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Abstract

A unified growth model is presented in which productivity growth is driven by learning-by-doing. We show that the growth rate of productivity is an increasing function of the share of capital. It is assumed that the industrial sector has a higher capital share than the agricultural sector and that the ability to substitute one output for the other slowly rises over time. Two distinct regimes of constant growth emerge, connected by a rapid transition in which the growth rates of population and income increase by an order of magnitude, indicative of simultaneous agricultural and industrial revolutions.

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JEL classifications: N10, O41

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1 Introduction

The preindustrial era was characterized by low rates of productivity growth, a heavy reliance on the agricultural sector both as a source of food and as a source of raw materials for industry, and low levels of fixed capital. Since the Industrial Revolution the growth rate of total factor productivity has increased by an order of magnitude, the relative size of the agricultural sector has shrunk considerably, and the economy has become reliant on vast quantities of fixed capital in the form of machines and infrastructure. This paper presents a unified growth model that ties together these observations based on Wrigley's thesis that a key enabler of the Industrial Revolution was a shift in the source of raw materials from agriculture to industry (Wrigley, 1988, 2010). In the present model this shift is sufficient to trigger all of the other changes we have come to associate with the transition to a modern economy.

Two mechanisms are employed to explain long-term changes in productivity: exogenous growth of knowledge, and endogenous learning-by-doing due to capital investment (Arrow, 1962). In a one-sector version of the model to be described in the next section, the growth rate of total factor productivity is proportional to the growth rate of knowledge, with the constant of proportionality an increasing function of the share of capital. If we accept that the capital share today is larger than it was in past centuries, this mechanism provides a possible explanation for the takeoff in productivity after the Industrial Revolution. The remaining task is to explain why the capital share suddenly increased during the Industrial Revolution. The approach taken here is to assume that there are *two* sectors, one having a high capital share (industry) and the other having a low capital share (agriculture), and to focus attention on the historical ability of firms to substitute the output of the high capital-share sector for that of the low capital-share sector in the construction of capital goods. The change in substitutability provides the link with Wrigley's thesis.

The model borrows from Hansen & Prescott (2002) in labeling the agricultural sector the "Malthus" sector, which has a low capital share and a heavy reliance on land, and in labeling the industrial sector the "Solow" sector, which has a larger capital share and no reliance on land. It is assumed that the two sectors are used at all times and produce two very different types of goods for two different markets. Capital goods are built using the outputs of both sectors via a Constant Elasticity of Substitution production function. Each of the two components of capital has its own embodied productivity factor, which grows by exogenous increases in knowledge and by learning-by-doing. Unlike in Hansen & Prescott, we assume that the parameters driving productivity growth are identical in both sectors. The elasticity of substitution (EoS) between the two outputs has a small initial value (less than one), and slowly increases over time as people learn to substitute industrial output for agricultural output in the production of capital goods. The rationale for assuming a constantly increasing EoS is that it is *always* profitable for a firm to introduce a new technique that expands the possibilities for substitution.

Preferences are modeled using a simple Cobb Douglas model of utility. Individuals exhibit preferences for consumption, investment, and childbearing that are assumed constant over time. These preferences translate into constant expenditure shares. Real earnings are largely determined by the output of the Solow sector because the consumer demand for Malthus output is assumed perfectly inelastic. Population dynamics is "Malthusian" at all times, so increases in income or decreases in the cost of Malthus output (food) lead to higher population growth rates.

The model makes the following predictions. When EoS is less than one, increases in EoS do not have an appreciable effect on the economy. The economy grows at a small rate that is independent of EoS. Once EoS surpasses the value of one, the economy shifts to a new regime in which the growth rate is much higher and is again approximately constant over time (independent of EoS). The explanation for the sudden transition is as follows. When EoS is less than one, the Malthus sector, with its low capital share and correspondingly low rate of productivity growth, acts as a brake on the economy because for a given level of capital there is a minimum required level of Malthus output. Hence, the learning-by-doing mechanism is never able to live up to its full potential. Once EoS is greater than one, the binding constraint of the Malthus sector is released. The output of the Solow sector can then be used directly in the production of new capital goods (for both sectors) and productivity growth accelerates. The passing of the EoS = 1 boundary marks the onset of the Industrial Revolution.

The demographic patterns produced by the model are broadly consistent with the historical record. The growth rate of population before the Industrial Revolution is very low because the economy is dominated by the Malthus sector. The growth rate of earnings is also very low because the Solow sector is constrained by the output of the Malthus sector. After the Industrial Revolution, the population growth rate increases because the Malthus sector is boosted by the Solow sector via lower prices for capital goods. Earnings also grow rapidly because the Solow sector grows faster than the Malthus sector due to the difference in their capital shares. Although population growth continues to be Malthusian in the modern era, the rate of growth of population reaches a constant value because the price of Malthus output increases in tandem with income. This last property suggests a reinterpretation of the "Malthus sector" as consisting of child-care services as well as agriculture.

The model presented in this paper is consistent with Wrigley's thesis that the Malthus sector acted as a material and energy constraint on the economy prior to industrialization (Wrigley, 1988, 2010). For example, farm plows used iron components from the preindustrial Solow sector (horseshoes, buckles, plowshares), but relied on horses from the Malthus sector, and the productivity of horses did not grow at modern rates. The preindustrial Solow sector contained things like iron foundries that were built using iron components, but that also required working capital derived from the Malthus sector such as wood for heat, and water or horses for mechanical energy. The supply of wood represented a very serious constraint since it depended on an exploitive activity that was almost preagricultural in nature, with very little invested capital. After the Industrial Revolution, Malthus output and Solow output became substitutes for the purposes of capital. Some of the most famous inventions of the Industrial Revolution, such as Watt's separate condenser for the steam engine and Cort's puddling process for iron production, had the effect of lessening the dependence of industry on land-based production.

The model is also broadly consistent with the view of Crafts (1995), who has analyzed the applicability of recent growth models to the Industrial Revolution. In Crafts view, the most promising approach is to combine exogenous shocks (inventions) with learning-by-doing effects (innovation). We have modeled increases in the elasticity of substitution between land-based output and industrial output as an exogenous process, and have then allowed the mechanism of learning-by-doing to follow through with its far-reaching effects on growth. The model is also consistent with the theme of Allen's study of the blast furnace industry, which showed that capital investment was a key driver of productivity improvements through learning effects (Allen, 1983).

As Lucas (2008) has pointed out, it is widely accepted amongst economists that the process of industrialization is an ongoing intellectual achievement. One of the defining characteristics of the Industrial Revolution is that it introduced a sharp discontinuity in the rate of growth of human knowledge. If we accept the premise that people were just as mentally capable 2,000 years ago as they are today, this discontinuity represents a deep mystery. The goal of unified growth theory is to develop a model that captures the dynamics of both the preindustrial economy and the modern economy without invoking any special exogenous shocks to explain the Industrial Revolution (Galor, 2011). The present model fits into that mold because the dynamical equations and their parameters are unchanging over time, yet a discontinuity emerges. The implication of this paper is that people have been striving to substitute the products of their hands for the products of nature for millennia, and this process finally reached a threshold of sustained capitalist learning about two hundred years ago.

The plan of the paper is as follows. In the next section, we briefly introduce a simple one-sector model designed to motivate the developments in this paper. Section 3 contains a description of the unified growth model. Section 4 contains the results of a simulation exercise, showing that our model is broadly consistent with the historical record of the Industrial Revolution. Section 5 analyses the asymptotic behavior of the model in two regimes, EoS < 1 and EoS > 1, showing that the model exhibits a rapid change in dynamics akin to a phase transition as EoS passes the value of one. In Section 6, we interpret the model and discuss its limitations and possible extensions. Finally, Section 7 concludes.

2 Malthus to Solow via Arrow

Amongst the many changes to occur during the Industrial Revolution, the two most relevant to the present discussion were the increase in the share of capital and the large increase in the rate of growth of Total Factor Productivity (TFP). The purpose of this section is to show that these two phenomena can be linked using a variant of Arrow's learning-by-doing model (Arrow, 1962). A simple toy model will demonstrate that even a modest increase in the share of capital can lead to a productivity takeoff.

The production function is given by

$$Y = AK^{\alpha}N^{1-\alpha}, \quad 0 < \alpha < 1, \tag{1}$$

where Y is output, A is TFP, K is capital, and N is labour. For the purposes of this exercise we will assume that labour is held fixed. The focus of our attention will be on the relationship between A and K. Let us also assume that savings are a fixed fraction s of output:

$$\dot{K} = sY - \delta K, \tag{2}$$

where δ is the rate of depreciation of capital. Productivity grows exogenously at a small rate g_0 and also grows due to gross investment in capital. The latter effect is inspired by Arrow's learning-by-doing model. Since savings are a fixed fraction of output we can state

$$\frac{d}{dt}\ln A = g_0 + \xi \frac{d}{dt}\ln Y, \tag{3}$$

where ξ is a parameter that captures the spillover effects of learning due to capital investment.

We wish to study the steady-state growth path associated with this model. We assume that all quantities grow exponentially at constant rates which we label $\{g_x\}$, where $\{x\}$ stands for whatever quantities are under consideration. Dividing both sides of equation (2) by K we have $g_K = sY/K - \delta$. Since g_K is assumed constant the capital/output ratio must be constant. Therefore, from (1) we find

$$g_A = (1 - \alpha) g_K, \tag{4}$$

and from equation (3) we see that

$$g_A = g_0 + \xi g_K. \tag{5}$$

Combining (4) and (5), we obtain the following expression for the growth rate of TFP

$$g_A = \frac{1 - \alpha}{1 - \alpha - \xi} g_0. \tag{6}$$

Notice that g_A is proportional to g_0, with the constant of proportionality being greater than 1 when $\xi > 0$. The learning-by-doing mechanism is acting as a lever on the exogenous growth rate.[1] Notice also that g_A is monotonically increasing in α (thereby linking the growth rate of TFP with the share of capital) and is essentially unbounded.

The growth rate of wages is

$$g_w = \frac{1}{1 - \alpha - \xi} g_0. \tag{7}$$

Consider the following numerical example: $g_0 = 0.1\%$ per annum, $\xi = 0.55$ and $\alpha = 0.1$, the latter corresponding to the share of capital in the Malthus sector of Hansen & Prescott's (2002) model. The resulting growth rate of wages is 0.29% per annum. If we increase α to 0.4 (the capital share in the Solow sector of Hansen & Prescott's model) the growth rate of wages increases to 2% per annum.

We now proceed to describe the unified growth model, incorporating the above learning mechanism in a two-sector framework.

[1] This model avoids the kind of "knife-edge" condition normally required in a purely endogenous model.

3 The Model

3.1 Production

Following Hansen & Prescott (2002), we define two sectors, a "Malthus" sector, which produces Y_{Mt} at time t, and a "Solow" sector, which produces Y_{St} at time t. The production functions are:

$$Y_{Mt} = K_{Mt}^{\phi} N_{Mt}^{\mu}, \tag{8}$$
$$Y_{St} = K_{St}^{\theta} N_{St}^{1-\theta}, \tag{9}$$

where K_{Mt} and K_{St} are the quantities of capital employed in the Malthus sector and Solow sector, respectively, and N_{Mt} and N_{St} are the quantities of labour employed in the Malthus sector and Solow sector, respectively. It is assumed that $\phi < \theta$, reflecting that the capital share in the Malthus sector is lower than the capital share in the Solow sector. It is also assumed that $\phi + \mu < 1$, reflecting the hidden presence of land in the Malthus sector, which is normalized to 1. There are no productivity factors in the above production functions. Instead, productivity is embodied in capital, as will be explained in the next subsection.

The production side of the economy solves the following optimization problem:

$$\max \left\{ p_t Y_{Mt} - w_{Mt} N_{Mt} - r_{Kt} K_{Mt} - r_{Lt} L_t \right\}, \quad L_t = 1, \tag{10}$$
$$\max \left\{ Y_{St} - w_{St} N_{St} - r_{Kt} K_{St} \right\}, \tag{11}$$

where w_{Mt} is the wage in the Malthus sector, w_{St} is the wage in the Solow sector, r_{Kt} is the rent on capital, r_{Lt} is the rent on land, and p_t is the price of goods produced by the Malthus sector. The output of the Solow sector acts as numeraire.

We assume that capital is perfectly mobile so there is a unique rent on capital. However, we distinguish two different wages w_{Mt} and w_{St}. The reason has to do with our treatment of land. Our focus is on capital and labor, so consistent with common practice in this field we assume that laborers in the Malthus sector own all of the land that they use for production. Hence total earnings in the Malthus sector consists of wages plus rents. We also assume that land has no value, so labour will freely migrate until the total income earned in the Malthus sector (wages plus land rents) is equal to the wage earned in the Solow sector.[2] Let us define total earnings y_t to be

$$y_t = w_{Mt} + \frac{r_{Lt}}{N_{Mt}} = w_{St}. \tag{12}$$

Competitive firms will then hire quantities of labour and capital such that

$$r_{Kt} = \frac{\partial Y_{St}}{\partial K_{St}} = \frac{\partial Y_{Mt}}{\partial K_{Mt}} p_t, \tag{13}$$

$$y_t = \frac{\partial Y_{St}}{\partial N_{St}} = \frac{Y_{Mt} - r_{Kt} K_{Mt}}{N_{Mt}} p_t, \tag{14}$$

or, in terms of labour and capital quantities:

$$r_{Kt} = \phi \frac{N_{Mt}^{\mu}}{K_{Mt}^{1-\phi}} p_t = \theta \frac{N_{St}^{1-\theta}}{K_{St}^{1-\theta}}, \tag{15}$$

$$y_t = (1-\phi) \frac{K_{Mt}^{\phi}}{N_{Mt}^{1-\mu}} p_t = (1-\theta) \frac{K_{St}^{\theta}}{N_{St}^{\theta}}. \tag{16}$$

3.2 Capital

Capital goods are constructed using two types of material: X_{Mt}, which is directly produced by the Malthus sector, and X_{St}, which is directly produced by the Solow sector.[3] That is, X_{Mt} is some portion of past output as produced according to equation (8), and X_{St} is some portion of past output as produced according to equation (9). The Malthus and Solow materials are combined to form total capital $K_t = K_{Mt} + K_{St}$ via a Constant Elasticity of Substitution (CES) production function:

$$K_t = \left[(1-\gamma)\left(A_{Mt} X_{Mt}\right)^{\rho_t} + \gamma \left(A_{St} X_{St}\right)^{\rho_t}\right]^{\frac{1}{\rho_t}}, \tag{17}$$

where A_{Mt} is the Malthus capital-augmenting productivity, A_{St} is the Solow capital-augmenting productivity, γ is the share of the Solow sector in capital goods pro-

[2] This modeling choice avoids the need to iteratively solve for the price of land, as was necessary in Hansen & Prescott (2002).

[3] Working capital can be thought of as a capital good having a high rate of depreciation.

duction, and ρ_t is a monotonic increasing function of the Elasticity of Substitution (EoS) σ_t:

$$\rho_t = 1 - \frac{1}{\sigma_t}. \tag{18}$$

The parameter ρ_t lies between $-\infty$ and 1, while σ_t lies between 0 and ∞. Note that there is no overall productivity factor outside the square brackets in equation (17). As will be shown later, this implies that if $\sigma_t < 1$ the growth-rate of productivity is constrained by the slowest-growing productivity factor.[4]

We assume that the suppliers of capital minimize the cost of each unit of capital by solving

$$\min \{p_t\, x_{Mt} + x_{St}\}, \tag{19}$$

subject to the constraint

$$0 = 1 - K_t\,(X_{Mt} = x_{Mt},\, X_{St} = x_{St}), \tag{20}$$

where K_t is given by (17). In a competitive economy, the price of capital q_t is equal to its marginal cost. Since the production function for capital is first-order homogeneous, marginal cost equals average cost. Defining C_t^* to be the minimum cost, x_{Mt}^* to be the optimal quantity of Malthus material and x_{St}^* to be the optimal quantity of Solow material, we have

$$q_t = C_t^* = p_t\, x_{Mt}^* + x_{St}^*. \tag{21}$$

The Lagrangian for this optimization problem is

$$\mathcal{L} = p_t x_{Mt} + x_{St} + \lambda(1 - K_t), \tag{22}$$

[4]We can offer an energy interpretation of equation (17) inspired by Wrigley (1988, 2010). Capital goods are low-entropy systems that require energy in the form of work for their construction, ongoing maintenance and operation. It is assumed that energy is embodied in $A_{Mt}X_{Mt}$ and/or $A_{St}X_{St}$. Traditionally, capital goods may have been produced using Solow output ($A_{St}X_{St}$) but the energy was embodied in $A_{Mt}X_{Mt}$. During industrialization energy became embodied in $A_{St}X_{St}$ as well, allowing substitution to take place. Historical examples include coal replacing wood as a source of heat for the iron industry, and steam engines replacing water wheels and horses for mechanical energy.

and the first-order conditions for cost minimization are

$$\frac{\partial \mathcal{L}}{\partial X_{Mt}} = 0, \quad \frac{\partial \mathcal{L}}{\partial X_{St}} = 0, \quad 1 - K_t = 0. \tag{23}$$

The solution for q_t, x^*_{Mt} and x^*_{St} is

$$q_t = \left\{ (1-\gamma)^{\frac{1}{1-\rho}} \left(\frac{p_t}{A_{Mt}} \right)^{-\frac{\rho}{1-\rho}} + \gamma^{\frac{1}{1-\rho}} \left(\frac{1}{A_{St}} \right)^{-\frac{\rho}{1-\rho}} \right\}^{-\frac{1-\rho}{\rho}}, \tag{24}$$

$$x^*_{Mt} = q_t^{\frac{1}{1-\rho}} (1-\gamma)^{\frac{1}{1-\rho}} \left(\frac{p_t}{A_{Mt}} \right)^{-\frac{\rho}{1-\rho}} \frac{1}{p_t}, \tag{25}$$

$$x^*_{St} = q_t^{\frac{1}{1-\rho}} \gamma^{\frac{1}{1-\rho}} \left(\frac{1}{A_{St}} \right)^{-\frac{\rho}{1-\rho}}. \tag{26}$$

We will assume that capital completely depreciates at the end of each time step and that a time step equals 35 years, consistent with Hansen & Prescott (2002). The interest rate is then

$$i = \left(\frac{r_{Kt}}{q_t} - 1 \right)^{\frac{1}{35}} - 1, \tag{27}$$

where r_{Kt} is given by equation (15).

3.3 Preferences

Building on Hansen & Prescott (2002), we assume an overlapping generations model. Each person lives for two periods of time, and the population includes a mix of young people and old people. Young people work to earn an income, which they use to purchase three things: manufactured goods from the Solow sector for their own consumption, capital goods for investment purposes, and Malthus output (e.g. food and child care) to support the raising of children. It is assumed that old people receive enough Malthus output in childhood to sustain them for life. Hence old people consume Solow goods only, which are paid for by the returns on their investments. Land has no value, so each generation simply confiscates it upon their entry into the Malthusian labour force.

We assume the following utility function

$$u_t = \alpha \log c_{1,t} + \beta \log c_{2,t+1} + (1 - \alpha - \beta) \log n_t, \tag{28}$$

where $c_{1,t}$ is consumption in the first period (the working period), $c_{2,t+1}$ is consumption in the second period, and n_t is the number of children born to each worker at the end of the time step (this is the new element not included in Hansen & Prescott's utility function). Children become part of the labor force in the next time step $t+1$. Assuming that children require one unit of output from the Malthus sector, which is sufficient to sustain them throughout life, the budget constraint is

$$y_t = c_{1,t} + q_t k_{t+1} + p_t n_t, \tag{29}$$

$$c_{2,t+1} = r_{t+1} k_{t+1}, \tag{30}$$

where y_t is the total income earned by young people in period 1 (wages and rent on land), k_{t+1} is the invested capital (savings) for period 2, and q_t and p_t are the prices of capital and Malthus output, respectively.

Maximizing utility, we have

$$c_{1,t} = \alpha y_t, \tag{31}$$

$$k_{t+1} = \beta \frac{y_t}{q_t}, \tag{32}$$

$$n_t = (1 - \alpha - \beta) \frac{y_t}{p_t}. \tag{33}$$

The first result says that young workers spend a fixed percentage of their earnings on output from the Solow sector. The second result states that savings are a fixed fraction of income. The third result describes Malthusian population dynamics: the higher the earnings and the lower the price of output from the Malthus sector, the more children.

The above results can be used to define real earnings. Consistent with our assumption of Cobb-Douglas utility, we define the cost-of-living index as the geometric average of the factor prices, with expenditure shares used as weights.[5] Real earn-

[5]Clark (2005) uses a weighted geometric average in his construction of a cost-of-living index

ings y_{Rt} are then defined as nominal earnings y_t divided by the cost-of-living index:

$$y_{Rt} \equiv \frac{y_t}{q_t^{\beta} p_t^{1-\alpha-\beta}}. \tag{34}$$

3.4 Productivity Growth

We now specify a dynamical process for the capital-augmenting productivity parameters A_{Mt} and A_{St}, and for the elasticity of substitution σ_t. Two processes are assumed for A_{Mt} and A_{St}. First, productivity grows exogenously at some small rate g_0. Second, we assume that productivity also increases as a side effect of gross capital investment (Arrow, 1962). The relative changes in $A_{M\,t}$ and $A_{S\,t}$ are thus driven by two factors, the first related to exogenous growth and the second related to total gross investment:

$$\frac{A_{M\,t+1}}{A_{M\,t}} = e^{g_0} \left(\frac{\tilde{X}_{M\,t+1}}{\tilde{X}_{M\,t}} \right)^{\xi}, \tag{35}$$

$$\frac{A_{S\,t+1}}{A_{S\,t}} = e^{g_0} \left(\frac{\tilde{X}_{S\,t+1}}{\tilde{X}_{S\,t}} \right)^{\xi}, \tag{36}$$

where ξ is a new parameter of $\mathcal{O}(1)$ that captures the spillover effects of learning, and

$$\tilde{X}_{M\,t} = \sum_{t_i \le t} X_{M\,t_i}, \tag{37}$$

$$\tilde{X}_{S\,t} = \sum_{t_i \le t} X_{S\,t_i}. \tag{38}$$

covering the years 1209-1869. Allen (2001) uses both geometric and arithmetic averages to compute inflation indices for several cities in Europe covering the years 1350 to 1750 and finds little difference in their values.

Here $X_{M\,t_i}$ and $X_{S\,t_i}$ are, respectively, the quantities of Malthus output and Solow output used in the construction of capital at time t_i:

$$X_{M\,t_i} = \frac{\beta y_{t_i} N_{t_i} x^*_{m\,t_i}}{q_{t_i}}, \tag{39}$$

$$X_{S\,t_i} = \frac{\beta y_{t_i} N_{t_i} x^*_{s\,t_i}}{q_{t_i}}. \tag{40}$$

Here we have made use of equation (32) and multiplied by $x^*_{m\,t_i}$ and $x^*_{s\,t_i}$, which are the Malthus and Solow components respectively of each unit of capital, and we have also multiplied by N_{t_i} to obtain total quantities. The above model of learning assumes that the efficiency of each component of capital grows in direct response to its use. Note that the parameters driving productivity growth (g_0, ξ) are identical in both sectors.

The final key parameter to consider is σ (or ρ). Since $0 < \sigma < \infty$, it is natural to assume an exponential growth process:

$$\sigma_t = \sigma_0 e^{g_\sigma t}, \tag{41}$$

where $\sigma_0 < 1$ and g_σ is a new growth parameter.

The rationale for assuming a constantly increasing σ_t is that it is always (temporarily) profitable for a capital-producing firm to introduce a new technique that expands the opportunities for substitution. To show this, it suffices to show that the cost of capital is a declining function of ρ_t (recall that ρ_t is monotonic increasing in σ_t). Application of the Envelope Theorem to the Lagrangian for cost minimization (22) results in

$$\frac{\partial C^*}{\partial \rho} = \frac{\partial \mathcal{L}}{\partial \rho} = -\lambda \frac{\partial K}{\partial \rho}.$$

K is of the form of a generalized mean, which has the property that it is a monotonic increasing function of ρ.[6] And the Lagrange multiplier λ is equal to C^*_t, which is

[6] The proof is contained in Hardy *et al.*, 1934, p. 26. See also La Grandville, 2009 (Appendix of Chapter 4).

positive.[7] Hence

$$\frac{\partial C^*}{\partial \rho} < 0.$$

Therefore, it is always profitable for a firm to introduce a new capital-goods production technique that embeds a higher elasticity of substitution than is currently prevailing.

3.5 The Equilibrium Path

The purpose of this section is to complete the set of equations required to find the equilibrium prices and allocations at each time step, and to describe the changes in the key variables over time. The initial conditions for the model are the quantities of labour and capital at time zero: N_0 and K_0. During each time step t, the economy optimally allocates labour and capital to the Malthus and Solow sectors. One complication is that in solving for the equilibrium allocations, one must take into account the optimal mix of materials required in the construction of capital goods to be used in the next time step. The resulting equilibrium income earned by young workers is the key determinant of the quantities of labour and capital in the next time step (N_{t+1} and K_{t+1}) via the preference equations (31) - (33). The end result is that we can take the total quantities of labour and capital as 'given' at the beginning of each time step.

The market-clearing conditions are

$$N_t = N_{Mt} + N_{St}, \tag{42}$$

$$K_t = K_{Mt} + K_{St}, \tag{43}$$

$$Y_{Mt} = (1 - \alpha - \beta)\frac{y_t}{p_t}N_t + \beta\frac{y_t}{q_t}N_t x^*_{Mt}, \tag{44}$$

$$Y_{St} = \alpha y_t N_t + r_{Kt}K_t + \beta\frac{y_t}{q_t}N_t x^*_{St}. \tag{45}$$

The first two conditions listed above simply equate total labour and capital with the supply of those factors at the beginning of each time step. The last two conditions

[7] One can use the first-order conditions listed in equation (23) to show that $\lambda = q_t$, which equals C^*_t, which in turn is positive assuming p_t is positive.

equate the supply and demand of final outputs. Equation (44) says that total Malthus output is equal to the amount required to support children plus the amount required in the construction of capital goods. Equation (45) says that total Solow output is equal to total consumption by young workers plus total consumption by old people (investment returns) plus the amount required for capital goods. We now have sufficient conditions to determine the equilibrium solution at each time step.[8]

Finally, from equations (32) and (33) the quantities of labour and capital at time $t + 1$ are

$$N_{t+1} = (1 - \alpha - \beta)\frac{y_t}{p_t}N_t, \tag{46}$$

$$K_{t+1} = \beta\frac{y_t}{q_t}N_t. \tag{47}$$

The model is now fully specified.

4 Simulation

The system of equations to be solved is a set of recursive-algebraic equations, consisting of two recursive equations, Eqs. (46) and (47), coupled to the algebraic constraints (15), (16) and (42)-(44). The productivity parameters evolve according to equations (35) and (36) and σ_t grows according to (41).

4.1 Solution Method

We will now introduce two new variables so as to derive a solution algorithm. These variables represent the fraction of population in the Solow sector

$$\eta_t = \frac{N_{st}}{N_t} \tag{48}$$

[8] In fact we have one too many equations! However it can be shown that one of (44) or (45) is redundant.

and the fraction of capital in the Solow sector

$$\kappa_t = \frac{K_{st}}{K_t}. \tag{49}$$

At the beginning of each time step, we know the values of A_{Mt}, A_{St}, N_t and K_t. In order to determine those quantities at time $t+1$ using (46) and (47), we need to know y_t, p_t and q_t. We note that y_t can be obtained from η_t and κ_t using equation (16). Hence, we need to solve the model equations for η_t, κ_t, p_t and q_t. The price of capital q_t can be expressed in terms of p_t using Eq. (24). The price of Malthus output p_t can in turn be obtained from η_t and κ_t using Eq. (15) in the form

$$p_t(\kappa_t, \eta_t) = \frac{\theta N_t^{1-\theta-\mu}}{\phi K_t^{\phi-\theta}} \frac{\eta_t^{1-\theta}(1-\kappa_t)^{1-\phi}}{(1-\eta_t)^{\mu}\kappa_t^{1-\theta}}. \tag{50}$$

We can also write κ_t in terms of η_t by dividing (15) by (16) to obtain

$$\kappa_t(\eta_t) = \frac{E(\eta_t)}{1+E(\eta_t)} \tag{51}$$

with $E = \frac{(1-\phi)\theta\eta_t}{\phi(1-\theta)(1-\eta_t)}$. Hence, we have just one remaining unknown: η_t, which can be obtained by solving the following equation, derived from Eqs. (8), (16) and (44):

$$\begin{aligned} G(\eta_t, p_t, q_t(p_t), x^*_{Mt}(p_t, q_t(p_t))) &= \frac{1}{p_t} - \frac{(1-\alpha-\beta)(1-\phi)}{p_t(1-\eta_t)} - \frac{\beta(1-\phi)x^*_{Mt}}{q_t(1-\eta_t)} \\ &= 0. \end{aligned} \tag{52}$$

A nonlinear root search technique must be used to find η_t. In the above equation, the quantity x^*_{Mt} can be expressed in terms of the other quantities, using Eq. (25).

The initial values K_0, N_0, A_{M0} and A_{S0} are arbitrary. The initial value of EoS is 0.01. We then let the system evolve according to the scheme described above.

4.2 Results

For various, yet realistic, parameters values (ϕ, μ, θ, γ, α, β, ξ, g_σ, σ_0, g_0), the numerical results exhibit qualitatively the same dynamics. For the figures and results listed below we have used the specific parameter values listed in Table 1 (to be justified in Section 5.3). Figure 1 shows the assumed evolution of σ_t, starting at the value of 0.01 and growing at the rate of 0.4% per year. As can be seen in Figure 2, after an initial period of adjustment, population and real earnings grow slowly and steadily until σ reaches the value of one. For $\sigma > 1$, population and real earnings grow much faster but still roughly at a constant rate. This transition occurs within a few time steps, equivalent to about 150 years.

The growth rates of population and real earnings are shown in Figure 3. The main observation is that the growth rates appear to settle to constant values for $\sigma < 1$ and for $\sigma > 1$. There is an order-of-magnitude increase in growth rates as the elasticity of substitution surpasses one, which can be interpreted as an Industrial Revolution.

Numerically, we find the following approximate values before and after the Industrial Revolution:

$$\begin{aligned} g_N^{before} &= 0.075\% \quad , \quad g_N^{after} = 0.53\%, \\ g_{yR}^{before} &= 0.3\% \quad , \quad g_{yR}^{after} = 1.9\%. \end{aligned}$$

Here, we have used real earnings growth, based on (34)

$$g_{yR} = g_y - \beta g_q - (1 - \alpha - \beta) g_p. \tag{53}$$

Notice that the ratio of the real earnings growth rate to the population growth rate is roughly the same in the two regimes.

The fractions of population and capital employed in the Solow sector are

$$\begin{aligned} \eta^{before} &= 55\% \quad , \quad \eta^{after} = 71\% \\ \kappa^{before} &= 95\% \quad , \quad \kappa^{after} = 98\%. \end{aligned}$$

Finally, using equation (27) the interest rate in the two regimes is

$$i^{before} = 1.4\% \quad , \quad i^{after} = 8.0\%.$$

Suffice it to say that the "after" values listed above (corresponding to the period after the Industrial Revolution) look reasonable, but some of the values before the Industrial Revolution are somewhat surprising. We note that before the Industrial Revolution the growth rate of real earnings is nonzero (but small), the fraction of labour and capital devoted to the Solow sector is high, and the interest rate is low. These observations will be addressed in the Discussion section.

This is only one specific example. A crucial result of our numerical study however is that for any set of realistic parameter values we find an order of magnitude jump in growth rates as σ exceeds one. Moreover, the growth rates are nearly independent of σ for $\sigma < 1$ and $\sigma > 1$. These results are consistent with the view that the Industrial Revolution was a sudden event when placed against the backdrop of recorded history. Even though innovation occurred over many centuries, the growth rates remained small (and nearly constant) until a crucial piece of innovation pushed σ passed one. In this sense, one can think of the Industrial Revolution as a phase transition.

5 Analytical Derivation of Growth Limits

In the previous simulation exercise, it was observed that there are clearly two regimes, corresponding to before the Industrial Revolution and after the Industrial Revolution, and that the growth rates of population and earnings appear to converge to constant values in each regime. In this section, we derive analytic formulas for the asymptotic growth rates in the two regimes. In what follows, we assume constant growth rates of y_t, p_t, K_t, N_t, q_t, A_{Mt} and A_{St}, represented by

$$y_t \sim e^{g_y t} \tag{54}$$

etc., thereby introducing new symbols g_y, g_p, g_K, g_N, g_q, g_{A_M} and g_{A_S}.

Substituting these exponential time dependencies into our equations, we can obtain several relations that are independent of σ_t, and hence valid in both regimes $\sigma < 1$ and $\sigma > 1$. Balancing terms in (46), we see that

$$g_y = g_p. \tag{55}$$

The above result is consistent with a constant population growth rate g_N (see equation (33)). From (16), this result implies that

$$\phi g_K = (1 - \mu) g_N. \tag{56}$$

The same equation also yields

$$g_y = \theta(g_K - g_N). \tag{57}$$

Substituting (56) into (57) leads to

$$g_y = \frac{\theta}{\phi}(1 - \mu - \phi) g_N, \tag{58}$$

implying that the ratio of the growth rate of nominal earnings to the growth rate of population is the same before and after the Industrial Revolution.

We can now derive an expression relating the growth rate of *real* earnings to the growth rate of population as follows. From (47) we have

$$g_K = g_y + g_N - g_q. \tag{59}$$

This, when combined with (57), implies that

$$g_q = -\frac{1 - \theta}{\theta} g_y. \tag{60}$$

Substituting this result along with (55) into (53), we obtain

$$g_{yR} = \left(\alpha + \frac{\beta}{\theta}\right) g_y. \tag{61}$$

Combining the above result with (58), we finally find

$$g_{yR} = \frac{(1-\mu-\phi)(\alpha\theta+\beta)}{\phi} g_N. \tag{62}$$

Hence, the ratio of the growth rate of real earnings to the growth rate of population is the same before and after the Industrial Revolution, consistent with the simulation results presented in Section 4.2.

Given the relationship between the real earnings growth and population growth, we can now concentrate our attention on deriving expressions for g_N before and after the Industrial Revolution. Combining (60) and (58), we have

$$g_N = -\frac{\phi}{(1-\mu-\phi)(1-\theta)} g_q. \tag{63}$$

So the task now is to determine g_q (which must be negative).

Pulling A_{St} out of the bracket in (24), we obtain

$$q_t = \frac{1}{A_{St}} \left\{ (1-\gamma)^{\frac{1}{1-\rho}} \left(\frac{p_t A_{St}}{A_{Mt}} \right)^{-\frac{\rho}{1-\rho}} + \gamma^{\frac{1}{1-\rho}} \right\}^{-\frac{1-\rho}{\rho}}. \tag{64}$$

Let us define $psm := \frac{p_t A_{St}}{A_{Mt}}$. The simulation exercise revealed that in the case $\sigma < 1$ ($\rho < 0$), where the Malthusian sector dominates, psm is larger than one but small. When σ is small, ρ is large and negative, in which case the first term containing psm in the curly brackets dominates over the second term and we have

$$q_t \sim \frac{p_t}{A_{Mt}}. \tag{65}$$

Hence to determine the asymptotic growth rates before the Industrial Revolution, we need to determine the difference in growth rates between p_t and A_{Mt}.

After the Industrial Revolution when $\sigma > 1$ ($0 < \rho < 1$), where the Solow sector dominates, psm approaches infinity. In that case, the second term in the curly brackets dominates over the first and we have

$$q_t \sim \frac{1}{A_{St}}. \tag{66}$$

So to determine the growth rates after the Industrial Revolution, we need only determine g_{A_S}.

We will now analyze the two regimes, starting with the second.

5.1 The Industrial Revolution: $\sigma > 1$

We have the setting described by Eq. (66), namely

$$g_q = -g_{A_S}. \tag{67}$$

We now need to utilize Eq. (36), rewritten as

$$35 g_{A_S} = 35 g_0 + \xi ln \left[1 + \frac{y_{t+1} N_{t+1} x^*_{St+1}/q_{t+1}}{\sum_{t_i \le t} y_{t_i} N_{t_i} x^*_{St_i}/q_{t_i}} \right], \tag{68}$$

where we have introduced the factor of 35 since a time step corresponds to 35 years and we wish to express all growth rates in annualized terms. The trick now is to write the sum as a geometric series, which the sum approaches asymptotically

$$\sum_{t_i \le t} y_{t_i} N_{t_i} x^*_{St_i}/q_{t_i} \to \sum_{t_i \le t} e^{35(g_y + g_N + g_{x^*_S} - g_q)t_i} = \frac{1 - e^{(\dots)(t+1)}}{1 - e^{(\dots)}}, \tag{69}$$

where the dots represent $g_y + g_N + g_{x^*_S} - g_q$. Note that the initialization constants cancel in the above expression. The logarithm simplifies dramatically as $t \to \infty$ and Eq. (68) becomes

$$g_{A_S} = g_0 + \xi(g_y + g_N + g_{x^*_S} - g_q). \tag{70}$$

From (26) we find that

$$\frac{x^*_{St}}{q_t} \sim (A_{St} q_t)^{\frac{\rho}{1-\rho}} \to 1 \tag{71}$$

since the assumption was that $q_t \sim 1/A_{St}$. Hence, there is no ρ dependency and $g_q = g_{x_S^*}$. Accordingly, (70) simplifies to

$$g_{A_S} = g_0 + \xi(g_y + g_N). \tag{72}$$

Starting with (63) and (67), in (72) we can express g_{A_S} in terms of g_N and substitute (58) for g_y, yielding

$$g_N^{after} = \frac{g_0}{\frac{1-\mu-\phi}{\phi}[1 - \theta(1+\xi)] - \xi}. \tag{73}$$

The above formula gives a population growth rate that matches numerical results very well, given sufficient simulation time. Analytical expressions for all other growth rates, e.g. g_{yR}, follow immediately by successive substitution into the previous equations.

5.2 Before the Industrial Revolution: $\sigma < 1$

The same type of analysis can now be applied to the Malthusian case

$$q_t \sim \frac{p_t}{A_{Mt}}. \tag{74}$$

Similarly, we end up with

$$g_{A_M} = g_0 + \xi(g_y + g_N - g_p), \tag{75}$$

leading to

$$g_N^{before} = \frac{g_0}{\frac{1-(1+\xi)\theta}{\phi}(1-\mu-\phi) + (1-\xi)(1-\theta-\mu) + (1+\xi)(1-\mu)\frac{\theta-\phi}{\phi} - \xi}. \tag{76}$$

Again, it matches numerical results very well when enough time is allowed for the system to evolve for constant small σ.

5.3 Calibration of Parameters and Consistency Checks

The analytical expressions for the growth rates of population, real earnings, etc. derived above can be used to calibrate the parameters of the model to historical data. The purpose of this section is to show how the parameter values listed in Table 1 were arrived at. Those values were used in the simulation study described in Section 4.2. We then describe some further consistency checks that were carried out, comparing model predictions to historical data.

We started by setting $\xi = 1$, which means that a percentage increase in a component of capital (either Malthus or Solow) leads to the same percentage increase in the embodied productivity of that component. The share of capital in the Solow sector, $\theta = 0.4$, was taken from Hansen & Prescott (2002), as was the share of labor in the Malthus sector, $\mu = 0.6$. We chose $\beta = 0.2$, corresponding to a savings rate of 20% which resembles the U.S. rate (Jones, 2002). We then chose $\alpha = 0.5$ in order to equate the growth rates of nominal earnings and real earnings (see equation (61)).

Next, we assumed the following population growth rates

$$g_N^{before} = 0.05\%,$$
$$g_N^{after} = 0.5\%.$$

The first number is close to the rate of growth of world population between 0 and 1700 based on data from Maddison (2007).[9] The figure of 0.5% is in the middle of the range of values for industrialized countries during the twentieth century.[10]

We calibrated ϕ and g_0 to the population growth rates listed above. To obtain ϕ, we divided Eq. (73) by (76) and set the ratio equal to ten. The only parameter appearing in this expression that had not already been determined above was ϕ. Solving, we obtained $\phi = 0.04$, which is less than the value of 0.1% used by Hansen

[9] A simple computation using Maddison data gives a growth rate of 0.058%. A similar exercise using world population data from Kremer (1993) over the period -10,000 BCE (after the start of the Holocene) to 1700 gives a growth rate of 0.043%.

[10] Based on Maddison data. At the high end of the range we have the “Western Offshoots” (U.S. Canada, Australia and New Zealand) which had growth rates above 1% per annum during the twentieth century. However those rates were heavily influenced by immigration. At the low end of the range we have Germany and the U.K., each with growth rates of about 0.36% per annum.

& Prescott (2002). This result will be explored in the Discussion section. We then used equation (73) to calibrate g_0 based on the assumption that $g_N^{after} = 0.5\%$, resulting in $g_0 = 0.004$. We then set $g_\sigma = 0.004$ as well.

Lastly, $\gamma = 0.5$ was chosen arbitrarily. This parameter does not appear in any of the formulas for growth rates.

These choices of parameters led to $g_{yR}^{after} = 1.8\%$ per year, which is close to the values for the U.S. and Europe during the twentieth century.[11] Recall that according to the simulation results, the growth rate of real earnings was 1.9% after the Industrial Revolution, which is higher than the theoretical asymptotic value of 1.8%. The reason for this difference is that the simulated growth rate overshoots somewhat and then slowly converges to the value of 1.8% (see Figure 3).

As a consistency check for the preindustrial epoch, we compared the model-predicted asymptotic growth rates of real earnings and prices: $g_{yR}^{before} = g_p = 0.18\%$ (recall equations (55), (58) and (62)) to historical data. Figure 4 shows population, real per-capita income and the price of food in terms of non-food items (a proxy for p_t) for England covering the years 1270 to 1870.[12] The challenge in comparing theory to data is that the English economy was not on a steady-state growth path during the late middle ages. The population suffered a heavy blow in the second half of the fourteenth century due to plague and famine, and did not fully recover until the mid-seventeenth century. Nevertheless the cumulative growth rate between 1270 and 1700 was 0.034% per annum, close to the value of 0.05% assumed above. The growth rates of real income and prices between 1270 and 1700 are 0.17% and 0.13% per annum, respectively, close to the model prediction.[13]

[11]According to Maddison data, the growth rate of real per-capita income during the twentieth century was 1.9% for the U.S. and 1.88% for Western Europe. Note that we are using real empirical GDP per capita as a proxy for earnings.

[12]Population and income data is based on the latest figures from Broadberry *et al.* (2010), who are in the process of constructing comprehensive estimates based on output measures. The food prices represent the terms of trade between agriculture and industry, and are courtesy of R. Allen. A price series published by O'Brien (1985) shows similar trends between 1500 and 1830, but does not cover the period prior to 1500.

[13]The close agreement between model and data should of course be treated with extreme caution. The point of the above exercise is merely to show that the model is consistent with this particular set of data. Notice that the price series does not keep up with real income growth after after 1700, a point to be addressed in the Discussion section below.

It should be noted that in the simulation exercise the growth rates had not yet reached their asymptotic limits before the Industrial Revolution. If we allow the growth rates to converge to their theoretical values before the Industrial Revolution (e.g. by setting σ_0 to a smaller value), the growth rates overshoot quite a bit after the Industrial Revolution (see Figure 5).

6 Discussion

The model presented above is consistent with the historical fact that between 1800 and 1900 the rates of growth of population and real earnings increased markedly in the western hemisphere (Maddison, 2007). We now proceed to examine the more detailed implications of the model and to identify its shortcomings.

First, it is worth pointing out a theoretical limitation of the model. It has been assumed that individual preferences are not influenced by the kind of factors that have been used (for example) in the unified growth models of Galor & Weil (2000) and Galor & Moav (2002). Human capital does not enter into the model, and there is no tradeoff between quantity of children and quality of children. Hence our model cannot account for a decline in fertility after the Industrial Revolution. Our fertility model is "Malthusian" at all times. Despite this restriction, the fertility model provides a mechanism by which real earnings could rise after industrialization without a corresponding explosion in population. The presence of two sectors allows for the taming of population growth because the consumption of manufactured goods coincides with a rise in the cost of childrearing. One of the contributions of this study, therefore, is to suggest that growth-theorists might simplify their models of the demographic transition by introducing a second sector. For example, one could add a human capital element to the present two-sector model.

One of the predictions of the present model is that the rate of growth of real earnings in the preindustrial era was positive. A benefit of using a two-sector model is that it allows for this possibility even when population dynamics is Malthusian. This seemingly contradictory result can be understood by noting from equation (55) that $g_y = g_p$, which implies that earnings growth is zero when measured in units of Malthusian output, but *real* earnings (which includes a component of industrial consumption) can still be rising. Hence, we can square the assumption of Malthusian population dynamics, which has much empirical support (Clark, 2007, Ashraf & Galor, 2008), with our intuition that real income must have been rising (albeit slowly) prior to industrialization. A citizen of England circa 1750, who had access to reading glasses and was able to travel by sailing ship to distant lands, was surely richer than a hunter-gatherer living 100,000 years ago.

In Section 5.3 it was mentioned that the predicted growth rate of food prices starts to diverge from historical data after 1700. The model might be salvaged by accounting for food imports during the process of industrialization, or by assuming that the percentage of childrearing costs going to food purchases was declining over time, e.g. there was a growing service sector. We have already seen hint that such an extension may be necessary because the calibrated value of ϕ (the share of capital in the Malthus sector) is lower than the value used by Hansen & Prescott (2002). Since the Malthus sector is tied to childrearing, this observation suggests the need for a third *labor-intensive* sector tied to childrearing.

In Section 4.2 it was observed that the simulated fraction of total population employed in the Solow sector prior to the Industrial Revolution seemed high (55%). Allen (2000) estimates that only 26% of the English population was employed outside of agriculture in 1500. However as Weisdorf (2006) has pointed out, during the preindustrial era agricultural workers spent a large fraction of their time producing non-agricultural goods such as clothing. Hence the empirical estimate of 26% may not completely reflect the extent of labor allocated to industry. Prior to industrialization, farming communities were largely self-sufficient. It was only after industrialization that farmers devoted their entire working hours to agricultural production and traded their produce for manufactured goods.

The predicted fraction of capital employed in the Solow sector before the Industrial Revolution (95%) seems *very* high, so this constitutes a strong prediction. However, if the share of capital (θ) was lower in the preindustrial era than it is today, the model predicts that the fraction of capital employed in the preindustrial Solow sector would also be lower. Given that an increasing share of capital is a key requirement of our model, it might be fruitful to explicitly model that dynamics. Also, if α was lower (less income spent on manufactured goods), the fraction of capital employed in the Solow sector would again be lower.

Section 4.2 also reported that the predicted interest rate rises from 1.4% to 8.0% over the course of the Industrial Revolution. The direction of this prediction is counterfactual (Clark, 2007) but is consistent with Hansen & Prescott (2002). A small modification of the model suffices to reverse this result: the parameter β can be made an increasing function of time (increasing savings rate). This modification has no effect on the growth rates of N_t or y_t but affects the path of real earnings.

The main characteristic of the model presented in this paper is that there are simultaneous sharp increases in the growth rates of population and real earnings when EoS surpasses the value of one. This prediction is consistent with the accepted view of historians that there were simultaneous agricultural and industrial revolutions in England in the late eighteenth and early nineteenth centuries. The model predicts an Industrial Revolution when EoS surpasses the value of one because the industrial sector is released from the constraints of agriculture. This is the theme of Wrigley's narrative of industrialization (Wrigley, 1988, 2010), his archetypical example being the replacement of horses with steam engines. What is less obvious from a historical point of view is how this event, EoS = 1, might have lead to an *agricultural* revolution.

The model predicts that when EoS surpasses one, the capital goods used in the Malthus sector (agriculture) are increasingly built out of Solow output. The best way to test this prediction is to look for historical links between agricultural improvements and industry. Adam Smith observed that agriculture tended to flourish in regions located near large towns and cities, and offered three possible explanations for that link.[14] First, he believed that towns spurred agricultural improvement because the demand for food was higher near towns than elsewhere. Second, he believed that towns and cities introduced good governance into the surrounding country. Third, he believed that urban merchants invested their wealth into the country because they saw opportunities that were not apparent to the country gen-

[14]Chapter IV, "How the Commerce of the Towns Contributed to the Improvement of the Country", Smith (1776). Jacobs (1970) postulated that cities came before agriculture and that cities have always been strong drivers of agricultural growth.

tlemen. Smith did not detail the forms that these investment might have taken, but one possibility is that urban capitalists were introducing capital goods into the countryside.

One example where manufacturing had a positive impact on agriculture during the eighteenth century was in the provisioning of plows (Brunt, 2003). Plowshares made of iron had about twice the efficiency of the older wooden plowshares (Sieferle, 2001), and were supplied by blacksmiths working in the neighboring towns. Brunt (1997) has conducted a cross-sectional study of wheat yields in the late eighteenth century, looking to uncover significant drivers of productivity. He has found that the two most significant improvement factors, both of which were forms of capital investment, were the planting of turnips the use of seed drills. Turnips assisted in the provisioning of humus to the soil, and can be thought of as a purely agricultural innovation. Seed drills were a manufactured product that allowed for more efficient weeding. Other factors of importance were also linked to industry: fertilizers such as marl and lime, and the use of drainage pipes. Marl and lime had to be transported from distant locations and so benefited from the development of the canal and railway networks (Mathew, 1993). Regarding drainage pipes, Brunt notes that their cost declined in the nineteenth century, leading to a large increase in use. This is a good example of a reduction in the cost of Solow output (higher p_t) leading to higher levels of capital. Taking a longer point of view, agriculture experienced huge increases in productivity in the twentieth century after the introduction of tractors, pesticides and industrial fertilizers.

Finally we should note that the model presented in this paper includes some restrictive assumptions that could be loosened. For example, the exogenous growth rates are assumed to be the same in the Malthus and Solow sectors. However there may be good historical reasons to believe that innovation occurred faster for manufactured goods than for agricultural goods. The workings of manufactured items might well have been understood by our ancestors, but the inner workings of agricultural goods such as horses were undoubtedly a complete mystery (physics came before biology). The learning-by-doing spillover parameter could also be made higher in the Solow sector than in the Malthus sector to reflect the tendency of industry to be located in densely populated towns and cities. The growth of EoS could be made endogenous by linking it to capital investment.

7 Conclusion

This paper has described a unified growth model based on Wrigley's thesis that a key enabler of the Industrial Revolution was a shift in the source of raw materials from agriculture to industry (Wrigley, 1988, 2010).

The main contributions of the paper are as follows. First, we have demonstrated the benefit of working with two sectors in developing a unified demographic model. Second, we have shown that Arrow's learning-by-doing model allows one to link an increase in the growth rate of productivity to an increase in the share of capital. Finally, we have proposed that the elasticity of substitution between industrial output and agricultural output was slowly rising over history, and that once it reached the critical value of one there were simultaneous agricultural and industrial revolutions.

The study has suggested several directions for future investigation. It would be fruitful to extend the model to include human capital and to include a service sector tied to childrearing in order to better capture the changing demographic patterns after the Industrial Revolution. As well, changes in the elasticity of substitution could be linked to investment decisions. Lastly, given the fact that the most important examples of substitution during the Industrial Revolution were related to energy use and conversion, future empirical work should include an explicit energy component.

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References

[1] Allen, R.C. (1983) Collective Invention. *The Journal of Economic Behavior and Organization* **4**: 1-24.

[2] Allen, R.C. (2000) Economic Structure and Agricultural Productivity in Europe, 1300-1800. *European Review of Economic History* **3**: 1-25.

[3] Allen, R.C. (2001) The Great Divergence in European Wages and Prices from the Middle Ages to the First World War. *Explorations in Economic History* **38**: 411-447.

[4] Arrow, K.J. (1962) The Economic Implications of Learning by Doing. *The Review of Economic Studies* **29**: 155-173.

[5] Ashraf, Q., Galor, O. (2011) Dynamics and Stagnation in the Malthusian Epoch. *The American Economic Review* **101**: 2003-41.

[6] Broadberry, S., Campbell, B., Klein, A., Overton, M., van Leeuwen, B. (2010) British Economic Growth, 1270-1870. Part of the Project "Constructing the National Income of Britain and Holland, c.1270/1500 to 1850", Leverhulme Trust.

[7] Brunt, L. (1997) Nature or Nurture? Explaining English Wheat Yields in the Agricultural Revolution. Discussion Paper. Nuffield College (University of Oxford).

[8] Brunt, L. (2003) Mechanical Innovation in the Industrial Revolution: The Case of Plough Design. *The Economic History Review* **56**: 444-477.

[9] Clark, G. (2005) The Condition of the Working Class in England, 1209-2004. *The Journal of Political Economy* **113**: 1307-1340.

[10] Clark, G. (2007) *A Farewell to Alms*. Princeton University Press, Princeton.

[11] Crafts, N.F.R. (1995) Exogenous or Endogenous Growth? The Industrial Revolution Reconsidered. *The Journal of Economic History* **55**: 745-772.

[12] Galor, O., Weil, D. (2000) Population, Technology and Growth: From Malthusian Stagnation to the Demographic Transition and Beyond. *The American Economic Review* **90**: 806-828.

[13] Galor, O., Moav, O. (2002) Natural Selection and the Origin of Economic Growth. *The Quarterly Journal of Economics* **117**: 1133-1191.

[14] Galor, O. (2011). *Unified Growth Theory.* Princeton University Press, Princeton.

[15] La Grandville, O. de, (2009). *Economic Growth: A Unified Approach.* Cambridge University Press, Cambridge.

[16] Hansen, G.D., Prescott, E.C. (2002) Malthus to Solow. *The American Economic Review* **92**: 1205-1217.

[17] Hardy, G.H., Littlewood, J. E. Polya, G. (1934). *Inequalities*, Second ed., Cambridge University Press, Cambridge.

[18] Jacobs, J. (1970) *The Economy of Cities*. Vintage, New York.

[19] Jones, C., (2002). *Introduction to Economic Growth*, Second ed., Norton & Company, New York.

[20] Kremer, M. (1993) Population Growth and Technological Change: One Million B.C. to 1990. *The Quarterly Journal of Economics* **108**: 681-716.

[21] Lucas, R.E. (2009) Ideas and Growth. *Economica* **76**: 1-19.

[22] Maddison, A. (2007) The contours of the world economy 1-2030 AD. Oxford University Press, Oxford. Data available from http://www.ggdc.net/MADDISON

[23] Mathew, W. (1993) Marling in British Agriculture: A Case of Partial identity. *Agriculture History Review* **41**: 97-110.

[24] O'Brien, P. (1985) Agriculture and the Home Market for English Industry, 1660-1820. *The English Historical Review* **100**: 773-800.

[25] Sieferle, R. (2001) *The Subterranean Forest: Energy Systems and the Industrial Revolution*, The White Horse Press, Cambridge.

[26] Smith, A., (1776). *The Wealth of Nations.*

[27] Weisdorf, J. (2006) , From Domestic Manufacture to Industrial Revolution: Long-Run Growth and Agricultural Development. *Oxford Economic Papers* **58**: 264-287.

[28] Wrigley, E.A. (1988) *Continuity Chance and Change. The Character of the Industrial Revolution in England.* Cambridge University Press, Cambridge, U.K.

[29] Wrigley, E.A. (2010) *Energy and the English Industrial Revolution.* Cambridge University Press, Cambridge, U.K.

Parameter	Value
g_0	0.004
g_σ	0.004
α	0.5
β	0.2 (Jones, 2002)
γ	0.5
μ	0.6 (Hansen & Prescott, 2002)
ϕ	0.04
σ_0	0.01
θ	0.4 (Hansen & Prescott, 2002)
ξ	1.0

Table 1: *Parameters used in the simulation exercise.*

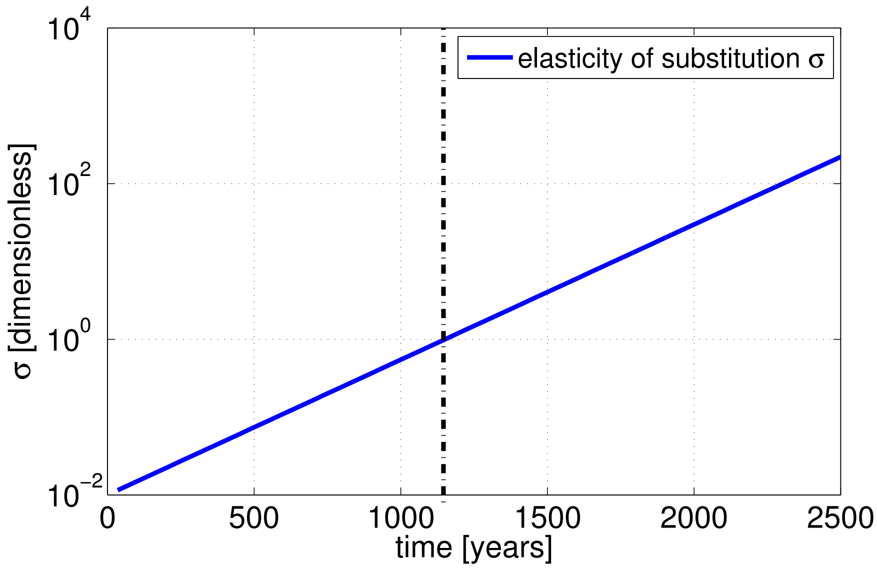

Figure 1: *The elasticity of substitution, σ, grows exponentially with time, starting at $\sigma_0 = 0.01$ and ending at $\sigma \gg 1$. It surpasses the value of one at $t = 1155$ years.*

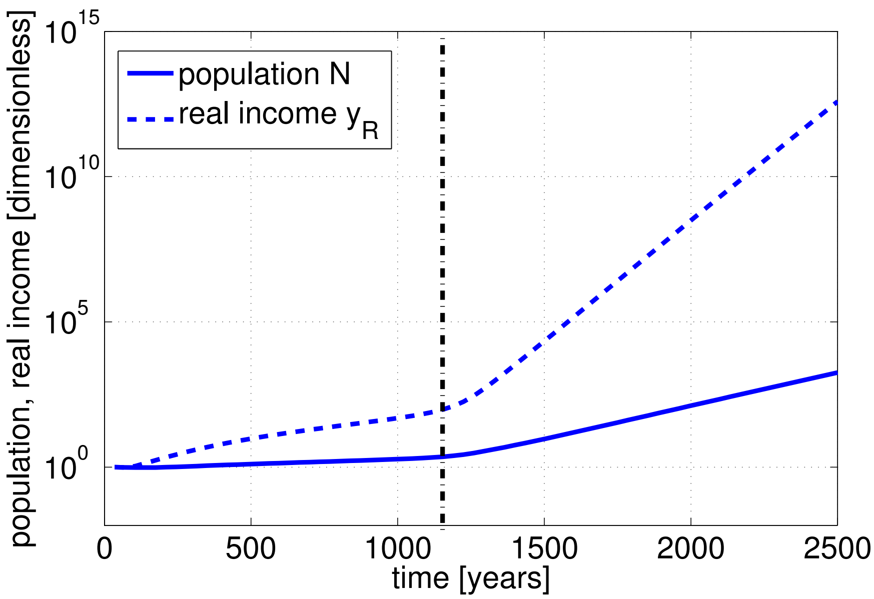

Figure 2: *Population and real income as a function of time, normalized by their respective values at $t = 0$. At $t = 1155$, equal to 33 time steps, σ passes one. Around this point in time, we can clearly observe a transition in dynamics between the two growth regimes.*

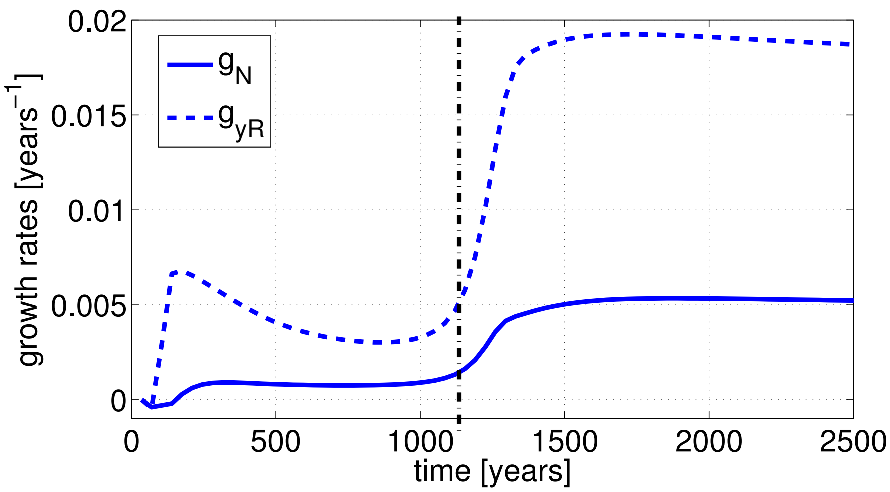

Figure 3: *Growth rates of population and real income versus time. After a rapid initial adjustment ($t < 200$) when the full growth model is switched on at $t = 0$, the population growth rate is nearly constant while the growth rate of real income converges from above to a constant value. However, as σ approaches one and eventually passes through one, these small growth rates change by an order of magnitude. This transition occurs within about four to five generations, equivalent to* 150 *years.*

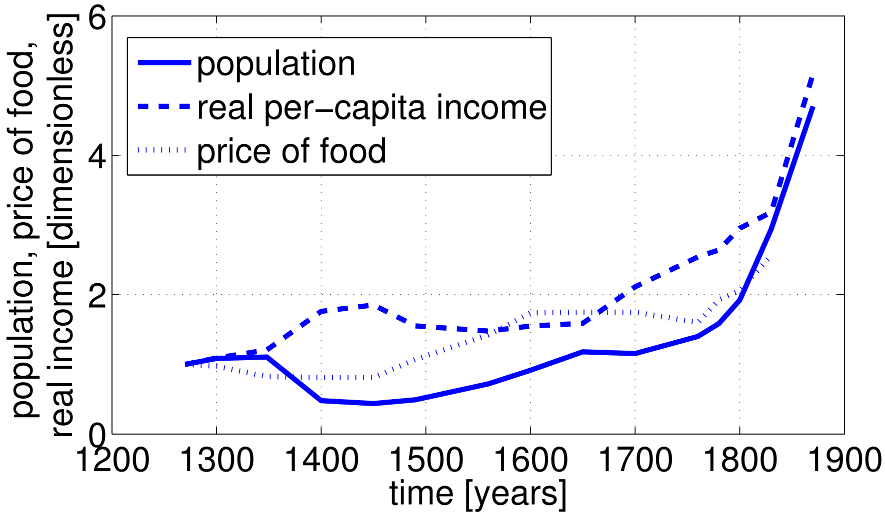

Figure 4: *Population, real per-capita income and the relative price of food (a proxy for p_t) for England, 1270-1879. Population and income data is from Broadberry* et al. *(2010). Food prices are in terms of non-food items (e.g. manufactured goods) and are courtesy of Robert Allen (Oxford University; personal communication). All series are normalized to the value of one in the year 1270.*

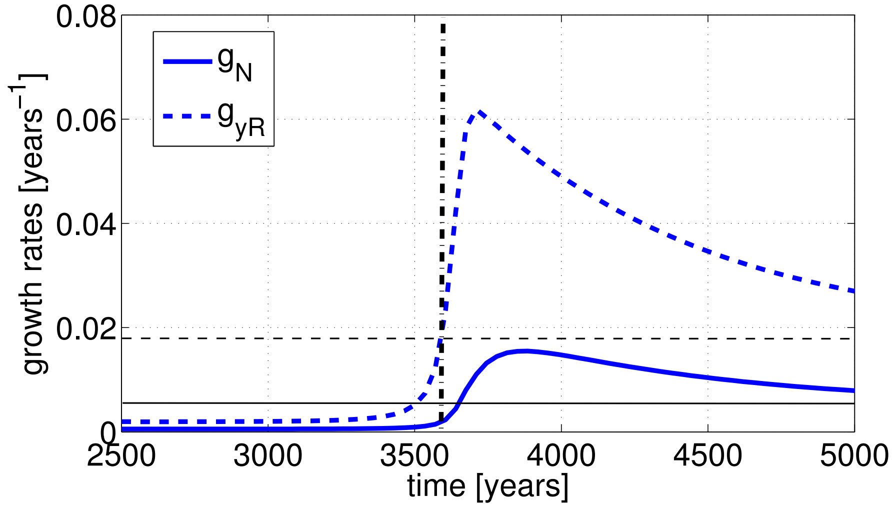

Figure 5: *Growth rates of population and real income versus time when the rates are allowed to reach their asymptotic values before the Industrial Revolution. The horizontal lines correspond to the theoretical long-term growth rates based on equations* (62) *and* (73).