}$$

Since $h(0)=-\rho<0$ and $\lim_{|r|\to\infty}h(r)=\infty$, we see that the equation $h(r)=0$ has two solutions r_1, r_2 such that $r_2<0<r_1$. We let r be a solution of this equation. To find a

particular solution $\varphi_1(k)$ of the non-homogenous equation,

$$\mathcal{A}_0\varphi_1(k)+\ln[(1-g_p-r_s^*)g_p{}^{\alpha/1-\alpha}k]=0\,, \tag{45}$$

We try,

$$\varphi_1(k)=C_5+C_6\ln k\,, \tag{46}$$

for some constants C_5, C_6 to be determined. Substituting (46) into (45) and applying (39), we have,

$$-\rho C_5-\rho C_6\ln k+\left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\sigma^2+b\right\}C_6$$
$$-\tfrac{1}{2}\sigma^2C_6+C_6\int_{\mathbb{R}_0}[\ln(\tfrac{1}{1+\gamma z})+\tfrac{\gamma z}{1+\gamma z}]\nu(dz)+\ln[(1-g_p-r_s^*)g_p{}^{\alpha/1-\alpha}k]=0\,.$$

which implies that,

$$C_6=\rho^{-1}, \tag{47}$$

Hence,

$$C_5=\rho^{-1}\begin{Bmatrix}\rho^{-1}\left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\sigma^2+b\right\}-\\ \tfrac{1}{2}\sigma^2\rho^{-1}+\rho^{-1}\int_{\mathbb{R}_0}\left(\ln\tfrac{1}{1+\gamma z}+\tfrac{\gamma z}{1+\gamma z}\right)\nu(dz)+\ln[(1-g_p-r_s^*)g_p{}^{\alpha/1-\alpha}]\end{Bmatrix}. \tag{48}$$

Consequently, for all constants C the function,

$$\varphi(k)=Ck^r+\rho^{-1}\ln k+C_5\,, \tag{49}$$

is a solution of the equation defined in (43) with C_5 given by (48). Thus, one can try to put,

$$\varphi(k)=\begin{cases}Ck^r+\rho^{-1}\ln k+C_5, & 0<k<k^*\\ g(k)\triangleq\ln(g_p{}^{\alpha/1-\alpha}k), & k\geq k^*\end{cases} \tag{50}$$

where $k^*>0$ and C remain to be determined. Continuity and differentiability of φ at $k=k^*$ give the following equations,

$$C(k^*)^r+\rho^{-1}\ln k^*+C_5=\ln(g_p{}^{\alpha/1-\alpha}k^*)\,, \tag{51}$$

$$Cr(k^*)^{r-1}+\rho^{-1}(k^*)^{-1}=(k^*)^{-1}. \tag{52}$$

By (52) we get,

$$(k^*)^r = \frac{1-\rho^{-1}}{rC}, \tag{53}$$

Inserting (53) into (51) produces,

$$k^* = g_p^{\frac{\alpha\rho}{(1-\alpha)(1-\rho)}} \exp(\tfrac{1}{r} + \tfrac{\rho C_5}{\rho-1}), \tag{54}$$

And by (52) we get,

$$C = \frac{\rho-1}{\rho r(k^*)^r}. \tag{55}$$

which implies that we should choose,

$$r = r_2 < 0, \tag{56}$$

in (44). To sum up, we have the following lemma,

LEMMA 5. *Under the above assumptions and constructions, if* $\sigma < 0$, $-1 < \gamma z < 0 \quad a.s.-\nu$, $(1-r)k^r > (k^*)^r$ *and*

$$-\rho \ln(1 - g_p - r_s^*)$$

$$\geq \mu g_p^{\alpha/1-\alpha}[(1-\tau_k) - (1+\tau_c)(1-g_p-r_s^*)] - n + \tfrac{1}{2}\sigma^2 + b + d,$$

where b *and* d *are defined in* (8) *and* (17)*, respectively. And,*

$$n - \tfrac{1}{2}\sigma^2 - \int_{\mathbb{R}} \gamma z \nu(dz)$$

$$< \mu g_p^{\alpha/1-\alpha}[(1-\tau_k) - (1+\tau_c)(1-g_p-r_s^*)]$$

$$\leq \rho + n - \tfrac{3}{2}\sigma^2 - \tfrac{1}{2}\int_{\mathbb{R}_0} [(\tfrac{1}{1+\gamma z})^2 - 1]\nu(dz) - \int_{\mathbb{R}} \gamma z \nu(dz)$$

Then we obtain the optimal $\mathcal{F}_t$ *− stopping time,* $\tau^* \triangleq \inf\{t \geq 0; k(t) = k^*\}$. *That is to say,* $g^*(s,k) = e^{-\rho s}[\frac{\rho-1}{\rho r(k^*)^r} k^r + \rho^{-1}\ln k + C_5]$ *is a supermeanvalued majorant of* $g(s,k)$ *with* k^* *given by* (54)*,* $k = k(0) > 0$, $r < 0$ *determined by* (44) *and* C_5 *given in* (48).

Proof. See Appendix D. ▌

REMARK. Lemma 5 is a natural correspondence to Lemma 2. And one can clearly and easily tell the differences between the two lemmas.

4.2. Government

Similar to section 3.3, and applying Lemma 4, the balanced budget constraint (24) can be expressed as follows,

$$\tau_k + \tau_c(1 - g_p - r_s^*) = g_p \,. \tag{25'}$$

CASE 1. *The goal of the government is to maximize the welfare of the representative agent*.

Hence, the stochastic optimal control problem facing the government can be written as follows,

$$\max_{\substack{0<\tau_c<1 \\ 0<\tau_k<1}} \mathbb{E}\left[\int_0^{\tau^*} e^{-\rho(s+t)} \ln\Big((1-g_p-r_s)g_p{}^{\alpha/1-\alpha}k(t)\Big)dt + e^{-\rho(s+\tau^*)}\ln\Big(g_p{}^{\alpha/1-\alpha}k(\tau^*)\Big)\right] \tag{57}$$

s.t.

$$dk(t) = k(t^-)\left[(\mu g_p{}^{\alpha/1-\alpha}r_s^* - n + \sigma^2 + b)dt - \sigma dB(t) - \int_{\mathbb{R}_0} \tfrac{\gamma z}{1+\gamma z}\widetilde{N}(dt,dz)\right]. \tag{58}$$

Solving the problem gives,

LEMMA 6. *Provided the balanced budget constraint given in* (25') *and the optimal control problem expressed in* (57) and (58)*, then the optimal capital income tax rate is* $\tau_k^* = g_p$ *and optimal consumption tax rate is zero. Moreover, the corresponding value function satisfies the following boundary condition,*

$$W(\tau^*, k(\tau^*)) = e^{-\rho(s+\tau^*)}(C_3 + \rho^{-1}\ln k^*) = e^{-\rho(s+\tau^*)}\ln(g_p{}^{\alpha/1-\alpha}k^*),$$

where τ^* *is defined in Lemma* 5*,* k^* *is given in* (54) *and* C_3 *is given in Lemma* 3.

REMARK. Lemma 6 is a natural correspondence to Lemma 3.

Noting that the proof is the same as that of Lemma 3, we take it omitted here. Applying Lemma 6 to Lemma 4, we get,

$$r_s^* = 1 - g_p - \tfrac{\rho}{\mu g_p{}^{\alpha/1-\alpha}} \,. \tag{59}$$

Inserting (59) into (58) produces the following optimal law of motion of capital accumulation,

$$dk(t) = k(t^-)\left\{[\mu g_p{}^{\alpha/1-\alpha}(1-g_p) - \rho - n + \sigma^2 + b]dt - \sigma dB(t) - \int_{\mathbb{R}_0}\tfrac{\gamma z}{1+\gamma z}\widetilde{N}(dt,dz)\right\} \tag{60}$$

Moreover, applying Lemma 6 and (59) to (44) and (48) shows,

$$h(r) = -\rho + r[\mu g_p{}^{\alpha/1-\alpha}(1-g_p) - \rho - n + \sigma^2 + b]$$
$$+\tfrac{1}{2}\sigma^2 r(r-1) + \int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^r - 1 + \tfrac{\gamma z r}{1+\gamma z}]\nu(dz) = 0\,, \tag{44'}$$

And,

$$C_5 = \rho^{-1}\begin{Bmatrix}\rho^{-1}[\mu g_p{}^{\alpha/1-\alpha}(1-g_p) - \rho - n + \\ \sigma^2 + b] - \tfrac{1}{2}\sigma^2\rho^{-1} + \rho^{-1}d + \ln\tfrac{\rho}{\mu}\end{Bmatrix}. \tag{48'}$$

where b and d are defined in (8) and (17), respectively. So k^* in (54) can be expressed as,

$$k^* = g_p{}^{\frac{\alpha\rho}{(1-\alpha)(1-\rho)}}\exp(\tfrac{1}{r} + \tfrac{\rho C_5}{\rho-1})\,, \tag{54'}$$

where $r < 0$ is a solution of (44') and C_5 is defined in (48'). Therefore, we conclude the following theorem,

THEOREM 2. *Based on Lemma* 4 *to Lemma* 6, *and provided the goal of the government is to choose tax policies so as to maximize the welfare of the representative agent, then the optimal stopping time given in Lemma* 5 *can be completely characterized as follows,*

$$\tau^* \triangleq \inf\{t \ge 0; k(t) = k^*\}\,, \tag{61}$$

where $k(t)$ *is determined by* (60) *and* k^* *is given in* (54').

REMARK. Theorem 2 is a natural correspondence to Theorem 1.

Furthermore, it follows from Theorem 2 that,

COROLLARY 1. *Suppose* $\int_{\mathbb{R}_0}(z^2 \wedge 1)\nu(dz) < \infty$, *where* $z^2 \wedge 1 \triangleq \min\{z^2, 1\}$, *and* $\int_{\mathbb{R}_0}(\frac{\gamma z}{1+\gamma z})^p \nu(dz) < \infty$ *for* $\forall p \in \mathbb{N}$ *and* $p \ge 2$. *Then the solution of* (60) *is in* $L^2(\Omega, \mathbb{P})$ *and*

$$\mathbb{E}\left[\sup_{0<t\le T}\left|k(t) - k^*\right|^p\right] \le \mathbf{\Psi}_{M(T),p}\left(1 + \mathbb{E}\left|k - k^*\right|^p\right),$$

where $k = k(0) > 0$, k^* *is given in* (54') *and*

$$\mathbf{\Psi}_{M(T),p} \triangleq \exp\Big\{M(T)\Big[1 + \left|\mu g_p{}^{\alpha/1-\alpha}(1-g_p) - \rho - n + \sigma^2 + b\right|^2 + \left|\sigma\right|^2 + \int_{\mathbb{R}_0}(\tfrac{\gamma z}{1+\gamma z})^2\nu(dz) +$$

$$\left|\mu g_p{}^{\alpha/1-\alpha}(1-g_p)-\rho-n+\sigma^2+b\right|^p+\left|\sigma\right|^p+\left(\int_{\mathbb{R}_0}(\tfrac{\gamma z}{1+\gamma z})^2\nu(dz)\right)^{p/2}+\int_{\mathbb{R}_0}(\tfrac{\gamma z}{1+\gamma z})^p\nu(dz)\Big]\Big\},$$

for some constant $M>0$ *and* M *depends on* T *with* $0<T\le\infty$.

Proof. See Appendix E. ▌

CASE 2. *The goal of the government is to minimize the optimal stopping time of the representative agent*.

As a matter of fact, we get the following interesting theorem,

THEOREM 3. *Suppose that* $\int_{\mathbb{R}_0}(z^2\wedge 1)\nu(dz)<\infty$ *and* $\int_{\mathbb{R}_0}(\frac{\gamma z}{1+\gamma z})^p\nu(dz)<\infty$ *for* $\forall p\in\mathbb{N}$ *and* $p\ge 2$, *then the solution of* (58) *is in* $L^2(\Omega,\mathbb{P})$ *and*

$$\mathbb{E}\left[\sup_{0<t\le T}\left|k(t)-k^*\right|^p\right]\le\widehat{\mathbf{\Psi}_{M(T),p}}\left(1+\mathbb{E}\left|k-k^*\right|^p\right),$$

where $k=k(0)>0$, k^* *is given in* (54'), $\widehat{\mathbf{\Psi}_{M(T),p}}$ *is minimized by letting* $\tau_k=g_p$ *and* $\tau_c=0$, *and* $\widehat{\mathbf{\Psi}_{M(T),p}}\to\mathbf{\Psi}_{M(T),p}$ *point-wise as* $\tau_c\to 0$ *or* $\tau_k\to g_p$.

Proof. See Appendix F. ▌

REMARK. In view of Theorem 1 and Theorem 2 shows that there is a technical difference between *the modified Radner fashion* and *the modified Ramsey fashion*, that is, the endogenous time can be explicitly computed and represented in the preference manifold of *the modified Radner fashion* shown in Theorem 1 while this cannot be realized in that of *the modified Ramsey fashion*. Therefore, noting that the endogenous time can not be explicitly represented as in Theorem 2, Corollary 1 and Theorem 3 are of crucial economic intuitions and implications. Specifically, Theorem 3 reveals that in the case of p ($\forall p\in\mathbb{N}$ and $p\ge 2$) order moment of uniform topology, the goal of the government expressed in the above Case 2 can be equivalently expressed as choosing optimal tax rates corresponding to *the best constant* $\widehat{\mathbf{\Psi}_{M(T),p}}\,|_{\tau_c^*=0,\tau_k^*=g_p}=\mathbf{\Psi}_{M(T),p}$. Obviously, one may choose different distance functions equipped with different topologies, thereby resulting different equivalent expressions of the above Case 2 and thus even different corresponding optimal tax rates. Finally, it is worthwhile emphasizing that the utility-optimal and sustainable terminal path level of capital stock per capita is usually not the same between the above two different preference manifolds, and thus the corresponding endogenous times may be not equivalent. That is to say, one type of preference manifold may imply a faster speed to its corresponding "economic maturity" than that of other types of preference manifolds[16].

5. THE EFFECT OF INFORMATION STRUCTURE

In this section, we will investigate the influences of different information structures on the endogenous time, and we will take preference manifold one in section 3 for example.

5.1. Definitions and Notations

First of all, besides the filtration $\mathcal{F} \triangleq \{\mathcal{F}_t\}_{0\le t\le T}$ introduced in section 3.2, we suppose that we are given another two filtrations $\mathfrak{H} \triangleq \{\mathcal{H}_t\}_{0\le t\le T}$ and $\mathfrak{M} \triangleq \{\mathcal{M}_t\}_{0\le t\le T}$ with,

$$\mathcal{H}_t \subseteq \mathcal{F}_t \subseteq \mathcal{M}_t, \ \ 0 \le t \le T .$$

which represent three kinds of information levels available to the agent at time t. And we give the following definitions[17],

DEFINITION 1. (Incomplete Information):

If the control of the agent is $\mathfrak{H}-$predictable, we say that the agent has incomplete information.

DEFINITION 2. (Complete Information):

If the control of the agent is $\mathcal{F}-$predictable, we say that the agent has complete information.

DEFINITION 3. (Perfect Information):

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[16] The current paper shows that different preference structures lead to different levels of "economic maturity" with a high probability and thus different speeds and paths of economic development. However, it is also possible that the convergence rate is equal between different economical systems with different preference manifolds although they have totally different sustainable terminal path levels of capital stock per capita.

[17] Similar definitions can be found in Miao (2009), who studies optimal consumption and portfolio choice in a Merton-style model with incomplete information when there is a distinction between ambiguity and risk.

If the control of the agent is $\mathfrak{M}-$*predictable, we say that the agent has* perfect information.

Moreover, based upon the above three definitions, we can define,

DEFINITION 4. (Symmetric Information):

For any two agents, if they share the same level of information, no matter it is incomplete, complete or perfect information, we say that the information is symmetric *between the two agents*.

DEFINITION 5. (Asymmetric Information):

For any two agents, if they don't share the same level of information, we say that the information is asymmetric *between them*.

5.2. Representative Agent

It is worth emphasizing that we focus on different information structures between the representative agent and the government, so the firm in this section is the same as that in section 3.1. And hence, we begin our analyses from the representative agent. Firstly, a little different from the SDE defined in (3), we introduce,

$$d^{-}L(t)=L(t)\left[ndt+\sigma d^{-}B(t)+\int_{\mathbb{R}_0}\gamma z\widetilde{N}(d^{-}t,dz)\right], \tag{3'}$$

where we have put $Z=\infty$ in (4) and $d^{-}B(t)$, $\widetilde{N}(d^{-}t,dz)$ denote forward integrals. Then, combining (3') with (5) and applying Itô-Ventzell formula for forward processes,

$$\begin{aligned}d^{-}k(t)=&\left\{\mu g_{p}^{\alpha/1-\alpha}[(1-\tau_{k})-(1+\tau_{c})(1-g_{p}-r_{s})]-n+\sigma^{2}+b_{0}\right\}k(t)dt\\&-\sigma k(t)d^{-}B(t)-k(t)\int_{\mathbb{R}_0}\tfrac{\gamma z}{1+\gamma z}\widetilde{N}(d^{-}t,dz),\end{aligned} \tag{62}$$

where,

$$b_{0}\triangleq\int_{\mathbb{R}_0}\tfrac{(\gamma z)^{2}}{1+\gamma z}\nu(dz), \tag{8'}$$

Hence, by (1'), (62) and Itô-Ventzell formula, we have,

$$d^{-}y(t)=g_{p}^{\alpha/1-\alpha}\left\{\mu g_{p}^{\alpha/1-\alpha}[(1-\tau_{k})-(1+\tau_{c})(1-g_{p}-r_{s})]-n+\sigma^{2}+b_{0}\right\}k(t)dt$$

$$-g_p{}^{\alpha/1-\alpha}\sigma k(t)d^{-}B(t)-g_p{}^{\alpha/1-\alpha}k(t)\int_{\mathbb{R}_0}\tfrac{\gamma z}{1+\gamma z}\widetilde{N}(d^{-}t,dz)$$

$$\triangleq \varsigma(t,\xi,\omega)dt+\varrho(t,\omega)d^{-}B(t)+\int_{\mathbb{R}_0}\theta(t,z,\omega)\widetilde{N}(d^{-}t,dz)\,, \tag{63}$$

where $\omega\in\Omega$ and,

$$\xi \triangleq 1-g_p-r_s\,, \tag{64}$$

And hence, a little different from (9') and (7'), we consider the following stochastic optimal control problem facing the representative agent,

$$\max_{0<r_s<1}\mathbb{E}\left[\int_0^{\hat{\tau}}e^{-\rho(s+t)}\ln\big(\xi y(t)\big)dt+U^{\hat{\tau}}\right], \tag{65}$$

s.t.

$$d^{-}y(t)=\varsigma(t,\xi,\omega)dt+\varrho(t,\omega)d^{-}B(t)+\int_{\mathbb{R}_0}\theta(t,z,\omega)\widetilde{N}(d^{-}t,dz)\,,$$

Thus, the corresponding Hamiltonian[18] can be expressed as follows,

$$\begin{aligned}H(t,y,\xi,\omega)=\exp(-\rho(s+t))\ln[\xi y(t)]+\mathit{\Gamma}(t)\big[\varsigma(t,\xi,\omega)+D_{t+}\varrho(t,\omega)\\ +\int_{\mathbb{R}_0}D_{t+,z}\theta(t,z,\omega)\nu(dz)\Big]+\varrho(t,\omega)D_t\mathit{\Gamma}(t)\\ +\int_{\mathbb{R}_0}D_{t,z}\mathit{\Gamma}(t)[\theta(t,z,\omega)+D_{t+,z}\theta(t,z,\omega)]\nu(dz)\,,\end{aligned} \tag{66}$$

where $\omega\in\Omega$, D_t, D_{t+}, $D_{t,z}$ and $D_{t+,z}$ denote Malliavin derivatives and,

$$\begin{aligned}\mathit{\Gamma}(t)&\triangleq\int_t^{\hat{\tau}}e^{-\rho(s+\lambda)}\tfrac{\partial}{\partial y}\ln[\xi y(\lambda)]d\lambda\\ &=\int_t^{\hat{\tau}}e^{-\rho(s+\lambda)}\tfrac{1}{y(\lambda)}d\lambda\,,\end{aligned} \tag{67}$$

Then we have the following proposition,

PROPOSITION 1. *Based on Definition* 1 *to Definition* 3, *and provided the above specifications, we have*:

(i) *If the representative agent has incomplete information, then the optimal savings rate is,*

$$\widehat{r_s(t)}=1-g_p-\frac{1}{\exp(\rho(s+t))\mu g_p{}^{2\alpha/1-\alpha}\mathbb{E}[(1+\tau_c)\widehat{k(t)}\widehat{\mathit{\Gamma}(t)}|\mathcal{H}_t]}\,, \tag{68}$$

where $\widehat{k(t)}\triangleq k^{(\widehat{r_s(t)})}(t)$ *and* $\widehat{\mathit{\Gamma}(t)}\triangleq\mathit{\Gamma}^{(\widehat{r_s(t)})}(t)$ *with* $\widehat{r_s(t)}$ $\mathfrak{H}-$*predictable.*

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[18] See Meyer-Brandis et al (2009).

(ii) *If the representative agent has complete information, then the optimal savings rate is,*

$$\widehat{r_s(t)} = 1 - g_p - \frac{1}{\exp(\rho(s+t))\mu g_p^{2\alpha/1-\alpha}\mathbb{E}[(1+\tau_c)\widehat{k(t)}\widehat{\Gamma(t)}|\mathcal{F}_t]}, \tag{69}$$

where $\widehat{k(t)} \triangleq k^{(\widehat{r_s(t)})}(t)$ *and* $\widehat{\Gamma(t)} \triangleq \Gamma^{(\widehat{r_s(t)})}(t)$ *with* $\widehat{r_s(t)}$ $\mathcal{F}-$*predictable.*

(iii) *If the representative agent has perfect information, then the optimal savings rate is,*

$$\widehat{r_s(t)} = 1 - g_p - \frac{1}{\exp(\rho(s+t))\mu g_p^{2\alpha/1-\alpha}\mathbb{E}[(1+\tau_c)\widehat{k(t)}\widehat{\Gamma(t)}|\mathcal{M}_t]}, \tag{70}$$

where $\widehat{k(t)} \triangleq k^{(\widehat{r_s(t)})}(t)$ *and* $\widehat{\Gamma(t)} \triangleq \Gamma^{(\widehat{r_s(t)})}(t)$ *with* $\widehat{r_s(t)}$ $\mathfrak{M}-$*predictable.*

Proof. See Appendix G. ▌

REMARK. Proposition 1 is a natural correspondence to Lemma 1. And we may easily tell the differences between Proposition 1 and Lemma 1, which reflects the fact that consideration of different information structures is not only necessary but also important.

Now, we are in the position to calculate the term $U^{\hat{\tau}}$ and the optimal stopping time $\hat{\tau}$ given in (65). That is, we are to solve the optimal stopping problem defined in (10) subject to the following SDE,

$$\begin{aligned} d^-\widehat{k(t)} &= \left\{\mu g_p^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\widehat{r_s(t)})]-n+\sigma^2+b_0\right\}\widehat{k(t)}dt \\ &\quad -\sigma\widehat{k(t)}d^-B(t) - \widehat{k(t)}\int_{\mathbb{R}_0}\tfrac{\gamma z}{1+\gamma z}\widetilde{N}(d^-t,dz), \end{aligned} \tag{71}$$

where $\widehat{r_s(t)}$ and $\widehat{k(t)}$ are given in Proposition 1, and b_0 is defined in (8’). It is easy to see that the construction of this problem is quite similar to that one in section 3.2, and rather, we have the following proposition,

PROPOSITION 2. *Conditional on the same assumptions and constructions as that of Lemma 2, if*

$$\mu g_p^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\widehat{r_s(0)})] < \rho+n-\sigma^2-b_0,$$

$$n-\tfrac{1}{2}\sigma^2-b_0-d$$

$$< \mu g_p^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\widehat{r_s(t)})]$$

$\leq \rho + n - \frac{1}{2}\sigma^2 - b_0 - d$, *a.e.*

And,

$$\left| \mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\widehat{r_s(t)})] - n + \tfrac{1}{2}\sigma^2 + b_0 + d \right| < \infty \text{, a.e.}$$

where b_0 *and* d *are defined in* (8') *and* (17), *respectively, and* $\widehat{r_s(t)}$ *is given in Proposition* 1. *Then we obtain the optimal* $\mathcal{F}_t$ *– stopping time* $\hat{\tau} \triangleq \inf\{t \geq 0; k(t) = \hat{k}\}$. *In other words,*

$$\hat{g}(s,k) = e^{-\rho s}\tfrac{1}{\beta}(\hat{k})^{-\beta}k^{\beta} = U^{\hat{\tau}},$$

which is a supermeanvalued majorant of $g(s,k)$ *with* $\hat{k}$ *given by* (22), *and* β *is a solution of,*

$$\widehat{h(\beta)} \triangleq -\rho + \beta\left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\widehat{r_s(0)})] - n + \sigma^2 + b_0\right\}$$
$$+\tfrac{1}{2}\sigma^2\beta(\beta-1) + \int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^{\beta} - 1 + \tfrac{\gamma z\beta}{1+\gamma z}]\nu(dz) = 0,$$

with $\widehat{r_s(0)}$ *determined by Proposition* 1.

Proof. See Appendix H. ▌

REMARK. Proposition 2 is a natural correspondence to Lemma 2. And a comparison of Proposition 2 and Lemma 2 shows that different information structures will intrinsically lead to different sustainable terminal path levels of capital stock per capita thanks to Proposition 1, where optimal savings rate strictly depends on the given level of information. Therefore, noting that the utility-optimal and sustainable terminal path level of capital stock per capita changed, thereby implying a different minimum time needed to "economic maturity" relative to Lemma 2.

5.3. Government

Firstly, similar to section 3.3, the balanced budget constraint defined in (24) can be expressed as follows,

$$\tau_k + \tau_c(1-g_p-\widehat{r_s(t)}) = g_p, \tag{25''}$$

And we specifically consider the following case, ASSUMPTION 1. *The goal of the government is to choose tax policies so as to maximize the welfare of the representative agent.*

Inserting (25'') into (71) gives,

$$d^- \widehat{k(t)} = \widehat{k(t)}\left[(\mu g_p{}^{\alpha/1-\alpha}\widehat{r_s(t)} - n + \sigma^2 + b_0)dt - \sigma d^- B(t) - \int_{\mathbb{R}_0} \tfrac{\gamma z}{1+\gamma z}\widetilde{N}(d^- t, dz)\right], \quad (72)$$

Thus applying Itô-Ventzell formula leads to,

$$d^- \widehat{y(t)} = g_p{}^{\alpha/1-\alpha}\widehat{k(t)}\left[(\mu g_p{}^{\alpha/1-\alpha}\widehat{r_s(t)} - n + \sigma^2 + b_0)dt - \sigma d^- B(t) - \int_{\mathbb{R}_0} \tfrac{\gamma z}{1+\gamma z}\widetilde{N}(d^- t, dz)\right]$$

$$\triangleq \hat{\varsigma}(t,\tau_c,\omega)dt + \hat{\varrho}(t,\omega)d^- B(t) + \int_{\mathbb{R}_0} \hat{\theta}(t,z,\omega)\widetilde{N}(d^- t, dz)\,, \quad (73)$$

Hence, the stochastic optimal control problem facing the government can be written as follows,

$$\max_{\substack{0<\tau_c<1 \\ 0<\tau_k<1}} \mathbb{E}\left[\int_0^{\hat{\tau}} e^{-\rho(s+t)} \ln\left((1-g_p - \widehat{r_s(t)})\widehat{y(t)}\right)dt + U^{\hat{\tau}}\right], \quad (74)$$

s.t.

$$d^- \widehat{y(t)} = \hat{\varsigma}(t,\tau_c,\omega)dt + \hat{\varrho}(t,\omega)d^- B(t) + \int_{\mathbb{R}_0} \hat{\theta}(t,z,\omega)\widetilde{N}(d^- t, dz)\,,$$

where $U^{\hat{\tau}}$ and $\hat{\tau}$ are given in Proposition 2, and $\widehat{r_s(t)}$ is given in Proposition 1. Accordingly, the corresponding Hamiltonian[19] amounts to,

$$\widehat{H}(t,\hat{y},\tau_c,\omega) = \exp(-\rho(s+t))\ln\left[(1-g_p - \widehat{r_s(t)})\widehat{y(t)}\right] + \widehat{\Gamma(t)}\big[\hat{\varsigma}(t,\tau_c,\omega) + D_{t+}\hat{\varrho}(t,\omega)$$

$$+\int_{\mathbb{R}_0} D_{t+,z}\hat{\theta}(t,z,\omega)\nu(dz)\Big] + \hat{\varrho}(t,\omega)D_t\widehat{\Gamma(t)}$$

$$+\int_{\mathbb{R}_0} D_{t,z}\widehat{\Gamma(t)}[\hat{\theta}(t,z,\omega) + D_{t+,z}\hat{\theta}(t,z,\omega)]\nu(dz)\,, \quad (75)$$

where $\omega \in \Omega$, D_t, D_{t+}, $D_{t,z}$ and $D_{t+,z}$ denote Malliavin derivatives and,

$$\widehat{\Gamma(t)} \triangleq \int_t^{\hat{\tau}} e^{-\rho(s+\lambda)} \tfrac{\partial}{\partial \hat{y}} \ln\left[(1-g_p - \widehat{r_s(\lambda)})\widehat{y(\lambda)}\right]d\lambda$$

$$= \int_t^{\hat{\tau}} e^{-\rho(s+\lambda)} \tfrac{1}{\widehat{y(\lambda)}}d\lambda\,, \quad (76)$$

where $\widehat{y(t)}$ is determined by SDE in (73). Therefore, the following proposition is derived,

[19] See Meyer-Brandis et al (2009).

PROPOSITION 3. *Based upon Assumption* 1 *and the above specifications, we establish,*

(i) *If the information is symmetric between the representative agent and the government, then the optimal consumption tax rate is zero and the optimal capital income tax rate is equal to* g_p.

(ii) *If the information is asymmetric between the representative agent and the government, and particularly, the representative agent gets more information than the government, then we obtain that the optimal consumption tax rate is zero and the optimal capital income tax rate is equal to* g_p.

(iii) *If the information is asymmetric between the representative agent and the government, and particularly, the government has more information than the representative agent, then,*

(iii-a) *If the government has perfect information while the representative agent has complete information, then we have,*

$$\mathbb{E}\left[(1+\tau_c^*)\widehat{k(t)}^*\widehat{\Gamma(t)}^*\middle|\mathcal{F}_t\right]=\mathbb{E}\left[\widehat{k(t)}^*\widehat{\Gamma(t)}^*\middle|\mathcal{M}_t\right], \tag{77}$$

where τ_c^* *denotes the optimal consumption tax rate and*

$$\widehat{k(t)}^* \triangleq \widehat{k(t)}^{(\tau_c^*)}, \widehat{\Gamma(t)}^* \triangleq \widehat{\Gamma(t)}^{(\tau_c^*)},$$

with τ_c^* $\mathfrak{M}-$*predictable.*

(iii-b) *If the government has perfect information while the representative agent has incomplete information, then we get,*

$$\mathbb{E}\left[(1+\tau_c^*)\widehat{k(t)}^*\widehat{\Gamma(t)}^*\middle|\mathcal{H}_t\right]=\mathbb{E}\left[\widehat{k(t)}^*\widehat{\Gamma(t)}^*\middle|\mathcal{M}_t\right], \tag{78}$$

where $\widehat{k(t)}^* \triangleq \widehat{k(t)}^{(\tau_c^*)}$ *and* $\widehat{\Gamma(t)}^* \triangleq \widehat{\Gamma(t)}^{(\tau_c^*)}$ *with* τ_c^* $\mathfrak{M}-$*predictable.*

(iii-c) *If the government has complete information while the representative agent has incomplete information, then we get,*

$$\mathbb{E}\left[(1+\tau_c^*)\widehat{k(t)}^*\widehat{\Gamma(t)}^*\middle|\mathcal{H}_t\right]=\mathbb{E}\left[\widehat{k(t)}^*\widehat{\Gamma(t)}^*\middle|\mathcal{F}_t\right], \tag{79}$$

where $\widehat{k(t)}^* \triangleq \widehat{k(t)}^{(\tau_c^*)}$ *and* $\widehat{\Gamma(t)}^* \triangleq \widehat{\Gamma(t)}^{(\tau_c^*)}$ *with* τ_c^* $\mathcal{F}-$*predictable.*

Proof. See Appendix I. ▌

REMARK. Proposition 3 is a natural correspondence to Lemma 3. And it is worth emphasizing that Proposition 3 itself is very interesting and important especially for the case where the government gets more information than the representative agent. Specifically, for the current endogenous growth economy, if the information is symmetric between the government and the representative agent or the representative agent gets more information than the government, then the optimal capital income tax rate is always equal to the exogenously given constant g_p. However, optimal tax rate on capital income may be *zero* when the government gets more information than the representative agent.

Thus, combining Proposition 3 with Proposition 1 gives the following corollary,

COROLLARY 2. (i) *For this case of symmetric information or the representative agent has more information, we have*:

(i-a) *If the representative agent has incomplete information, then the optimal savings rate is,*

$$\widehat{r_s(t)}^* = 1 - g_p - \frac{1}{\exp(\rho(s+t))\mu g_p{}^{2\alpha/1-\alpha}\mathbb{E}[\widehat{k(t)}^*\widehat{\Gamma(t)}^*|\mathcal{H}_t]}, \tag{80}$$

(i-b) *If the representative agent has complete information, then the optimal savings rate is,*

$$\widehat{r_s(t)}^* = 1 - g_p - \frac{1}{\exp(\rho(s+t))\mu g_p{}^{2\alpha/1-\alpha}\mathbb{E}[\widehat{k(t)}^*\widehat{\Gamma(t)}^*|\mathcal{F}_t]}, \tag{81}$$

(i-c) *If the representative agent has perfect information, then the optimal savings rate is,*

$$\widehat{r_s(t)}^* = 1 - g_p - \frac{1}{\exp(\rho(s+t))\mu g_p{}^{2\alpha/1-\alpha}\mathbb{E}[\widehat{k(t)}^*\widehat{\Gamma(t)}^*|\mathcal{M}_t]}, \tag{82}$$

(ii) *For the case of asymmetric information and particularly the government has more information*:

(ii-a) *If the government has perfect information while the representative agent has complete information, then the optimal savings rate is,*

$$\widehat{r_s(t)}^* = 1 - g_p - \frac{1}{\exp(\rho(s+t))\mu g_p{}^{2\alpha/1-\alpha}\mathbb{E}[\widehat{k(t)}^*\widehat{\Gamma(t)}^*|\mathcal{M}_t]}, \tag{83}$$

(ii-b) *If the government has perfect information while the representative agent* *has incomplete information, then the optimal savings rate is,*

$$\widehat{r_s(t)}^* = 1 - g_p - \frac{1}{\exp(\rho(s+t))\mu g_p{}^{2\alpha/1-\alpha}\mathbb{E}[\widehat{k(t)}^* \widehat{\Gamma(t)}^* | \mathcal{M}_t]}, \tag{84}$$

(ii-c) *If the government has complete information while the representative agent has incomplete information, then the optimal savings rate is,*

$$\widehat{r_s(t)}^* = 1 - g_p - \frac{1}{\exp(\rho(s+t))\mu g_p{}^{2\alpha/1-\alpha}\mathbb{E}[\widehat{k(t)}^* \widehat{\Gamma(t)}^* | \mathcal{F}_t]}, \tag{85}$$

Now, by Proposition 2, we have,

$$U^{\hat{\tau}} = e^{-\rho s}\tfrac{1}{\beta}(\hat{k})^{-\beta} k^{\beta},$$

Thus,

$$U^{\hat{\tau}} = e^{-\rho s}\tfrac{1}{\beta}(\hat{k})^{-\beta} k^{\beta} = e^{-\rho(s+\hat{\tau})}\ln(g_p{}^{\alpha/1-\alpha}\hat{k})$$

$$\Leftrightarrow \hat{\tau} = \rho^{-1}\ln[\beta(\tfrac{\hat{k}}{k})^{\beta}\ln(g_p{}^{\alpha/1-\alpha}\hat{k})], \tag{86}$$

Therefore, we conclude the following theorem,

THEOREM 4. *Based on the above propositions and Corollary* 2, *we have,*

(i) *The corresponding optimal stopping time is given by* (86), *where* $k = k(0) > 0$, $\hat{k}$ *is given in* (22) *and* β *is a solution of,*

$$\widehat{h(\beta)}^* \triangleq -\rho + \beta\left(\mu g_p{}^{\alpha/1-\alpha}\widehat{r_s(0)}^* - n + \sigma^2 + b_0\right) + \tfrac{1}{2}\sigma^2\beta(\beta-1) + \int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^{\beta} - 1 + \tfrac{\gamma z\beta}{1+\gamma z}]\nu(dz) = 0, \tag{87}$$

where $\widehat{r_s(0)}^*$ *is determined by* (80).

(ii) *The optimal stopping time is given by* (86), *where* $k = k(0) > 0$, $\hat{k}$ *is given in* (22), *and* β *is a solution of* (87) *with* $\widehat{r_s(0)}^*$ *determined by* (81).

(iii) *The optimal stopping time is given by* (86), *where* $k = k(0) > 0$, $\hat{k}$ *is given in* (22), *and* β *is a solution of* (87) *with* $\widehat{r_s(0)}^*$ *determined by* (82).

(iv) *The optimal stopping time is given by* (86), *where* $k = k(0) > 0$, $\hat{k}$ *is given in* (22), *and* β *is a solution of,*

$$\widehat{h(\beta)}^* \triangleq -\rho + \beta\left\{\mu g_p{}^{\alpha/1-\alpha}\left[(1-\tau_k^*) - (1+\tau_c^*)(1 - g_p - \widehat{r_s(0)}^*)\right] - n + \sigma^2 + b_0\right\}$$

$$+\tfrac{1}{2}\sigma^2\beta(\beta-1)+\int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^{\beta}-1+\tfrac{\gamma z\beta}{1+\gamma z}]\nu(dz)=0\,, \tag{88}$$

where τ_c^* *is determined by* (77), τ_k^* *is determined by* (77) *and* (25''), *and* $\widehat{r_s(0)}^*$ *is determined by* (83).

(v) *The optimal stopping time is given by* (86), *where* $k=k(0)>0$, $\hat{k}$ *is given in* (22) , *and* β *is a solution of* (88), *where* τ_c^* *is determined by* (78), τ_k^* *is determined by* (78) *and* (25''), *and* $\widehat{r_s(0)}^*$ *is determined by* (84).

(vi) *The optimal stopping time is given by* (86), *where* $k=k(0)>0$, $\hat{k}$ *is given in* (22) , *and* β *is a solution of* (88), *where* τ_c^* *is determined by* (79), τ_k^* *is determined by* (79) *and* (25''), *and* $\widehat{r_s(0)}^*$ *is determined by* (85).

REMARK. This theorem shows that different information structures lead to different endogenous times directly on the one hand and indirectly by leading to different utility-optimal and sustainable terminal path levels of capital stock per capita on the other hand. That is to say, information constraint is of crucial importance in determining the minimum time needed to "economic maturity". The economic implication is that certain level of information would make the economy reach its "maturity" *faster* than other levels of information, and also certain kind of *information structure* would make the economy reach its "economic maturity" much faster than other kinds of information structure. Accordingly, Theorem 4 implies that the issue of information constraint consists of at least two parts: one is that the absolute quantity of information is nontrivial and the other is that the distributive functions of information among the agents are also of great importance from the viewpoint of economic development. All in all, this theorem provides us with an efficient mechanism to build a close linkage between the micro-information-structure and the macro-economic-development.

6. LOCAL SENSITIVITY ANALYSES

In this section, we will make local sensitivity analyses of optimal consumption strategy of the representative agent with respect to the initial level of capital stock per capita. And, in particular, we will take preference manifold one discussed in section 3 for example. In order to make local sensitivity analyses, some preparations should be firstly supplied. And, specifically, the following theorem and corresponding corollary are employed to prove our results.

For any given Itô-Lévy process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\begin{cases} dQ(t) = \vartheta(t)dt + \mathcal{E}(t)dB(t) + \int_{\mathbb{R}_0} \psi(t,z)\widetilde{N}(dt,dz), \quad t \in [0,T] \\ Q(0) = q \in \mathbb{R} \end{cases}$$

Thus, the following theorem is established,

THEOREM 5. (Representation Theorem for Functions of Jump Diffusions)[20]:

Let $\Phi : \mathbb{R} \to \mathbb{R}$ *be a function with Fourier transform,*

$$\widehat{\Phi(\lambda)} = \tfrac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda x}\Phi(x)dx, \quad \lambda \in \mathbb{R}$$

satisfying the Fourier inversion property,

$$\Phi(\kappa) = \int_{\mathbb{R}} e^{i\lambda\kappa}\widehat{\Phi(\lambda)}d\lambda, \quad \kappa \in \mathbb{R}$$

Then,

$$\Phi\left(Q^q(t)\right) = \int_{\mathbb{R}} \widehat{\Phi(\lambda)} \exp^{\diamond}\{\Lambda_\lambda^q(t)\}d\lambda, \quad t \in [0,T]$$

where,

$$\Lambda_\lambda^q(t) \triangleq i\lambda q + \int_0^t i\lambda\mathcal{E}(s)dB(s) + \int_0^t \int_{\mathbb{R}_0} [e^{i\lambda\psi(s,z)} - 1]\widetilde{N}(ds,dz)$$

$$+ \int_0^t \left\{ i\lambda\vartheta(s) - \tfrac{1}{2}\lambda^2\mathcal{E}^2(s) + \int_{\mathbb{R}_0} [e^{i\lambda\psi(s,z)} - 1 - i\lambda\psi(s,z)]\nu(dz) \right\} ds, t \in [0,T].$$

Moreover, we have,

COROLLARY 3[21]. *Let* Φ *be a real function as in Theorem* 5*, then we have,*

$$\mathbb{E}\left[\Phi\left(Q^q(t)\right)\right] = \int_{\mathbb{R}} \widehat{\Phi(\lambda)} \exp(i\lambda q + \mathbf{\Phi}_\lambda(t))d\lambda,$$

where

$$\mathbf{\Phi}_\lambda(t) \triangleq \int_0^t \left\{ i\lambda\vartheta(s) - \tfrac{1}{2}\lambda^2\mathcal{E}^2(s) + \int_{\mathbb{R}_0} [e^{i\lambda\psi(s,z)} - 1 - i\lambda\psi(s,z)]\nu(dz) \right\} ds.$$

[20] See Theorem 14.13 in pp. 259 of Di Nunno et al (2009).
[21] See Corollary 14.14 in pp. 259 of Di Nunno et al (2009).

Now, we begin our local sensitivity analyses. Firstly, inserting (29) into (26) produces,

$$\begin{cases} dk(t) = k(t)\left\{[\mu g_p{}^{\alpha/1-\alpha}(1-g_p) - \rho - n + \sigma^2 + b]dt - \sigma dB(t) - \int_{\mathbb{R}_0} \frac{\gamma z}{1+\gamma z}\widetilde{N}(dt,dz)\right\} \\ k(0) = k > 0, \quad t \in [0, \hat{\tau}] \end{cases} \tag{89}$$

And by (29), the optimal consumption strategy is given by,

$$c(t) = \tfrac{\rho}{\mu} k(t)\,, \ \ t \in [0, \hat{\tau}]. \tag{90}$$

where $k(t)$ is determined by (89). Noting that μ, σ and γz are deterministic and suppose that $\gamma z > -1 + \varepsilon$ for a.a. z, for some $\varepsilon > 0$, and,

$$\int_0^{\hat{\tau}} \left[\left|\mu g_p{}^{\alpha/1-\alpha}(1-g_p) - \rho - n + \sigma^2 + b\right| + \sigma^2 + \int_{\mathbb{R}_0} (\tfrac{\gamma z}{1+\gamma z})^2 \nu(dz)\right] dt < \infty\,, \tag{91}$$

By the Itô formula for Lévy processes, the solution of (89) is given as follows,

$$\begin{aligned} k^k(t) &= k\exp\Big\{[\mu g_p{}^{\alpha/1-\alpha}(1-g_p) - \rho - n + \tfrac{1}{2}\sigma^2 + b + d]t \\ &\qquad -\sigma B(t) + \int_0^t \int_{\mathbb{R}_0} \ln\tfrac{1}{1+\gamma z}\widetilde{N}(ds,dz)\Big\} \\ &\triangleq \exp\big(Q^q(t)\big), \end{aligned} \tag{92}$$

where b and d are defined in (8) and (17), respectively, and,

$$dQ^q(t) = \vartheta(t)dt + \mathcal{E}(t)dB(t) + \int_{\mathbb{R}_0} \psi(t,z)\widetilde{N}(dt,dz)\,, \tag{93}$$

with

$$q \triangleq \ln k\,, \tag{94}$$

$$\vartheta(t) \triangleq \mu g_p{}^{\alpha/1-\alpha}(1-g_p) - \rho - n + \tfrac{1}{2}\sigma^2 + b + d\,, \tag{95}$$

$$\mathcal{E}(t) \triangleq -\sigma\,, \tag{96}$$

$$\psi(t,z) \triangleq \ln\tfrac{1}{1+\gamma z}\,, \tag{97}$$

If $h:\mathbb{R}\to\mathbb{R}$, then by (90),

$$\mathbb{E}\big[h(c(t))\big] = \mathbb{E}\Big[h\big(\tfrac{\rho}{\mu}k^k(t)\big)\Big] = \mathbb{E}\Big[h\big(\tfrac{\rho}{\mu}\exp\big(Q^q(t)\big)\big)\Big] = \mathbb{E}\Big[\Phi\big(Q^q(t)\big)\Big], \tag{98}$$

where

$$\Phi(\kappa) \triangleq h\left(\tfrac{\rho}{\mu}\exp(\kappa)\right), \text{ for } \kappa \in \mathbb{R} \tag{99}$$

If Φ satisfies the conditions of Theorem 5, then by Corollary 3,

$$\begin{aligned}\tfrac{d}{dk}\mathbb{E}\left[h(c(t))\right] &= \tfrac{d}{dk}\mathbb{E}\left[\Phi\left(Q^{\ln k}(t)\right)\right] \\ &= \tfrac{d}{dk}\int_{\mathbb{R}} \widehat{\Phi(\lambda)}\exp(i\lambda\ln k + \mathbf{\Phi}_\lambda(t))d\lambda \\ &= \int_{\mathbb{R}} \widehat{\Phi(\lambda)}\tfrac{i\lambda}{k}\exp(i\lambda\ln k + \mathbf{\Phi}_\lambda(t))d\lambda, \end{aligned} \tag{100}$$

where,

$$\mathbf{\Phi}_\lambda(t) \triangleq \int_0^t \left\{ i\lambda[\mu g_p{}^{\alpha/1-\alpha}(1-g_p) - \rho - n + \tfrac{1}{2}\sigma^2 + b + d] - \tfrac{1}{2}\lambda^2\sigma^2 + \int_{\mathbb{R}_0}[e^{i\lambda\ln\frac{1}{1+\gamma z}} - 1 - i\lambda\ln\tfrac{1}{1+\gamma z}]\nu(dz)\right\}ds \tag{101}$$

Noting that we focus on *local* sensitivity analyses, and without loss of any generality, we put,

$$h(\kappa) \triangleq \mathbf{1}_{[\frac{\rho}{\mu}\underline{M},\frac{\rho}{\mu}\overline{M}]}(\kappa), \text{ with } 0 < \underline{M} < \overline{M} < \infty. \tag{102}$$

which combines with (99) produces,

$$\Phi(\kappa) \triangleq \mathbf{1}_{[\frac{\rho}{\mu}\underline{M},\frac{\rho}{\mu}\overline{M}]}\left(\tfrac{\rho}{\mu}\exp(\kappa)\right), \quad \kappa \in \mathbb{R} \tag{103}$$

and,

$$2\pi\widehat{\Phi(\lambda)} = \int_{\mathbb{R}} e^{-i\lambda\kappa}\Phi(\kappa)d\kappa = \int_{\mathbb{R}} e^{-i\lambda\kappa}\mathbf{1}_{[\frac{\rho}{\mu}\underline{M},\frac{\rho}{\mu}\overline{M}]}\left(\tfrac{\rho}{\mu}\exp(\kappa)\right)d\kappa = \tfrac{1}{i\lambda}\left(\underline{M}^{-i\lambda} - \overline{M}^{-i\lambda}\right), \tag{104}$$

Substituting (104) into (100) leads to,

$$\Delta \triangleq \tfrac{d}{dk}\mathbb{E}\left[\mathbf{1}_{[\frac{\rho}{\mu}\underline{M},\frac{\rho}{\mu}\overline{M}]}(c(t))\right] = \int_{\mathbb{R}} \tfrac{1}{2\pi k}\left(\underline{M}^{-i\lambda} - \overline{M}^{-i\lambda}\right)\exp(i\lambda\ln k + \mathbf{\Phi}_\lambda(t))d\lambda, \tag{105}$$

where $\mathbf{\Phi}_\lambda(t)$ is defined in (101). Moreover, by Di Nunno et al (2009)[22] we see that, if for some $\delta > 0$,

$$\lambda^2\int_0^{\hat{\tau}}\left[\sigma^2 + \int_{\mathbb{R}_0}\left(1-\cos(\lambda\ln\tfrac{1}{1+\gamma z})\right)\nu(dz)\right]ds \geq \delta\lambda^2, \tag{106}$$

Then the integral in (105) converges. Therefore, the following theorem has been established,

THEOREM 6. *Based on preference manifold one introduced in section* 3, *if* $\gamma z > -1+\varepsilon$ *for a.a.* z, *for some* $\varepsilon > 0$, *and* (91), (106) *are fulfilled, and also,*

[22] See pp. 262.

$$\int_{\mathbb{R}} \left| \widehat{\Phi(\lambda)} \lambda \exp(\operatorname{Re} \mathbf{\Phi}_\lambda(t)) \right| d\lambda < \infty ,$$

Then we get,

$$\Delta \triangleq \tfrac{d}{dk} \mathbb{E}\left[\mathbf{1}_{[\frac{\rho}{\mu}\underline{M}, \frac{\rho}{\mu}\overline{M}]} \left(c(t) \right) \right] = \int_{\mathbb{R}} \tfrac{1}{2\pi k} \left(\underline{M}^{-i\lambda} - \overline{M}^{-i\lambda} \right) \exp(i\lambda \ln k + \mathbf{\Phi}_\lambda(t)) d\lambda ,$$

with $\mathbf{\Phi}_\lambda(t)$ *given in* (101).

REMARK. It is well-known that sensitivity analyses have been widely applied in literatures of finance. Theorem 6 shows that this kind of analysis method can be naturally brought into macroeconomic analyses. And so by Theorem 6, we can tell the extent of the dependence of optimal consumption strategy on the initial conditions of the corresponding economical system. That is to say, we can show how much would the optimal consumption change for a given scale change of the initial conditions of the economical system. Most importantly, local sensitivity analyses can be applied to different preference manifolds corresponding to different endogenous times. Therefore, we have been supplied an appropriate variable instrument to tell the differences between different preference manifolds and hence different minimum times to "economic maturity". Finally, we need to argue that different economical systems may share the same level of Δ, while different levels of Δ absolutely correspond to different economical systems.

7. CONCLUDING REMARKS

The major goal of the current paper is to determine the minimum time needed to reach "economic maturity" for an underdeveloped economy in the background of stochastic endogenous growth. And the major novelties can be summarized as follows: first, the minimum time to "economic maturity" and the sustainable and utility-optimal terminal path level of capital stock per capita are simultaneously and endogenously determined; second, the endogenous time can be *explicitly computed* in some conditions, specifically for the criterion or the preference of *the modified Radner fashion*, which will completely support comparative static analyses; third, two kinds of preference manifolds are simultaneously incorporated into our model and the resulting different endogenous times are comparatively studied for the first time, which, in other words, implies that there may exist a one-to-one correspondence between the preference manifold and the endogenous time; forth, the effects of the endogenous time with respect to optimal fiscal policies and different information structures are thoroughly explored for the first time to the best of our knowledge in the background of general equilibrium framework; and fifth, local sensitivity analyses[23] of the optimal consumption strategy with respect to initial level of capital stock per capita are incorporated into the unbalanced macroeconomic models.

Finally, it would be clear that the methodology introduced here can be easily employed to compute the optimal stopping times in finance. Noting that a considerable number of literatures (see, Myneni, 1992; Shepp and Shiryaev, 1993; Hobson, 1998; Guo and Shepp, 2001; Avram et al, 2004; Choi et al 2004; Alili and Kyprianou, 2005) have been devoted to the issue of optimal stopping problems in finance, the advantage of the current method is that it will support the explicit computation[24] of the corresponding optimal stopping times in certain conditions and therefore to further analyze the influences of other parameters, e.g., those reflect different financial institutions and different preferences of information structure, on the optimal stopping times.

APPENDIX

A. Proof of Lemma 1

Applying the maximization operator in (11) yields,

$$-\exp(-\rho(s+t))\tfrac{1}{1-g_p-r_s}+V_k(t,k(t))k(t)\mu g_p^{\alpha/1-\alpha}(1+\tau_c)=0$$

$$\Leftrightarrow 1-g_p-r_s=\frac{1}{\exp(\rho(s+t))V_k(t,k(t))k(t)\mu g_p^{\alpha/1-\alpha}(1+\tau_c)}, \tag{A.1}$$

Substituting (A.1) into (11) gives,

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[23] One can easily tell the differences between the method used here and those in empirical literatures, see, Kydland and Prescott, 1982; Levine and Renelt, 1992; Canova, 1995; and Fernández and Rogerson, 1998.

[24] That is to say, a simple formula of the optimal stopping time can be derived in certain conditions.

$$-V_t(t,k(t))-\tfrac{1}{2}\sigma^2k^2(t)V_{kk}(t,k(t))$$

$$-\int_{\mathbb{R}_0}\Big[V\Big(t,k(t)-\tfrac{\gamma z}{1+\gamma z}k(t)\Big)-V(t,k(t))+\tfrac{\gamma z}{1+\gamma z}k(t)V_k(t,k(t))\Big]\nu(dz)$$

$$=-\exp(-\rho(s+t))\Big[\ln\Big(e^{\rho(s+t)}V_k(t,k(t))\mu(1+\tau_c)\Big)+1\Big]$$

$$+V_k(t,k(t))k(t)[\mu g_p{}^{\alpha/1-\alpha}(1-\tau_k)-n+\sigma^2+b], \tag{A.2}$$

Naturally, one can try,

$$V(t,k(t))=\exp(-\rho(s+t))[C_1+C_2\ln k(t)], \tag{A.3}$$

for some constants C_1, C_2 to be determined. Hence, by (A.3),

$$V_t(t,k(t))=-\rho\exp(-\rho(s+t))[C_1+C_2\ln k(t)], \tag{A.4}$$

$$V_k(t,k(t))=C_2\exp(-\rho(s+t))k^{-1}(t), \tag{A.5}$$

$$V_{kk}(t,k(t))=-C_2\exp(-\rho(s+t))k^{-2}(t), \tag{A.6}$$

Inserting (A.3)-(A.6) into (A.2) produces,

$$\rho C_1+\rho C_2\ln k(t)+\tfrac{1}{2}\sigma^2C_2-C_2\int_{\mathbb{R}_0}\Big(\ln\tfrac{1}{1+\gamma z}+\tfrac{\gamma z}{1+\gamma z}\Big)\nu(dz)$$

$$=-\ln C_2+\ln k(t)-\ln[\mu(1+\tau_c)]+C_2[\mu g_p{}^{\alpha/1-\alpha}(1-\tau_k)-n+\sigma^2+b]-1, \tag{A.7}$$

which implies that,

$$C_2=\rho^{-1}, \tag{A.8}$$

And combining (A.7) with (A.8) leads to,

$$C_1=\rho^{-1}\begin{Bmatrix}\ln\frac{\rho}{\mu(1+\tau_c)}+\rho^{-1}[\mu g_p{}^{\alpha/1-\alpha}(1-\tau_k)-n+\sigma^2+b]-1\\ -\frac{1}{2}\sigma^2\rho^{-1}+\rho^{-1}\int_{\mathbb{R}_0}\Big(\ln\frac{1}{1+\gamma z}+\frac{\gamma z}{1+\gamma z}\Big)\nu(dz)\end{Bmatrix}. \tag{A.9}$$

Thus, it follows from (A.1), (A.5) and (A.8) that,

$$\hat{r}_s=1-g_p-\tfrac{\rho}{\mu g_p{}^{\alpha/1-\alpha}(1+\tau_c)}, \tag{A.10}$$

And by (A.3), (A.8) and (12), we obtain,

$$V(\hat{\tau},k(\hat{\tau}))=\exp(-\rho(s+\hat{\tau}))[C_1+\rho^{-1}\ln k(\hat{\tau})]=U^{\hat{\tau}}. \tag{A.11}$$

where C_1 is given in (A.9). ▌

B. Proof of Lemma 2

It follows from the "Integro-variational inequalities for optimal stopping" (see, Theorem 2.2, pp. 29) of Øksendal and Sulem (2005), we are to prove,

(i) We need to prove that $\phi \geq g$ on D, i.e.,

$$Ck^{\beta} \geq \ln({g_p}^{\alpha/1-\alpha}k) \quad \text{for} \quad 0 < k < \hat{k} \tag{B.1}$$

Define $l(k) \triangleq Ck^{\beta} - \ln({g_p}^{\alpha/1-\alpha}k)$. By our chosen values of C and $\hat{k}$, we see that $l(\hat{k}) = l'(\hat{k}) = 0$. Moreover, noting that $l''(k) = C\beta(\beta-1)k^{\beta-2} + k^{-2}$. Thus, if we put $\beta > 1$, we get $l''(k) > 0$ for $0 < k < \hat{k}$, and also we have $l(k) > 0$ for all $0 < k < \hat{k}$. Notice by (14) that,

$$h(1) = -\rho + \mu {g_p}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)] - n + \sigma^2 + b < 0$$

$$\Leftrightarrow \mu {g_p}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)] < n + \rho - \sigma^2 - b, \tag{B.2}$$

Thus, (B.1) follows as long as (B.2) is fulfilled.

(ii) Outside D we have $\phi(s,k) = e^{-\rho s}\ln({g_p}^{\alpha/1-\alpha}k)$ and by (16),

$$\mathcal{A}g(s,k) = e^{-\rho s}\Big\{-\rho\ln({g_p}^{\alpha/1-\alpha}k) + \mu {g_p}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]$$

$$-n + \tfrac{1}{2}\sigma^2 + b + d\Big\} \leq 0 \qquad \text{for all } k \geq \hat{k}$$

$$\Leftrightarrow k \geq {g_p}^{-\alpha/1-\alpha}\exp\left\{\frac{\mu {g_p}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)] - n + \frac{1}{2}\sigma^2 + b + d}{\rho}\right\}, \quad \forall k \geq \hat{k}$$

$$\Leftrightarrow \hat{k} \geq {g_p}^{-\alpha/1-\alpha}\exp\left\{\frac{\mu {g_p}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)] - n + \frac{1}{2}\sigma^2 + b + d}{\rho}\right\}.$$

which holds by (20).

(iii) To check if $\hat{\tau} < \infty$ almost surely. It is easy to see that one can choose parameters such that the geometric Lévy diffusion process defined in (7'') satisfies the "At most linear growth" and "Lipschitz continuity" conditions, thereby implying a unique càdlàg (right continuous with left limits, i.e., RCLL processes) strong solution $k(t)$.

Then by (4), (8) and Itô formula, we obtain,

$$d\ln k(t) = \Big\{\mu {g_p}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)] - n + \tfrac{1}{2}\sigma^2 + b\Big\}dt$$

$$-\sigma dB(t)+\int_{\mathbb{R}}\Big(\ln\tfrac{1}{1+\gamma z}+\tfrac{\gamma z}{1+\gamma z}\Big)\nu(dz)dt+\int_{\mathbb{R}_0}\ln(\tfrac{1}{1+\gamma z})\widetilde{N}(dt,dz)$$

$$=\Big\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]-n+\tfrac{1}{2}\sigma^2\Big\}dt$$

$$-\sigma dB(t)+\Big[\int_{\mathbb{R}}\tfrac{\gamma z}{1+\gamma z}\nu(dz)+\int_{\mathbb{R}}\tfrac{(\gamma z)^2}{1+\gamma z}\nu(dz)\Big]dt+\int_{\mathbb{R}_0}\ln(\tfrac{1}{1+\gamma z})N(dt,dz)$$

$$=\Big\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]-n+\tfrac{1}{2}\sigma^2+\int_{\mathbb{R}}\gamma z\nu(dz)\Big\}dt$$

$$-\sigma dB(t)+\int_{\mathbb{R}_0}\ln(\tfrac{1}{1+\gamma z})N(dt,dz).$$

Hence, we get,

$$k(t)=k\exp\Big(\Big\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]-n+\tfrac{1}{2}\sigma^2$$

$$+\int_{\mathbb{R}}\gamma z\nu(dz)\Big\}t-\sigma B(t)+\int_0^t\int_{\mathbb{R}_0}\ln(\tfrac{1}{1+\gamma z})N(ds,dz)\Big) \tag{B.3}$$

We see that if,

$$\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]>n-\tfrac{1}{2}\sigma^2-\int_{\mathbb{R}}\gamma z\nu(dz) \tag{B.4}$$

$$\gamma z<0 \quad \text{a.s.}-\nu \tag{B.5}$$

And,

$$\sigma<0. \tag{B.6}$$

by the law of the iterated logarithm of Brownian motion, then we have,

$$\lim_{t\to\infty}k(t)=\infty \quad \text{a.s.}$$

And particularly, $\hat{\tau}<\infty$ almost surely.

(iv) Noting from (22) that $\hat{k}<\infty$, thus $[0,\hat{k}]$ is compact set by Heine-Borel theorem. Accordingly, ϕ is bounded on $[0,\hat{k}]$ via applying the fact that $\phi\in C^2(\mathbb{R}^2)$ and the well-known Weierstrass theorem. So, it suffices to check that,

$\{e^{-\rho\tau}\ln[g_p{}^{\alpha/1-\alpha}k(\tau)]\}_{\tau\in\mathcal{T}}$ is uniformly integrable on $[\hat{k},\infty)$.

where $\mathcal{T}$ denotes the set of admissible stopping time and the uniform topology is naturally induced by the norm, which is induced by inner product, of Hilbert space $L^2(\Omega,\mathcal{F},\mathbb{P})$. For this to hold, it suffices to show that there exists a constant $M<\infty$ such that

$$\mathbb{E}\{e^{-2\rho\tau}[\ln(g_p{}^{\alpha/1-\alpha}k(\tau))]^2\} \le M \quad \text{for all } \tau \in \mathcal{T} \text{ and } k(\tau) \ge \hat{k}\,. \tag{B.7}$$

Since

$$0 < \ln[g_p{}^{\alpha/1-\alpha}k(t)] < g_p{}^{\alpha/1-\alpha}k(t) \quad \text{on} \quad [\hat{k},\infty)\,.$$

Hence, by (4) and (B.3), we have,

$$\mathbb{E}\{e^{-2\rho\tau}[\ln(g_p{}^{\alpha/1-\alpha}k(\tau))]^2\}$$

$$\le g_p{}^{2\alpha/1-\alpha}\mathbb{E}[e^{-2\rho\tau}k(\tau)^2]$$

$$= g_p{}^{2\alpha/1-\alpha}k^2\mathbb{E}\Big[\exp\Big(\Big\{2\int_{\mathbb{R}}\gamma z\nu(dz) + 2\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]$$

$$-2n+3\sigma^2-2\rho\Big\}\tau + 2\int_0^\tau\int_{\mathbb{R}_0}\ln(\tfrac{1}{1+\gamma z})N(ds,dz)\Big)\Big]$$

$$= g_p{}^{2\alpha/1-\alpha}k^2\mathbb{E}\Big[\exp\Big(\Big\{2\int_{\mathbb{R}}\gamma z\nu(dz) + 2\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]$$

$$-2n+3\sigma^2-2\rho+2\int_{\mathbb{R}}\ln(\tfrac{1}{1+\gamma z})\nu(dz)\Big\}\tau + 2\int_0^\tau\int_{\mathbb{R}_0}\ln(\tfrac{1}{1+\gamma z})\widetilde{N}(ds,dz)\Big)\Big] \tag{B.8}$$

$$= g_p{}^{2\alpha/1-\alpha}k^2\mathbb{E}\Big[\exp\Big(\Big\{2\int_{\mathbb{R}}\gamma z\nu(dz) + 2\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]$$

$$-2n+3\sigma^2-2\rho+2\int_{\mathbb{R}}\ln(\tfrac{1}{1+\gamma z})\nu(dz)\Big\}\tau + \int_0^\tau\int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^2-1-2\ln(\tfrac{1}{1+\gamma z})]\nu(dz)ds\Big)\Big] \tag{B.9}$$

$$= g_p{}^{2\alpha/1-\alpha}k^2\mathbb{E}\Big[\exp\Big(\Big\{2\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]-2n+3\sigma^2-2\rho$$

$$+\int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^2-1]\nu(dz)+2\int_{\mathbb{R}}\gamma z\nu(dz)\Big\}\tau\Big)\Big].$$

We conclude that if,

$$2\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]$$

$$\le 2n+2\rho-3\sigma^2-\int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^2-1]\nu(dz)-2\int_{\mathbb{R}}\gamma z\nu(dz)\,, \tag{B.10}$$

Then (B.7) holds and so does (iv). Specifically, from (B.8) to (B.9), we have used the following fact. For the following equation,

$$dX(t) = X(t^-)\int_{\mathbb{R}_0}(e^{\psi(t,z)}-1)\widetilde{N}(dt,dz)\,,\quad X(0)=1\,. \tag{B.11}$$

which has the solution,

$$X(t) = \exp\left\{\int_0^t\int_{\mathbb{R}_0}\psi(s,z)N(ds,dz) - \int_0^t\int_{\mathbb{R}_0}(e^{\psi(s,z)}-1)\nu(dz)ds\right\}$$

$$= \exp\left\{\int_0^t\int_{\mathbb{R}_0}\psi(s,z)\widetilde{N}(ds,dz) - \int_0^t\int_{\mathbb{R}_0}[e^{\psi(s,z)}-1-\psi(s,z)]\nu(dz)ds\right\} \tag{B.12}$$

Suppose

$$\int_0^t\int_{\mathbb{R}_0}(e^{\psi(s,z)}-1)^2\nu(dz)ds < \infty\,,$$

Then by (B.11) we see that $\mathbb{E}[X(t)]=1$ and hence by (B.12) we obtain,

$$\mathbb{E}\left[\exp\left(\int_0^t\int_{\mathbb{R}_0}\psi(s,z)\widetilde{N}(ds,dz)\right)\right] = \exp\left\{\int_0^t\int_{\mathbb{R}_0}[e^{\psi(s,z)}-1-\psi(s,z)]\nu(dz)ds\right\}$$

If we put $\psi(s,z)=\ln(\frac{1}{1+\gamma z})^2$, then (B.9) follows.

(v) We need to prove that,

$$\mathbb{E}^k\left[\left|\phi(k(\tau))\right| + \int_0^{\hat{\tau}}\left(\left|\hat{\mathcal{A}}\phi(k(t))\right| + \left|-\sigma k(t)\phi_k(k(t))\right|^2\right.\right.$$

$$\left.\left.+\int_{\mathbb{R}_0}\left|\phi(k(t)-\tfrac{\gamma z}{1+\gamma z}k(t))-\phi(k(t))\right|^2\nu(dz)\right)dt\right] < \infty \qquad \text{for } \forall\tau\in\mathcal{T} \tag{B.13}$$

where $\phi(k(t))=Ck(t)^\beta$ with C given in (23) and β satisfying $h(\beta)=0$ in (14). Noting that,

$$\hat{\mathcal{A}}\phi(k(t)) = \left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]-n+\sigma^2+b\right\}k(t)\tfrac{\partial\phi}{\partial k}$$

$$+\tfrac{1}{2}\sigma^2 k(t)^2\tfrac{\partial^2\phi}{\partial k^2} + \int_{\mathbb{R}_0}[\phi(k(t)-\tfrac{\gamma z}{1+\gamma z}k(t))-\phi(k(t))+\tfrac{\gamma z}{1+\gamma z}k(t)\tfrac{\partial\phi}{\partial k}]\nu(dz)$$

$$=[h(\beta)+\rho]\phi(k(t))$$

$$=\rho\phi(k(t))\,. \tag{B.14}$$

$$\left|-\sigma k(t)\phi_k(k(t))\right|^2 = (C\sigma\beta)^2 k(t)^{2\beta}\,, \tag{B.15}$$

And

$$\int_{\mathbb{R}_0}\left|\phi(k(t)-\tfrac{\gamma z}{1+\gamma z}k(t))-\phi(k(t))\right|^2\nu(dz)$$

$$=\int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^{\beta}-1]^2\nu(dz)[\phi(k(t))]^2, \tag{B.16}$$

Consequently, given,

$$\int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^{\beta}-1]^2\nu(dz)<\infty, \tag{B.17}$$

and via applying (iii), (B.13) follows as long as we show that $\mathbb{E}^k[k(t)^{2\beta}]<\infty$ almost everywhere on $[0,\hat{\tau}]$. In particular, here we have $\beta>1$ by (B.2). Obviously, our following proof is similar to that of (iv). By (4) and (B.3), we have,

$$\mathbb{E}^k[k(t)^{2\beta}]$$

$$=k^{2\beta}\mathbb{E}^k\Big[\exp\Big(\Big\{2\beta\int_{\mathbb{R}}\gamma z\nu(dz)+2\beta\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]$$

$$-2\beta n+(\beta+2\beta^2)\sigma^2\Big\}t+2\beta\int_0^t\int_{\mathbb{R}_0}\ln(\tfrac{1}{1+\gamma z})N(ds,dz)\Big)\Big]$$

$$=k^{2\beta}\mathbb{E}^k\Big[\exp\Big(\Big\{2\beta\int_{\mathbb{R}}\gamma z\nu(dz)+2\beta\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]$$

$$-2\beta n+(\beta+2\beta^2)\sigma^2+2\beta\int_{\mathbb{R}}\ln(\tfrac{1}{1+\gamma z})\nu(dz)\Big\}t+2\beta\int_0^t\int_{\mathbb{R}_0}\ln(\tfrac{1}{1+\gamma z})\widetilde{N}(ds,dz)\Big)\Big]$$

$$=k^{2\beta}\mathbb{E}^k\Big[\exp\Big(\Big\{2\beta\int_{\mathbb{R}}\gamma z\nu(dz)+2\beta\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]$$

$$-2\beta n+(\beta+2\beta^2)\sigma^2+2\beta\int_{\mathbb{R}}\ln(\tfrac{1}{1+\gamma z})\nu(dz)\Big\}t+\int_0^t\int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^{2\beta}-1-2\beta\ln(\tfrac{1}{1+\gamma z})]\nu(dz)ds\Big)\Big]$$

$$=k^{2\beta}\mathbb{E}^k\Big[\exp\Big(\Big\{2\beta\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]-2\beta n+(\beta+2\beta^2)\sigma^2$$

$$+\int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^{2\beta}-1]\nu(dz)+2\beta\int_{\mathbb{R}}\gamma z\nu(dz)\Big\}t\Big)\Big].$$

Consequently, we show that if,

$$\Big|2\beta\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]+2\beta\int_{\mathbb{R}}\gamma z\nu(dz)$$

$$-2\beta n+(\beta+2\beta^2)\sigma^2+\int_{\mathbb{R}_0}[(\tfrac{1}{1+\gamma z})^{2\beta}-1]\nu(dz)\Big|<\infty, \tag{B.18}$$

Then we get $\mathbb{E}^k[k(t)^{2\beta}] < \infty$ almost surely. ▌

C. Proof of Lemma 3

Performing the maximization in (27) produces,

$$-\exp(-\rho(s+t))\tfrac{1}{1-g_p-\hat{r}_s}\tfrac{\partial \hat{r}}{\partial \tau_c} + W_k(t,k(t))k(t)\mu g_p{}^{\alpha/1-\alpha}\tfrac{\partial \hat{r}}{\partial \tau_c} = 0, \quad \text{(C.1)}$$

$$-\exp(-\rho(s+t))\tfrac{1}{1-g_p-\hat{r}_s}\tfrac{\partial \hat{r}}{\partial \tau_k} + W_k(t,k(t))k(t)\mu g_p{}^{\alpha/1-\alpha}\tfrac{\partial \hat{r}}{\partial \tau_k} = 0. \quad \text{(C.2)}$$

Noting by Lemma 1 and (25) that $\tfrac{\partial \hat{r}}{\partial \tau_c} \neq 0$ and $\tfrac{\partial \hat{r}}{\partial \tau_k} \neq 0$, so (C.1) and (C.2) becomes,

$$1 - g_p - \hat{r}_s = \tfrac{1}{\exp(\rho(s+t))W_k(t,k(t))k(t)\mu g_p{}^{\alpha/1-\alpha}}, \quad \text{(C.3)}$$

Substituting (C.3) into (27) gives rise to,

$$\begin{aligned}
&-W_t(t,k(t)) - \tfrac{1}{2}\sigma^2 k^2(t) W_{kk}(t,k(t)) \\
&-\int_{\mathbb{R}_0}\Big[W\Big(t,k(t)-\tfrac{\gamma z}{1+\gamma z}k(t)\Big) - W(t,k(t)) + \tfrac{\gamma z}{1+\gamma z}k(t)W_k(t,k(t))\Big]\nu(dz) \\
&= -\exp(-\rho(s+t))\Big[\ln\Big(e^{\rho(s+t)}W_k(t,k(t))\mu\Big)+1\Big] \\
&\quad + W_k(t,k(t))k(t)[\mu g_p{}^{\alpha/1-\alpha}(1-g_p) - n + \sigma^2 + b],
\end{aligned} \quad \text{(C.4)}$$

If we choose $W(t,k(t))$ of the following form,

$$W(t,k(t)) = \exp(-\rho(s+t))[C_3 + C_4 \ln k(t)], \quad \text{(C.5)}$$

for some constants C_3, C_4 to be determined. Then,

$$W_t(t,k(t)) = -\rho\exp(-\rho(s+t))[C_3 + C_4 \ln k(t)], \quad \text{(C.6)}$$

$$W_k(t,k(t)) = C_4 \exp(-\rho(s+t))k^{-1}(t), \quad \text{(C.7)}$$

$$W_{kk}(t,k(t)) = -C_4 \exp(-\rho(s+t))k^{-2}(t), \quad \text{(C.8)}$$

Inserting (C.5)-(C.8) into (C.4) yields,

$$\begin{aligned}
&\rho C_3 + \rho C_4 \ln k(t) + \tfrac{1}{2}\sigma^2 C_4 - C_4\int_{\mathbb{R}_0}\Big(\ln\tfrac{1}{1+\gamma z} + \tfrac{\gamma z}{1+\gamma z}\Big)\nu(dz) \\
&= -\ln C_4 + \ln k(t) - \ln\mu + C_4[\mu g_p{}^{\alpha/1-\alpha}(1-g_p) - n + \sigma^2 + b] - 1,
\end{aligned} \quad \text{(C.9)}$$

which implies that,

$$C_4 = \rho^{-1}, \quad \text{(C.10)}$$

And hence,

$$C_3 = \rho^{-1} \begin{Bmatrix} \ln \frac{\rho}{\mu} + \rho^{-1}[\mu g_p{}^{\alpha/1-\alpha}(1-g_p) - n + \sigma^2 + b] - 1 \\ -\frac{1}{2}\sigma^2\rho^{-1} + \rho^{-1}\int_{\mathbb{R}_0} \left(\ln \frac{1}{1+\gamma z} + \frac{\gamma z}{1+\gamma z}\right)\nu(dz) \end{Bmatrix}. \tag{C.11}$$

Thus, by (C.7) and (C.10), (C.3) becomes,

$$1 - g_p - \hat{r}_s = \frac{\rho}{\mu g_p{}^{\alpha/1-\alpha}}, \tag{C.12}$$

which combining with Lemma 1 shows that,

$$\tau_c^* = 0. \tag{C.13}$$

Hence, by (25), we have,

$$\tau_k^* = g_p. \tag{C.14}$$

And by (C.5), (C.10), (28) and Lemma 2, we obtain,

$$\begin{aligned} W(\hat{\tau}, k(\hat{\tau})) &= \exp(-\rho(s+\hat{\tau}))[C_3 + \rho^{-1}\ln k(\hat{\tau})] \\ &= \exp(-\rho(s+\hat{\tau}))(C_3 + \rho^{-1}\ln \hat{k}) \\ &= U^{\hat{\tau}}. \end{aligned}$$

where C_3 is given in (C.11). ▌

D. Proof of Lemma 5

(i) We need to prove that $\phi \geq g$ on region D, i.e.,

$$Ck^r + \rho^{-1}\ln k + C_5 \geq \ln(g_p{}^{\alpha/1-\alpha}k) \quad \text{for } 0 < k < k^* \tag{D.1}$$

Define $\zeta(k) \triangleq Ck^r + \rho^{-1}\ln k + C_5 - \ln(g_p{}^{\alpha/1-\alpha}k)$. By our chosen values of C and k^*, we see that $\zeta(k^*) = \zeta'(k^*) = 0$. And,

$$\zeta''(k) = \underbrace{Cr(r-1)k^{r-2}}_{>0} + \underbrace{(1-\rho^{-1})k^{-2}}_{<0},$$

by (55) and (56). And using (55), we obtain,

$$\zeta''(k) > 0 \Leftrightarrow (1-r)k^r > (k^*)^r, \tag{D.2}$$

where $k = k(0) > 0$ and C_5 is defined in (48). Thus, as long as (D.2) holds, we have $\zeta''(k)>0$ for all $0<k<k^*$, and also we have $\zeta(k)>0$ for all $0<k<k^*$. Therefore, (D.1) follows as long as (D.2) is satisfied.

(ii) Outside of D, we have $\varphi(k)=\ln({g_p}^{\alpha/1-\alpha}k)$, and by (39),

$$\mathcal{A}_0\varphi(k)=-\rho\ln({g_p}^{\alpha/1-\alpha}k)$$

$$+\left\{\mu {g_p}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\tfrac{1}{2}\sigma^2+b+d\right\}$$

$$\forall k\geq k^*$$

$$\Leftrightarrow k\geq {g_p}^{-\alpha/1-\alpha}\exp\left\{\tfrac{\mu {g_p}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\frac{1}{2}\sigma^2+b+d}{\rho}\right\}, \qquad \forall k\geq k^*$$

$$\Leftrightarrow k^*\geq {g_p}^{-\alpha/1-\alpha}\exp\left\{\tfrac{\mu {g_p}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\frac{1}{2}\sigma^2+b+d}{\rho}\right\}. \tag{D.3}$$

Combining (42) with (D.3) shows that,

$${g_p}^{-\alpha/1-\alpha}(1-g_p-r_s^*)^{1/(\rho-1)}\exp\left(\tfrac{\mu {g_p}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\frac{1}{2}\sigma^2+b+d}{\rho-1}\right)$$

$$\geq {g_p}^{-\alpha/1-\alpha}\exp\left\{\tfrac{\mu {g_p}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\frac{1}{2}\sigma^2+b+d}{\rho}\right\}$$

$$\Leftrightarrow -\rho\ln(1-g_p-r_s^*)$$

$$\geq \mu {g_p}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s^*)]-n+\tfrac{1}{2}\sigma^2+b+d, \tag{D.4}$$

Thus, (D.3) follows as long as (D.4) holds.

It is easy to check that the remaining proof is quite similar to that of Lemma 2, so we take it omitted. ▌

E. Proof of Corollary 1

Firstly, we introduce a Lévy process $Z(t)$ and denote by $\Delta Z(s)$ the jump of $Z(t)$ at time s, i.e., $\Delta Z(s)\triangleq Z(s)-Z(s-)$. Then, combining with the SDE defined in (60) shows that the corresponding Lévy process $Z(t)$ has the following Lévy decomposition,

$$Z(t)=-\sigma B(t)+[\mu {g_p}^{\alpha/1-\alpha}(1-g_p)-\rho-n+\sigma^2+b]t$$

$$+\int_{\mathbb{R}_0}\tfrac{\gamma z}{1+\gamma z}\widetilde{N}(dt,dz)+\sum_{0<s\leq t}\Delta Z(s)\mathbf{1}_{\{|\Delta Z(s)|\geq 1\}}, \tag{E.1}$$

where $\mathbf{1}_{\{|\Delta Z(s)|\geq 1\}}$ denotes the indicator function of the set $\{\omega \in \Omega; |\Delta Z(s,\omega)| \geq 1\}$. Moreover, we define a sequence of stopping times as follows,

$$\tau_m \triangleq \inf\{t \geq 0; |k(t)| > m > 0\}. \tag{E.2}$$

Hence, it is easy to see that τ_m is increasing with respect to m, i.e., $\lim_{m\to\infty} \tau_m = \infty$ almost surely. And we put,

$$k^{\tau_m -}(t) \triangleq k(t)\mathbf{1}_{\{t<\tau_m\}} + k(\tau_m -)\mathbf{1}_{\{t\geq\tau_m\}}, \tag{E.3}$$

Then if we suppose that,

$$\int_{\mathbb{R}_0} (z^2 \wedge 1)\nu(dz) < \infty,$$

And

$$\int_{\mathbb{R}_0} (\tfrac{\gamma z}{1+\gamma z})^p \nu(dz) < \infty.$$

for $\forall p \in \mathbb{N}$ and $p \geq 2$. We can apply Lemma 4.1 and Lemma 5.1 of Protter and Talay (1997) to produce,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0<s\leq t}\left|k^{\tau_m -}(s) - k^*\right|^p\right] \\
&= C_p\left(\mathbb{E}\left[\sup_{0<s\leq t}\left|\int_0^s k^{\tau_m -}(\lambda -)dZ(\lambda)\right|^p\right] + \mathbb{E}\left|k(0)-k^*\right|^p\right) \\
&\leq \Big[1 + \left|\mu g_p^{\alpha/1-\alpha}(1-g_p) - \rho - n + \sigma^2 + b\right|^2 + |\sigma|^2 + \textstyle\int_{\mathbb{R}_0} (\frac{\gamma z}{1+\gamma z})^2 \nu(dz) \\
&\quad + \left|\mu g_p^{\alpha/1-\alpha}(1-g_p) - \rho - n + \sigma^2 + b\right|^p + |\sigma|^p + \left(\textstyle\int_{\mathbb{R}_0} (\frac{\gamma z}{1+\gamma z})^2 \nu(dz)\right)^{p/2} \\
&\quad + \textstyle\int_{\mathbb{R}_0} (\frac{\gamma z}{1+\gamma z})^p \nu(dz)\Big] \displaystyle\int_0^T \mathbb{E}\left|k^{\tau_m -}(\lambda -)\right|^p d\lambda + C_p \mathbb{E}\left|k(0)-k^*\right|^p,
\end{aligned} \tag{E.4}$$

where $C_p > 0$ is a constant depends on p, k^* is given in (54'). And noting that the right hand side of (E.4) is finite because $\left|k(\tau_m -)\right| \leq m < \infty$, and by triangle inequality,

$$\left|k^{\tau_m -}(\lambda -)\right| \leq \left|k^{\tau_m -}(\lambda -) - k^*\right| + \left|k^*\right|,$$

Thus, applying Gronwall's lemma to (E.4) leads to,

$$\mathbb{E}\left[\sup_{0<s\le t}\left|k^{\tau_m -}(s)-k^*\right|^p\right]\le \mathbf{\Psi}_{M(T),p}\left(1+\mathbb{E}\left|k(0)-k^*\right|^p\right), \tag{E.5}$$

where,

$$\mathbf{\Psi}_{M(T),p} \triangleq \exp\Big\{M(T)\Big[1+\left|\mu g_p{}^{\alpha/1-\alpha}(1-g_p)-\rho-n+\sigma^2+b\right|^2+\left|\sigma\right|^2+\int_{\mathbb{R}_0}(\tfrac{\gamma z}{1+\gamma z})^2\nu(dz)+$$

$$\left|\mu g_p{}^{\alpha/1-\alpha}(1-g_p)-\rho-n+\sigma^2+b\right|^p+\left|\sigma\right|^p+\left(\int_{\mathbb{R}_0}(\tfrac{\gamma z}{1+\gamma z})^2\nu(dz)\right)^{p/2}+\int_{\mathbb{R}_0}(\tfrac{\gamma z}{1+\gamma z})^p\nu(dz)\Big]\Big\}.$$

with $M(T)>0$ and $0<T\le\infty$. Noting that the right hand side of (E.5) is independent of m, so employment of Fatou's lemma and Levi lemma gives the result in our theorem. ▌

F. Proof of Theorem 3

The proof is the same as that of Corollary 1. Hence, combining with (58), we see that,

$$\widehat{\mathbf{\Psi}_{M(T),p}} \triangleq \exp\Big\{M(T)\Big[1+\left|\mu g_p{}^{\alpha/1-\alpha}r_s^*-n+\sigma^2+b\right|^2+\left|\sigma\right|^2+\int_{\mathbb{R}_0}(\tfrac{\gamma z}{1+\gamma z})^2\nu(dz)+$$

$$\left|\mu g_p{}^{\alpha/1-\alpha}r_s^*-n+\sigma^2+b\right|^p+\left|\sigma\right|^p+\left(\int_{\mathbb{R}_0}(\tfrac{\gamma z}{1+\gamma z})^2\nu(dz)\right)^{p/2}+\int_{\mathbb{R}_0}(\tfrac{\gamma z}{1+\gamma z})^p\nu(dz)\Big]\Big\}, \tag{F.1}$$

where $M(T)>0$ and $0<T\le\infty$. So, $\widehat{\mathbf{\Psi}_{M(T),p}}$ is an increasing function of r_s^*, which is itself an increasing function of τ_c by Lemma 4. Hence, $\widehat{\mathbf{\Psi}_{M(T),p}}$ is minimized by sending τ_c to zero. And by the balanced budget constraint given in (25'), we see that $\tau_k=g_p$. Thus, substituting $\tau_c=0$ into Lemma 4 shows that,

$$r_s^*=1-g_p-\frac{\rho}{\mu g_p{}^{\alpha/1-\alpha}}, \tag{F.2}$$

Inserting (F.2) into (F.1) gives $\mathbf{\Psi}_{M(T),p}$ defined in Corollary 1. And this completes the proof. ▌

G. Proof of Proposition 1

By (66) and (63), we have,

$$\frac{\partial H}{\partial \xi} = \exp(-\rho(s+t))\frac{1}{\xi} - \mu g_p^{\,2\alpha/1-\alpha}(1+\tau_c)k(t)\Gamma(t)\,, \tag{G.1}$$

Thus, if the representative agent has incomplete information, the corresponding first order condition (FOC) is,

$$\mathbb{E}[\tfrac{\partial H}{\partial \xi} \mid \mathcal{H}_t]_{\xi=\hat{\xi}} = 0\,, \tag{G.2}$$

Substituting (G.1) into (G.2) shows,

$$\exp(-\rho(s+t))\frac{1}{\hat{\xi}} = \mu g_p^{\,2\alpha/1-\alpha}\mathbb{E}[(1+\tau_c)k(t)\Gamma(t) \mid \mathcal{H}_t]_{\xi=\hat{\xi}}$$

$$\Leftrightarrow \hat{\xi} = \frac{1}{\exp(\rho(s+t))\mu g_p^{\,2\alpha/1-\alpha}\mathbb{E}[(1+\tau_c)k(t)\Gamma(t)|\mathcal{H}_t]_{\xi=\hat{\xi}}}\,, \tag{G.3}$$

Combining (G.3) with (64) gives,

$$\widehat{r_s(t)} = 1 - g_p - \frac{1}{\exp(\rho(s+t))\mu g_p^{\,2\alpha/1-\alpha}\mathbb{E}[(1+\tau_c)\widehat{k(t)}\widehat{\Gamma(t)}|\mathcal{H}_t]}\,, \tag{G.4}$$

where,

$$\widehat{k(t)} \triangleq k^{(\widehat{r_s(t)})}(t) \quad \text{and} \quad \widehat{\Gamma(t)} \triangleq \Gamma^{(\widehat{r_s(t)})}(t)\,, \tag{G.5}$$

with $\widehat{r_s(t)}$ $\mathfrak{H}-$ predictable. And hence (G.4)-(G.5) give the result in (i) of the proposition. Noting that the proof of (ii) and (iii) is quite similar to the above one, so we take it omitted. ▌

H. Proof of Proposition 2

The proof of Proposition 2 proceeds as follows,

(i) To prove that $\phi \geq g$ on region D. This proof is quite similar to that of Lemma 2. Hence, we argue that (i) follows as long as,

$$\mu g_p^{\,\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\widehat{r_s(0)})] < \rho + n - \sigma^2 - b_0\,, \tag{H.1}$$

(ii) To prove that outside D we always have $\mathcal{A}g(s,k) \leq 0$ for $\forall k \geq \hat{k}$. This proof is the same as that of Lemma 2.

(iii) To check that $\hat{\tau} < \infty$ almost surely. By (71) and Itô-Ventzell formula, we have,

$$\begin{aligned} d^- \ln \widehat{k(t)} = & \left\{\mu g_p^{\,\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\widehat{r_s(t)})] - n + \tfrac{1}{2}\sigma^2 + b_0 + d\right\}dt \\ & -\sigma d^- B(t) + \int_{\mathbb{R}_0} \ln(\tfrac{1}{1+\gamma z})\widetilde{N}(d^- t, dz)\,, \end{aligned} \tag{H.2}$$

Hence, we get,

$$\widehat{k(t)} = k\exp\Big(\int_0^t \Big\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\widehat{r_s(\lambda)})] - n + \tfrac{1}{2}\sigma^2 + b_0 + d\Big\} d\lambda$$

$$+\int_0^t (-\sigma) d^- B(\lambda) + \int_0^t \int_{\mathbb{R}_0} \ln(\tfrac{1}{1+\gamma z}) \widetilde{N}(d^-\lambda, dz)\Big), \qquad \text{(H.3)}$$

It follows from Duality formula for forward integrals that,

$$\mathbb{E}\Big[\int_0^T (-\sigma) d^- B(t)\Big] = \mathbb{E}\Big[\int_0^T D_{t+}(-\sigma) dt\Big] = 0, \ \ \forall T > 0 \qquad \text{(H.4)}$$

And,

$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} \ln(\tfrac{1}{1+\gamma z}) \widetilde{N}(d^- t, dz)\Big] = \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} D_{t+,z} \ln(\tfrac{1}{1+\gamma z}) \nu(dz) dt\Big] = 0, \ \ \forall T > 0 \qquad \text{(H.5)}$$

So, we get by Lebesgue monotone convergence theorem,

$$\Big|\int_0^\infty (-\sigma) d^- B(t)\Big| < \infty \ \ \text{a.e.} \qquad \text{(H.6)}$$

And

$$\Big|\int_0^\infty \int_{\mathbb{R}_0} \ln(\tfrac{1}{1+\gamma z}) \widetilde{N}(d^- t, dz)\Big| < \infty \ \ \text{a.e.} \qquad \text{(H.7)}$$

Hence, by (H.3), we see that if,

$$n - \tfrac{1}{2}\sigma^2 - b_0 - d$$

$$< \mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\widehat{r_s(t)})], \text{ a.e.} \qquad \text{(H.8)}$$

Then we get,

$$\lim_{t\to\infty} \widehat{k(t)} = \infty \quad \text{a.s.}$$

And also, $\hat{\tau} < \infty$ almost surely.

(iv) Similar to that of Lemma 2, we need to prove that,

$$\{e^{-\rho\tau} \ln[g_p{}^{\alpha/1-\alpha}\widehat{k(\tau)}]\}_{\tau\in\mathcal{T}} \text{ is uniformly integrable on } [\hat{k},\infty). \qquad \text{(H.9)}$$

By (H.3) we get,

$$\mathbb{E}\Big[e^{-2\rho\tau}\widehat{k(\tau)}^2\Big]$$

$$= k^2\mathbb{E}\Big[\exp\Big(2\int_0^\tau \Big\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\widehat{r_s(t)})] - n + \tfrac{1}{2}\sigma^2 + b_0 + d - \rho\Big\} dt$$

$$+2\int_0^{\tau}(-\sigma)d^{-}B(t)+2\int_0^{\tau}\int_{\mathbb{R}_0}\ln(\tfrac{1}{1+\gamma z})\widetilde{N}(d^{-}t,dz)\Big)\Big], \tag{H.10}$$

So combining (H.10) with (H.6) and (H.7) shows that, if

$$\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\widehat{r_s(t)})]$$

$$\leq \rho+n-\tfrac{1}{2}\sigma^2-b_0-d \text{ , a.e.} \tag{H.11}$$

Then (H.9) holds and so does (iv).

(v) Similar to the proof of Lemma 2, we need to prove that $\mathbb{E}\left[\widehat{k(t)}^{2\beta}\right]<\infty$ almost everywhere on $[0,\hat{\tau}]$ with $\hat{\tau}<\infty$ a.s. and $1<\beta<\infty$. By (H.3), we see that,

$$\mathbb{E}\left[\widehat{k(t)}^{2\beta}\right]$$

$$=k^{2\beta}\mathbb{E}\Big[\exp\Big(2\beta\int_0^t\Big\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\widehat{r_s(\lambda)})]-n+\tfrac{1}{2}\sigma^2+b_0+d\Big\}d\lambda$$

$$+2\beta\int_0^t(-\sigma)d^{-}B(\lambda)+2\beta\int_0^t\int_{\mathbb{R}_0}\ln(\tfrac{1}{1+\gamma z})\widetilde{N}(d^{-}\lambda,dz)\Big)\Big],$$

Thus, applying (H.6) and (H.7), we find if,

$$\left|\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\widehat{r_s(t)})]-n+\tfrac{1}{2}\sigma^2+b_0+d\right|<\infty \text{ , a.e.} \tag{H.12}$$

Then (v) follows. ▌

I. Proof of Proposition 3

The proof will be naturally divided into two parts.

(i) Symmetric information:

For instance, if both the representative agent and the government have complete information, then by (69) in Proposition 1, we get,

$$\widehat{r_s(t)}=1-g_p-\frac{1}{\exp(\rho(s+t))\mu g_p{}^{2\alpha/1-\alpha}(1+\tau_c)\mathbb{E}[\widehat{k(t)}\widehat{\Gamma(t)}|\mathcal{F}_t]}, \tag{I.1}$$

By (75), (73) and (I.1), we see that,

$$\frac{\partial\widehat{H}}{\partial\tau_c}=-\exp(-\rho(s+t))\frac{1}{1-g_p-\widehat{r_s(t)}}\frac{\partial\widehat{r_s(t)}}{\partial\tau_c}+\mu g_p{}^{2\alpha/1-\alpha}\widehat{k(t)}\widehat{\Gamma(t)}\frac{\partial\widehat{r_s(t)}}{\partial\tau_c}, \tag{I.2}$$

Since we have the following FOC,

$$\mathbb{E}\left[\tfrac{\partial\widehat{H}}{\partial\tau_c}\Big|\mathcal{F}_t\right]_{\tau_c=\tau_c^*}=0, \tag{I.3}$$

which combining with (I.2) implies that,

$$\exp(-(s+t))\mathbb{E}\left[\frac{1}{1-g_p-\widehat{r_s(t)}^*}\Big|\mathcal{F}_t\right]=\mu g_p{}^{2\alpha/1-\alpha}\mathbb{E}\left[\widehat{k(t)}^*\widehat{\Gamma(t)}^*\Big|\mathcal{F}_t\right], \tag{I.4}$$

where $\widehat{r_s(t)}^*\triangleq\widehat{r_s(t)}^{(\tau_c^*)}$, $\widehat{k(t)}^*\triangleq\widehat{k(t)}^{(\tau_c^*)}$ and $\widehat{\Gamma(t)}^*\triangleq\widehat{\Gamma(t)}^{(\tau_c^*)}$ with τ_c^* $\mathcal{F}-$predictable. Then, inserting (I.1) into (I.4) and applying the law of iterated expectation,

$$(1+\tau_c^*)\mathbb{E}\left[\widehat{k(t)}^*\widehat{\Gamma(t)}^*\Big|\mathcal{F}_t\right]=\mathbb{E}\left[\widehat{k(t)}^*\widehat{\Gamma(t)}^*\Big|\mathcal{F}_t\right], \tag{I.5}$$

which yields,

$$\tau_c^*=0, \tag{I.6}$$

Hence, by (25''), we obtain,

$$\tau_k^*=g_p, \tag{I.7}$$

(ii) Asymmetric information:

(ii-a) As usual, suppose that the representative agent has private information, that is, the representative agent has more information than the government. For example, the representative agent has perfect information while the government has complete information, then by (70) in Proposition 1 we get,

$$\widehat{r_s(t)}=1-g_p-\frac{1}{\exp(\rho(s+t))\mu g_p{}^{2\alpha/1-\alpha}(1+\tau_c)\mathbb{E}[\widehat{k(t)}\widehat{\Gamma(t)}|\mathcal{M}_t]}, \tag{I.8}$$

And the corresponding FOC is given in (I.3). Thus, combining (I.2) and (I.3) with (I.8), we obtain by the law of iterated expectation,

$$(1+\tau_c^*)\mathbb{E}\left[\mathbb{E}\left[\widehat{k(t)}\widehat{\Gamma(t)}\Big|\mathcal{M}_t\right]\Big|\mathcal{F}_t\right]_{\tau_c=\tau_c^*}=\mathbb{E}\left[\widehat{k(t)}^*\widehat{\Gamma(t)}^*\Big|\mathcal{F}_t\right], \tag{I.9}$$

$$\Leftrightarrow (1+\tau_c^*)\mathbb{E}\left[\widehat{k(t)}^*\widehat{\Gamma(t)}^*\Big|\mathcal{F}_t\right]=\mathbb{E}\left[\widehat{k(t)}^*\widehat{\Gamma(t)}^*\Big|\mathcal{F}_t\right]$$

which implies that,

$$\tau_c^*=0, \tag{I.10}$$

Hence, by (25''), we obtain,

$$\tau_k^*=g_p, \tag{I.11}$$

Similarly, it is easy to show that (I.10) and (I.11) follow for other cases as long as the representative agent gets more information than the government.

(ii-b) Suppose that the government has more information than the representative agent. For example, the government has perfect information while the representative agent has complete information. Then the FOC is,

$$\mathbb{E}\left[\tfrac{\partial \widehat{H}}{\partial \tau_c} \middle| \mathcal{M}_t\right]_{\tau_c=\tau_c^*} = 0, \tag{I.12}$$

Then combining (I.2), (I.12) with (69) given in Proposition 1, we get by the law of iterated expectation,

$$\mathbb{E}\left[\mathbb{E}\left[(1+\tau_c)\widehat{k(t)}\widehat{\Gamma(t)} \middle| \mathcal{F}_t\right] \middle| \mathcal{M}_t\right]_{\tau_c=\tau_c^*} = \mathbb{E}\left[\widehat{k(t)}^* \widehat{\Gamma(t)}^* \middle| \mathcal{M}_t\right]$$

$$\Leftrightarrow \mathbb{E}\left[(1+\tau_c^*)\widehat{k(t)}^* \widehat{\Gamma(t)}^* \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\widehat{k(t)}^* \widehat{\Gamma(t)}^* \middle| \mathcal{M}_t\right], \tag{I.13}$$

which gives the desired result. Noting that the proof of other cases is similar to this one, so we take it omitted. ▌

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