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Dai, Darong

Department of Economics, School of Business, Nanjing University

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Darong Dai

Department of Economics, School of Business, Nanjing University, Nanjing 210093, P. R. China

E-mail: daidarong998@163.com.

Abstract

In the current paper, we study the asymmetric normal-form game between two heterogeneous groups of populations by employing the stochastic replicator dynamics driven by Lévy process. A new game equilibrium, i.e., the game equilibrium of a stochastic differential cooperative game on time, is derived by introducing optimal-stopping technique into evolutionary game theory, which combines with the Pareto optimal standard leads us to the existence of Pareto optimal endogenous matching. Moreover, stability of the Pareto optimal endogenous matching is confirmed by essentially using the well-known Girsanov Theorem.

Keywords: Stochastic differential cooperative game on time; Endogenous matching; Stability; Fairness; Adaptive learning.

JEL classification: C62; C70; C78.

1. Introduction

It is convincing to argue that people live in a highly *structured* society consists of groups rather than individuals, which implies that random matching will not always provide us with compelling approximation to reality when we are concerned with the interactions among the players. In fact, Ellison (1993) shows that local interaction will have very important and also different implications in equilibrium selection relative to that of uniform interaction or random matching. So, given the importance of non-random matching in equilibrium selection, we express the *motivation* of the present paper as follows, i.e., can we directly prove the existence and stability of certain non-random matching that is Pareto optimal and also endogenously determined in a given game situation? If we can, what are the conditions we will rely on? In other words, the major goal of the present exploration is not to study any exogenously given matching mechanism but to find out the optimal matching mechanism in a given game situation, and to prove its stability.[1]

In two pioneering papers, Kandori et al. (1993) and Young (1993) prove that the trial-and-error learning processes of the players will definitely converge to one particular pure-strategy Nash equilibrium, which is named as the *long run equilibrium* by Kandori et al. and the *convention* by Young. From the perspective of multiple-equilibrium problem, they provide us with an equilibrium selection device, under which the players are correctly predicted to play a particular Nash equilibrium. However, we can also evaluate their contribution from the following view of point, i.e., provided a particular Nash equilibrium, they prove that there exists a pattern of learning mechanism that will definitely lead the players to play the given Nash equilibrium. To summarize, they confirm the existence of certain type of learning mechanism, based upon which the players' behavior will be uniquely predicted in the long run. Instead of emphasizing *micro-strategy*, we focus on *macro-structure* and it is confirmed that there exist certain macro-structure under which one particular Pareto optimal Nash equilibrium will be definitely played by the players. Obviously, in order to derive much more comprehensive understanding of the strategic behaviors of the individuals in a given society, micro-strategy and macro-structure should be explored as a whole. Accordingly, the present study also examines the internal relationship between the micro-strategy and the macro-structure by analyzing the internal relationship between the learning mechanism and the matching mechanism. That is to say, the Pareto optimal endogenous matching as well as the Pareto-optimal Nash equilibrium can be regarded as the limit of the learning processes of the players in some sense by noting that there exists a one-to-one correspondence between the Pareto optimal endogenous matching and the Pareto optimal Nash equilibrium. Furthermore, if we argue that different matching mechanisms imply different Nash equilibria, we have demonstrated that there exists certain learning mechanism under which one particular matching mechanism will be achieved, that is, one particular micro-strategy implies one particular macro-structure under certain relatively weak conditions. To conclude, we indeed prove the following two important and also interesting claims: first, provided a particular Nash equilibrium, we show the existence of one matching mechanism such that the given

[1] That is to say, in an artificial world, we can employ the matching mechanism to lead the players to play the Pareto optimal Nash equilibrium regardless of the enforcement cost. And in this sense, matching mechanism plays the role of equilibrium selection device.

Nash equilibrium will be endogenously chosen by the players as a rational prediction; second, given a particular matching mechanism, we demonstrate the existence of certain learning mechanism so that the given matching mechanism will be endogenously established by all the players spontaneously in the long run. We therefore believe that the present study has supplied an interesting and also relatively complete characterization of the internal relationship between the micro-strategy employed by the players and the macro-structure facing the players.

In the paper, we are encouraged to study the asymmetric normal-form games between two heterogeneous groups of populations under the modified framework of evolutionary game theory.[2] Each of the two groups is assumed to have countable many pure strategies. Hyper-rational assumptions (see, Aumann, 1976) about the players broadly used in classical non-cooperative game theory will be dropped in the present model, instead, the players or individuals play the game following certain adaptive learning processes arising from the stochastic replicator dynamics driven by Lévy processes for the first time.[3] On the contrary, the strategies themselves are supposed to be smart and rational enough to optimize their fitness[4], which directly depend on the stochastic replicator dynamics or the learning processes of the players, following the classical *as if* methodology from the perspective of posteriori. And the corresponding control variables of these fitness-optimization problems[5] are chosen to be stochastic stopping times or stopping rules, which reasonably reflects the fact that strategies themselves are no longer suitable for the roles of control variables as in the best-response correspondences of Nash equilibria because "strategies" of the players' strategies will not be well-defined through the traditional approach. Luckily, noting that the optimal stopping rules are partially determined and completely characterized by the learning processes of the players, the optimal stopping rules as a whole may be exactly one of the Nash equilibra, no matter it is a mixed-strategy Nash equilibrium or a pure-strategy Nash equilibrium, of the original normal-form games derived from the best-response approach.[6] Generally speaking, the

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[2] It will be without loss of nay generality when focusing on the case of two heterogeneous groups of populations by noting that two-sided markets broadly exist in reality, for instance, the marriage market and the labor market.

[3] We extend the pioneering stochastic replicator dynamics of Foster and Young (1990) and Fudenberg and Harris (1992) to Lévy processes by emphasizing the role of *jumps* in learning processes. Binmore and Samuelson (1999) show the importance of drift in equilibrium selection, and here we point out that jumps not only really happen in social and biological evolution but also play crucial role in equilibrium selection.

[4] We prefer fitness to payoff because payoff will neglect some important and even determinant factors in equilibrium selection. Fitness will be a much more complete characterization of the objective of the player than payoff. Fitness not only focuses on the game itself like payoff but also pays attention to other factors, such as the environment where the game happens and also the importance of the game to the players by noting that players usually are faced with many different or alternative games at the same time, which often leads to the fact that there exist substitutive and complementary relationship between these games from the perspective of the players, thereby violating the usually implicit assumption that each game is regarded as an isolated one. To sum up, the concept of fitness will capture much more relevant factors of the game situations facing the players, including objective factors like payoff structure and also subjective factors like the degree of game participation.

[5] In the current model, we do not incorporate inter-temporal consideration like that of Matsui and Matsuyama (1995) into the present optimization problem because we insist that the present case is of independent interest.

[6] One major difference between the traditional non-cooperative game theory and evolutionary game theory is that we do not give dynamics characterizing the evolution of the strategy distribution in priori in the former case. In the traditional approach, individually and decentralized rational choice leads to the game equilibrium, which implies that strategies themselves are suitable control variables for the best response problems to derive Nash equilibria. However, in evolutionary game theory, we give dynamics reflecting the learning processes of the players to characterize the evolution of total strategy distribution over the populations, then we study certain

optimal stopping rules as a whole will not be equal to anyone of the Nash equilibria, that is, there exists certain difference between the both. However, it is confirmed that it is just the difference between the optimal stopping rules as a whole and the Pareto optimal Nash equilibrium of the original normal-form game that established our Pareto optimal endogenous matching. We, hence, to the best of our knowledge, enrich the matching rule widely used in evolutionary game theory by naturally adding into economic-welfare implications for the first time.[7]

Moreover, it is shown that the well-known random matching (e.g., Maynard Smith, 1982; Fudenberg and Levine, 1993; Ellison, 1994; Okuno-Fujiwara and Postlewaite, 1995; Weibull, 1995) just represents one special and extreme case of the current endogenous matching and we supply the conditions under which the random matching will be *asymptotically Pareto efficient*.[8] Thus, proving the existence of Pareto optimal endogenous matching would be regarded as one innovation of the present paper by noticing the above facts.

Up to the present step of our story, we have been provided with a Pareto-optimal endogenous matching in the current game situation by solving the above fitness-optimization problems of the

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limiting distribution, i.e., sending the number of population to infinity, sending the time to infinity or sending the mistakes to zero, and we finally compare it with the classical solution concept like Nash equilibrium. Therefore, if we argue that the traditional approach focuses on strategy-space dimension, evolutionary game theory pays relatively more attention to evolutionary-time dimension (see, Binmore and Samuelson, 1997, 1999; Binmore et al., 1995). Evolutionary game theory prefers to study the basin of attraction (see, Ellison, 2000), the steady state (Fudenberg and Levine, 1993) or the rest point of the evolutionary dynamics or learning processes, we argue that *optimal stopping rule* plays the similar role except that rest point of dynamics only depends on the properties of the dynamics themselves while optimal stopping rule adding into a rational constraint. To summarize, the traditional approach emphasizes *micro-strategy* from the perspective of individual choice while evolutionary game theory focuses more on *macro-structure* from the viewpoint of group evolution, and the method introduced in the present paper will supply a *linkage* between the both, that is to say, optimal stopping rule is partially determined and completely characterized by macro-structure while it is also partially determined by micro-strategy.

7 Existing studies usually focus on the enforcement or reputation mechanism for given matching mechanism (see, Kandori, 1992, for instance). Then they explore the corresponding welfare implications of the enforcement or reputation mechanism. We, however, directly examine the welfare implications of the matching mechanism by noting that it will lead us to the Pareto efficient equilibrium.

8 For any given game, different matching patterns imply different payoffs for the players. Rather, if we let the payoffs corresponding to random matching, which does work in a perfect world with well-mixed population, denote the benchmark, the payoffs defined by any non-random matching would be regarded as certain perturbations to the benchmark-payoffs by noting the linearity of von Neuman-Morgenstern payoff functions. And in this sense, we argue that random matching just represents a special case where we have sent the payoff- perturbations to zero. However, why we argue that non-random matching, especially endogenous matching, is of crucial importance? Besides the argument of Ellison (1993), we point out the following problem, that is, random matching usually leads the dynamics or learning processes to equilibrium that is not Pareto efficient and even Pareto inefficient (see, Weibull, 1995, for instance). Consequently, in order to prevent the dynamics or learning processes from being attracted into the Pareto inefficient rest point, we introduce mutations or perturbations into the dynamics or the learning processes to produce efficient equilibrium (see, Canning, 1992; Binmore and Samuelson, 1999). And we show that non-random matching mechanism will be a suitable choice. Furthermore, it is easily noticed that most of the existing literatures (see, Fudenberg and Levine, 1993; Kandori et al., 1993; Young, 1993, and among others) employ the random matching to study the game played by a *large* population of players, however, for the games in reality, random matching is much more suitable for the case that consists of *small* population of players, for example, in a village or in a community. Let us consider a gift-giving game in a village or in a community, and it is reasonably to suppose that the players will interact with each other *equally* thanks to the reputation effect or enforcement effect. Now, let us consider the same gift-giving game with the players coming from two *isolated* villages or communities, we can easily find that the interacting frequency in each village or community will be much higher than that between the two villages or communities. That is to say, this is an imperfect world and people live in a highly structured society. To sum up, if we study the game played by a small population of players, random matching really works, however, if we study the game played by a large population of players, random matching should not be directly applied to the whole population, and the population should be divided into many sub-populations (see, Young (1993)) and we apply random matching to each sub-population while non-random matching will be suitable for the interactions between these sub-populations.

strategies and then *smoothing* the possible or potential difference between the corresponding optimal stopping rules as a whole and the Pareto optimal Nash equilibrium of the original normal-form game. In other words, the Pareto optimal Nash equilibrium of the original asymmetric normal-form game can be actually achieved by the two heterogeneous groups of populations as a rational solution of the above fitness-optimization problems given the existence of the Pareto optimal endogenous matching. Now, we proceed to the next step of demonstrating the stability of the Pareto optimal matching given its existence. Noting that there exists a one-to-one correspondence between the Pareto optimal matching and the Pareto optimal Nash equilibrium, we then just need to prove the stability of the Pareto optimal Nash equilibrium, and also it will be confirmed that this equivalent transformation will apparently and greatly lower the technical requirement. Indeed, we prove that the adaptive learning processes will uniformly and robustly converge to the above Pareto optimal Nash equilibrium as the time approaching infinity while the errors or stochastic perturbations in the learning processes always exist except that they are reasonably controlled in certain region following from the martingale property. That is to say, the learning processes will robustly converge to the modified[9] optimal stopping rules as a whole, i.e., the Pareto optimal Nash equilibrium, in the sense of uniform topology as long as the adaptive learning processes exhibit martingale property, which, however, can be established by applying the well-known Girsanov Theorem under certain weak conditions[10]. Accordingly, the present paper not only proves the stability of the Pareto optimal endogenous matching but also confirms the following important and also interesting byproduct, i.e., we claim: the adaptive learning processes of the individuals will uniformly and robustly converge to the Pareto optimal Nash equilibrium, which is exactly the *rational solution* of the above fitness-optimization problems of the strategies given the Pareto optimal endogenous matching, of the original normal-form game as long as they exhibit martingale property.[11] We, hence, argue that this conclusion would be regarded as one major contribution of the current study when compared with existing literatures, for example, first, existing literatures[12] (see, Canning, 1992; Young, 1993; Kandori et al., 1993) proved the similar convergence essentially requiring that the errors or perturbations approach zero; second, existing literatures showed that their learning processes will either converge to a mixed-strategy Nash equilibrium (e.g., Fudenberg and Kreps, 1993; Benaïm and

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9 It is modified by the Pareto optimal endogenous matching.

10 That is, the Novikov conditions are assumed to be fulfilled.

11 What's the aspiration of this conclusion? We emphasize the following three points: first, endogenous matching mechanism will meet the gap between the Pareto optimal Nash equilibrium and the limiting behavior of evolutionary dynamics or adaptive learning processes, and this Pareto optimal endogenous matching mechanism is exactly the matching mechanism that requires the least information, especially in computation; second, one can directly model stochastic learning processes with martingale property in future research; finally, our argument is in line with some of the existing studies (see, Harsanyi, 1973; Canning, 1992; Fudenberg and Kreps, 1993; Binmore and Samuelson, 1999) by noting that martingale process itself is a stochastic process, i.e., there exist persistent stochastic perturbations in the corresponding learning processes and hence the payoffs.

12 Canning (1992) shows that, under certain regularity conditions, the stationary distribution of the perturbed process converges to a stationary distribution of the unperturbed one. Kandori et al. (1993) show that the stochastic evolutionary learning process defined on symmetric 2×2 games selects the risk dominant Nash equilibrium when the mistake probability is small. In his seminal paper, Young (1993) shows that the adaptive dynamics defined by random sampling will converge almost surely to a pure strategy Nash equilibrium, which he specifically names as the stochastically stable equilibrium, when the likelihood of mistakes goes to zero, otherwise, then the limiting distribution will occasionally switches from one pure strategy Nash equilibrium to another pure strategy Nash equilibrium.

Hirsch, 1999; Ellison and Fudenberg, 2000, and among others) or a pure-strategy Nash equilibrium (see, Young, 1993; Kandori et al., 1993) depending on the types of learning processes they specified[13] while the current exploration confirms that convergence always happens under weak conditions and also the limit will be a Pareto optimal Nash equilibrium given the endogenous matching mechanism, thus, we supply a *unified* framework by introducing the endogenous matching; third, convergence of the learning processes not only implies the Pareto-dominant equilibrium in coordination games but can also yield cooperation equilibrium in PD games by slightly modifying the endogenous matching mechanism,[14] which reflects that matching mechanism as well as learning mechanism should be paid at least equal attention to in our study;[15] forth, the convergence result implicitly argues that the learning approach need not to be absolutely different from the traditional rational-approach, otherwise, we can tell why the difference exists and what forms the difference, and finally we demonstrate that the endogenous matching mechanism will provide us with a practical bridge that links the learning approach and the traditional approach, thereby effectively meeting the so-called unbridgeable gap; last but not least, our robust convergence happens in a persistently non-stationary environment and in the sense of uniform topology, and hence it is obviously much stronger than that of existing studies (see, Kandori et al., 1993; Young, 1993; Fudenberg and Kreps, 1993; Benaïm and Hirsch, 1999) after a quick check.

Although the major contribution of the present limited study has been expressed above, the following innovations are also worth noticing in some sense. Indeed, the existence and stability of the Pareto optimal endogenous matching are not necessarily independent of each other. For instance, on the one hand, one can easily find that the stability assertion intimately depends on the characterization of the existence result by checking the details of the following proof in Appendix C.[16] On the other hand, the expected fitness of the strategies will also exhibit martingale property if the corresponding adaptive learning processes are martingale processes by noting the mathematical conclusion that martingale property keeps invariant under affine transformation. To conclude, stability produces existence in turn in the following sense, i.e., stability of the Pareto optimal endogenous matching implies the existence of Pareto optimal matching with *fairness* because the game between different strategies will become a *fair-game* after the *martingale-payoffs* being incorporated into the game-situations. And this is why the word "fairness" specifically appears in the title of the paper.

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13 For example, one major difference between Fudenberg and Kreps's (1993) model and Young's (1993) model is that Fudenberg and Kreps use a generalization of fictitious play where the players asymptotically choose the best replies to other players' past actions based upon the entire historical frequencies, while the players base their decisions on limited information in Young's model. Moreover, in contrast to the model of Fudenberg and Kreps, the players do not always optimize in Young's model.

14 Noting that the risk-dominant equilibrium usually has a larger basin of attraction than the Pareto dominant equilibrium, and both-defect is the only Nash equilibrium in PD games, the endogenous matching mechanism truly plays a key role in equilibrium selection.

15 It is compelling that matching mechanism would be regarded as an equilibrium selection device in some sense. Indeed, the current study is concerned with the case that matching leads to unique equilibrium. Moreover, the stability of the matching mechanism implies the stability of the equilibrium selection.

16 However, similar to that of Young (1993), the existence of the endogenous matching does not supply a sufficient condition for the convergence and hence stability of the endogenous matching.

Furthermore, the present paper defines a much stronger stability concept of equilibrium strategy and hence matching pattern by naturally combining the traditional interpretation and the evolutionary interpretation. Because the strategies are proposed to be *as if* rational "players" in some sense, it is easily discovered below that the optimal stopping rules as a whole is computed by satisfying the following two constraints, i.e., individually-rational solution and no blocks exist[17], which implies certain stability from the concept of Nash bargaining solution (see, Nash, 1950) in cooperative game theory. However, we do not stop here by just focusing on rational requirements of stability from the viewpoint of *micro-strategy*. We also emphasize the importance of evolutionary interpretation from the group level as a whole, i.e., the *macro-structure*. We do this by building up new adaptive-learning processes via introducing exogenous perturbations into the original learning processes, and we prove that the new learning processes and the original learning processes will converge to the same equilibrium as the exogenous perturbations approaching zero. We should specifically emphasize that the original learning processes themselves are driven by Lévy processes, i.e., including both diffusion terms and jump terms. Thus, in order to build up new learning processes, all we have to do is to disturb the original drift terms, diffusion terms and the original jump terms. Consequently, the new adaptive learning processes are also driven by Lévy processes. To summarize, we check the stability from the viewpoint of macro-structure by not essentially changing the errors or perturbations existing in the original learning processes, and we take limit just by sending the exogenous perturbations to zero.

In the next section, some well-known examples of crucial importance in non-cooperative game theory will be presented and discussed to help capture some basic ideas and intuitions of the formal model. Section 3 will construct the formal model, introduce some basic concepts and prove the key theorem of the present paper. Section 4 is used to demonstrate the stability of the Pareto optimal endogenous matching. There is a brief concluding section. All proofs, unless otherwise noted in the text, appear in the Appendix.

2. Examples

Before constructing formal models, we will first introduce some well-known and also simple examples of non-cooperative game theory in the current section. And it is believed that these examples would help a lot in understanding the formulation in the following section and capturing the economic intuitions behind the model for the readers. Moreover, it is worth noting that these examples would have effectively reflected some ideas of the formulation, they are nevertheless far

[17] Here, we interpret "no blocks exist" in the following sense, that is, everybody is coordinated to follow the optimal stopping rule in each group of populations and potential departure is avoided.

away from representing the whole story of the current investigation.

EXAMPLE 1—Classical Prisoner's Dilemma.

		Player 2	
		C	D
Player 1	C	a, a	c, d
	D	d, c	b, b

Figure 1 - Symmetric PD Game

In Figure 1, C and D are used to denote strategies cooperation and defection, respectively. And it is assumed that $d > a > b > c$. As usual, the entries in the matrix represent corresponding payoffs of player 1 and player 2, respectively, for any given strategy choices. For instance, (c, d) implies that player 1 will get payoff c if she chooses strategy C given player 2 chooses strategy D, and vice versa. And it is well known that (D, D) is the only Nash equilibrium in one-short situations and for rational players with common knowledge (see, Aumann, 1976) although (C, C) strictly Pareto dominates it. And most of excellent existing literatures have been devoted to searching for possible mechanism, i.e., enforcement mechanism[18], rational mechanism[19], learning mechanism[20] and evolutionary mechanism[21], so that (C, C) will be actually chosen, thereby breaking through the so-called dilemma. In the current paper, we study the mechanism of endogenous matching. Indeed, one can easily note below that our approach is equivalent to that of disturbing payoff to some extent and in some sense. Noting that pure strategy stands for one special case of mixed strategy, i.e., all other pure strategies are given zero-probability weights while the chosen pure strategy with one-probability weight. Moreover, by noting the symmetry, we just consider the case that player 2 chooses to randomize his strategy choice between C and D, i.e., she chooses C with probability p and chooses D with probability $1-p$ with $0 \le p \le 1$. Now, as the players are involved in a structured society, we introduce endogenous matching by adding ε to p while adding $-\varepsilon$ to $1-p$, that is to say, we get a new weighted form $(p+\varepsilon, 1-p-\varepsilon)$. And we define the generalized

18 See, Fudenberg and Maskin (1986), Kandori (1992), Ellison (1994) and among others.

19 See, Kreps et al. (1982), Andreoni and Samuelson (2006).

20 See, Selten and Stoecker (1986), Kirchkamp and Nagel (2007).

21 See, Axelrod (1984), Fudenberg and Maskin (1990), Young and Foster (1991), Nowak et al. (2004), Imhof (2005), Imhof and Nowak (2006) and among others.

expected payoffs of player 1 by,

$$EU_1(\text{C},\ (p+\varepsilon,1-p-\varepsilon)) \triangleq a(p+\varepsilon)+c(1-p-\varepsilon) = \left[ap+c(1-p)\right] +(a-c)\varepsilon\ ,$$

And,

$$EU_1(\text{D},\ (p+\varepsilon,1-p-\varepsilon)) \triangleq d(p+\varepsilon)+b(1-p-\varepsilon) = \left[dp+b(1-p)\right]+(d-b)\varepsilon\ ,$$

where $(a-c)\varepsilon$ and $(d-b)\varepsilon$ could be regarded as perturbations of the expected payoffs of strategy C and strategy D, respectively, for player 1 and provided that player 2 randomizes between strategy C and strategy D by $(p,1-p)$ with $0 \le p \le 1$. Letting $EU_1($ C, $(p+\varepsilon,\ 1-p-\varepsilon)) = EU_1($ D, $(p+\varepsilon,1-p-\varepsilon))$, we obtain $\varepsilon = \left[(a-d)p+(c-b)(1-p)\right]/(d-b+c-a)$. Noting that (C, C) Pareto dominates (D, D), we put $p=1$ and then we get the *Pareto optimal endogenous matching*[22] $\left((\varepsilon^*,-\varepsilon^*),(\varepsilon^*,-\varepsilon^*)\right)$ with $\varepsilon^* = (a-d)/(d-b+c-a)$. And thus, we call (C, C) the *induced Pareto optimal game equilibrium*, i.e., an equilibrium induced by the above Pareto optimal endogenous matching $\left((\varepsilon^*,-\varepsilon^*),(\varepsilon^*,-\varepsilon^*)\right)$.

EXAMPLE 2—Asymmetric Prisoner's Dilemma.

		Player 2	
		C	D
Player 1	C	a_{11}, b_{11}	a_{12}, b_{12}
	D	a_{21}, b_{21}	a_{22}, b_{22}

Figure 2 - Asymmetric PD Game

Noting that Figure 2 shows an asymmetric PD game, we then must have $a_{21} > a_{11} > a_{22} > a_{12}$ and $b_{12} > b_{11} > b_{22} > b_{21}$. Here we specifically study the asymmetric PD game because asymmetric games

[22] One can easily tell the difference between the definition of matching in the current paper and that in existing literatures. That is to say, we use the word "matching" in a generalized fashion because our "endogenous matching" really plays similar role as that of matching usually used and understood. Moreover, one can also interpret our definition of matching in the following way, i.e., we call the "disturbance" itself that essentially induces new matching pattern as the definition of endogenous matching because we only care about the "disturbance" in most cases.

themselves play a very important role in non-cooperative game theory. In particular, asymmetric games usually appear in evolutionary game theory where we study the replicator dynamics of interacting heterogeneous populations in stationary or fluctuating environments[23]. Again, (D, D) is the unique strictly Nash equilibrium while (C, C) Pareto (or payoff) dominates (D, D). Now, suppose that player 2 randomizes between C and D, i.e., she chooses C with probability p_2 while she employs D with probability $1-p_2$ with $0 \le p_2 \le 1$. And, as the players are involved in a structured society rather than a well-mixed population, we introduce endogenous matching by adding ε_2 to p_2 while adding $-\varepsilon_2$ to $1-p_2$, then we get a new weighted form $(p_2+\varepsilon_2, 1-p_2-\varepsilon_2)$. So, the generalized expected payoffs of player 1 can be expressed as

$$EU_1(\text{C},\ (p_2+\varepsilon_2, 1-p_2-\varepsilon_2))$$

$$\triangleq a_{11}(p_2+\varepsilon_2)+a_{12}(1-p_2-\varepsilon_2)=\left[a_{11}p_2+a_{12}(1-p_2)\right]+(a_{11}-a_{12})\varepsilon_2,$$

And,

$$EU_1(\text{D},\ (p_2+\varepsilon_2, 1-p_2-\varepsilon_2))$$

$$\triangleq a_{21}(p_2+\varepsilon_2)+a_{22}(1-p_2-\varepsilon_2)=\left[a_{21}p_2+a_{22}(1-p_2)\right]+(a_{21}-a_{22})\varepsilon_2,$$

We then get $\varepsilon_2^* = (a_{11}-a_{21})/(a_{21}-a_{22}+a_{12}-a_{11})$ by letting $EU_1(\text{C},(p_2+\varepsilon_2, 1-p_2-\varepsilon_2)) = EU_1$ $(\text{D},\ (p_2+\varepsilon_2, 1-p_2-\varepsilon_2))$ and sending p_2 to 1. Similarly, assume that player 1 randomizes between C and D, that is, she chooses C with probability p_1 and D with probability $1-p_1$ with $0 \le p_1 \le 1$. We incorporate endogenous matching here by adding ε_1 to p_1 while adding $-\varepsilon_1$ to $1-p_1$, thereby implying a new weighted form $(p_1+\varepsilon_1, 1-p_1-\varepsilon_1)$. Thus, player 2 exhibits the following generalized expected payoffs, i.e.,

$$EU_2(\text{C},(p_1+\varepsilon_1, 1-p_1-\varepsilon_1)) \triangleq b_{11}(p_1+\varepsilon_1)+b_{21}(1-p_1-\varepsilon_1)=\left[b_{11}p_1+b_{21}(1-p_1)\right]+(b_{11}-b_{21})\varepsilon_1,$$

And,

[23] See, Foster and Young (1990), Weibull (1995), Cabrales (2000), and among others.

$$EU_2\ (\mathrm{D},(p_1+\varepsilon_1,1-p_1-\varepsilon_1)) \triangleq b_{12}(p_1+\varepsilon_1)+b_{22}(1-p_1-\varepsilon_1)=\left[b_{12}p_1+b_{22}(1-p_1)\right]+(b_{12}-b_{22})\varepsilon_1,$$

We then get $\varepsilon_1^* = (b_{11}-b_{12})\times 1/(b_{12}-b_{22}+b_{21}-b_{11})$ by letting $EU_2(\mathrm{C},(p_1+\varepsilon_1,1-p_1-\varepsilon_1)) = EU_2$ $(\mathrm{D},\ (p_1+\varepsilon_1,1-p_1-\varepsilon_1))$ and sending p_1 to 1. Consequently, we obtain the corresponding *Pareto optimal endogenous matching* rule denoted $\left((\varepsilon_1^*,-\varepsilon_1^*),(\varepsilon_2^*,-\varepsilon_2^*)\right)$ with $\varepsilon_1^* = \frac{b_{11}-b_{12}}{b_{12}-b_{22}+b_{21}-b_{11}}$ and $\varepsilon_2^* =$ $(a_{11}-a_{21})/(a_{21}-a_{22}+a_{12}-a_{11})$. And also, (C, C) can be regarded as the *induced Pareto optimal game equilibrium* in the current game situation.

EXAMPLE 3—Symmetric Coordination Game.

		Player 2	
		I	II
Player 1	I	a,a	c,d
	II	d,c	b,b

Figure 3 - Symmetric Coordination Game

We suppose that $a>d$, $b>c$, $a-d>b-c$ and $b>a$ in Figure 3. Rather, Figure 3 reveals a classical symmetric coordination game with the pure strategy space $\{\mathrm{I},\ \mathrm{II}\}$. There are three strictly Nash equilibria in the game, i.e., two pure-strategy Nash equilibra, denoted (I, I) and (II, II), and one mixed-strategy Nash equilibrium $((\hat{p},1-\hat{p}),(\hat{p},1-\hat{p}))$ with $\hat{p}=\frac{b-c}{a-c+b-d}$. Notice that we suppose that $a-d>b-c$, which implies that equilibrium (I, I) strictly *risk dominates* (see, Harsanyi and Selten, 1988) equilibrium (II, II). Nonetheless, equilibrium (II, II) strictly *Pareto* (*or payoff*) *dominates* equilibrium (I, I) since it is assumed that $b>a$. Then we are faced with one of the most important issues in non-cooperative game theory, i.e., *multiple-equilibrium problem.* Indeed, players in this type of coordination game situations have to face the *tradeoff* between risk and payoff before making strategy choice, which also broadly exists in other economic contexts. As usual, some of the literatures[24] prefer risk-dominant equilibrium while some others[25] prefer Pareto-dominant equilibrium. Obviously, the present paper prefers Pareto dominant equilibrium from the viewpoint of economic welfare and our endogenous matching rule will definitely support our preference. Noting that the computation algorithm is just the same as that of Example 1 and Example 2, we omit it. It follows from symmetry

24 See, Carlsson and van Damme (1993), Young (1993), Kandori et al. (1993), Matsui and Matsuyama (1995) and among others.
25 See, Harsanyi and Selten (1988), Aumann and Sorin (1989), Matsui (1991), Anderlini (1999).

that the *Pareto optimal endogenous matching* can be given by $\left((-\varepsilon^*,\varepsilon^*),(-\varepsilon^*,\varepsilon^*)\right)$ with the element denoted by $\varepsilon^* = (c-b)/(b-d+a-c)$. And also, we call the Pareto dominant equilibrium (II, II) *induced game equilibrium* in the current sense.

EXAMPLE 4—Asymmetric Coordination Game.

		Player 2 L	Player 2 R
Player 1	T	a_{11}, b_{11}	a_{12}, b_{12}
Player 1	B	a_{21}, b_{21}	a_{22}, b_{22}

Figure 4 - Asymmetric Coordination Game

In Figure 4, $a_{11} > a_{21}$, $a_{22} > a_{12}$, $b_{11} > b_{12}$, $b_{22} > b_{21}$, $(a_{11} - a_{21})(b_{11} - b_{12}) > (a_{22} - a_{12})(b_{22} - b_{21})$, $a_{22} > a_{11}$ and $b_{22} > b_{11}$, thus we get that equilibrium (T, L) strictly risk dominates (B, R) while (B, R) strictly Pareto dominates (T, L). Accordingly, similar to the above examples, the corresponding *Pareto optimal endogenous matching* reads as follows, i.e., $\left((-\varepsilon_1^*,\varepsilon_1^*),(-\varepsilon_2^*,\varepsilon_2^*)\right)$ with $\varepsilon_1^* = \frac{b_{21}-b_{22}}{b_{22}-b_{12}+b_{11}-b_{21}}$ and $\varepsilon_2^* = \frac{a_{12}-a_{22}}{a_{22}-a_{21}+a_{11}-a_{12}}$. Then, (B, R) can be seen as an *induced game equilibrium* in the present case.

3. Formulation

3.1. Set-up and Assumptions

Let $A_{I_1\times I_2}$ be the payoff matrix for row-players and $B_{I_1\times I_2}$ be the payoff matrix for column-players with $A_{I_1\times I_2}$, $B_{I_1\times I_2} \in \mathbb{R}^{I_1\times I_2}$, and I_1, $I_2 \geq 1$. Here, and throughout the current paper, we study the replicator dynamics of $I_1 \times I_2$ normal form games between two groups of populations. Put $\sum_{i_1=1}^{I_1} M^{i_1}(t) \triangleq M(t)$, where $M^{i_1}(t)$ denotes the number of strategy-i_1 players at period t. Similarly, let $\sum_{i_2=1}^{I_2} N^{i_2}(t) \triangleq N(t)$, where $N^{i_2}(t)$ denotes the number of strategy-i_2 players at period t.

We let $X^{i_1}(t) \triangleq M^{i_1}(t)/M(t)$, $Y^{i_2}(t) \triangleq N^{i_2}(t)/N(t)$ denote the frequencies of strategies i_1 and i_2, respectively, with $i_1 = 1,2,...,I_1$ and $i_2 = 1,2,...,I_2$. Therefore, the average payoffs of strategy i_1 and strategy i_2 are given by $u\left(i_1, Y(t)\right) \triangleq \bar{e}_{i_1}^T A Y(t)$ and $u\left(i_2, X(t)\right) \triangleq \tilde{e}_{i_2}^T B^T X(t)$, respectively, with the superscript "T" denoting transpose, and $X(t) \triangleq \left(X^1(t),..., X^{i_1}(t), ..., X^{I_1}(t)\right)^T$, $Y(t) \triangleq \left(Y^1(t),...,Y^{i_2}(t), \ ...,Y^{I_2}(t)\right)^T$, and also $\bar{e}_{i_1} = (0,...,1,...,0)^T$, $\tilde{e}_{i_2} = (0,...,1,...,0)^T$, where the i_1-th entry and i_2-th entry are ones, respectively, for $i_1 = 1,2,...,I_1$ and $i_2 = 1,2,...,I_2$.

Specifically, in the current paper, we employ the following *endogenous matching* by incorporating two vectors, i.e., $\bar{\rho} \triangleq \left(\bar{\rho}^1,...,\bar{\rho}^{i_1},...,\bar{\rho}^{I_1}\right)^T \in \mathbb{R}^{I_1}$ and $\tilde{\rho} \triangleq (\tilde{\rho}^1,...,\tilde{\rho}^{i_2} \ ,...,\tilde{\rho}^{I_2})^T \in \mathbb{R}^{I_2}$ with $\sum_{i_1=1}^{I_1} \bar{\rho}^{i_1} = 0$ and $\sum_{i_2=1}^{I_2} \tilde{\rho}^{i_2} = 0$, into the present model. Now, the generalized average payoffs of strategies i_1 and i_2 are rewritten as $u\left(i_1, Y(t) + \tilde{\rho}\right) \triangleq \bar{e}_{i_1}^T A\left(Y(t) + \tilde{\rho}\right) = \bar{e}_{i_1}^T A Y(t) + \bar{e}_{i_1}^T A \tilde{\rho}$ and $u\left(i_2, X(t) + \bar{\rho}\right) \triangleq \tilde{e}_{i_2}^T B^T \left(X(t) + \bar{\rho}\right) = \tilde{e}_{i_2}^T B^T X(t) + \tilde{e}_{i_2}^T B^T \bar{\rho}$, respectively, for $i_1 = 1,2,...,I_1$ and $i_2 = 1,2,...,I_2$. In other words, $u\left(i_1, Y(t) + \tilde{\rho}\right)$ and $u\left(i_2, X(t) + \bar{\rho}\right)$ can be seen as $\bar{e}_{i_1}^T A \tilde{\rho}$-perturbation and $\tilde{e}_{i_2}^T B^T \bar{\rho}$-perturbation of $u\left(i_1, Y(t)\right)$ and $u\left(i_2, X(t)\right)$, respectively. Moreover, as is discussed thoroughly below, both of the above payoff-perturbations are actually endogenously determined, that is, the present paper analyzes the case of endogenous matching. We emphasize here again that the two vectors remaining to be determined are named as endogenous matching not because they themselves represent a matching pattern but just because they are the key or essential factors that induce new matching pattern. In other words, we only care about what induce new matching pattern from existing matching pattern because existing matching is given and new matching pattern is also given following from the Pareto optimal standard. Therefore, we without great loss of generality name the two vectors, to be determined, as the endogenous matching is for the sake of convenience and also with the purpose of capturing the essence of the problem facing us. To sum up, we argue that, in the present case and in some sense, what the present state is will not be important, what the goal will be is also not important, the only important matter we care about is what we need to lead us from the present state to our goal. Here, the endogenous matching defined by the two vectors meets our need.

We now denote by $\left(\Omega^{(W^{i_\beta})}, \mathcal{F}^{(W^{i_\beta})}, \left\{\mathcal{F}_t^{(W^{i_\beta})}\right\}_{0\le t\le \tau^{i_\beta}(\omega)}, \mathbb{P}^{(W^{i_\beta})}\right)$ the filtered probability space with $\mathbb{F}^{(W^{i_\beta})} \triangleq \left\{\mathcal{F}_t^{(W^{i_\beta})}\right\}_{0\le t\le \tau^{i_\beta}(\omega)}$ the $\mathbb{P}^{(W^{i_\beta})}-$ augmented filtration generated by $d_\beta-$ dimensional standard Brownian motion $\left(W^{i_\beta}(t), 0\le t\le \tau^{i_\beta}(\omega)\right)$ with $\mathcal{F}^{(W^{i_\beta})} \triangleq \mathcal{F}_{\tau^{i_\beta}(\omega)}^{(W^{i_\beta})}$, $\omega \in \Omega^{(W^{i_\beta})}$ and $\tau^{i_\beta}(\omega)$ a stopping time, to be endogenously determined. Moreover, we define

$$\begin{aligned}\tilde{N}^{i_\beta}\left(dt, dz^{i_\beta}\right) &\triangleq \left(\tilde{N}_1^{i_\beta}\left(dt, dz_1^{i_\beta}\right), ..., \tilde{N}_{n_\beta}^{i_\beta}\left(dt, dz_{n_\beta}^{i_\beta}\right)\right)^T \\ &\triangleq \left(N_1^{i_\beta}\left(dt, dz_1^{i_\beta}\right) - \nu_1^{i_\beta}\left(dz_1^{i_\beta}\right)dt, ..., N_{n_\beta}^{i_\beta}\left(dt, dz_{n_\beta}^{i_\beta}\right) - \nu_{n_\beta}^{i_\beta}\left(dz_{n_\beta}^{i_\beta}\right)dt\right)^T,\end{aligned}$$

in which $\left\{N_{l_\beta}^{i_\beta}\right\}_{l_\beta=1}^{n_\beta}$ are independent Poisson random measures with Lévy measures $\nu_{l_\beta}^{i_\beta}$ coming from n_β independent (1-dimensional) Lévy processes $\eta_1^{i_\beta}(t) \triangleq \int_0^t \int_{\mathbb{R}_0} z_1^{i_\beta} \tilde{N}_1^{i_\beta}\left(ds, dz_1^{i_\beta}\right)$, …, $\eta_{n_\beta}^{i_\beta}(t) \triangleq \int_0^t \int_{\mathbb{R}_0} z_{n_\beta}^{i_\beta} \tilde{N}_{n_\beta}^{i_\beta}\left(ds, dz_{n_\beta}^{i_\beta}\right)$ with $\mathbb{R}_0 \triangleq \mathbb{R} - \{0\}$, and then the corresponding stochastic basis is given by $\left(\Omega^{(\tilde{N}^{i_\beta})}, \mathcal{F}^{(\tilde{N}^{i_\beta})}, \left\{\mathcal{F}_t^{(\tilde{N}^{i_\beta})}\right\}_{0\le t\le \tau^{i_\beta}(\omega)}, \mathbb{P}^{(\tilde{N}^{i_\beta})}\right)$ with $\mathbb{F}^{(\tilde{N}^{i_\beta})} \triangleq \left\{\mathcal{F}_t^{(\tilde{N}^{i_\beta})}\right\}_{0\le t\le \tau^{i_\beta}(\omega)}$ the $\mathbb{P}^{(\tilde{N}^{i_\beta})}-$ augmented filtration and $\mathcal{F}^{(\tilde{N}^{i_\beta})} \triangleq \mathcal{F}_{\tau^{i_\beta}(\omega)}^{(\tilde{N}^{i_\beta})}$, $\omega \in \Omega^{(\tilde{N}^{i_\beta})}$ and $\tau^{i_\beta}(\omega)$ a stopping time, to be endogenously determined. Thus, we are provided with a new stochastic basis $\left(\Omega^{i_\beta}, \mathcal{F}^{i_\beta}, \left\{\mathcal{F}_t^{i_\beta}\right\}_{0\le t\le \tau^{i_\beta}(\omega)}, \mathbb{P}^{i_\beta}\right)$, where $\Omega^{i_\beta} \triangleq \Omega^{(W^{i_\beta})} \times \Omega^{(\tilde{N}^{i_\beta})}$, $\mathcal{F}^{i_\beta} \triangleq \mathcal{F}^{(W^{i_\beta})} \otimes \mathcal{F}^{(\tilde{N}^{i_\beta})}$, $\mathcal{F}_t^{i_\beta} \triangleq \mathcal{F}_t^{(W^{i_\beta})} \otimes \mathcal{F}_t^{(\tilde{N}^{i_\beta})}$, $\mathbb{P}^{i_\beta} \triangleq \mathbb{P}^{(W^{i_\beta})} \otimes \mathbb{P}^{(\tilde{N}^{i_\beta})}$ and $\mathbb{F}^{i_\beta} \triangleq \left\{\mathcal{F}_t^{i_\beta}\right\}_{0\le t\le \tau^{i_\beta}(\omega)}$ denotes the corresponding filtration satisfying the well-known "usual conditions". Here, and throughout the current paper, $\mathbb{E}^{i_\beta}$ is used to denote the expectation operator with respect to (w. r. t.) the probability law $\mathbb{P}^{i_\beta}$ for $\forall\ i_\beta = 1, 2, ..., I_\beta$ and for $\beta = 1,2$. Naturally, we have stochastic basis $\left(\Omega^{\beta}, \mathcal{F}^{\beta}, \left\{\mathcal{F}_t^{\beta}\right\}_{0\le t\le \tau^{\beta}(\omega)}, \mathbb{P}^{\beta}\right)$ with $\Omega^{\beta} \triangleq \times_{i_\beta}^{I_\beta} \Omega^{i_\beta}$, $\mathcal{F}^{\beta} \triangleq \otimes_{i_\beta}^{I_\beta} \mathcal{F}^{i_\beta}$, $\mathcal{F}_t^{\beta} \triangleq \otimes_{i_\beta}^{I_\beta} \mathcal{F}_t^{i_\beta}$, $\mathbb{P}^{\beta} \triangleq \otimes_{i_\beta}^{I_\beta} \mathbb{P}^{i_\beta}$, $\tau^{\beta}(\omega) \triangleq \vee_{i_\beta}^{I_\beta} \tau^{i_\beta}(\omega) \triangleq \vee_{i_\beta}^{I_\beta} \overline{\tau}^{i_\beta}(\omega)$ if $\beta = 1$, and $\tau^{\beta}(\omega) \triangleq \vee_{i_\beta}^{I_\beta} \tau^{i_\beta}(\omega) \triangleq \vee_{i_\beta}^{I_\beta} \tilde{\tau}^{i_\beta}(\omega)$ if $\beta = 2$ with $\omega \in \Omega^{\beta}$, $\mathbb{F}^{\beta} \triangleq \left\{\mathcal{F}_t^{\beta}\right\}_{0\le t\le \tau^{\beta}(\omega)}$ denoting the corresponding filtration satisfying the usual conditions, and $\mathbb{E}^{\beta}$ is used to denote the expectation operator w. r. t. the probability law $\mathbb{P}^{\beta}$ for $\beta = 1,2$. Furthermore, we are led to the following probability space $\left(\Omega, \mathcal{F}, \left\{\mathcal{F}_t\right\}_{0\le t\le \tau(\omega)}, \mathbb{P}\right)$ with $\Omega \triangleq \times_{\beta=1}^{2} \Omega^{\beta}$, $\mathcal{F} \triangleq \otimes_{\beta=1}^{2} \mathcal{F}^{\beta}$, $\mathcal{F}_t \triangleq \otimes_{\beta=1}^{2} \mathcal{F}_t^{\beta}$, $\mathbb{P} \triangleq \otimes_{\beta=1}^{2} \mathbb{P}^{\beta}$, $\tau(\omega) \triangleq \vee_{\beta=1}^{2} \tau^{\beta}(\omega)$ with $\omega \in \Omega$, $\mathbb{F} \triangleq \left\{\mathcal{F}_t\right\}_{0\le t\le \tau(\omega)}$ denoting the corresponding filtration satisfying the usual conditions, and $\mathbb{E}$ is used to denote the expectation operator w. r. t. the probability law $\mathbb{P}$.

We now define the canonical Lebesgue measure μ on measure space $\left(\mathbb{R}_+, \mathfrak{B}\left(\mathbb{R}_+\right)\right)$ with $\mathbb{R}_+ \triangleq [0,\infty)$, $\mathbb{R}_{++} \triangleq (0,\infty)$ and $\mathfrak{B}\left(\mathbb{R}_+\right)$ the Borel sigma-algebra, and also the corresponding regular properties about Lebesgue measure are supposed to be fulfilled. Thus, we can define the following product measure spaces $\left(\Omega^{i_\beta} \times \mathbb{R}_+, \mathbb{F}^{i_\beta} \otimes \mathfrak{B}\left(\mathbb{R}_+\right)\right)$ and $\left(\Omega^{\beta} \times \mathbb{R}_+, \mathbb{F}^{\beta} \otimes \mathfrak{B}\left(\mathbb{R}_+\right)\right)$ with corresponding product measures $\mu \otimes \mathbb{P}^{i_\beta}$ and $\mu \otimes \mathbb{P}^{\beta}$, respectively, for $\forall\ i_\beta = 1,2,\ldots,I_\beta$ and for $\beta = 1,2$.

Now, based upon the probability space $\left(\Omega^{i_\beta}, \mathcal{F}^{i_\beta}, \mathbb{F}^{i_\beta}, \mathbb{P}^{i_\beta}\right)$ for $\beta = 1,2$, and following Fudenberg and Harris (1992), Cabrales (2000), Imhof (2005), Benaïm et al (2008), Hofbauer and Imhof (2009), the stochastic replicator dynamics[26] of the two groups of populations can be respectively given as follows,

$$dM^{i_1}(t) = M^{i_1}(t)\left[\overline{e}_{i_1}^{T} AY(t)dt + \sum_{k_1=1}^{d_1} \overline{\sigma}_{i_1 k_1}(t) dW_{k_1}^{i_1}(t) + \sum_{l_1=1}^{n_1} \int_{\mathbb{R}_0} \overline{\gamma}_{i_1 l_1}\left(t, z_{l_1}^{i_1}\right) \tilde{N}_{l_1}^{i_1}\left(dt, dz_{l_1}^{i_1}\right)\right],$$

[26] Throughout, the stochastic replicator dynamics will help us to construct adaptive learning processes for the players following the argument of Gale et al. (1995), Binmore et al. (1995), Börgers and Sarin (1997), and Cabrales (2000). Thus, we will take indifference between the stochastic replicator dynamics and the adaptive learning processes.

$$dN^{i_2}(t) = N^{i_2}(t)\left[\tilde{e}_{i_2}^T B^T X(t)dt + \sum_{k_2=1}^{d_2} \tilde{\sigma}_{i_2k_2}(t)dW_{k_2}^{i_2}(t) + \sum_{l_2=1}^{n_2} \int_{\mathbb{R}_0} \tilde{\gamma}_{i_2l_2}\left(t, z_{l_2}^{i_2}\right) \tilde{N}_{l_2}^{i_2}\left(dt, dz_{l_2}^{i_2}\right)\right].$$

where $M^{i_1}(t)$ is assumed to be $\mathbb{F}^{i_1} \otimes \mathfrak{B}(\mathbb{R}_+)-$adapted, $N^{i_2}(t)$ is $\mathbb{F}^{i_2} \otimes \mathfrak{B}(\mathbb{R}_+)-$adapted, $Y(t)$ is also assumed to be $\mathbb{F}^2 \otimes \mathfrak{B}(\mathbb{R}_+)-$ adapted, $X(t)$ is $\mathbb{F}^1 \otimes \mathfrak{B}(\mathbb{R}_+)-$ adapted, $\bar{\sigma}_{i_1k_1}(t)$ and $\bar{\gamma}_{i_1l_1}\left(t, z_{l_1}^{i_1}\right)$ are $\mathbb{F}^{i_1} \otimes \mathfrak{B}(\mathbb{R}_+)-$progressively measurable, and $\tilde{\sigma}_{i_2k_2}(t)$ and $\tilde{\gamma}_{i_2l_2}\left(t, z_{l_2}^{i_2}\right)$ are $\mathbb{F}^{i_2} \otimes \mathfrak{B}\left(\mathbb{R}_+\right)-$progressively measurable, for $\forall\ i_1 = 1, 2, ..., I_1$, $\forall\ i_2 = 1, 2, ..., I_2$, $\forall\ k_1 = 1, 2, ..., d_1$, $\forall\ k_2 = 1, 2, ..., d_2$, $\forall\ l_1 = 1, 2, ..., n_1$ and $\forall\ l_2 = 1, 2, ..., n_2$. Thus, based upon the above SDEs, we get the following proposition,

PROPOSITION 1: *The Lyapunov exponents of the above SDEs are,*

$$\lim_{t\to\infty} \frac{1}{t} \log M^{i_1}(t) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \left\{ \bar{e}_{i_1}^T AY(s) - \tfrac{1}{2} \sum_{k_1=1}^{d_1} \bar{\sigma}_{i_1k_1}^2(s) \right.$$
$$\left. + \sum_{l_1=1}^{n_1} \int_{\mathbb{R}_0} \left[\log\left(1 + \bar{\gamma}_{i_1l_1}\left(s, z_{l_1}^{i_1}\right)\right) - \bar{\gamma}_{i_1l_1}\left(s, z_{l_1}^{i_1}\right) \right] \nu_{l_1}^{i_1}\left(dz_{l_1}^{i_1}\right) \right\} ds \quad a.s.$$

And,

$$\lim_{t\to\infty} \frac{1}{t} \log N^{i_2}(t) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \left\{ \tilde{e}_{i_2}^T B^T X(s) - \tfrac{1}{2} \sum_{k_2=1}^{d_2} \tilde{\sigma}_{i_2k_2}^2(s) \right.$$
$$\left. + \sum_{l_2=1}^{n_2} \int_{\mathbb{R}_0} \left[\log\left(1 + \tilde{\gamma}_{i_2l_2}\left(s, z_{l_2}^{i_2}\right)\right) - \tilde{\gamma}_{i_2l_2}\left(s, z_{l_2}^{i_2}\right) \right] \nu_{l_2}^{i_2}\left(dz_{l_2}^{i_2}\right) \right\} ds \quad a.s.$$

respectively, for $M^{i_1}(0) \neq 0$, $N^{i_2}(0) \neq 0$ *and* $\forall\ i_1 = 1, 2, ..., I_1$, $\forall\ i_2 = 1, 2, ..., I_2$. *Hence, the above SDEs are almost surely exponentially stable if and only if,*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \left\{ \bar{e}_{i_1}^T AY(s) - \tfrac{1}{2} \sum_{k_1=1}^{d_1} \bar{\sigma}_{i_1k_1}^2(s) \right.$$
$$\left. + \sum_{l_1=1}^{n_1} \int_{\mathbb{R}_0} \left[\log\left(1 + \bar{\gamma}_{i_1l_1}\left(s, z_{l_1}^{i_1}\right)\right) - \bar{\gamma}_{i_1l_1}\left(s, z_{l_1}^{i_1}\right) \right] \nu_{l_1}^{i_1}\left(dz_{l_1}^{i_1}\right) \right\} ds < 0.$$

And,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \left\{ \tilde{e}_{i_2}^T B^T X(s) - \tfrac{1}{2} \sum_{k_2=1}^{d_2} \tilde{\sigma}_{i_2k_2}^2(s) \right.$$
$$\left. + \sum_{l_2=1}^{n_2} \int_{\mathbb{R}_0} \left[\log\left(1 + \tilde{\gamma}_{i_2l_2}\left(s, z_{l_2}^{i_2}\right)\right) - \tilde{\gamma}_{i_2l_2}\left(s, z_{l_2}^{i_2}\right) \right] \nu_{l_2}^{i_2}\left(dz_{l_2}^{i_2}\right) \right\} ds < 0.$$

respectively, for $\forall\ i_1 = 1, 2, ..., I_1$ *and* $\forall\ i_2 = 1, 2, ..., I_2$.

PROOF: See Appendix A. ▌

So, we directly provide the following assumption,

ASSUMPTION 1: *Here, and throughout the current paper, we suppose that,*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\Big\{\overline{e}_{i_1}^T AY(s)-\tfrac{1}{2}\sum\nolimits_{k_1=1}^{d_1}\overline{\sigma}_{i_1k_1}^2(s)$$
$$+\sum\nolimits_{l_1=1}^{n_1}\int_{\mathbb{R}_0}\Big[\log\Big(1+\overline{\gamma}_{i_1l_1}\big(s,z_{l_1}^{i_1}\big)\Big)-\overline{\gamma}_{i_1l_1}\big(s,z_{l_1}^{i_1}\big)\Big]\nu_{l_1}^{i_1}\big(dz_{l_1}^{i_1}\big)\Big\}ds<0.$$

And,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\Big\{\tilde{e}_{i_2}^T B^T X(s)-\tfrac{1}{2}\sum\nolimits_{k_2=1}^{d_2}\tilde{\sigma}_{i_2k_2}^2(s)$$
$$+\sum\nolimits_{l_2=1}^{n_2}\int_{\mathbb{R}_0}\Big[\log\Big(1+\tilde{\gamma}_{i_2l_2}\big(s,z_{l_2}^{i_2}\big)\Big)-\tilde{\gamma}_{i_2l_2}\big(s,z_{l_2}^{i_2}\big)\Big]\nu_{l_2}^{i_2}\big(dz_{l_2}^{i_2}\big)\Big\}ds<0.$$

always hold.

Now, based upon Proposition 1 and Assumption 1, we, without loss of generality, introduce the following technical assumption,

ASSUMPTION 2: *Throughout the current paper, both* $M(t)$ *and* $N(t)$*, sufficiently large, are assumed to be finite constants.*

REMARK 3.1: Some of existing literatures (see, Nowak et al., 2004; Imhof and Nowak, 2006, and among others) have confirmed that Assumption 2 has very important implications. That is to say, on the one hand, Assumption 2 is used to make things much easier from the viewpoint of pure mathematics; and also, Assumption 2 is indeed without loss of any generality in the sense of economic and biological intuitions on the other hand (see, Fudenberg and Levine, 1993; Young, 1993; Binmore et al., 1995; Binmore and Samuelson, 1997).

Notice from Assumption 2 that the sizes of the two populations are finite constants, based on Itô's rule one can easily find,

$$dX^{i_1}(t)=X^{i_1}(t)\left[\overline{e}_{i_1}^T AY(t)dt+\sum_{k_1=1}^{d_1}\overline{\sigma}_{i_1k_1}(t)dW_{k_1}^{i_1}(t)+\sum_{l_1=1}^{n_1}\int_{\mathbb{R}_0}\overline{\gamma}_{i_1l_1}\big(t,z_{l_1}^{i_1}\big)\tilde{N}_{l_1}^{i_1}\big(dt,dz_{l_1}^{i_1}\big)\right]$$
$$\triangleq X^{i_1}(t)\left[\overline{e}_{i_1}^T AY(t)dt+\overline{\sigma}^{i_1}(t)dW^{i_1}(t)+\int_{\mathbb{R}_0^{n_1}}\overline{\gamma}^{i_1}\big(t,z^{i_1}\big)\tilde{N}^{i_1}\big(dt,dz^{i_1}\big)\right],$$
$$dY^{i_2}(t)=Y^{i_2}(t)\left[\tilde{e}_{i_2}^T B^T X(t)dt+\sum_{k_2=1}^{d_2}\tilde{\sigma}_{i_2k_2}(t)dW_{k_2}^{i_2}(t)+\sum_{l_2=1}^{n_2}\int_{\mathbb{R}_0}\tilde{\gamma}_{i_2l_2}\big(t,z_{l_2}^{i_2}\big)\tilde{N}_{l_2}^{i_2}\big(dt,dz_{l_2}^{i_2}\big)\right]$$
$$\triangleq Y^{i_2}(t)\left[\tilde{e}_{i_2}^T B^T X(t)dt+\tilde{\sigma}^{i_2}(t)dW^{i_2}(t)+\int_{\mathbb{R}_0^{n_2}}\tilde{\gamma}^{i_2}\big(t,z^{i_2}\big)\tilde{N}^{i_2}\big(dt,dz^{i_2}\big)\right], \quad (1)$$

subject to the initial conditions, i.e., $W^{i_1}(0)=(0,\ldots,0)^T$ $\mathbb{P}^{i_1}-$a.s., $W^{i_2}(0)=(0,\ldots,0)^T$ $\mathbb{P}^{i_2}-$a.s., $X(0)=\big(X^1(0),\ldots,X^{i_1}(0),\ldots,X^{I_1}(0)\big)^T\triangleq\big(x^1,\ldots,x^{i_1},\ldots,x^{I_1}\big)^T\triangleq x>0$ $\mathbb{P}^1-$a.s., $Y(0)=\big(Y^1(0),$

$..., Y^{i_2}(0),..., Y^{I_2}(0)\Big)^T \triangleq \Big(y^1,..., y^{i_2},..., y^{I_2}\Big)^T \triangleq y > 0$ $\mathbb{P}^2-$ a.s., $X^{i_1}(t)$ is assumed to be $\mathbb{F}^{i_1} \otimes \mathfrak{B}(\mathbb{R}_+)-$adapted, and $Y^{i_2}(t)$ is assumed to be $\mathbb{F}^{i_2} \otimes \mathfrak{B}(\mathbb{R}_+)-$adapted, for $\forall\ i_1 = 1,2,..., I_1$ and $\forall\ i_2 = 1,2,..., I_2$. Moreover, with a little abuse of notations, we put $\bar{\sigma}^{i_1}(0) = \Big(\bar{\sigma}_{i_1 1}(0), ..., \bar{\sigma}_{i_1 k_1}(0),..., \bar{\sigma}_{i_1 d_1}(0)\Big)^T \triangleq \Big(\bar{\sigma}_{i_1 1}, ..., \bar{\sigma}_{i_1 k_1},..., \bar{\sigma}_{i_1 d_1}\Big)^T \triangleq \bar{\sigma}^{i_1}$, $\bar{\gamma}^{i_1}\left(0, z^{i_1}\right) = \Big(\bar{\gamma}_{i_1 1}\left(0, z_1^{i_1}\right),..., \bar{\gamma}_{i_1 l_1}\left(0, z_{l_1}^{i_1}\right),..., \bar{\gamma}_{i_1 n_1}\left(0, z_{n_1}^{i_1}\right)\Big)^T \triangleq \Big(\bar{\gamma}_{i_1 1}\left(z_1^{i_1}\right),..., \bar{\gamma}_{i_1 l_1}\left(z_{l_1}^{i_1}\right),..., \bar{\gamma}_{i_1 n_1}\left(z_{n_1}^{i_1}\right)\Big)^T \triangleq \bar{\gamma}^{i_1}\left(z^{i_1}\right)$, $\tilde{\sigma}^{i_2}(0) = \Big(\tilde{\sigma}_{i_2 1}(0), ..., \tilde{\sigma}_{i_2 k_2}(0),..., \tilde{\sigma}_{i_2 d_2}(0)\Big)^T \triangleq \Big(\tilde{\sigma}_{i_2 1}, ..., \tilde{\sigma}_{i_2 k_2},..., \tilde{\sigma}_{i_2 d_2}\Big)^T \triangleq \tilde{\sigma}^{i_2}$, and $\tilde{\gamma}^{i_2}\left(0, z^{i_2}\right) = \Big(\tilde{\gamma}_{i_2 1}\left(0, z_1^{i_2}\right),..., \tilde{\gamma}_{i_2 l_2}\left(0, z_{l_2}^{i_2}\right),..., \tilde{\gamma}_{i_2 n_2}\left(0, z_{n_2}^{i_2}\right)\Big)^T \triangleq \Big(\tilde{\gamma}_{i_2 1}\left(z_1^{i_2}\right),..., \tilde{\gamma}_{i_2 l_2}\left(z_{l_2}^{i_2}\right),..., \tilde{\gamma}_{i_2 n_2}\left(z_{n_2}^{i_2}\right)\Big)^T \triangleq \tilde{\gamma}^{i_2}\left(z^{i_2}\right)$, for $\forall\ i_1 = 1,2,..., I_1$ and $\forall\ i_2 = 1,2, ..., I_2$. Moreover, we have,

$$dX(t) = f^1\left(X(t)\right)dt + g^1\left(X(t)\right)dW^1(t) + \int_{\mathbb{R}_0^{I_1 n_1}} h^1\left(X(t), z^1\right)\tilde{N}^1\left(dt, dz^1\right),$$

$$dY(t) = f^2\left(Y(t)\right)dt + g^2\left(Y(t)\right)dW^2(t) + \int_{\mathbb{R}_0^{I_2 n_2}} h^2\left(Y(t), z^2\right)\tilde{N}^2\left(dt, dz^2\right), \tag{2}$$

with $X(t) \triangleq \left(X^1(t),..., X^{i_1}(t),..., X^{I_1}(t)\right)^T$ and $Y(t) \triangleq \left(Y^1(t),..., Y^{i_2}(t),..., Y^{I_2}(t)\right)^T$.

Next, we are in the position to introduce some necessary assumptions,

ASSUMPTION 3: *The initial conditions* $X^{i_1}(0) = x^{i_1} > 0$, $Y^{i_2}(0) = y^{i_2} > 0$, $X(0) = x > 0$ *and* $Y(0) = y > 0$ *are all supposed to be deterministic and bounded for* $\forall\ i_1 = 1, 2, ..., I_1$ *and* $\forall\ i_2 = 1, 2, ..., I_2$. *Furthermore,* $\bar{\sigma}^{i_1} \neq 0$ $\mathbb{P}^{i_1}-$ *a.s.,* $\tilde{\sigma}^{i_2} \neq 0$ $\mathbb{P}^{i_2}-$ *a.s.,* $\bar{\gamma}_{i_1 l_1}\left(t, z_{l_1}^{i_1}\right) > -1 + \varepsilon_{l_1}^{i_1}$ $\mu \otimes \mathbb{P}^{i_1}-$ *a.e., and* $\tilde{\gamma}_{i_2 l_2}\left(t, z_{l_2}^{i_2}\right) > -1 + \varepsilon_{l_2}^{i_2}$ $\mu \otimes \mathbb{P}^{i_2}-$ *a.e., for* $\forall\ \varepsilon_{l_1}^{i_1} > 0$, $\varepsilon_{l_2}^{i_2} > 0$ *and for* $\forall\ i_1 = 1, 2, ..., I_1$; $i_2 = 1, 2, ..., I_2$; $l_1 = 1, 2, ..., n_1$ *and* $l_2 = 1, 2, ..., n_2$.

ASSUMPTION 4: *The following linear growth and local Lipschitz continuity conditions are fulfilled, respectively,,*

$$\left\|f^1(x)\right\|_2^2 + \left\|g^1(x)\right\|_2^2 + \int_{\mathbb{R}_0} \sum_{l_1=1}^{I_1 n_1} \left\|h^1\left(x, z_{l_1}^1\right)\right\|_2^2 \nu_{l_1}^1\left(dz_{l_1}^1\right) \leq C^1\left(1 + \|x\|_2^2\right),$$

$$\left\|f^2(y)\right\|_2^2+\left\|g^2(y)\right\|_2^2+\int_{\mathbb{R}_0}\sum_{l_2=1}^{I_2 n_2}\left\|h^2\left(y,z_{l_2}^2\right)\right\|_2^2 \nu_{l_2}^2\left(dz_{l_2}^2\right)\leq C^2\left(1+\left\|y\right\|_2^2\right),$$

for some constants C^1, $C^2<\infty$. *And,*

$$\left\|f^1(x)-f^1(\widehat{x})\right\|_2^2\vee\left\|g^1(x)-g^1(\widehat{x})\right\|_2^2\vee$$

$$\int_{\mathbb{R}_0}\sum_{l_1=1}^{I_1 n_1}\left\|h^{1(l_1)}\left(x,z^1\right)-h^{1(l_1)}\left(\widehat{x},z^1\right)\right\|_2^2 \nu_{l_1}^1\left(dz_{l_1}^1\right)\leq L_{R^1}^1\left\|x-\widehat{x}\right\|_2^2,$$

$$\left\|f^2(y)-f^2(\widehat{y})\right\|_2^2\vee\left\|g^2(y)-g^2(\widehat{y})\right\|_2^2\vee$$

$$\int_{\mathbb{R}_0}\sum_{l_2=1}^{I_2 n_2}\left\|h^{2(l_2)}\left(y,z^2\right)-h^{2(l_2)}\left(\widehat{y},z^2\right)\right\|_2^2 \nu_{l_2}^2\left(dz_{l_2}^2\right)\leq L_{R^2}^2\left\|y-\widehat{y}\right\|_2^2,$$

for any given constants R^1, $R^2>0$ *with,* $\left\|x\right\|_2\vee\left\|\widehat{x}\right\|_2\leq R^1$, $\left\|y\right\|_2\vee\left\|\widehat{y}\right\|_2\leq R^2$, *and constants* $L_{R^1}^1$, $L_{R^2}^2<\infty$ *that depend only on* R^1 *and* R^2, *respectively, for all* x, $\widehat{x}\in\mathbb{R}_{++}^{I_1}$ *and* y, $\widehat{y}\in\mathbb{R}_{++}^{I_2}$ *with* $h^{1(l_1)}\left(x,z^1\right)$, $h^{1(l_1)}\left(\widehat{x},z^1\right)$ *representing the* l_1*-th columns of matrixes* $h^1\left(x,z^1\right)$, $h^1\left(\widehat{x},z^1\right)$, *respectively, and* $h^{2(l_2)}\left(y,z^2\right)$, $h^{2(l_2)}\left(\widehat{y},z^2\right)$ *denoting the* l_2*-th columns of matrixes* $h^2\left(y,z^2\right)$, $h^2\left(\widehat{y},z^2\right)$, *respectively, for* $\forall\ l_1=1,2,\ ...,\ n_1$ *and* $l_2=1,2,\ ...,\ n_2$.

REMARK 3.2: (i) Provided Assumption 4, the existence and uniqueness of strong solutions of the Lévy SDEs given in (2) are ensured, respectively.

(ii) Assumption 4 is indeed weak in the following sense, local Lipschitz continuity conditions can be naturally satisfied for any C^1 functions or correspondences thanks to the Mean Value Theorem.

(iii) Here, and throughout the current paper, $|\cdot|$ is used to denote absolute value, $\|\cdot\|_2$ is used to represent both Euclidean vector norm and the Frobenius (or trace) matrix norm, and $<\cdot>$ is used to denote the scalar product.

3.2. Stochastic Differential Cooperative Game on Time

Now, as in the model of Nowak et al (2004), and Imhof and Nowak (2006), we define the following generalized expected discounted fitness functions[27],

$$\bar{f}_{i_1}\left(t, Y(t)\right) \triangleq \mathbb{E}^2_{(s,y)}\left[\exp\left(-\bar{\theta}^{i_1} t\right)\left\{1-\bar{w}^{i_1}+\bar{w}^{i_1}\left[\bar{e}^T_{i_1} A\left(Y(t)+\tilde{\rho}\right)\right]\right\}\right],$$

$$\tilde{f}_{i_2}\left(t, X(t)\right) \triangleq \mathbb{E}^1_{(s,x)}\left[\exp\left(-\tilde{\theta}^{i_2} t\right)\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\tilde{e}^T_{i_2} B^T\left(X(t)+\bar{\rho}\right)\right]\right\}\right].$$

with $\bar{\theta}^{i_1}$, $\tilde{\theta}^{i_2} \in [0,1]$ ($\forall\ i_1 = 1, 2, ..., I_1;\ i_2 = 1, 2, ..., I_2$) denoting the discounted factors, $\bar{w}^{i_1}$, $\tilde{w}^{i_2}$ $\in [0,1]$ ($\forall\ i_1 = 1, 2, ..., I_1;\ i_2 = 1, 2, ..., I_2$) the parameters that measure the contributions of the matrix payoffs of the game to the fitness of the corresponding strategies, and $\mathbb{E}^2_{(s,y)}$, $\mathbb{E}^1_{(s,x)}$ representing the expectation operators w. r. t. the complete probability law $\mathbb{P}^2$, $\mathbb{P}^1$ with depending on initial conditions $(s,y) \in \mathbb{R}_+ \times [0,1]^{I_2}$ and $(s,x) \in \mathbb{R}_+ \times [0,1]^{I_1}$, respectively. Thus, the problem, after technically modifying the above generalized expected discounted fitness functions, facing us can be expressed as follows,[28]

PROBLEM 1 (*Stochastic Differential Cooperative Game on Time*)[29]: We need to demonstrate that there

[27] Indeed, we are not necessary restricted to this type of specification of fitness functions, that is, one can employ much more general form of fitness functions. So, we employ this form of fitness functions is just for the sake of simplicity and tractability in determining the optimal stopping rule defined in the following Problem 1. Moreover, employing this kind of fitness functions is without loss of generality and will provide us with a suitable example to introduce the modeling-idea of the paper and to help the reader intuitively capture the logic implied by the formulation. Generally speaking, much more general form of objective function will prevent us from finding out the "smooth fit" conditions and hence determining the boundary explicitly in a given stochastic optimal-stopping problem from the viewpoint of mathematical technique, and so we leave this branching of exploration to future research.

[28] In a certain sense, Problem 1 defines a stochastic differential cooperative game on time, which hence could be regarded as a natural correspondence to the traditional Dynkin games.

[29] It should be emphasized here that we usually view game equilibrium in evolutionary game theory not from the viewpoint of individual choice since we have dropped the hyper-rational hypothesis widely used in classical non- cooperative game theory but directly from the viewpoint of strategies themselves, which can be analogically compared to different genes or cultures, following the standard biological evolution theory that strategies will have much more opportunities to survive in competition if they have much higher fitness or one strategy finally survives in posteriori just because it exhibits much higher fitness than any other competitive strategies in priori. In other words, if we employ the classical *as if* methodology, we directly argue that those strategies are regarded as successful in posteriori just because they are much more "rational" than any other unsuccessful strategies and they have been always maximizing their fitness in priori. So, the optimization problem defined in Problem 1 corresponds to the strategies but not directly to the individuals, who indeed follow adaptive learning mechanism in the current case, although we can apply the definition to both without bringing out troubles because the fitness defined in Problem 1 has an equivalent relationship with the payoffs of the individuals when they only employ pure strategies. As matter of fact, there exists certain subtle action-and-reaction or determination-and-redetermination relationship between "rational" strategies and naive individuals, that is, on the one hand, one strategy can be seen as a successful strategy or "smart" strategy just because as if this strategy has successfully attracted much more individuals in the population to use it in games, while on the other hand, more and more individuals will actually employ those "smart" strategies through imitation and learning since they find that these strategies provide them with much higher payoffs. Thus, those individuals have used those unsuccessful strategies will be finally replaced by new-born generations of the remaining individuals given the number of the population is a constant in the present model. So, in the limit, if it exists, of the successful adaptive learning processes, the population will consists of those lucky individuals equipped with successful genes which are represented by successful strategies. Then, one can easily find that the limit, if it exists, of the successful adaptive learning processes and those successful strategies will appear at the same time from the viewpoint of posteriori. Moreover, noting that we have put *as if* assumption on those

exist two vectors of $\mathcal{F}-$stopping times $\overline{\tau}^*(\omega) \triangleq \left(\overline{\tau}^{1*}(\omega),...,\overline{\tau}^{i_1*}(\omega),...,\overline{\tau}^{I_1*}(\omega)\right)^T$ and $\tilde{\tau}^*(\omega) \triangleq \left(\tilde{\tau}^{1*}(\omega),...,\tilde{\tau}^{i_2*}(\omega),...,\tilde{\tau}^{I_2*}(\omega)\right)^T$ with $\omega \in \Omega$ such that,

$$\overline{f}_{i_1}^*\left(\overline{\tau}^{i_1*}(\omega), Y\left(\overline{\tau}^{i_1*}(\omega)\right)\right)$$

$$\triangleq \sup_{\overline{\tau}^{i_1}(\omega)\leq\infty} \mathbb{E}_{(s,y)}\left[\exp\left(-\overline{\theta}^{i_1}\overline{\tau}^{i_1}(\omega)\right)\left\{1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[\overline{e}_{i_1}^T A\left(Y\left(\overline{\tau}^{i_1}(\omega)\right)+\tilde{\rho}\right)\right]\right\}\right],$$

$$\triangleq \mathbb{E}_{(s,y)}\left[\exp\left(-\overline{\theta}^{i_1}\overline{\tau}^{i_1*}(\omega)\right)\left\{1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[\overline{e}_{i_1}^T A\left(Y\left(\overline{\tau}^{i_1*}(\omega)\right)+\tilde{\rho}\right)\right]\right\}\right].$$

And simultaneously,

$$\tilde{f}_{i_2}^*\left(\tilde{\tau}^{i_2*}(\omega), X\left(\tilde{\tau}^{i_2*}(\omega)\right)\right)$$

$$\triangleq \sup_{\tilde{\tau}^{i_2}(\omega)\leq\infty} \mathbb{E}_{(s,x)}\left[\exp\left(-\tilde{\theta}^{i_2}\tilde{\tau}^{i_2}(\omega)\right)\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\tilde{e}_{i_2}^T B^T\left(X\left(\tilde{\tau}^{i_2}(\omega)\right)+\overline{\rho}\right)\right]\right\}\right],$$

$$\triangleq \mathbb{E}_{(s,x)}\left[\exp\left(-\tilde{\theta}^{i_2}\tilde{\tau}^{i_2*}(\omega)\right)\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\tilde{e}_{i_2}^T B^T\left(X\left(\tilde{\tau}^{i_2*}(\omega)\right)+\overline{\rho}\right)\right]\right\}\right].$$

with $\overline{\tau}^{i_1*}(\omega)=\overline{\tau}^{k_1*}(\omega)$ ($\forall\ i_1 \neq k_1, i_1, k_1 = 1, 2, ..., I_1$) $\mathbb{P}-$a.s., $\tilde{\tau}^{i_2*}(\omega)=\tilde{\tau}^{k_2*}(\omega)$ ($\forall\ i_2 \neq k_2$, i_2, $k_2 = 1, 2, ..., I_2$) $\mathbb{P}-$a.s., $\mathbb{E}_{(s,y)}$ and $\mathbb{E}_{(s,x)}$ stand for the expectation operators depending on initial

strategies. Therefore, we conclude that the limit of the successful learning processes of those lucky individuals just forms the solution of the fitness-optimization problems of those smart strategies. In other words, the successful adaptive learning processes will definitely converge to the rational solution of the fitness-optimization problems of these smart strategies. Correspondingly, the limit of the unsuccessful adaptive learning processes of those replaced individuals just forms the solution of those problems of those unsuccessful strategies, and these problems are, *as if*, not always fitness-optimization problems, or these strategies always made mistakes in attracting the individuals from the viewpoint of posteriori. And in the current model, these replicator dynamics defined in (2) will be employed as the adaptive learning processes of the individuals while Problem 1 corresponds to the *as if* fitness-optimization problems of the strategies. Rather, in Theorem 2 of the following section 4, we have proved that the learning processes will uniformly and robustly converge to the solution of the fitness-optimization problems under certain weak conditions. Nonetheless, up to the present step, the above story has not been completed. Noting that the fitness-optimization problems are defined for the strategies, thus the corresponding control variables naturally will not be these strategies themselves any more. And the present paper shows that stopping times or stopping rules will be suitable and useful control variables by noticing the following important fact, i.e., these fitness functions of the strategies directly and heavily depend on the learning processes or replicator dynamics of the players. Accordingly, the essence of Problem 1 can be expressed as follows, that is, searching for optimal stopping rules for the learning processes of the players according to the interest of these strategies, and also the essence of the competition between the two heterogeneous groups of populations is the competition between two kinds of learning processes, and finally the essence of the competition between the two kinds of learning processes will be valued and characterized by the competition between the column strategies and row strategies, the coordination among the column strategies and the coordination among the row strategies, in the original normal-form games. And in Theorem 2 of the following section 4, we have proved that the learning processes will uniformly converge to the optimal stopping rules determined by Problem 1 by essentially used the well-known Girsanov Theorem. We will emphasize in the following section again that the robust convergence from the learning processes to the rational solutions no longer depends on the requirement that errors, noises or stochastic disturbances approach zero, instead, the errors, noises or stochastic disturbances will always exist except that they are controlled in certain region following the *martingale* property.

conditions or information (s, y) and (s, x), respectively.

REMARK 3.3: (i) Indeed, $\overline{w}^{i_1}$, $\tilde{w}^{i_2} \in [0,1]$ ($\forall\ i_1 = 1, 2, ..., I_1$; $i_2 = 1, 2, ..., I_2$) can be regarded as objective parameters that measure the intensity of evolutionary selection (see, Ohtsuki et al., 2007), and also, this specification reflects the idea that, in reality, individuals or players inclined to use different strategies may feel different levels of importance of the corresponding game payoff to their fitness, thereby determining different degrees of participation which in turn will greatly affect the strategy choice of the players.

(ii) In order to capture the idea behind Problem 1 intuitively, we give the following slightly modified expression of Problem 1,

On the one hand, for strategy i_1, we call the stopping rule $\overline{\tau}^{i_1 *}(\omega)$ is *individually rational* if and only if,

$$\overline{\tau}^{i_1 *}(\omega) \triangleq \inf\left\{t \geq 0; Y(t) = y_{i_1}^*\left(x^*, \tilde{\rho}\right)\right\} \in \underset{\overline{\tau}^{i_1}(\omega) \leq \infty}{\arg\sup}\left[\overline{f}_{i_1}\left(\overline{\tau}^{i_1}(\omega), Y\left(\overline{\tau}^{i_1}(\omega)\right)\right)\right]$$

for $\forall\ i_1 = 1, 2, ..., I_1$. Moreover, we argue that the vector of stopping rules $\overline{\tau}^*(\omega) \triangleq \left(\overline{\tau}^{1*}(\omega), ..., \overline{\tau}^{i_1 *}(\omega), ..., \overline{\tau}^{I_1 *}(\omega)\right)^T$ defines a *stable equilibrium* if and only if,

$$\inf\left\{t \geq 0; Y(t) = y_{i_1}^*\left(x^*, \tilde{\rho}\right)\right\} \triangleq \overline{\tau}^{i_1 *}(\omega) = \overline{\tau}^{k_1 *}(\omega) \triangleq \inf\left\{t \geq 0; Y(t) = y_{k_1}^*\left(x^*, \tilde{\rho}\right)\right\}$$

for $\forall\ i_1 \neq k_1, i_1, k_1 = 1, 2, ..., I_1$. That is, $y_{i_1}^*\left(x^*, \tilde{\rho}\right) = y_{k_1}^*\left(x^*, \tilde{\rho}\right) \equiv y^*\left(x^*, \tilde{\rho}\right) \triangleq y^*$ for $\forall\ i_1 \neq k_1$, $i_1, k_1 = 1, 2, ..., I_1$. To conclude, "individually rational" requires there exists a boundary that defines a stopping rule for each strategy, and meanwhile "stable equilibrium" requires that these boundaries should be equal to each other from the viewpoint of group-level or collective behavior.

On the other hand, for strategy i_2, we call the stopping rule $\tilde{\tau}^{i_2 *}(\omega)$ is *individually rational* if and only if,

$$\tilde{\tau}^{i_2 *}(\omega) \triangleq \inf\left\{t \geq 0; X(t) = x_{i_2}^*\left(y^*, \overline{\rho}\right)\right\} \in \underset{\tilde{\tau}^{i_2}(\omega) \leq \infty}{\arg\sup}\left[\tilde{f}_{i_2}\left(\tilde{\tau}^{i_2}(\omega), X\left(\tilde{\tau}^{i_2}(\omega)\right)\right)\right]$$

for $\forall\ i_2 = 1, 2, ..., I_2$. Furthermore, we name the vector of stopping rules $\tilde{\tau}^*(\omega) \triangleq \left(\tilde{\tau}^{1*}(\omega), ..., \tilde{\tau}^{i_2 *}(\omega), ..., \tilde{\tau}^{I_2 *}(\omega)\right)^T$ a *stable equilibrium* if and only if,

$$\inf\left\{t \geq 0; X(t) = x_{i_2}^*\left(y^*, \overline{\rho}\right)\right\} \triangleq \tilde{\tau}^{i_2 *}(\omega) = \tilde{\tau}^{k_2 *}(\omega) \triangleq \inf\left\{t \geq 0; X(t) = x_{k_2}^*\left(y^*, \overline{\rho}\right)\right\}$$

for $\forall\ i_2 \neq k_2, i_2, k_2 = 1, 2, ..., I_2$. That is, $x_{i_2}^*\left(y^*, \bar{\rho}\right) = x_{k_2}^*\left(y^*, \bar{\rho}\right) \equiv x^*\left(y^*, \bar{\rho}\right) \triangleq x^*$ for $\forall\ i_2 \neq k_2$, $i_2, k_2 = 1, 2, ..., I_2$. To sum up, "individually rational" requirement is equivalent to the existence of one boundary for each strategy while "stable equilibrium" implying that these boundaries should take the same value, otherwise, conflict always exists and any given equilibrium consists of stopping rules will not be stable.

(iii) Rather, the game defined in Problem 1 can be regarded as consisting of three sub-games, i.e., the sub-game between the two groups of populations, the sub-game in column group and the sub-game in row group. The sub-game between the two groups of populations reflects the idea of Nash equilibrium or best-response correspondence, i.e., determining y^* given x^* and vice versa. However, for the sub-games in each group, the key issue is about coordination such that the stable equilibrium will be finally achieved. To summarize, for each decentralized and rational individual and in the group level, the corresponding Nash equilibrium or best-response strategy y^* (given x^*) (or x^* for any given y^*) need not be unique. However, there always exist only one population distribution y^* (given x^*) and only one population distribution x^* (given y^*) for each group, respectively, in each period. Accordingly, the in-group bargaining and coordination will finally lead us to the unique stable equilibrium denoted $\left(x^*\left(y^*, \bar{\rho}\right), y^*\left(x^*, \tilde{\rho}\right)\right)$.

(iv) In other words, the algorithm of Problem 1 actually implies a non-cooperative game sub-problem first and then a cooperative game sub-problem, that is, the Nash equilibrium $\left(x_{i_1}^*\left(y^*, \bar{\rho}\right), y_{i_2}^*\left(x^*, \tilde{\rho}\right)\right)$ for $\forall\ i_1 = 1, 2, ..., I_1$ and $\forall\ i_2 = 1, 2, ..., I_2$, which may be not unique in theory, will be determined by the non-cooperative game between the two heterogeneous groups of populations thanks to the best-response strategy, and finally the unique stable equilibrium $\left(x^*\left(y^*, \bar{\rho}\right), y^*\left(x^*, \tilde{\rho}\right)\right)$ will be derived via the cooperative game in each of the two groups. Noting that people live in a structured society and the game problem facing us in reality is rare a pure non-cooperative game problem or a pure cooperative game problem, for instance and relatively speaking, the game problem between two countries will be usually interpreted as a non-cooperative game problem while the game problem in each country will be thus interpreted as a cooperative game problem, we argue that Problem 1 supplies a good approximation to reality. To be much more exact and much deeper, there exist many *social equilibria* facing us need be established by employing both non-cooperative game theory and cooperative game theory, for instance, in the field of international trade and for any two given democratic trading countries denoted α and β, both country α and country β consist of two departments, i.e., export department and import department, thus, given the possible optimal strategy of country β, export department of country α will have a best-response strategy denoted α^* according to its interest consideration, and also import department of country α will have a best-response strategy denoted α_*, and vice versa. Then we obtain the following four possible or potential Nash equilibria, i.e., (α^*,β^*), (α^*,β_*), (α_*,β^*) and (α_*,β_*) from the above between-country non-cooperative game. Obviously, the Nash equilibria are not necessary to be unique from the viewpoint of pure game theory. However, there always exists unique equilibrium in reality, that is, only one foreign-policy equilibrium will be established and enforced. Then, we turn to the in-country cooperative game and this is naturally related to the bargaining and coordination problem. Finally, we will get the unique stable foreign-policy equilibrium denoted $(\bar{\alpha},\bar{\beta})$, in which $\bar{\alpha}$ and $\bar{\beta}$ should be the Nash bargaining solutions (see, Nash, 1950) or Shapley Values (see, Shapley, 1953) of export department and import department of country α and country β, respectively.

(v) It follows from Problem 1 that,

$$\bar{f}_{i_1}^*\left(\bar{\tau}^{i_1*}(\omega),Y\left(\bar{\tau}^{i_1*}(\omega)\right)\right)$$

$$\triangleq \sup_{\bar{\tau}^{i_1}(\omega)\le\infty} \mathbb{E}\left[\exp\left(-\bar{\theta}^{i_1}\bar{\tau}^{i_1}(\omega)\right)\left\{1-\bar{w}^{i_1}+\bar{w}^{i_1}\left[\bar{e}_{i_1}^T A\left(Y\left(\bar{\tau}^{i_1}(\omega)\right)+\tilde{\rho}\right)\right]\right\}\middle|\mathcal{F}_s\right],$$

And simultaneously,

$$\tilde{f}_{i_2}^*\left(\tilde{\tau}^{i_2*}(\omega),X\left(\tilde{\tau}^{i_2*}(\omega)\right)\right)$$

$$\triangleq \sup_{\tilde{\tau}^{i_2}(\omega)\le\infty} \mathbb{E}\left[\exp\left(-\tilde{\theta}^{i_2}\tilde{\tau}^{i_2}(\omega)\right)\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\tilde{e}_{i_2}^T B^T\left(X\left(\tilde{\tau}^{i_2}(\omega)\right)+\bar{\rho}\right)\right]\right\}\middle|\mathcal{F}_s\right]$$

for $\forall\ i_1 = 1,2,...,I_1$ and $\forall\ i_2 = 1,2,\ ...,I_2$. And this implies that we have proposed complete-information and symmetric-information assumption in Problem 1. Alternatively, if we are given another two filtrations denoted $\overline{\mathbb{F}} \triangleq \left\{\overline{\mathcal{F}}_t\right\}_{0\le t\le \tau(\omega)}$ and $\tilde{\mathbb{F}} \triangleq \left\{\tilde{\mathcal{F}}_t\right\}_{0\le t\le \tau(\omega)}$ with $\overline{\mathcal{F}}_t \subseteq \mathcal{F}_t \subseteq \tilde{\mathcal{F}}_t$ for $\forall 0 \le t \le \tau(\omega)$, Problem 1 can be extended to include *incomplete information*, for example, we consider the following optimization problem,

$$\overline{f}_{i_1}^*\left(\overline{\tau}^{i_1*}(\omega), Y\left(\overline{\tau}^{i_1*}(\omega)\right)\right)$$

$$\triangleq \sup_{\overline{\tau}^{i_1}(\omega)\le\infty} \mathbb{E}\left[\exp\left(-\overline{\theta}^{i_1}\overline{\tau}^{i_1}(\omega)\right)\left\{1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[\overline{e}_{i_1}^T A\left(Y\left(\overline{\tau}^{i_1}(\omega)\right)+\tilde{\rho}\right)\right]\right\}\middle|\overline{\mathcal{F}}_s\right],$$

And simultaneously,

$$\tilde{f}_{i_2}^*\left(\tilde{\tau}^{i_2*}(\omega), X\left(\tilde{\tau}^{i_2*}(\omega)\right)\right)$$

$$\triangleq \sup_{\tilde{\tau}^{i_2}(\omega)\le\infty} \mathbb{E}\left[\exp\left(-\tilde{\theta}^{i_2}\tilde{\tau}^{i_2}(\omega)\right)\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\tilde{e}_{i_2}^T B^T\left(X\left(\tilde{\tau}^{i_2}(\omega)\right)+\overline{\rho}\right)\right]\right\}\middle|\overline{\mathcal{F}}_s\right]$$

for $\forall\ i_1 = 1,2,\ldots,I_1$ and $\forall\ i_2 = 1,2,\ \ldots,I_2$. And also, the case of *asymmetric information* can be easily constructed, for instance, we now consider the following optimization problem,

$$\overline{f}_{i_1}^*\left(\overline{\tau}^{i_1*}(\omega), Y\left(\overline{\tau}^{i_1*}(\omega)\right)\right)$$

$$\triangleq \sup_{\overline{\tau}^{i_1}(\omega)\le\infty} \mathbb{E}\left[\exp\left(-\overline{\theta}^{i_1}\overline{\tau}^{i_1}(\omega)\right)\left\{1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[\overline{e}_{i_1}^T A\left(Y\left(\overline{\tau}^{i_1}(\omega)\right)+\tilde{\rho}\right)\right]\right\}\middle|\overline{\mathcal{F}}_s\right],$$

And simultaneously,

$$\tilde{f}_{i_2}^*\left(\tilde{\tau}^{i_2*}(\omega), X\left(\tilde{\tau}^{i_2*}(\omega)\right)\right)$$

$$\triangleq \sup_{\tilde{\tau}^{i_2}(\omega)\le\infty} \mathbb{E}\left[\exp\left(-\tilde{\theta}^{i_2}\tilde{\tau}^{i_2}(\omega)\right)\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\tilde{e}_{i_2}^T B^T\left(X\left(\tilde{\tau}^{i_2}(\omega)\right)+\overline{\rho}\right)\right]\right\}\middle|\tilde{\mathcal{F}}_s\right]$$

for $\forall\ i_1 = 1,2,\ldots,I_1$ and $\forall\ i_2 = 1,2,\ \ldots,I_2$. Moreover, we can set problems implying that there exists asymmetric information in each group, for example, $\forall k_1 \in \left\{1,2,\ \ \ldots,I_1\right\}$, we now consider the following case,

$$\overline{f}_{i_1}^*\left(\overline{\tau}^{i_1*}(\omega), Y\left(\overline{\tau}^{i_1*}(\omega)\right)\right)$$

$$\triangleq \sup_{\overline{\tau}^{i_1}(\omega)\leq\infty} \mathbb{E}\left[\exp\left(-\overline{\theta}^{i_1}\overline{\tau}^{i_1}(\omega)\right)\left\{1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[\overline{e}_{i_1}^T A\left(Y\left(\overline{\tau}^{i_1}(\omega)\right)+\tilde{\rho}\right)\right]\right\}\middle|\overline{\mathcal{F}}_s\right]$$

for $\forall\ i_1 = 1,2,\ldots,k_1$. And,

$$\overline{f}_{i_1}^*\left(\overline{\tau}^{i_1*}(\omega), Y\left(\overline{\tau}^{i_1*}(\omega)\right)\right)$$

$$\triangleq \sup_{\overline{\tau}^{i_1}(\omega)\leq\infty} \mathbb{E}\left[\exp\left(-\overline{\theta}^{i_1}\overline{\tau}^{i_1}(\omega)\right)\left\{1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[\overline{e}_{i_1}^T A\left(Y\left(\overline{\tau}^{i_1}(\omega)\right)+\tilde{\rho}\right)\right]\right\}\middle|\mathcal{F}_s\right]$$

for $\forall\ i_1 = k_1, k_1+1,\ldots,I_1$. That is to say, filtration $\overline{\mathbb{F}} \triangleq \left\{\overline{\mathcal{F}}_t\right\}_{0\leq t\leq\tau(\omega)}$ implies *incomplete information* while filtration $\tilde{\mathbb{F}} \triangleq \left\{\tilde{\mathcal{F}}_t\right\}_{0\leq t\leq\tau(\omega)}$ implies *private information* or *inside information*. All in all, Problem 1 can be naturally extended to study much more cases appear in society by employing more complicated mathematical techniques.

DEFINITION 1 (*Pareto Optimal Endogenous Matching and Induced Nash Equilibrium*): The solution, if it exists, to Problem 1 defines a game equilibrium, denoted

$$\left(x^*(y^*,\overline{\rho}) \triangleq \left(x^{1*}(y^*,\overline{\rho}),\ldots,x^{i_1*}(y^*,\overline{\rho}),\ldots,x^{I_1*}(y^*,\overline{\rho})\right)^T,\right.$$

$$\left. y^*(x^*,\tilde{\rho}) \triangleq \left(y^{1*}(x^*,\tilde{\rho}),\ \ldots, y^{i_2*}(x^*,\tilde{\rho}),\ldots, y^{I_2*}(x^*,\tilde{\rho})\right)^T\right).$$

with $\sum_{i_1}^{I_1} x^{i_1*}(y^*,\overline{\rho}) = 1$ and $\sum_{i_2}^{I_2} y^{i_2*}(x^*,\tilde{\rho}) = 1$, induced by stochastic group evolution and rational individual choice corresponding to the very general normal form game situations. Suppose that we are provided with a Pareto optimal Nash equilibrium denoted by $\left(\hat{x} \triangleq (\hat{x}^1,\ldots,\hat{x}^{i_1},\ldots,\hat{x}^{I_1})^T,\right.$ $\left.\hat{y} \triangleq (\hat{y}^1,\ldots,\hat{y}^{i_2},\ldots,\hat{y}^{I_2})^T\right)$ with $\sum_{i_1}^{I_1}\hat{x}^{i_1} = 1$ and $\sum_{i_2}^{I_2}\hat{y}^{i_2} = 1$ in the original normal form game, then we arrive at the *Pareto optimal endogenous matching* by solving the following equations, i.e., $x^*(y^*,\overline{\rho}) = \hat{x}$ and $y^*(x^*,\tilde{\rho}) = \hat{y}$, and we represent the corresponding Pareto optimal endogenous matching by $(\overline{\rho}^*,\tilde{\rho}^*)$. Moreover, we call the Pareto optimal Nash equilibrium $(\hat{x},\hat{y})$ *induced Nash equilibrium* in the current game situations and in some sense.

REMARK 3.4: (i) Here, and throughout the current paper, we study the game equilibrium by employing evolutionary game theory under uncertainty, which implies that the game equilibrium is characterized from the viewpoint of group level, thereby leading to a case where classical optimal control theory is not suitable for rational individual choice while stochastic optimal stopping theory is powerful and hence plays a crucial role in proving and characterizing the existence of the induced game equilibrium, and hence the Pareto optimal endogenous matching.

(ii) Specifically, it is worth noting that there exists intrinsic relationship between the endogenous matching and the broadly applied random matching (see, Ellison, 1994; and Weibull, 1995, for instance). Notice that the present endogenous matching could be naturally viewed as certain perturbation of the perfect world with well-mixed population to some extent and in some sense, random matching indeed represents a special case of the endogenous matching studied in the paper. In other words, if we suppose that individuals or players play the game in a perfect world rather than a *structured society*, random-matching hypothesis is quite appropriate and also random matching itself would be regarded as endogenously determined, i.e., determined by the corresponding game environment. Generally speaking and to the best of our knowledge, random matching is just employed as an exogenous matching mechanism which does not imply any welfare standard or will be implied by any welfare standard in existing studies (see, Ellison, 1994; and Weibull, 1995, and among others). Nevertheless, as an extreme case of the endogenous matching studied here, random matching itself indeed yields economic-welfare implications. For example, if we can establish that $\lim_{\bar{\rho}\to 0} x^*(y^*,\bar{\rho}) = \hat{x}$ and $\lim_{\tilde{\rho}\to 0} y^*(x^*,\tilde{\rho}) = \hat{y}$, we can definitely call the corresponding random matching *asymptotically Pareto efficient* (or *Pareto optimal*). As is well known, people live in a structured society and thus random matching only works as certain limit of the endogenous matching. And random matching will be supplied with much richer economic intuitions and implications as long as it is studied in a way intimately related to the present endogenous matching. All in all, game rule is implied by the society structure[30] in some sense and the society structure rather implies certain economic-welfare implication, so our study of endogenous matching deepens the present study of matching theory.

(iii) Intuitively, we name the matching mechanism here *endogenous* matching just because it is determined by other parameters of the model, say, the payoff structures, the discount factors, the parameters that measure the contribution of the payoffs to the fitness, and also the stochastic volatility. In other words, endogenous matching can be written as a function of the above parameters, and, if motivated, we can even take comparative static analyses after complicated computations.

3.3. Existence of Pareto Optimal Endogenous Matching

To do this, we now define $\tilde{Z}(t) \triangleq \left(s+t, X(t)\right)$ for $\forall\, t \in \mathbb{R}_+$ with $\tilde{Z}(0) \triangleq (s,x) \in \mathbb{R}_+ \times \left[0,1\right]^{l_1}$,

[30] It should include both spatial structure and division structure of any given mature market.

and $\bar{Z}(t) \triangleq \left(s+t, Y(t)\right)$ for $\forall\ t \in \mathbb{R}_+$ with $\bar{Z}(0) \triangleq (s, y) \in \mathbb{R}_+ \times \left[0,1\right]^{I_2}$. And also we let,

$$\nabla\tilde{f}(s,x) \triangleq \left(\tfrac{\partial \tilde{f}}{\partial x^1}(s,x),\ldots,\tfrac{\partial \tilde{f}}{\partial x^{i_1}}(s,x),\ldots,\tfrac{\partial \tilde{f}}{\partial x^{I_1}}(s,x)\right)^T,$$

$$\bar{\gamma}_{l_1}(x) \triangleq \left(x^1\bar{\gamma}_{1l_1}\left(z_{l_1}^1\right),\ \ldots\ ,x^{i_1}\bar{\gamma}_{i_1l_1}\left(z_{l_1}^{i_1}\right),\ldots,x^{I_1}\bar{\gamma}_{I_1l_1}\left(z_{l_1}^{I_1}\right)\right)^T,$$

$$\nabla\bar{f}(s,y) \triangleq \left(\tfrac{\partial \bar{f}}{\partial y^1}(s,y),\ldots,\tfrac{\partial \bar{f}}{\partial y^{i_2}}(s,y),\ldots,\tfrac{\partial \bar{f}}{\partial y^{I_2}}(s,y)\right)^T,$$

And,

$$\tilde{\gamma}_{l_2}(y) \triangleq \left(y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right),\ldots,y^{i_2}\tilde{\gamma}_{i_2l_2}\left(z_{l_2}^{i_2}\right),\ldots,y^{I_2}\tilde{\gamma}_{I_2l_2}\left(z_{l_2}^{I_2}\right)\right)^T.$$

Then the characteristic operators of $\tilde{Z}(t)$ and $\bar{Z}(t)$ can be respectively given by,

$$\begin{aligned}\mathcal{A}\tilde{f}(s,x) = {}& \frac{\partial \tilde{f}}{\partial s}(s,x) + \sum_{i_1=1}^{I_1} x^{i_1}\left(\bar{e}_{i_1}^T A y\right)\frac{\partial \tilde{f}}{\partial x^{i_1}}(s,x) + \frac{1}{2}\sum_{i_1=1}^{I_1}\left(x^{i_1}\right)^2\left(\bar{\sigma}^{i_1}\right)^T\bar{\sigma}^{i_1}\frac{\partial^2 \tilde{f}}{\partial (x^{i_1})^2}(s,x)\\ &+\sum_{k_1=1}^{I_1}\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\tilde{f}\left(s,x+\bar{\gamma}_{l_1}(x)\right)-\tilde{f}(s,x)-\left\langle\nabla\tilde{f}(s,x),\bar{\gamma}_{l_1}(x)\right\rangle\right\}\nu_{l_1}^{k_1}\left(dz_{l_1}^{k_1}\right),\\ &\qquad\qquad\forall\ \tilde{f}\in C^2\left(\mathbb{R}^{I_1+1}\right).\end{aligned}$$

And,

$$\begin{aligned}\mathcal{A}\bar{f}(s,y) = {}& \frac{\partial \bar{f}}{\partial s}(s,y) + \sum_{i_2=1}^{I_2} y^{i_2}\left(\tilde{e}_{i_2}^T B^T x\right)\frac{\partial \bar{f}}{\partial y^{i_2}}(s,y) + \frac{1}{2}\sum_{i_2=1}^{I_2}\left(y^{i_2}\right)^2\left(\tilde{\sigma}^{i_2}\right)^T\tilde{\sigma}^{i_2}\frac{\partial^2 \bar{f}}{\partial (y^{i_2})^2}(s,y)\\ &+\sum_{k_2=1}^{I_2}\int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\{\bar{f}\left(s,y+\tilde{\gamma}_{l_2}(y)\right)-\bar{f}(s,y)-\left\langle\nabla\bar{f}(s,y),\tilde{\gamma}_{l_2}(y)\right\rangle\right\}\nu_{l_2}^{k_2}\left(dz_{l_2}^{k_2}\right),\\ &\qquad\qquad\forall\ \bar{f}\in C^2\left(\mathbb{R}^{I_2+1}\right).\end{aligned}$$

Furthermore, we let $\sum_{i_1=1}^{I_1-1} x^{i_1} = \bar{\delta}_1$, then $x^{I_1} = 1-\bar{\delta}_1$ with $0 \le \bar{\delta}_1 \le 1$ by noting that $\sum_{i_1=1}^{I_1} x^{i_1} = 1$. Let $\sum_{i_1=1}^{I_1-2} x^{i_1} = \bar{\delta}_2$, then we can get $x^{I_1-1} = \bar{\delta}_1 - \bar{\delta}_2$ with $0 \le \bar{\delta}_2 \le \bar{\delta}_1 \le 1$. Inductively, we let $\sum_{i_1=1}^{I_1-(I_1-2)} x^{i_1} = \bar{\delta}_{I_1-2}$, then we have $x^3 = x^{I_1-(I_1-3)} = \bar{\delta}_{I_1-3} - \bar{\delta}_{I_1-2}$ with $0 \le \bar{\delta}_{I_1-2} \le \bar{\delta}_{I_1-3} \le \ldots \le \bar{\delta}_1 \le 1$; let $\sum_{i_1=1}^{I_1-(I_1-1)} x^{i_1} = \bar{\delta}_{I_1-1}$, i.e., $x^1 = \bar{\delta}_{I_1-1}$, then we get $x^2 = x^{I_1-(I_1-2)} = \bar{\delta}_{I_1-2} - \bar{\delta}_{I_1-1}$ with $0 \le \bar{\delta}_{I_1-1} \le$

$\bar{\delta}_{I_1-2} \le \bar{\delta}_{I_1-3} \le ... \le \bar{\delta}_1 \le 1$. And without loss of any generality, we put $\bar{\delta}_0 \equiv 1$. Then we obtain,

$$u\left(i_2, x+\bar{\rho}\right) = \tilde{e}_{i_2}^T B^T (x+\bar{\rho}) = \left(b_{i_2 1} - b_{i_2 2}\right) x^1 + b_{i_2 2}\bar{\delta}_{I_1-2} + \sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1} - \bar{\delta}_{I_1-i_1+1}\right) + \tilde{e}_{i_2}^T B^T \bar{\rho}.$$

Similarly, notice that $\sum_{i_2=1}^{I_2} y^{i_2} \equiv 1$ and let $\sum_{i_2=1}^{I_2-1} y^{i_2} = \tilde{\delta}_1$, then we have $y^{I_2} = 1 - \tilde{\delta}_1$ with $0 \le \tilde{\delta}_1 \le 1$. Let $\sum_{i_2=1}^{I_2-2} y^{i_2} = \tilde{\delta}_2$, then we see that $y^{I_2-1} = \tilde{\delta}_1 - \tilde{\delta}_2$ with $0 \le \tilde{\delta}_2 \le \tilde{\delta}_1 \le 1$. Inductively, let $\sum_{i_2=1}^{I_2-(I_2-2)} y^{i_2} = \tilde{\delta}_{I_2-2}$, then we have $y^3 = y^{I_2-(I_2-3)} = \tilde{\delta}_{I_2-3} - \tilde{\delta}_{I_2-2}$ with $0 \le \tilde{\delta}_{I_2-2} \le \tilde{\delta}_{I_2-3} \le ... \le \tilde{\delta}_1 \le 1$; let $\sum_{i_2=1}^{I_2-(I_2-1)} y^{i_2} = \tilde{\delta}_{I_2-1}$, i.e., $y^1 = \tilde{\delta}_{I_2-1}$, then it follows that $y^2 = y^{I_2-(I_2-2)} = \tilde{\delta}_{I_2-2} - \tilde{\delta}_{I_2-1}$ with $0 \le \tilde{\delta}_{I_2-1} \le \tilde{\delta}_{I_2-2} \le \tilde{\delta}_{I_2-3} \le ... \le \tilde{\delta}_1 \le 1$. And we, without loss of any generality, put $\tilde{\delta}_0 \equiv 1$. Then we get,

$$u\left(i_1, y+\tilde{\rho}\right) = \bar{e}_{i_1}^T A(y+\tilde{\rho}) = \left(a_{i_1 1} - a_{i_1 2}\right) y^1 + a_{i_1 2}\tilde{\delta}_{I_2-2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2} - \tilde{\delta}_{I_2-i_2+1}\right) + \bar{e}_{i_1}^T A\tilde{\rho}.$$

Therefore, the discounted fitness functions in Problem 1 can be rewritten as,

$$\begin{aligned}\bar{f}_{i_1}\left(s, y^1\right) = \exp\left(-\bar{\theta}^{i_1} s\right)& \\ \times\left\{1 - \bar{w}^{i_1} + \bar{w}^{i_1}\left[\left(a_{i_1 1} - a_{i_1 2}\right) y^1 + a_{i_1 2}\tilde{\delta}_{I_2-2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2} - \tilde{\delta}_{I_2-i_2+1}\right) + \bar{e}_{i_1}^T A\tilde{\rho}\right]\right\},& \\ \forall\ i_1 = 1, 2, ..., I_1.&\end{aligned}$$

$$\begin{aligned}\tilde{f}_{i_2}\left(s, x^1\right) = \exp\left(-\tilde{\theta}^{i_2} s\right)& \\ \times\left\{1 - \tilde{w}^{i_2} + \tilde{w}^{i_2}\left[\left(b_{i_2 1} - b_{i_2 2}\right) x^1 + b_{i_2 2}\bar{\delta}_{I_1-2} + \sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1} - \bar{\delta}_{I_1-i_1+1}\right) + \tilde{e}_{i_2}^T B^T \bar{\rho}\right]\right\},& \\ \forall\ i_2 = 1, 2, ..., I_2. \quad (3)&\end{aligned}$$

with $0 \le \tilde{\delta}_{I_2-1} \le \tilde{\delta}_{I_2-2} \le \tilde{\delta}_{I_2-3} \le ... \le \tilde{\delta}_1 \le \tilde{\delta}_0 \equiv 1$ and $0 \le \bar{\delta}_{I_1-1} \le \bar{\delta}_{I_1-2} \le \bar{\delta}_{I_1-3} \le ... \le \bar{\delta}_1 \le \bar{\delta}_0 \equiv 1$. And inspection of the fitness functions given in (3) reveals that one can just define $\tilde{Z}(t) \triangleq \left(s+t, X^1(t)\right)$

for $\forall\ t \in \mathbb{R}_+$ with $\tilde{Z}(0) \triangleq \left(s, x^1\right) \in \mathbb{R}_+ \times [0,1]$, and $\bar{Z}(t) \triangleq \left(s+t, Y^1(t)\right)$ for $\forall\ t \in \mathbb{R}_+$ with $\bar{Z}(0) \triangleq \left(s, y^1\right) \in \mathbb{R}_+ \times [0,1]$. And hence the corresponding characteristic operators of $\tilde{Z}(t)$ and $\bar{Z}(t)$ are respectively given by,

$$\mathcal{A}\tilde{f}\left(s,x^1\right)=\frac{\partial\tilde{f}}{\partial s}\left(s,x^1\right)+x^1\left(\bar{e}_1^T Ay\right)\frac{\partial\tilde{f}}{\partial x^1}\left(s,x^1\right)+\frac{1}{2}\left(x^1\right)^2\left(\bar{\sigma}^1\right)^T\bar{\sigma}^1\frac{\partial^2\tilde{f}}{\partial (x^1)^2}\left(s,x^1\right)$$

$$+\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\tilde{f}\left(s,x^1+x^1\bar{\gamma}_{1l_1}\left(z_{l_1}^1\right)\right)-\tilde{f}\left(s,x^1\right)-x^1\bar{\gamma}_{1l_1}\left(z_{l_1}^1\right)\tfrac{\partial\tilde{f}}{\partial x^1}\left(s,x^1\right)\right\}\nu_{l_1}^1\left(dz_{l_1}^1\right),$$

$$\forall\ \tilde{f}\in C^2\left(\mathbb{R}^2\right).$$

And,

$$\mathcal{A}\bar{f}\left(s,y^1\right)=\frac{\partial\bar{f}}{\partial s}\left(s,y^1\right)+y^1\left(\tilde{e}_1^T B^T x\right)\frac{\partial\bar{f}}{\partial y^1}\left(s,y^1\right)+\frac{1}{2}\left(y^1\right)^2\left(\tilde{\sigma}^1\right)^T\tilde{\sigma}^1\frac{\partial^2\bar{f}}{\partial (y^1)^2}\left(s,y^1\right)$$

$$+\int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\{\bar{f}\left(s,y^1+y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\right)-\bar{f}\left(s,y^1\right)-y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\tfrac{\partial\bar{f}}{\partial y^1}\left(s,y^1\right)\right\}\nu_{l_2}^1\left(dz_{l_2}^1\right)$$

$$\forall\ \bar{f}\in C^2\left(\mathbb{R}^2\right).$$

Therefore, based upon the above assumptions and specifications, the following theorem is derived,

THEOREM 1: *There exists a unique solution to Problem* 1 *under very weak conditions, and accordingly the existence of the Pareto optimal endogenous matching is confirmed provided that we are given a Pareto optimal Nash equilibrium* $\left(\hat{x}, \hat{y}\right)$, *which is given in Definition* 1.

PROOF: See Appendix B. ▌

REMARK 3.5: (i) It is especially worth noting that Theorem 1 not only shows the existence of the Pareto optimal endogenous matching and induced Nash equilibrium given by Definition 1 but also provides us with the *explicit time length* needed so that the Pareto optimal endogenous matching and also the induced Nash equilibrium can be achieved by decentralized players. Moreover, it is also worth emphasizing that our conclusion holds true for any Pareto optimal *strategy combination* of very general normal form games although we have only considered Pareto optimal *Nash equilibrium* in Theorem 1. For instance, (cooperation, cooperation) is a Pareto optimal strategy combination in PD games although it is generally not a Nash equilibrium at all. Obviously, our endogenous matching rule can lead us to cooperation in PD games.

(ii) Furthermore, it follows from proving the existence of the Pareto optimal endogenous matching that we have provided an *algorithm* for computing equilibria in asymmetric normal-form games with countable many strategies and players. Actually, we just need to solve a group of linear equations.

4. Stability

In what follows, we are encouraged to show the stability of the Pareto optimal endogenous matching. And we do so by first giving the following definition,

DEFINITION 2 (Unit Simplex of Evolutionary Dynamics): Here, and throughout the present paper, we put $\bar{\Delta} \triangleq \left\{ y \in \mathbb{R}_+^{I_2}; \sum_{i_2=1}^{I_2} y^{i_2} = 1 \right\}$ and $\tilde{\Delta} \triangleq \left\{ x \in \mathbb{R}_+^{I_1}; \sum_{i_1=1}^{I_1} x^{i_1} = 1 \right\}$ as the *unit simplexes of the evolutionary dynamics* defined in (2). Moreover, we let $\operatorname{int}\left(\bar{\Delta}\right)$ and $\operatorname{int}\left(\tilde{\Delta}\right)$ denote the interiors of $\bar{\Delta}$ and $\tilde{\Delta}$, respectively.

4.1. Convergence of Induced Game Equilibrium

In the present section, we consider the convergence of the induced Nash equilibrium, denoted $x^*(y^*, \bar{\rho}) = \hat{x} \in \tilde{\Delta}$ and $y^*(x^*, \tilde{\rho}) = \hat{y} \in \bar{\Delta}$. It follows from (1) that,

$$dX^{i_1}(t) = X^{i_1}(t)\left[\bar{e}_{i_1}^T AY(t)dt + \bar{\sigma}^{i_1}(t)dW^{i_1}(t) + \int_{\mathbb{R}_0^{n_1}} \bar{\gamma}^{i_1}\left(t, z^{i_1}\right)\tilde{N}^{i_1}\left(dt, dz^{i_1}\right) \right],$$

$$dY^{i_2}(t) = Y^{i_2}(t)\left[\tilde{e}_{i_2}^T B^T X(t)dt + \tilde{\sigma}^{i_2}(t)dW^{i_2}(t) + \int_{\mathbb{R}_0^{n_2}} \tilde{\gamma}^{i_2}\left(t, z^{i_2}\right)\tilde{N}^{i_2}\left(dt, dz^{i_2}\right) \right]. \tag{1'}$$

for $\forall\ i_1 = 1, 2, ..., I_1$ and $i_2 = 1, 2, ..., I_2$. Now, we need the following assumptions,

ASSUMPTION 5: *It is reasonable to suppose that there exist processes* $\bar{\Theta}^{i_1}\left(t, z^{i_1}\right) \triangleq \left(\bar{\Theta}_1^{i_1}\left(t, z_1^{i_1}\right), ..., \bar{\Theta}_{l_1}^{i_1}\left(t, z_{l_1}^{i_1}\right), ..., \bar{\Theta}_{n_1}^{i_1}\left(t, z_{n_1}^{i_1}\right)\right)^T \in \mathbb{R}^{n_1}$ *with* $\bar{\Theta}_{l_1}^{i_1}\left(t, z_{l_1}^{i_1}\right) \le 1$ *and* $\bar{\vartheta}^{i_1}(t) \in \mathbb{R}^{d_1}$ *that are* $\mathbb{F}^{i_1}-$*predictable such that for any* $T > 0$,

- $\bar{\sigma}^{i_1}(t)\bar{\vartheta}^{i_1}(t)+\sum_{l_1=1}^{n_1}\int_{\mathbb{R}_0}\bar{\gamma}_{i_1 l_1}\left(t,z_{l_1}^{i_1}\right)\bar{\Theta}_{l_1}^{i_1}\left(t,z_{l_1}^{i_1}\right)\nu_{l_1}^{i_1}\left(dz_{l_1}^{i_1}\right)=\bar{e}_{i_1}^{T}AY(t)$, $\mu\otimes\mathbb{P}^{i_1}-a.e.$
- $\int_0^T\left\|\bar{\vartheta}^{i_1}(t)\right\|_2^2 dt<\infty$, $\mathbb{P}^{i_1}-a.s.$
- $\sum_{l_1=1}^{n_1}\int_0^T\int_{\mathbb{R}_0}\left[\left|\log\left(1+\bar{\Theta}_{l_1}^{i_1}\left(t,z_{l_1}^{i_1}\right)\right)\right|+\left\|\bar{\Theta}^{i_1}\left(t,z^{i_1}\right)\right\|_2^2\right]\nu_{l_1}^{i_1}\left(dz_{l_1}^{i_1}\right)dt<\infty$, $\mathbb{P}^{i_1}-a.s.$

for a.a. $(t,\omega)\in[0,T]\times\Omega^{i_1}$, *and for* $\forall\ i_1=1,2,...,I_1$.

ASSUMPTION 6: *It is, similar to Assumption* 5, *assumed that there exist processes* $\tilde{\Theta}^{i_2}\left(t,z^{i_2}\right)\triangleq\left(\tilde{\Theta}_1^{i_2}\left(t,z_1^{i_2}\right),...,\ \tilde{\Theta}_{l_2}^{i_2}\left(t,z_{l_2}^{i_2}\right),...,\tilde{\Theta}_{n_2}^{i_2}\left(t,z_{n_2}^{i_2}\right)\right)^T\in\mathbb{R}^{n_2}$ *with* $\tilde{\Theta}_{l_2}^{i_2}\left(t,z_{l_2}^{i_2}\right)\le 1$ *and* $\tilde{\vartheta}^{i_2}(t)\in\mathbb{R}^{d_2}$ *that are* $\mathbb{F}^{i_2}-$*predictable such that for any* $T>0$,

- $\tilde{\sigma}^{i_2}(t)\tilde{\vartheta}^{i_2}(t)+\sum_{l_2=1}^{n_2}\int_{\mathbb{R}_0}\tilde{\gamma}_{i_2 l_2}\left(t,z_{l_2}^{i_2}\right)\tilde{\Theta}_{l_2}^{i_2}\left(t,z_{l_2}^{i_2}\right)\nu_{l_2}^{i_2}\left(dz_{l_2}^{i_2}\right)=\tilde{e}_{i_2}^{T}B^{T}X(t)$, $\mu\otimes\mathbb{P}^{i_2}-a.e.$
- $\int_0^T\left\|\tilde{\vartheta}^{i_2}(t)\right\|_2^2 dt<\infty$, $\mathbb{P}^{i_2}-a.s.$
- $\sum_{l_2=1}^{n_2}\int_0^T\int_{\mathbb{R}_0}\left[\left|\log\left(1+\tilde{\Theta}_{l_2}^{i_2}\left(t,z_{l_2}^{i_2}\right)\right)\right|+\left\|\tilde{\Theta}^{i_2}\left(t,z^{i_2}\right)\right\|_2^2\right]\nu_{l_2}^{i_2}\left(dz_{l_2}^{i_2}\right)dt<\infty$, $\mathbb{P}^{i_2}-a.s.$

for a.a. $(t,\omega)\in[0,T]\times\Omega^{i_2}$, *and for* $\forall\ i_2=1,2,...,I_2$.

Now, letting,

$$\begin{aligned}\bar{\Psi}^{i_1}(t)\triangleq\exp\Bigg\{&\sum_{l_1=1}^{n_1}\int_0^t\int_{\mathbb{R}_0}\left[\left|\log\left(1+\bar{\Theta}_{l_1}^{i_1}\left(s,z_{l_1}^{i_1}\right)\right)\right|+\bar{\Theta}_{l_1}^{i_1}\left(s,z_{l_1}^{i_1}\right)\right]\nu_{l_1}^{i_1}\left(dz_{l_1}^{i_1}\right)ds\\&-\int_0^t\bar{\vartheta}^{i_1}(s)dW^{i_1}(s)-\int_0^t\left\|\bar{\vartheta}^{i_1}(s)\right\|_2^2 ds\\&+\sum_{l_1=1}^{n_1}\int_0^t\int_{\mathbb{R}_0}\log\left(1-\bar{\Theta}_{l_1}^{i_1}\left(s,z_{l_1}^{i_1}\right)\right)\tilde{N}_{l_1}^{i_1}\left(ds,dz_{l_1}^{i_1}\right)\Bigg\},\end{aligned}$$

And,

$$\tilde{\Psi}^{i_2}(t)\triangleq\exp\Bigg\{\sum_{l_2=1}^{n_2}\int_0^t\int_{\mathbb{R}_0}\left[\left|\log\left(1+\tilde{\Theta}_{l_2}^{i_2}\left(s,z_{l_2}^{i_2}\right)\right)\right|+\tilde{\Theta}_{l_2}^{i_2}\left(s,z_{l_2}^{i_2}\right)\right]\nu_{l_2}^{i_2}\left(dz_{l_2}^{i_2}\right)ds$$

$$-\int_0^t \tilde{\vartheta}^{i_2}(s)dW^{i_2}(s) - \int_0^t \left\| \tilde{\vartheta}^{i_2}(s) \right\|_2^2 ds$$

$$\left. + \sum_{l_2=1}^{n_2} \int_0^t \int_{\mathbb{R}_0} \log\left(1 - \tilde{\Theta}_{l_2}^{i_2}\left(s, z_{l_2}^{i_2}\right)\right) \tilde{N}_{l_2}^{i_2}\left(ds, dz_{l_2}^{i_2}\right) \right\},$$

And then we define new measures $\mathbb{Q}^{i_1}$ and $\mathbb{Q}^{i_2}$ on $\mathcal{F}_T^{i_1}$ and $\mathcal{F}_T^{i_2}$, respectively, by,

$$d\mathbb{Q}^{i_1}(\omega) = \bar{\Psi}^{i_1}(\omega, T) d\mathbb{P}^{i_1}(\omega),$$

$$d\mathbb{Q}^{i_2}(\omega) = \tilde{\Psi}^{i_2}(\omega, T) d\mathbb{P}^{i_2}(\omega),$$

i.e., $\bar{\Psi}^{i_1}(\omega, T)$ and $\tilde{\Psi}^{i_2}(\omega, T)$ are the well-known Radon-Nikodym derivatives. By Assumption 5 and 6, $\bar{\Psi}^{i_1}(\omega, T)$ and $\tilde{\Psi}^{i_2}(\omega, T)$ satisfy the following Novikov conditions, respectively,

$$\mathbb{E}^{i_1}\left[\exp\left(\tfrac{1}{2} \int_0^T \left\| \bar{\vartheta}^{i_1}(t) \right\|_2^2 dt + \sum_{l_1=1}^{n_1} \int_0^T \int_{\mathbb{R}_0} \left\{ \left(1 - \bar{\Theta}_{l_1}^{i_1}\left(t, z_{l_1}^{i_1}\right)\right) \right.\right.\right.$$

$$\left.\left.\left. \times \log\left(1 - \bar{\Theta}_{l_1}^{i_1}\left(t, z_{l_1}^{i_1}\right)\right) + \bar{\Theta}_{l_1}^{i_1}\left(t, z_{l_1}^{i_1}\right) \right\} \nu_{l_1}^{i_1}\left(dz_{l_1}^{i_1}\right) dt \right) \right] < \infty,$$

And,

$$\mathbb{E}^{i_2}\left[\exp\left(\tfrac{1}{2} \int_0^T \left\| \tilde{\vartheta}^{i_2}(t) \right\|_2^2 dt + \sum_{l_2=1}^{n_2} \int_0^T \int_{\mathbb{R}_0} \left\{ \left(1 - \tilde{\Theta}_{l_2}^{i_2}\left(t, z_{l_2}^{i_2}\right)\right) \right.\right.\right.$$

$$\left.\left.\left. \times \log\left(1 - \tilde{\Theta}_{l_2}^{i_2}\left(t, z_{l_2}^{i_2}\right)\right) + \tilde{\Theta}_{l_2}^{i_2}\left(t, z_{l_2}^{i_2}\right) \right\} \nu_{l_2}^{i_2}\left(dz_{l_2}^{i_2}\right) dt \right) \right] < \infty.$$

Thus, according to the well-known Girsanov Theorem for Lévy processes, $\mathbb{Q}^{i_1}$ and $\mathbb{Q}^{i_2}$ are new probability measures on $\mathcal{F}_T^{i_1}$ and $\mathcal{F}_T^{i_2}$, respectively, and $X^{i_1}(t)$, $Y^{i_2}(t)$ will be martingales w. r. t. the probability laws $\mathbb{Q}^{i_1}$ and $\mathbb{Q}^{i_2}$, respectively. We will denote by $\mathbb{E}_{\mathbb{Q}^{i_1}}^{i_1}$, $\mathbb{E}_{\mathbb{Q}^{i_2}}^{i_2}$ the expectation operators w. r. t. the probability laws $\mathbb{Q}^{i_1}$ and $\mathbb{Q}^{i_2}$, respectively, for $\forall\ i_1 = 1,\ 2,\ ...,I_1$ and $i_2 = 1, 2,\ ..., I_2$. Moreover, we let $\mathbb{Q} \triangleq \mathbb{Q}^1 \otimes \mathbb{Q}^2$ with $\mathbb{Q}^1 \triangleq \otimes_{i_1=1}^{I_1} \mathbb{Q}^{i_1}$ and $\mathbb{Q}^2 \triangleq \otimes_{i_2=1}^{I_2} \mathbb{Q}^{i_2}$. And now we are given a stochastic basis $\left(\Omega, \mathcal{F}, \mathbb{F} \triangleq \left\{\mathcal{F}_t\right\}_{0 \le t \le T}, \mathbb{Q}\right)$ with $\mathbb{E}_{\mathbb{Q}}$ representing the corresponding expectation operator. Define,

$$\tilde{N}^{i_1}_{l_1\mathbb{Q}^{i_1}}\left(ds, dz^{i_1}_{l_1}\right) \triangleq \bar{\Theta}^{i_1}_{l_1}\left(s, z^{i_1}_{l_1}\right)\nu^{i_1}_{l_1}\left(dz^{i_1}_{l_1}\right)ds + \tilde{N}^{i_1}_{l_1}\left(ds, dz^{i_1}_{l_1}\right),$$

$$dW^{i_1}_{\mathbb{Q}^{i_1}}(s) \triangleq \bar{\vartheta}^{i_1}(s)ds + dW^{i_1}(s).$$

And,

$$\tilde{N}^{i_2}_{l_2\mathbb{Q}^{i_2}}\left(ds, dz^{i_2}_{l_2}\right) \triangleq \tilde{\Theta}^{i_2}_{l_2}\left(s, z^{i_2}_{l_2}\right)\nu^{i_2}_{l_2}\left(dz^{i_2}_{l_2}\right)ds + \tilde{N}^{i_2}_{l_2}\left(ds, dz^{i_2}_{l_2}\right),$$

$$dW^{i_2}_{\mathbb{Q}^{i_2}}(s) \triangleq \tilde{\vartheta}^{i_2}(s)ds + dW^{i_2}(s).$$

for $\forall\ i_1 = 1, 2, ..., I_1$; $i_2 = 1, 2, ..., I_2$; $l_1 = 1, 2, ..., n_1$ and $l_2 = 1, 2, ..., n_2$. Thus, with these new compensated Poisson random measures and Brownian motions, (1') can be rewritten as follows,

$$dX^{i_1}(t) = X^{i_1}(t)\left[\bar{\sigma}^{i_1}(t)dW^{i_1}_{\mathbb{Q}^{i_1}}(t) + \int_{\mathbb{R}^{n_1}_0}\bar{\gamma}^{i_1}\left(t, z^{i_1}\right)\tilde{N}^{i_1}_{\mathbb{Q}^{i_1}}\left(dt, dz^{i_1}\right)\right],$$

$$dY^{i_2}(t) = Y^{i_2}(t)\left[\tilde{\sigma}^{i_2}(t)dW^{i_2}_{\mathbb{Q}^{i_2}}(t) + \int_{\mathbb{R}^{n_2}_0}\tilde{\gamma}^{i_2}\left(t, z^{i_2}\right)\tilde{N}^{i_2}_{\mathbb{Q}^{i_2}}\left(dt, dz^{i_2}\right)\right]. \quad (1'')$$

for $\forall\ i_1 = 1, 2, ..., I_1$ and $i_2 = 1, 2, ..., I_2$. Now, we slightly modify Problem 1 given in section 3 and give,

PROBLEM 2 (*Stochastic Differential Cooperative Game on Time*): To solve Problem 1 subject to the new stochastic differential dynamics given by (1'').

Hence, employing the similar proof of Theorem 1, we derive,

COROLLARY 1: *There exists a unique solution to Problem* 2 *under very weak conditions, and accordingly the existence of the Pareto optimal endogenous matching is confirmed provided that we are given a Pareto optimal Nash equilibrium* $(\hat{x}, \hat{y})$, *and we still denote it by* $x^*(y^*, \bar{\rho}) = \hat{x} \in \tilde{\Delta}$ *and* $y^*(x^*, \tilde{\rho}) = \hat{y} \in \bar{\Delta}$ *as are given in Definition* 1.

Therefore, based upon the above assumptions and constructions, we derive the following theorem,

THEOREM 2: *Provided Corollary* 1 *and for* $\forall\ x^*(y^*, \bar{\rho}) = \hat{x} \in \tilde{\Delta}$ *and* $y^*(x^*, \tilde{\rho}) = \hat{y} \in \bar{\Delta}$. *Then we always get that both* $X(t)$ *and* $Y(t)$ *converge in* $L^1(\mathbb{Q})$ *with their limits belonging to the space* $L^1(\mathbb{Q})$, *and particularly, we arrive at,*

(i) $X(t)$ *uniformly converges to* $x^*(y^*, \bar{\rho})$ $\mathbb{Q}-$*a.s., or equivalently,*

$$\lim_{t'\to\infty}\mathbb{Q}\left(\bigcup_{t=t'}^{\infty}\left[\left\|X(t) - x^*(y^*, \bar{\rho})\right\|_2 \geq \varepsilon\right]\right) = 0.$$

for $\forall \varepsilon \in \mathbb{R}_{++}$.

(ii) $Y(t)$ *uniformly converges to* $y^*(x^*, \tilde{\rho})$ $\mathbb{Q}-$ *a.s., or equivalently,*

$$\lim_{t' \to \infty} \mathbb{Q}\left(\bigcup_{t=t'}^{\infty} \left[\left\| Y(t) - y^*(x^*, \tilde{\rho}) \right\|_2 \geq \varepsilon \right] \right) = 0.$$

for $\forall \varepsilon \in \mathbb{R}_{++}$.

PROOF: See Appendix C. ▌

REMARK 4.1: (i) It is especially worth emphasizing that Theorem 2 holds true for any $x^*(y^*, \overline{\rho}) = \hat{x} \in \tilde{\Delta}$ and any $y^*(x^*, \tilde{\rho}) = \hat{y} \in \overline{\Delta}$ no matter $(\hat{x}, \hat{y})$ is a completely mixed-strategy equilibrium, a non-completely mixed-strategy equilibrium or a pure-strategy equilibrium in the original normal-form games. However, for $x^*(y^*, \overline{\rho}) = \hat{x} \in \operatorname{int}(\tilde{\Delta})$ and $y^*(x^*, \tilde{\rho}) = \hat{y} \in \operatorname{int}(\overline{\Delta})$, we usually need to prove that there exists a unique invariant probability measure on the spaces $\operatorname{int}(\tilde{\Delta})$ and $\operatorname{int}(\overline{\Delta})$, respectively. For more details, one can refer to Theorem 2.1 of Imhof (2005), Theorem 3.1 of Benaïm et al. (2008) and Theorem 5 of Schreiber et al (2011). Moreover, it follows from Theorem 1 that $(\hat{x}, \hat{y})$ need not be a Nash equilibrium, for instance, (cooperation, cooperation) in PD games, and even not an evolutionary stable strategy (ESS) of the original game thanks to the specification of Problem 1. Noting that Theorem 2.1 of Imhof (2005) only holds for interior ESS and Theorem 3.1 of Benaïm et al. (2008) only holds for the attractor of the corresponding replicator dynamics, we argue that the method employed in demonstrating Theorem 2 has enriched existing literatures.

(ii) As is well known, mixed equilibria are usually interpreted as the limits of some learning process arising from fictitious play with randomly perturbed payoffs in the manner of Harsanyi's (1973) purification theorem (e.g., Fudenberg and Kreps, 1993; Benaïm and Hirsch, 1999; Ellison and Fudenberg, 2000, and among others). Nonetheless, there exist some problems that prevent the convergence of learning mechanisms to a mixed-strategy Nash equilibrium (see, Jordan, 1993). Moreover, Benaïm and Hirsch's (1999) study shows that there are robust parameter values giving probability zero of convergence for Jordan's 3×2 matching game. Obviously, our results, which essentially based upon the specification of Problem 1 and the martingale property of the corresponding replicator dynamics, applied to much broader cases. Last but not least, our conclusion in Theorem 2 proved the convergence of game equilibrium in the sense of *uniform topology*, which is much stronger than that of existing literatures.

(iii) One may notice that we take the limits in Theorem 2 just by sending the time to infinity but not through letting the diffusion terms or the jumps terms in replicator dynamics or the adaptive learning processes approach zero, instead, noises, errors or stochastic disturbances of the replicator dynamics always exist except that they are reasonably controlled in certain region according to the martingale property.

4.2. Stable Endogenous Matching

It follows from (1) that,

$$dX(t) = X(t) \circ F^1\left(Y(t)\right)dt + X(t) \circ G^1(t)dW^1(t) + \int_{\mathbb{R}_0^{l_1 n_1}} X(t) \circ H^1\left(t, z^1\right)\tilde{N}^1\left(dt, dz^1\right) \tag{4}$$

$$dY(t) = Y(t) \circ F^2\left(X(t)\right)dt + Y(t) \circ G^2(t)dW^2(t) + \int_{\mathbb{R}_0^{l_2 n_2}} Y(t) \circ H^2\left(t, z^2\right)\tilde{N}^2\left(dt, dz^2\right) \tag{5}$$

where $\circ$ denotes the Hadamard product. Now, we introduce the following Lévy SDEs,

$$d\hat{X}(t) = \hat{X}(t) \circ \hat{F}^1\left(Y(t)\right)dt + \hat{X}(t) \circ \hat{G}^1(t)dW^1(t) + \int_{\mathbb{R}_0^{l_1 n_1}} \hat{X}(t) \circ \hat{H}^1\left(t, z^1\right)\tilde{N}^1\left(dt, dz^1\right) \tag{6}$$

$$d\hat{Y}(t) = \hat{Y}(t) \circ \hat{F}^2\left(X(t)\right)dt + \hat{Y}(t) \circ \hat{G}^2(t)dW^2(t) + \int_{\mathbb{R}_0^{l_2 n_2}} \hat{Y}(t) \circ \hat{H}^2\left(t, z^2\right)\tilde{N}^2\left(dt, dz^2\right) \tag{7}$$

where we have used the following assumption,

ASSUMPTION 7: *For any* $\varsigma^1 > 0$, $\varsigma^2 > 0$, *we suppose that,*

$$\sup_{y\in\mathbb{R}^{l_2}}\left\|F^1(y) - \hat{F}^1(y)\right\|_2 \vee \sup_{t\in\mathbb{R}_+}\left\|G^1(t) - \hat{G}^1(t)\right\|_2 \vee \sup_{(t,z^1)\in\mathbb{R}_+\times\mathbb{R}_0^{l_1 n_1}}\left\|H^1\left(t, z^1\right) - \hat{H}^1\left(t, z^1\right)\right\|_2 \le \varsigma^1,$$

And,

$$\sup_{x\in\mathbb{R}^{l_1}}\left\|F^2(x) - \hat{F}^2(x)\right\|_2 \vee \sup_{t\in\mathbb{R}_+}\left\|G^2(t) - \hat{G}^2(t)\right\|_2 \vee \sup_{(t,z^2)\in\mathbb{R}_+\times\mathbb{R}_0^{l_2 n_2}}\left\|H^2\left(t, z^2\right) - \hat{H}^2\left(t, z^2\right)\right\|_2 \le \varsigma^2.$$

In other words, we call (6), (7) ς^1- and ς^2-perturbations of (4) and (5), respectively. Moreover, we give,

ASSUMPTION 8: *We suppose that there exist constants* K^1, $K^2 < \infty$ *and* $\hat{K}^1$, $\hat{K}^2 < \infty$ *such that,*

$$\sup_{y\in\mathbb{R}^{l_2}}\left\|F^1(y)\right\|_2^2 \vee \sup_{t\in\mathbb{R}_+}\left\|G^1(t)\right\|_2^2 \vee \sup_{(t,z^1)\in\mathbb{R}_+\times\mathbb{R}_0^{l_1 n_1}}\left\|H^1\left(t, z^1\right)\right\|_2^2 \le K^1,$$

$$\sup_{x\in\mathbb{R}^{l_1}}\left\|F^2(x)\right\|_2^2 \vee \sup_{t\in\mathbb{R}_+}\left\|G^2(t)\right\|_2^2 \vee \sup_{(t,z^2)\in\mathbb{R}_+\times\mathbb{R}_0^{l_2 n_2}}\left\|H^2\left(t, z^2\right)\right\|_2^2 \le K^2,$$

And also,

$$\int_{\mathbb{R}_0^{I_1 n_1}} \nu^1\left(dz^1\right) \triangleq \sum_{l_1=1}^{I_1 n_1} \int_{\mathbb{R}_0} \nu_{l_1}^1\left(dz_{l_1}^1\right) < \hat{K}^1,$$

$$\int_{\mathbb{R}_0^{I_2 n_2}} \nu^2\left(dz^2\right) \triangleq \sum_{l_2=1}^{I_2 n_2} \int_{\mathbb{R}_0} \nu_{l_2}^2\left(dz_{l_2}^2\right) < \hat{K}^2.$$

ASSUMPTION 9: *To ensure that the replicator dynamics given in* (2) *remain on* $\tilde{\Delta}$ *and* $\bar{\Delta}$, *i.e.,* $\tilde{\Delta}$ *and* $\bar{\Delta}$ *are invariant, respectively, we assume that for each* $x \in \tilde{\Delta}$, *the drift vector* $f^1(x)$, *the diffusion terms* $g^{1(k_1)}(x)$ $(k_1 = 1, 2, ..., I_1 d_1)$ *and the jump terms* $h^{1(l_1)}\left(x, z^1\right)$ $(l_1 = 1, 2, ..., I_1 n_1)$ *are elements of the tangent space* $T\tilde{\Delta} \triangleq \left\{\bar{r} \in \mathbb{R}^{I_1}; \sum_{i_1=1}^{I_1} \bar{r}^{i_1} = 0\right\}$ *of* $\tilde{\Delta}$, *and also for each* $y \in \bar{\Delta}$, *the drift vector* $f^2(y)$, *the diffusion terms* $g^{2(k_2)}(y)$ $(k_2 = 1, 2, ..., I_2 d_2)$ *and the jump terms* $h^{2(l_2)}\left(y, z^2\right)$ $(l_2 = 1, 2, ..., I_2 n_2)$ *are elements of the tangent space* $T\bar{\Delta} \triangleq \left\{\tilde{r} \in \mathbb{R}^{I_2}; \sum_{i_2=1}^{I_2} \tilde{r}^{i_2} = 0\right\}$ *of* $\bar{\Delta}$.

Then, the following proposition is established,

PROPOSITION 2: *Based upon the above constructions and assumptions, suppose* $X(0) = \hat{X}(0)$ *and* $Y(0) = \hat{Y}(0)$, *then we have,*

(i) $\mathbb{E}^1\left[\lim_{T\to\infty} \sup_{0\le t\le T} \left\|X(t) - \hat{X}(t)\right\|_2^2\right] \to 0$ *as* $\varsigma^1 \to 0$.

(ii) $\mathbb{E}^2\left[\lim_{T\to\infty} \sup_{0\le t\le T} \left\|Y(t) - \hat{Y}(t)\right\|_2^2\right] \to 0$ *as* $\varsigma^2 \to 0$.

PROOF: See Appendix D. ▌

REMARK 4.2: It should be pointed out here that in the proof of Proposition 2, we have implicitly used the following facts or assumptions, i.e., the speed of ς approaching zero is much faster than that of time T approaching infinity and also $0 \times \infty \equiv 0$. Moreover, we can get the same conclusion by taking the limit as $\varsigma \to 0$ first and then as $T \to \infty$.

Provided the above preparations, the following stability theorem of Pareto- optimality endogenous matching can be established,

THEOREM 3 (Stable Endogenous Matching): *No matter the corresponding Pareto optimal Nash* *equilibrium is a completely mixed strategy Nash equilibrium, a non-completely mixed strategy Nash equilibrium, or just a pure strategy Nash equilibrium, there always exists stable endogenous matching that is Pareto optimal based upon Theorem* 2 *and Proposition* 2.

PROOF: We take $\hat{X}(t)$ for example. And indeed, to prove the theorem, one only need to notice the following facts, i.e., for $\forall p \geq 2$ and $p \in \mathbb{N}$,

$$\begin{aligned}\left\|\hat{X}(t)-x^{*}\left(y^{*},\bar{\rho}\right)\right\|_{2}^{p} &= \left\|\hat{X}(t)-X(t)+X(t)-x^{*}\left(y^{*},\bar{\rho}\right)\right\|_{2}^{p} \\ &\leq 2^{p-1}\left(\left\|\hat{X}(t)-X(t)\right\|_{2}^{p}+\left\|X(t)-x^{*}\left(y^{*},\bar{\rho}\right)\right\|_{2}^{p}\right)\end{aligned}$$

And also, most importantly, proving the stability of the Pareto optimal endogenous matching $\bar{\rho}^{*}$ is equivalent to prove the stability of $x^{*}\left(y^{*},\bar{\rho}^{*}\right)$ by noting that there exists a one-to-one correspondence between $x^{*} \triangleq x^{*}\left(y^{*},\bar{\rho}^{*}\right)$ and $\bar{\rho}^{*}$. Then, combining Proposition 2 with Theorem 2 will easily confirm Theorem 3. The details are left to the interested reader. $\blacksquare$

REMARK 4.3: Theorem 3 combines with Theorem 1 actually supplies us a standard of stability much stronger than that of existing literatures. For example, classical non-cooperative game theory builds the strategic stability of equilibria on the basis of rational assumption, no matter it is common knowledge, forward induction or backward induction (see, Kohlberg and Mertens, 1986; and van Damme, 1987, and among others). On the other hand, evolutionary game theory builds up its stability of equilibria upon the concepts of evolutionary stable strategy (ESS) (e.g., Maynard Smith, 1982; Axelrod, 1984; Fudenberg and Maskin, 1990; Samuelson and Zhang, 1992; Weibull (1995) and references therein) or stochastically stable equilibrium (e.g., Foster and Young, 1990; Young and Foster, 1991; Fudenberg and Harris, 1992; and Young, 1993, and among others). In the current study, Theorem 1 follows the classical individually-rational assumption, i.e., expected payoffs or fitness functions are maximized, and also Theorem 3 confirms stability of the endogenous matching from the view of point of evolutionary interpretation, i.e., the continuous time Markov process arising from the replicator dynamics will stably converge to the Pareto optimal equilibrium in the sense of *uniform topology* and so we argue that the endogenous matching is stable because there exists a one-to-one correspondence between the game equilibrium and the matching. Noting that individual choice usually bases on rational decision while group of populations as a whole follows stochastic evolution, we emphasize again that Theorem 1 and Theorem 3 indeed provide us with a complete characterization of the stability of the Pareto optimal endogenous matching. To summarize, the present paper introduces the following stronger definition of stability, that is, we call a given equilibrium or matching stable if and only if it is both individually rational and it satisfies group-level stochastic evolutionary stability.

5. Conclusion

What's the directing philosophy insisted by the present exploration? Rather, throughout the present paper, we insist and emphasize the following philosophy, i.e., relatively speaking, both what the present state is and what the goal will be are not at all important, the only thing that does matter is what we will need to lead us from the present state to the goal. Indeed, in the present framework and given the problem facing us, the Pareto optimal endogenous matching defined and derived by us just plays the key and essential role in leading us from any given present state to our given goal. We therefore argue that the definition of endogenous matching employed by the paper is not only for the sake of convenience of expression but also reflecting the above practical philosophy.

In the present paper, we are encouraged to study the asymmetric normal-form games between two heterogeneous groups of populations by incorporating stochastic optimal stopping theory into the stochastic replicator dynamics for the first time, thereby defining a *stochastic differential cooperative game on time*. It is demonstrated that optimal stopping theory will play a crucial role in studying the endogenous matching from the viewpoint of evolutionary game theory. Existence of Pareto optimal endogenous matching has been proved and also the corresponding stability is confirmed by employing two important standards: individually-rational standard and stochastic evolutionary stable standard.

Finally, the current paper can be naturally extended in the following ways, first, asymmetric information can be introduced into the present model to capture much more economic implications; second, stochastic differential cooperative game can be explored based upon the present framework; third, specific mechanism, say, reputation mechanism or searching mechanism, can be incorporated into the model to support any other pattern of endogenous matching; forth, our approach can be easily extended to include multiple priors (see, Riedel, 2009, for instance) and also to explore the evolutionary equilibria on graphs (see, Ohtsuki et al., 2007, and among others).

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Appendix

A. Proof of Proposition 1

We take $M^{i_1}(t)$ for example. For any $M^{i_1}(0) \neq 0$, we, by applying Itô's rule to $M^{i_1}(t)$, arrive at,

$$\begin{aligned}\log M^{i_1}(t) = \log M^{i_1}(0) + \int_0^t \Big\{ & \overline{e}_{i_1}^T AY(s) - \tfrac{1}{2}\sum\nolimits_{k_1=1}^{d_1} \overline{\sigma}_{i_1 k_1}^2(s) \\ & + \sum\nolimits_{l_1=1}^{n_1} \int_{\mathbb{R}_0} \left[\log\left(1+\overline{\gamma}_{i_1 l_1}\left(s, z_{l_1}^{i_1}\right)\right) - \overline{\gamma}_{i_1 l_1}\left(s, z_{l_1}^{i_1}\right) \right] \nu_{l_1}^{i_1}\left(dz_{l_1}^{i_1}\right) \Big\} ds \\ & + \int_0^t \sum\nolimits_{k_1=1}^{d_1} \overline{\sigma}_{i_1 k_1}(s) dW_{k_1}^{i_1}(s) + \int_0^t \int_{\mathbb{R}_0} \sum\nolimits_{l_1=1}^{n_1} \log\left(1+\overline{\gamma}_{i_1 l_1}\left(s, z_{l_1}^{i_1}\right)\right) \tilde{N}_{l_1}^{i_1}\left(ds, dz_{l_1}^{i_1}\right) \end{aligned} \tag{A.1}$$

Applying the classical Large Number Theorem of martingales reveals that,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \sum_{k_1=1}^{d_1}\bar{\sigma}_{i_1k_1}(s)dW_{k_1}^{i_1}(s)=0 \quad \text{a.s.}$$

And,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\log\left(1+\bar{\gamma}_{i_1l_1}\left(s,z_{l_1}^{i_1}\right)\right)\tilde{N}_{l_1}^{i_1}\left(ds,dz_{l_1}^{i_1}\right)=0 \quad \text{a.s.}$$

Dividing both sides of (A.1) by t and then letting $t\to\infty$ we hence obtain the desired assertion in Proposition 1. ▌

B. Proof of Theorem 1

STEP 1: For strategy i_1, $\forall\ i_1=1,2,...,I_1$. Notice that,

$$\begin{aligned}
\mathcal{A}\bar{f}_{i_1}\left(s,y^1\right) &= -\bar{\theta}^{i_1}\exp\left(-\bar{\theta}^{i_1}s\right)\\
&\times\left\{1-\bar{w}^{i_1}+\bar{w}^{i_1}\left[\left(a_{i_1 1}-a_{i_1 2}\right)y^1+a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\bar{e}_{i_1}^T A\tilde{\rho}\right]\right\}\\
&+y^1\left(\tilde{e}_1^T B^T x\right)\exp\left(-\bar{\theta}^{i_1}s\right)\bar{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)\geq 0\\
&\Leftrightarrow\left(\tilde{e}_1^T B^T x-\bar{\theta}^{i_1}\right)\bar{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)y^1\\
&\geq\bar{\theta}^{i_1}\left(1-\bar{w}^{i_1}\right)+\bar{\theta}^{i_1}\bar{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\bar{e}_{i_1}^T A\tilde{\rho}\right].
\end{aligned}$$

Case 1.1: $\begin{cases}\bar{\theta}^{i_1}\left(1-\bar{w}^{i_1}\right)+\bar{\theta}^{i_1}\bar{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\bar{e}_{i_1}^T A\tilde{\rho}\right]<0\\ \operatorname{sgn}\left(\tilde{e}_1^T B^T x-\bar{\theta}^{i_1}\right)=\operatorname{sgn}\left(a_{i_1 2}-a_{i_1 1}\right)\end{cases}$

Then,

$$\mathcal{A}\bar{f}_{i_1}\left(s,y^1\right)\geq 0$$

$$\Leftrightarrow y^1\leq\frac{\bar{\theta}^{i_1}\left(1-\bar{w}^{i_1}\right)+\bar{\theta}^{i_1}\bar{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\bar{e}_{i_1}^T A\tilde{\rho}\right]}{\left(\tilde{e}_1^T B^T x-\bar{\theta}^{i_1}\right)\bar{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)}.$$

Hence, we have,

$$U^{i_1}=\left\{\left(s,y^1\right);y^1\le\frac{\bar{\theta}^{i_1}\left(1-\bar{w}^{i_1}\right)+\bar{\theta}^{i_1}\bar{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\bar{e}_{i_1}^T A\tilde{\rho}\right]}{\left(\tilde{e}_1^T B^T x-\bar{\theta}^{i_1}\right)\bar{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)}\right\}. \quad \text{(B.1)}$$

And it is natural to guess that the continuation region D^{i_1} has the following form,

$$D^{i_1}\left(y_{i_1}^{1*}\right)=\left\{\left(s,y^1\right);0\le y^1\le y_{i_1}^{1*}\right\}.$$

where,

$$y_{i_1}^{1*}\ge\frac{\bar{\theta}^{i_1}\left(1-\bar{w}^{i_1}\right)+\bar{\theta}^{i_1}\bar{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\bar{e}_{i_1}^T A\tilde{\rho}\right]}{\left(\tilde{e}_1^T B^T x-\bar{\theta}^{i_1}\right)\bar{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)}. \quad \text{(B.2)}$$

Notice that the generator of $\bar{Z}(t)$ is given by,

$$\mathcal{A}\bar{\phi}_{i_1}\left(s,y^1\right)=\frac{\partial\bar{\phi}_{i_1}}{\partial s}+y^1\left(\tilde{e}_1^T B^T x\right)\frac{\partial\bar{\phi}_{i_1}}{\partial y^1}+\frac{1}{2}\left(y^1\right)^2\left(\tilde{\sigma}^1\right)^T\tilde{\sigma}^1\frac{\partial^2\bar{\phi}_{i_1}}{\partial(y^1)^2}$$

$$+\int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\{\bar{\phi}_{i_1}\left(s,y^1+y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\right)-\bar{\phi}_{i_1}\left(s,y^1\right)-y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\frac{\partial\bar{\phi}_{i_1}}{\partial y^1}\left(s,y^1\right)\right\}\nu_{l_2}^1\left(dz_{l_2}^1\right)$$

for $\forall\bar{\phi}_{i_1}\left(s,y^1\right)\in C^2\left(\mathbb{R}^2\right)$. If we try a function $\bar{\phi}_{i_1}$ of the following form,

$$\bar{\phi}_{i_1}\left(s,y^1\right)=\exp\left(-\bar{\theta}^{i_1}s\right)\left(y^1\right)^{\bar{\lambda}^{i_1}} \quad \text{for some constant } \bar{\lambda}^{i_1}\in\mathbb{R}.$$

We then get,

$$\mathcal{A}\bar{\phi}_{i_1}\left(s,y^1\right)=\exp\left(-\bar{\theta}^{i_1}s\right)\Big[-\bar{\theta}^{i_1}\left(y^1\right)^{\bar{\lambda}^{i_1}}+\left(\tilde{e}_1^T B^T x\right)y^1\bar{\lambda}^{i_1}\left(y^1\right)^{\bar{\lambda}^{i_1}-1}$$

$$+\frac{1}{2}\left(\tilde{\sigma}^1\right)^T\tilde{\sigma}^1\left(y^1\right)^2\bar{\lambda}^{i_1}\left(\bar{\lambda}^{i_1}-1\right)\left(y^1\right)^{\bar{\lambda}^{i_1}-2}$$

$$+\int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\{\left[y^1+y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\right]^{\bar{\lambda}^{i_1}}-\left(y^1\right)^{\bar{\lambda}^{i_1}}-\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)y^1\bar{\lambda}^{i_1}\left(y^1\right)^{\bar{\lambda}^{i_1}-1}\right\}\nu_{l_2}^1\left(dz_{l_2}^1\right)\Big]$$

$$=\exp\left(-\bar{\theta}^{i_1}s\right)\left(y^1\right)^{\bar{\lambda}^{i_1}}\bar{h}_{i_1}\left(\bar{\lambda}^{i_1}\right).$$

where,

$$\overline{h}_{i_1}\left(\overline{\lambda}^{i_1}\right)\triangleq-\overline{\theta}^{i_1}+\left(\tilde{e}_1^T B^T x\right)\overline{\lambda}^{i_1}+\frac{1}{2}\left(\tilde{\sigma}^1\right)^T\tilde{\sigma}^1\overline{\lambda}^{i_1}\left(\overline{\lambda}^{i_1}-1\right)$$

$$+\int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\{\left[1+\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\right]^{\overline{\lambda}^{i_1}}-1-\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\overline{\lambda}^{i_1}\right\}\nu_{l_2}^1\left(dz_{l_2}^1\right).$$

Note that,

$$\overline{h}_{i_1}(1)=\tilde{e}_1^T B^T x-\overline{\theta}^{i_1} \quad \text{and} \quad \lim_{\overline{\lambda}^{i_1}\to\infty}\overline{h}_{i_1}\left(\overline{\lambda}^{i_1}\right)=\infty .$$

Therefore, if we assume that,

$$\tilde{e}_1^T B^T x<\overline{\theta}^{i_1}, \tag{B.3}$$

Then we find that there exists $\overline{\lambda}^{i_1}>1$ such that,

$$\overline{h}_{i_1}\left(\overline{\lambda}^{i_1}\right)=0. \tag{B.4}$$

with this value of $\overline{\lambda}^{i_1}$ we put,

$$\overline{\phi}_{i_1}\left(s,y^1\right)=\begin{cases} e^{-\overline{\theta}^{i_1}s}\overline{C}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}}, 0\le y^1\le y_{i_1}^{1*} \\ e^{-\overline{\theta}^{i_1}s}\left\{1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[\left(a_{i_1 1}-a_{i_1 2}\right)y^1+a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}\right]\right\}, y_{i_1}^{1*}\le y^1\le 1\end{cases}$$

for some constant $\overline{C}^{i_1}>0$, to be determined. We, without loss of any generality, guess that the value function is C^1 at $y^1=y_{i_1}^{1*}$ and this leads us to the following "high contact" conditions,

$$\overline{C}^{i_1}\left(y_{i_1}^{1*}\right)^{\overline{\lambda}^{i_1}}=1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[\left(a_{i_1 1}-a_{i_1 2}\right)y_{i_1}^{1*}+a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}\right]$$

(continuity at $y^1=y_{i_1}^{1*}$)

$$\overline{C}^{i_1}\overline{\lambda}^{i_1}\left(y_{i_1}^{1*}\right)^{\overline{\lambda}^{i_1}-1}=\overline{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right) \quad \text{(differentiability at } y^1=y_{i_1}^{1*}\text{)}$$

Combining the above equations shows that,

$$\frac{\overline{C}^{i_1}\left(y_{i_1}^{1*}\right)^{\overline{\lambda}^{i_1}}}{\overline{C}^{i_1}\overline{\lambda}^{i_1}\left(y_{i_1}^{1*}\right)^{\overline{\lambda}^{i_1}-1}}=\frac{1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[\left(a_{i_1 1}-a_{i_1 2}\right)y_{i_1}^{1*}+a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}\right]}{\overline{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)}$$

$$\Leftrightarrow\ y_{i_1}^{1*} = \frac{\overline{\lambda}^{i_1}\left\{1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}\right]\right\}}{\left(1-\overline{\lambda}^{i_1}\right)\overline{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)}. \tag{B.5}$$

And this gives,

$$\overline{C}^{i_1} = \frac{\overline{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)}{\overline{\lambda}^{i_1}\left(y_{i_1}^{1*}\right)^{\overline{\lambda}^{i_1}-1}}. \tag{B.6}$$

Hence, by (B.4), (B.5) and (B.6), we can define,

$$\overline{\overline{f}}_{i_1}\left(s, y^1\right) \triangleq \exp\left(-\overline{\theta}^{i_1} s\right)\overline{C}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}}.$$

And then we are in the position to prove that,

$$\overline{\overline{f}}_{i_1}\left(s, y^1\right) \triangleq \exp\left(-\overline{\theta}^{i_1} s\right)\overline{C}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}} = \overline{f}_{i_1}^*\left(s, y^1\right).$$

in which $\overline{f}_{i_1}^*\left(s, y^1\right)$ is a supermeanvalued majorant of $\overline{f}_{i_1}\left(s, y^1\right)$. Firstly, noting that,

$$\begin{aligned}
&\mathcal{A}\overline{f}_{i_1}\left(s, y^1\right) = -\overline{\theta}^{i_1}\exp\left(-\overline{\theta}^{i_1} s\right) \\
&\quad\times\left\{1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[\left(a_{i_1 1}-a_{i_1 2}\right)y^1 + a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}\right]\right\} \\
&\quad + y^1\left(\tilde{e}_1^T B^T x\right)\exp\left(-\overline{\theta}^{i_1} s\right)\overline{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right) \le 0,\quad \forall y^1 \ge y_{i_1}^{1*}. \\
&\quad \Leftrightarrow \left(\tilde{e}_1^T B^T x - \overline{\theta}^{i_1}\right)\overline{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)y^1 \\
&\quad \le \overline{\theta}^{i_1}\left(1-\overline{w}^{i_1}\right)+\overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}\right],\quad \forall y^1 \ge y_{i_1}^{1*}.
\end{aligned}$$

$$\Leftrightarrow\ y_{i_1}^{1*} \ge \frac{\overline{\theta}^{i_1}\left(1-\overline{w}^{i_1}\right)+\overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}\right]}{\left(\tilde{e}_1^T B^T x-\overline{\theta}^{i_1}\right)\overline{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)}.$$

which holds by (B.2). Secondly, to prove,

$$\overline{C}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}} \ge 1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[\left(a_{i_1 1}-a_{i_1 2}\right)y^1 + a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}\right],$$

$$\text{for}\ \ \forall 0 \le y^1 \le y_{i_1}^{1*}.$$

Define

$$\bar{\xi}^{i_1}\left(y^1\right) \triangleq \bar{C}^{i_1}\left(y^1\right)^{\bar{\lambda}^{i_1}} - 1 + \bar{w}^{i_1}$$
$$-\bar{w}^{i_1}\left[\left(a_{i_1 1} - a_{i_1 2}\right)y^1 + a_{i_1 2}\tilde{\delta}_{I_2-2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2} - \tilde{\delta}_{I_2-i_2+1}\right) + \bar{e}_{i_1}^T A\tilde{\rho}\right].$$

Then with our chosen values of $\bar{C}^{i_1}$ and $\bar{\lambda}^{i_1}$, we see that $\bar{\xi}^{i_1}\left(y_{i_1}^{1*}\right) = \bar{\xi}^{i_1\prime}\left(y_{i_1}^{1*}\right) = 0$. Furthermore, noting that $\bar{\xi}^{i_1\prime\prime}\left(y^1\right) = \bar{C}^{i_1}\bar{\lambda}^{i_1}\left(\bar{\lambda}^{i_1} - 1\right)\left(y^1\right)^{\bar{\lambda}^{i_1}-2}$, and hence $\bar{\xi}^{i_1\prime\prime}\left(y^1\right) > 0$ holds for $\forall 0 \le y^1 \le y_{i_1}^{1*}$ given $\bar{\lambda}^{i_1} > 1$ in (B.4), that is, $\bar{\xi}^{i_1}\left(y^1\right) > 0$ follows for $\forall\ 0 \le y^1 \le y_{i_1}^{1*}$. And this completes the short proof.

Case 1.2: $$\begin{cases} \bar{\theta}^{i_1}\left(1 - \bar{w}^{i_1}\right) + \bar{\theta}^{i_1}\bar{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2} - \tilde{\delta}_{I_2-i_2+1}\right) + \bar{e}_{i_1}^T A\tilde{\rho}\right] > 0 \\ \operatorname{sgn}\left(\tilde{e}_1^T B^T x - \bar{\theta}^{i_1}\right) = \operatorname{sgn}\left(a_{i_1 1} - a_{i_1 2}\right) \end{cases}$$

It is easy to see that the proof is quite similar to that of case 1.1, so we take it omitted.

STEP 2: For strategy i_2, $\forall\ i_2 = 1, 2, \ldots, I_2$. Notice that,

$$\mathcal{A}\tilde{f}_{i_2}\left(s, x^1\right) = -\tilde{\theta}^{i_2}\exp\left(-\tilde{\theta}^{i_2}s\right)$$
$$\times\left\{1 - \tilde{w}^{i_2} + \tilde{w}^{i_2}\left[\left(b_{i_2 1} - b_{i_2 2}\right)x^1 + b_{i_2 2}\bar{\delta}_{I_1-2} + \sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1} - \bar{\delta}_{I_1-i_1+1}\right) + \tilde{e}_{i_2}^T B^T\bar{\rho}\right]\right\}$$
$$+x^1\left(\bar{e}_1^T Ay\right)\exp\left(-\tilde{\theta}^{i_2}s\right)\tilde{w}^{i_2}\left(b_{i_2 1} - b_{i_2 2}\right) \ge 0$$
$$\Leftrightarrow \left(\bar{e}_1^T Ay - \tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1} - b_{i_2 2}\right)x^1$$
$$\ge \tilde{\theta}^{i_2}\left(1 - \tilde{w}^{i_2}\right) + \tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\bar{\delta}_{I_1-2} + \sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1} - \bar{\delta}_{I_1-i_1+1}\right) + \tilde{e}_{i_2}^T B^T\bar{\rho}\right].$$

Case 2.1: $\begin{cases} \tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T \overline{\rho}\right]<0 \\ \operatorname{sgn}\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)=\operatorname{sgn}\left(b_{i_2 2}-b_{i_2 1}\right) \end{cases}$

Hence,

$$\mathcal{A}\tilde{f}_{i_2}\left(s,x^1\right)\geq 0$$

$$\Leftrightarrow x^1 \leq \frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T \overline{\rho}\right]}{\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}.$$

Then, we have,

$$U^{i_2}=\left\{\left(s,x^1\right);x^1 \leq \frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T \overline{\rho}\right]}{\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}\right\}. \quad \text{(B.7)}$$

So it is natural to guess that the continuation region D^{i_2} has the following form,

$$D^{i_2}\left(x_{i_2}^{1*}\right)=\left\{\left(s,x^1\right);0\leq x^1 \leq x_{i_2}^{1*}\right\}.$$

where,

$$x_{i_2}^{1*} \geq \frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T \overline{\rho}\right]}{\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}. \quad \text{(B.8)}$$

Notice that the generator of $\tilde{Z}(t)$ is given by,

$$\begin{aligned}\mathcal{A}\tilde{\phi}_{i_2}\left(s,x^1\right)=&\frac{\partial\tilde{\phi}_{i_2}}{\partial s}+x^1\left(\overline{e}_1^T Ay\right)\frac{\partial\tilde{\phi}_{i_2}}{\partial x^1}+\frac{1}{2}\left(x^1\right)^2\left(\overline{\sigma}^1\right)^T\overline{\sigma}^1\frac{\partial^2\tilde{\phi}_{i_2}}{\partial(x^1)^2}\\ &+\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\tilde{\phi}_{i_2}\left(s,x^1+x^1\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\right)-\tilde{\phi}_{i_2}\left(s,x^1\right)-x^1\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\tfrac{\partial\tilde{\phi}_{i_2}}{\partial x^1}\left(s,x^1\right)\right\}\nu_{l_1}^1\left(dz_{l_1}^1\right)\end{aligned}$$

for $\forall\tilde{\phi}_{i_2}\left(s,x^1\right)\in C^2\left(\mathbb{R}^2\right)$. If we choose $\tilde{\phi}_{i_2}\left(s,x^1\right)=\exp\left(-\tilde{\theta}^{i_2}s\right)\left(x^1\right)^{\tilde{\lambda}^{i_2}}$ for some constant $\tilde{\lambda}^{i_2}\in\mathbb{R}$. Then we get,

$$\mathcal{A}\tilde{\phi}_{i_2}\left(s,x^1\right)=\exp\left(-\tilde{\theta}^{i_2}s\right)\Bigg[-\tilde{\theta}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}}+\left(\overline{e}_1^T Ay\right)x^1\tilde{\lambda}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}-1}$$

$$+\frac{1}{2}\left(\overline{\sigma}^1\right)^T\overline{\sigma}^1\left(x^1\right)^2\tilde{\lambda}^{i_2}\left(\tilde{\lambda}^{i_2}-1\right)\left(x^1\right)^{\tilde{\lambda}^{i_2}-2}$$

$$+\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\left[x^1+x^1\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\right]^{\tilde{\lambda}^{i_2}}-\left(x^1\right)^{\tilde{\lambda}^{i_2}}-\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)x^1\tilde{\lambda}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}-1}\right\}\nu_{l_1}^1\left(dz_{l_1}^1\right)\Bigg]$$

$$=\exp\left(-\tilde{\theta}^{i_2}s\right)\left(x^1\right)^{\tilde{\lambda}^{i_2}}\tilde{h}_{i_2}\left(\tilde{\lambda}^{i_2}\right).$$

where,

$$\tilde{h}_{i_2}\left(\tilde{\lambda}^{i_2}\right)\triangleq-\tilde{\theta}^{i_2}+\left(\overline{e}_1^T Ay\right)\tilde{\lambda}^{i_2}+\frac{1}{2}\left(\overline{\sigma}^1\right)^T\overline{\sigma}^1\tilde{\lambda}^{i_2}\left(\tilde{\lambda}^{i_2}-1\right)$$

$$+\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\left[1+\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\right]^{\tilde{\lambda}^{i_2}}-1-\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\tilde{\lambda}^{i_2}\right\}\nu_{l_1}^1\left(dz_{l_1}^1\right).$$

Noting that,

$$\tilde{h}_{i_2}(1)=\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\quad\text{and}\quad\lim_{\tilde{\lambda}^{i_2}\to\infty}\tilde{h}_{i_2}\left(\tilde{\lambda}^{i_2}\right)=\infty.$$

Consequently, if we suppose that,

$$\overline{e}_1^T Ay<\tilde{\theta}^{i_2}, \tag{B.9}$$

Thus, it is easily seen that there exists $\tilde{\lambda}^{i_2}>1$ such that,

$$\tilde{h}_{i_2}\left(\tilde{\lambda}^{i_2}\right)=0. \tag{B.10}$$

with this value of $\tilde{\lambda}^{i_2}$ we put,

$$\tilde{\phi}_{i_2}\left(s,x^1\right)=\begin{cases}e^{-\tilde{\theta}^{i_2}s}\tilde{C}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}},0\le x^1\le x_{i_2}^{1*}\\ e^{-\tilde{\theta}^{i_2}s}\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\overline{\rho}\right]\right\},x_{i_2}^{1*}\le x^1\le 1\end{cases}$$

in which $\tilde{C}^{i_2}>0$ is some constant that remains to be determined. If we require that $\tilde{\phi}_{i_2}$ is continuous at $x^1=x_{i_2}^{1*}$ we get the following equation,

$$\tilde{C}^{i_2}\left(x_{i_2}^{1*}\right)^{\tilde{\lambda}^{i_2}}=1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x_{i_2}^{1*}+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\overline{\rho}\right], \tag{B.11}$$

If we require that $\tilde{\phi}_{i_2}$ is differentiable at $x^1 = x_{i_2}^{1*}$ we get the additional equation,

$$\tilde{C}^{i_2}\tilde{\lambda}^{i_2}\left(x_{i_2}^{1*}\right)^{\tilde{\lambda}^{i_2}-1} = \tilde{w}^{i_2}\left(b_{i_2 1} - b_{i_2 2}\right). \tag{B.12}$$

So, combining equation (B.11) and equation (B.12) yields,

$$\frac{\tilde{C}^{i_2}\left(x_{i_2}^{1*}\right)^{\tilde{\lambda}^{i_2}}}{\tilde{C}^{i_2}\tilde{\lambda}^{i_2}\left(x_{i_2}^{1*}\right)^{\tilde{\lambda}^{i_2}-1}} = \frac{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x_{i_2}^{1*}+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\overline{\rho}\right]}{\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}$$

$$\Leftrightarrow x_{i_2}^{1*} = \frac{\tilde{\lambda}^{i_2}\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\overline{\rho}\right]\right\}}{\left(1-\tilde{\lambda}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}. \tag{B.13}$$

And this produces,

$$\tilde{C}^{i_2} = \frac{\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}{\tilde{\lambda}^{i_2}\left(x_{i_2}^{1*}\right)^{\tilde{\lambda}^{i_2}-1}}. \tag{B.14}$$

Then, by applying equation (B.10), equation (B.13) and equation (B.14), we are in the position to prove that $\tilde{f}_{i_2}^*\left(s,x^1\right)=\exp\left(-\tilde{\theta}^{i_2}s\right)\tilde{C}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}}$ is a supermeanvalued majorant of $\tilde{f}_{i_2}\left(s,x^1\right)$. Firstly, noting that,

$$\begin{aligned}
\mathcal{A}\tilde{f}_{i_2}\left(s,x^1\right) &= -\tilde{\theta}^{i_2}\exp\left(-\tilde{\theta}^{i_2}s\right)\\
&\quad\times\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\overline{\rho}\right]\right\}\\
&\quad+x^1\left(\overline{e}_1^T Ay\right)\exp\left(-\tilde{\theta}^{i_2}s\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)\le 0,\quad \forall x^1\ge x_{i_2}^{1*}\\
&\Leftrightarrow \left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)x^1\\
&\le \tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\overline{\rho}\right],\quad \forall x^1\ge x_{i_2}^{1*}\\
&\Leftrightarrow x^1 \ge \frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\overline{\rho}\right]}{\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)},\quad \forall x^1\ge x_{i_2}^{1*}
\end{aligned}$$

$$\Leftrightarrow x_{i_2}^{1*} \geq \frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T \overline{\rho}\right]}{\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}$$

which holds by (B.8). Secondly, to show that,

$$\tilde{C}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}} \geq 1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T \overline{\rho}\right],$$

$$\text{for } \forall 0 \leq x^1 \leq x_{i_2}^{1*}.$$

Define

$$\tilde{\xi}^{i_2}\left(x^1\right) \triangleq \tilde{C}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}}-1+\tilde{w}^{i_2}$$
$$-\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T \overline{\rho}\right].$$

Then with our chosen values of $\tilde{C}^{i_2}$ and $\tilde{\lambda}^{i_2}$, we see that $\tilde{\xi}^{i_2}\left(x_{i_2}^{1*}\right)=\tilde{\xi}^{i_2\prime}\left(x_{i_2}^{1*}\right)=0$. Furthermore, noting that $\tilde{\xi}^{i_2\prime\prime}\left(x^1\right)=\tilde{C}^{i_2}\tilde{\lambda}^{i_2}\left(\tilde{\lambda}^{i_2}-1\right)\left(x^1\right)^{\tilde{\lambda}^{i_2}-2}$, and hence $\tilde{\xi}^{i_2\prime\prime}\left(x^1\right)>0$ holds for $\forall 0 \leq x^1 \leq x_{i_2}^{1*}$ given $\tilde{\lambda}^{i_2}>1$ in (B.10), that is, $\tilde{\xi}^{i_2}\left(x^1\right)>0$ follows for $\forall 0 \leq x^1 \leq x_{i_2}^{1*}$. And hence the desired result is established.

Case 2.2: $\begin{cases} \tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T \overline{\rho}\right]>0 \\ \operatorname{sgn}\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)=\operatorname{sgn}\left(b_{i_2 1}-b_{i_2 2}\right) \end{cases}$

Similar to case 1.2 and we take the proof of case 2.2, which is quite similar to that of case 2.1, omitted.

STEP 3: The existence of the Pareto optimal endogenous matching.

It follows from the requirements of Problem 1 that $y_1^{1*}=y_2^{1*}=\ldots=y_{i_1}^{1*}=\ldots=y_{I_1}^{1*}$ with $y_{i_1}^{1*}$ defined in (B.5). Let $y_{i_1}^{1*}=y_{k_1}^{1*}$ $(\forall i_1 \neq k_1, i_1, k_1=1,2,\ldots,I_1)$, then one can easily see that,

$$\overline{\Sigma}_{i_1 k_1,23}\tilde{\delta}_{I_2-2}+\overline{\Sigma}_{i_1 k_1,34}\tilde{\delta}_{I_2-3}+\ldots+\overline{\Sigma}_{i_1 k_1,I_2-1,I_2}\tilde{\delta}_1=\overline{\Gamma}_{i_1 k_1}.$$

where,

$$\bar{\Sigma}_{i_1k_1,j_2,j_2+1} \triangleq \frac{\bar{\lambda}^{i_1}\left(a_{i_1j_2}-a_{i_1,j_2+1}\right)}{\left(1-\bar{\lambda}^{i_1}\right)\left(a_{i_11}-a_{i_12}\right)}-\frac{\bar{\lambda}^{k_1}\left(a_{k_1j_2}-a_{k_1,j_2+1}\right)}{\left(1-\bar{\lambda}^{k_1}\right)\left(a_{k_11}-a_{k_12}\right)}.$$

$$\bar{\Gamma}_{i_1k_1} \triangleq \frac{\bar{\lambda}^{k_1}\left[\left(1-\bar{w}^{k_1}\right)+\bar{w}^{k_1}\left(a_{k_1I_2}+\bar{e}_{k_1}^{T}A\tilde{\rho}\right)\right]}{\left(1-\bar{\lambda}^{k_1}\right)\left(a_{k_11}-a_{k_12}\right)\bar{w}^{k_1}}-\frac{\bar{\lambda}^{i_1}\left[\left(1-\bar{w}^{i_1}\right)+\bar{w}^{i_1}\left(a_{i_1I_2}+\bar{e}_{i_1}^{T}A\tilde{\rho}\right)\right]}{\left(1-\bar{\lambda}^{i_1}\right)\left(a_{i_11}-a_{i_12}\right)\bar{w}^{i_1}}.$$

$$\forall i_1 \neq k_1,\ \ i_1,k_1=1,2,...,I_1;\ \ j_2=2,3,...,I_2-1.$$

Accordingly, we have,

$$\begin{bmatrix} \bar{\Sigma}_{12,23} & \bar{\Sigma}_{12,34} & \cdots & \bar{\Sigma}_{12,I_2-1,I_2} \\ \bar{\Sigma}_{23,23} & \bar{\Sigma}_{23,34} & \cdots & \bar{\Sigma}_{23,I_2-1,I_2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\Sigma}_{I_1-1,I_1,23} & \bar{\Sigma}_{I_1-1,I_1,34} & \cdots & \bar{\Sigma}_{I_1-1,I_1,I_2-1,I_2} \end{bmatrix}_{(I_1-1)\times(I_2-2)} \begin{bmatrix} \tilde{\delta}_{I_2-2} \\ \tilde{\delta}_{I_2-3} \\ \vdots \\ \tilde{\delta}_1 \end{bmatrix}_{(I_2-2)\times 1} = \begin{bmatrix} \bar{\Gamma}_{12} \\ \bar{\Gamma}_{23} \\ \vdots \\ \bar{\Gamma}_{I_1-1,I_1} \end{bmatrix}_{(I_1-1)\times 1}.$$

which implies that,

$$\tilde{\delta} = \bar{\Sigma}^{+}\bar{\Gamma}. \tag{B.15}$$

where "$+$" denotes Moore-Penrose generalized inverse.

Similarly, we obtain $x_1^{1*} = x_2^{1*} = ... = x_{i_2}^{1*} = ... = x_{I_2}^{1*}$ with $x_{i_2}^{1*}$ defined in (B.13) according to Problem 1. Now, let $x_{i_2}^{1*} = x_{k_2}^{1*}$ ($\forall i_2 \neq k_2, i_2,k_2 = 1,2,...,I_2$), then we get,

$$\tilde{\Sigma}_{i_2k_2,23}\bar{\delta}_{I_1-2}+\tilde{\Sigma}_{i_2k_2,34}\bar{\delta}_{I_1-3}+...+\tilde{\Sigma}_{i_2k_2,I_1-1,I_1}\bar{\delta}_1=\tilde{\Gamma}_{i_2k_2}.$$

where,

$$\tilde{\Sigma}_{i_2k_2,j_1,j_1+1} \triangleq \frac{\tilde{\lambda}^{i_2}\left(b_{i_2j_1}-b_{i_2,j_1+1}\right)}{\left(1-\tilde{\lambda}^{i_2}\right)\left(b_{i_21}-b_{i_22}\right)}-\frac{\tilde{\lambda}^{k_2}\left(b_{k_2j_1}-b_{k_2,j_1+1}\right)}{\left(1-\tilde{\lambda}^{k_2}\right)\left(b_{k_21}-b_{k_22}\right)}.$$

$$\tilde{\Gamma}_{i_2k_2} \triangleq \frac{\tilde{\lambda}^{k_2}\left[\left(1-\tilde{w}^{k_2}\right)+\tilde{w}^{k_2}\left(b_{k_2I_1}+\tilde{e}_{k_2}^{T}B^{T}\bar{\rho}\right)\right]}{\left(1-\tilde{\lambda}^{k_2}\right)\left(b_{k_21}-b_{k_22}\right)\tilde{w}^{k_2}}-\frac{\tilde{\lambda}^{i_2}\left[\left(1-\tilde{w}^{i_2}\right)+\tilde{w}^{i_2}\left(b_{i_2I_1}+\tilde{e}_{i_2}^{T}B^{T}\bar{\rho}\right)\right]}{\left(1-\tilde{\lambda}^{i_2}\right)\left(b_{i_21}-b_{i_22}\right)\tilde{w}^{i_2}}.$$

$$\forall i_2 \neq k_2,\ \ i_2,k_2=1,2,...,I_2;\ \ j_1=2,3,...,I_1-1.$$

Consequently, we obtain,

$$\begin{bmatrix} \tilde{\Sigma}_{12,23} & \tilde{\Sigma}_{12,34} & \cdots & \tilde{\Sigma}_{12,I_1-1,I_1} \\ \tilde{\Sigma}_{23,23} & \tilde{\Sigma}_{23,34} & \cdots & \tilde{\Sigma}_{23,I_1-1,I_1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\Sigma}_{I_2-1,I_2,23} & \tilde{\Sigma}_{I_2-1,I_2,34} & \cdots & \tilde{\Sigma}_{I_2-1,I_2,I_1-1,I_1} \end{bmatrix}_{(I_2-1)\times(I_1-2)} \begin{bmatrix} \overline{\delta}_{I_1-2} \\ \overline{\delta}_{I_1-3} \\ \vdots \\ \overline{\delta}_1 \end{bmatrix}_{(I_1-2)\times 1} = \begin{bmatrix} \tilde{\Gamma}_{12} \\ \tilde{\Gamma}_{23} \\ \vdots \\ \tilde{\Gamma}_{I_2-1,I_2} \end{bmatrix}_{(I_2-1)\times 1}.$$

which leads us to the following equation,

$$\overline{\delta} = \tilde{\Sigma}^{+}\tilde{\Gamma}. \tag{B.16}$$

where " $+$ " stands for the Moore-Penrose generalized inverse.

Consequently, by equations in (B.16) and (B.15), we get $y^{2*} = \tilde{\delta}_{I_2-2} - y^{1*}$, $y^{3*} = \tilde{\delta}_{I_2-3} - \tilde{\delta}_{I_2-2}$, $\ldots$, $y^{I_2*} = 1 - \tilde{\delta}_1$ and $x^{2*} = \overline{\delta}_{I_1-2} - x^{1*}, x^{3*} = \overline{\delta}_{I_1-3} - \overline{\delta}_{I_1-2}, \ldots, x^{I_1*} = 1 - \overline{\delta}_1$ with $y^{1*} \equiv y_1^{1*} = y_2^{1*} = \ldots = y_{i_1}^{1*} = \ldots = y_{I_1}^{1*}$ and $x^{1*} \equiv x_1^{1*} = x_2^{1*} = \ldots = x_{i_2}^{1*} = \ldots = x_{I_2}^{1*}$. So, we obtain the corresponding game equilibrium, denoted by

$$\Big(x^*(y^*, \overline{\rho}) \triangleq \left(x^{1*}(y^*, \overline{\rho}), \ldots, x^{i_1*}(y^*, \overline{\rho}), \ldots, x^{I_1*}(y^*, \overline{\rho}) \right)^T,$$
$$y^*(x^*, \tilde{\rho}) \triangleq \left(y^{1*}(x^*, \tilde{\rho}), \ldots, y^{i_2*}(x^*, \tilde{\rho}), \ldots, y^{I_2*}(x^*, \tilde{\rho}) \right)^T \Big)$$

with $\sum_{i_1}^{I_1} x^{i_1*}(y^*, \overline{\rho}) = 1$ and $\sum_{i_2}^{I_2} y^{i_2*}(x^*, \tilde{\rho}) = 1$, but noting that this game equilibrium may be not the Pareto optimal equilibrium of the original normal form games thanks to the stochastic factors, and this is why we need to choose appropriate values of $\overline{\rho}$ and $\tilde{\rho}$ such that the original Pareto optimal Nash equilibrium $(\hat{x}, \hat{y})$ will be absolutely chosen by the players.

To summarize, we get the following theorem,

THEOREM 1': *If we are provided that the following inequalities hold, that is,* $\tilde{e}_1^T B^T x < \overline{\theta}^{i_1}$ *in* (B.3) *and* $\overline{e}_1^T A y < \tilde{\theta}^{i_2}$ *in* (B.9)*, then Problem* 1 *is solved as long as we have* $\tilde{\delta} = \overline{\Sigma}^{+}\overline{\Gamma}$ *in* (B.15) *and* $\overline{\delta} = \tilde{\Sigma}^{+}\tilde{\Gamma}$ *in* (B.16)*. That is to say, the existence of the Pareto optimal endogenous matching is confirmed just via putting* $x^*(y^*, \overline{\rho}) = \hat{x}$ *and* $y^*(x^*, \tilde{\rho}) = \hat{y}$*, in which* $(\hat{x}, \hat{y})$ *is the given Pareto optimal Nash equilibrium in the corresponding normal form games.*

Therefore, Theorem 1 is established thanks to Theorem 1'. ▌

C. Proof of Theorem 2

The technique used here is mainly developed by Dai (2012).

By the Doob's Martingale Inequality,

$$\mathbb{Q}\left(\sup_{0\le t\le T}\left|X^{i_1}(t)\right|\ge\Phi^{i_1}\right)\le\frac{1}{\Phi^{i_1}}\mathbb{E}_{\mathbb{Q}}\left[\left|X^{i_1}(T)\right|\right]=\frac{X^{i_1}(0)}{\Phi^{i_1}}\triangleq\frac{x^{i_1}}{\Phi^{i_1}},\ \ \forall\Phi^{i_1}>0,\ \ \forall T>0. \tag{C.1}$$

Without loss of any generality, we put $\Phi^{i_1}=2^{k_1}$ for $\forall k_1\in\mathbb{N}$, then we get,

$$\mathbb{Q}\left(\sup_{0\le t\le T}\left|X^{i_1}(t)\right|\ge 2^{k_1}\right)\le\frac{1}{2^{k_1}}x^{i_1},\ \ \forall k_1\in\mathbb{N},\ \ \forall T>0.$$

By using the well-known Borel-Cantelli Lemma, we arrive at,

$$\mathbb{Q}\left(\sup_{0\le t\le T}\left|X^{i_1}(t)\right|\ge 2^{k_1}\, i.m.k_1\right)=0.$$

where $i.m.k_1$ stands for "infinitely many k_1". So for a.a. $\omega\in\Omega$, there exists $\hat{k}_1(\omega)$ such that,

$$\sup_{0\le t\le T}\left|X^{i_1}(t)\right|<2^{k_1},\ \text{a.s. for}\ \ k_1\ge\hat{k}_1(\omega),\ \ \forall T>0.$$

i.e.,

$$\lim_{T\to\infty}\sup_{0\le t\le T}\left|X^{i_1}(t)\right|\le 2^{k_1},\ \text{a.s. for}\ \ k_1\ge\hat{k}_1(\omega).$$

Consequently, $X^{i_1}(t)=X^{i_1}(t,\omega)$ is uniformly bounded for $t\in[0,T]$, $\forall T>0$ and for a.a. $\omega\in\Omega$. Hence, it is ensured that $X^{i_1}(t)=X^{i_1}(t,\omega)$ converges a.s. and the corresponding limit belongs to $L^1(\mathbb{Q})$ thanks to the Doob's Martingale Convergence Theorem. Moreover, by applying the Kolmogorov's Inequality, we arrive at,

$$\mathbb{Q}\left(\sup_{0\le t\le T}\left|X^{i_1}(t)\right|\ge\Phi^{i_1}\right)\le\frac{1}{(\Phi^{i_1})^2}\operatorname{var}_{\mathbb{Q}}\left[\left|X^{i_1}(T)\right|\right],\ \ \forall 0<\Phi^{i_1}<\infty,\ \ \forall T>0.$$

It follows from (C.1) that,

$$\frac{1}{(\Phi^{i_1})^2}\operatorname{var}_{\mathbb{Q}}\left[\left|X^{i_1}(T)\right|\right]\le\frac{x^{i_1}}{\Phi^{i_1}}\ \ \Leftrightarrow\ \ \operatorname{var}_{\mathbb{Q}}\left[\left|X^{i_1}(T)\right|\right]\le\Phi^{i_1}x^{i_1},\ \ \forall T>0. \tag{C.2}$$

Noting that,

$$\operatorname{var}_{\mathbb{Q}}\left[\left|X^{i_1}(T)\right|\right]=\mathbb{E}_{\mathbb{Q}}\left[\left|X^{i_1}(T)\right|^2\right]-(x^{i_1})^2,\ \ \forall T>0.$$

We get by (C.2),

$$\mathbb{E}_{\mathbb{Q}}\left[\left|X^{i_1}(T)\right|^2\right] \le \left(\Phi^{i_1} + x^{i_1}\right)x^{i_1} < \infty,\ \ \forall 0 < \Phi^{i_1} < \infty,\ \ \forall T > 0.$$

which yields,

$$\sup_{T \ge 0} \mathbb{E}_{\mathbb{Q}}\left[\left|X^{i_1}(T)\right|^2\right] \le \left(\Phi^{i_1} + x^{i_1}\right)x^{i_1} < \infty.$$

Accordingly, $X^{i_1}(t) = X^{i_1}(t,\omega)$ converges in $L^1(\mathbb{Q})$ by using the Doob's Martingale Convergence Theorem again.

Furthermore, it is easily seen that $X^{i_1}(t) - x^{i_1 *}(y^*, \bar{\rho})$, in which $x^{i_1 *}(y^*, \bar{\rho})$ is given by Corollary 1, is also an $\mathcal{F}_t -$martingale w. r. t. $\mathbb{Q}$. Thus, applying the Doob's Martingale Inequality again implies that,

$$\mathbb{Q}\left(\sup_{0 \le t \le T}\left|X^{i_1}(t) - x^{i_1 *}(y^*, \bar{\rho})\right| \ge \frac{\varepsilon^{i_1}}{I_1}\right) \le \frac{I_1}{\varepsilon^{i_1}}\mathbb{E}_{\mathbb{Q}}\left[\left|X^{i_1}(T) - x^{i_1 *}(y^*, \bar{\rho})\right|\right],\ \ \forall \varepsilon^{i_1} \in \mathbb{R}_{++}, \forall T > 0. \quad \text{(C.3)}$$

Provided that $\tilde{\tau}^{i_2 *}(\omega) \triangleq \inf\left\{t \ge 0; X^{i_1}(t) = x^{i_1 *}(y^*, \bar{\rho})\right\}$ ($\forall\ i_1 = 1, 2, ..., I_1$ and $\forall\ i_2 = 1, 2,\ ..., I_2$) determined by Corollary 1, we see that there exists $\upsilon^{i_1} > 0$ such that the martingale inequality in (C.3) holds for $\forall\, \tau \in B^{i_1}_{\upsilon^{i_1}}\left(\tilde{\tau}^{i_2 *}(\omega)\right) \triangleq \left\{\tau \ge 0; \left|\tau - \tilde{\tau}^{i_2 *}(\omega)\right| \le \upsilon^{i_1}\right\}$ for $i_1 = 1, 2, ..., I_1$ and $\forall\ i_2 = 1, 2,$ $..., I_2$ by using Doob's Optional Sampling Theorem. Thus, $\left|X^{i_1}(\tau) - x^{i_1 *}(y^*, \bar{\rho})\right|$ is uniformly bounded on the compact set $B^{i_1}_{\upsilon^{i_1}}\left(\tilde{\tau}^{i_2 *}(\omega)\right)$ by applying the Heine-Borel Theorem, Weierstrass Theorem and Triangle Inequality. Therefore, we, without loss of generality, set up $\upsilon^{i_1} = 2^{-k_0}$ for $\forall k_0 \in \mathbb{N}$ and employ the continuity of martingale w. r. t. time t for any given $\omega \in \Omega$ so that for $\forall T^{i_1}_{k_0} \in B^{i_1}_{\upsilon^{i_1}}\left(\tilde{\tau}^{i_2 *}(\omega)\right)$, based upon the well-known Lebesgue Dominated Convergence Theorem, we are led to,

$$\limsup_{k_0 \to \infty} \mathbb{Q}\left(\sup_{0 \le t \le T^{i_1}_{k_0}}\left|X^{i_1}(t) - x^{i_1 *}(y^*, \bar{\rho})\right| \ge \frac{\varepsilon^{i_1}}{I_1}\right) \le \frac{I_1}{\varepsilon^{i_1}}\limsup_{k_0 \to \infty} \mathbb{E}_{\mathbb{Q}}\left[\left|X^{i_1}(T^{i_1}_{k_0}) - x^{i_1 *}(y^*, \bar{\rho})\right|\right] = 0.$$

almost surely. And this implies that,

$$\limsup_{k_0\to\infty} \mathbb{Q}\left(\sup_{0\le t\le T_{k_0}^{i_1}} \left| X^{i_1}(t) - x^{i_1 *}(y^*,\overline{\rho}) \right| < \frac{\varepsilon^{i_1}}{I_1} \right) \ge 1, \text{ a.s.}$$

Letting $\varepsilon^{i_1} = 2^{-k} I_1$, $\forall k \in \mathbb{N}$, we get,

$$\limsup_{k_0\to\infty} \mathbb{Q}\left(\sup_{0\le t\le T_{k_0}^{i_1}} \left| X^{i_1}(t) - x^{i_1 *}(y^*,\overline{\rho}) \right| < 2^{-k} \right) = 1, \text{ a.s.}, \quad \forall k \in \mathbb{N}$$

It follows from the well-known Fatou's Lemma that,

$$\mathbb{Q}\left(\sup_{0\le t\le \tilde{\tau}^{i_2 *}(\omega)} \left| X^{i_1}(t) - x^{i_1 *}(y^*,\overline{\rho}) \right| < 2^{-k} \right) = 1, \text{ a.s.}, \quad \forall k \in \mathbb{N}$$

Then, applying the Borel-Cantelli Lemma again implies that,

$$\mathbb{Q}\left(\sup_{0\le t\le \tilde{\tau}^{i_2 *}(\omega)} \left| X^{i_1}(t) - x^{i_1 *}(y^*,\overline{\rho}) \right| < 2^{-k}\, i.m.k \right) = 1$$

in which $i.m.k$ stands for "infinitely many k". So for a.a. $\omega \in \Omega$, there exists $k^*(\omega) \in \mathbb{N}$ such that,

$$\sup_{0\le t\le \tilde{\tau}^{i_2 *}(\omega)} \left| X^{i_1}(t) - x^{i_1 *}(y^*,\overline{\rho}) \right| < 2^{-k} \quad \text{for } \forall k \ge k^*(\omega).$$

That is,

$$\sup_{0\le t\le \tilde{\tau}^{i_2 *}(\omega)} \left| X^{i_1}(t) - x^{i_1 *}(y^*,\overline{\rho}) \right| < \frac{\varepsilon^{i_1}}{I_1}, \quad \mathbb{Q}-\text{a.s.}$$

for $i_1 = 1,2,\ldots,I_1$ and $\forall\ i_2 = 1,2,\ \ldots, I_2$. Now, we define the supremum norm $\|x\|_\infty \triangleq \max_{i_1} \left| x^{i_1} \right|$ equipped with uniform topology. Thus, one may easily obtain,

$$\begin{aligned}
&\sup_{0\le t\le \tilde{\tau}^{i_2 *}(\omega)} \left\| X(t) - x^*(y^*,\overline{\rho}) \right\|_2 \\
&\le I_1 \sup_{0\le t\le \tilde{\tau}^{i_2 *}(\omega)} \left\| X(t) - x^*(y^*,\overline{\rho}) \right\|_\infty \\
&= I_1 \sup_{0\le t\le \tilde{\tau}^{i_2 *}(\omega)} \max_{i_1\in\{1,2,\ldots,I_1\}} \left| X^{i_1}(t) - x^{i_1 *}(y^*,\overline{\rho}) \right| \\
&= I_1 \sup_{0\le t\le \tilde{\tau}^{i_2 *}(\omega)} \left| X^{\hat{i}_1}(t) - x^{\hat{i}_1 *}(y^*,\overline{\rho}) \right| \\
&\le I_1 \frac{\varepsilon^{\hat{i}_1}}{I_1} = \varepsilon^{\hat{i}_1}, \quad \mathbb{Q}-\text{a.s.}
\end{aligned}$$

Notice the arbitrariness of $\varepsilon^{\hat{i}_1}$, we get,

$$\lim_{\tilde{\tau}^{i_2*}(\omega)\to\infty} \sup_{0\le t\le\tilde{\tau}^{i_2*}(\omega)} \left\| X(t) - x^*(y^*,\overline{\rho}) \right\|_2 = 0, \ \mathbb{Q}-\text{a.s.}$$

That is to say,

$$\mathbb{Q}\left(\bigcup_{k=1}^{\infty} \bigcap_{t'=0}^{\infty} \bigcup_{t=t'}^{\infty} \left[\left\| X(t) - x^*(y^*,\overline{\rho}) \right\|_2 \ge \frac{1}{k} \right] \right) = 0.$$

Equivalently, for $\forall k \in \mathbb{N}$, we arrive at,

$$\mathbb{Q}\left(\bigcap_{t'=0}^{\infty} \bigcup_{t=t'}^{\infty} \left[\left\| X(t) - x^*(y^*,\overline{\rho}) \right\|_2 \ge \frac{1}{k} \right] \right) = 0.$$

i.e., for $\forall \varepsilon > 0$,

$$\lim_{t'\to\infty} \mathbb{Q}\left(\bigcup_{t=t'}^{\infty} \left[\left\| X(t) - x^*(y^*,\overline{\rho}) \right\|_2 \ge \varepsilon \right] \right) = 0.$$

which gives the desired assertion in (i). Notice that the proof of (ii) will be quite similar to that of (i), we will take it omitted. And hence we have completed the whole proof. ▌

D. Proof of Proposition 2

Provided the SDEs defined in (4) and (6), and it follows from Assumption 9 that for $\forall\ 2 < p < \infty$, $\forall T > 0$, we have,

$$\mathbb{E}^1\left[\sup_{0\le t\le T} \left\| X(t) \right\|_2^p \right] \vee \mathbb{E}^1\left[\sup_{0\le t\le T} \left\| \hat{X}(t) \right\|_2^p \right] \le 1, \tag{D.1}$$

where,

$$X(t) = X(0) + \int_0^t X(s)\circ F^1\left(Y(s)\right)ds + \int_0^t X(s)\circ G^1(s)dW^1(s) + \int_0^t \int_{\mathbb{R}_0^{l_{1m}}} X(s)\circ H^1\left(s,z^1\right)\tilde{N}^1\left(ds,dz^1\right)$$

$$\hat{X}(t) = \hat{X}(0) + \int_0^t \hat{X}(s)\circ \hat{F}^1\left(Y(s)\right)ds + \int_0^t \hat{X}(s)\circ \hat{G}^1(s)dW^1(s) + \int_0^t \int_{\mathbb{R}_0^{l_{1m}}} \hat{X}(s)\circ \hat{H}^1\left(s,z^1\right)\tilde{N}^1\left(ds,dz^1\right)$$

Here, and throughout the current proof, we suppose that $X(0) = \hat{X}(0)$. Moreover, suppose that $\left\| X(t) \right\|_2 \vee \left\| \hat{X}(t) \right\|_2 \le E$ for $\forall t \in \mathbb{R}_+$ and $E < \infty$. Indeed, one just need to let $E \ge 1$. In what follows, we first define the following stopping times,

$$\tau_E \triangleq \inf\left\{ t \ge 0; \left\| X(t) \right\|_2 \ge E \right\},\ \hat{\tau}_E \triangleq \inf\left\{ t \ge 0; \left\| \hat{X}(t) \right\|_2 \ge E \right\},\ \tau_E^0 \triangleq \tau_E \wedge \hat{\tau}_E.$$

By the Young Inequality (see, Higham et al., 2003) and for any $S > 0$,

$$\mathbb{E}^1\left[\sup_{0\le t\le T}\left\|X(t)-\hat{X}(t)\right\|_2^2\right]$$

$$=\mathbb{E}^1\left[\sup_{0\le t\le T}\left\|X(t)-\hat{X}(t)\right\|_2^2\chi_{\{\tau_E>T,\hat{\tau}_E>T\}}\right]+\mathbb{E}^1\left[\sup_{0\le t\le T}\left\|X(t)-\hat{X}(t)\right\|_2^2\chi_{\{\tau_E\le T,or\hat{\tau}_E\le T\}}\right]$$

$$\le\mathbb{E}^1\left[\sup_{0\le t\le T}\left\|X\left(t\wedge\tau_E^0\right)-\hat{X}\left(t\wedge\tau_E^0\right)\right\|_2^2\chi_{\{\tau_E^0>T\}}\right]+\frac{2S}{p}\mathbb{E}^1\left[\sup_{0\le t\le T}\left\|X(t)-\hat{X}(t)\right\|_2^p\right]$$

$$+\frac{1-\frac{2}{p}}{S^{\frac{2}{p-2}}}\mathbb{P}^1\left(\tau_E\le T,or\hat{\tau}_E\le T\right),\tag{D.2}$$

It follows from (D.1) that,

$$\mathbb{P}^1\left(\tau_E\le T\right)=\mathbb{E}^1\left[\chi_{\{\tau_E\le T\}}\frac{\left\|X\left(\tau_E\right)\right\|_2^p}{E^p}\right]\le\frac{1}{E^p}\mathbb{E}^1\left[\sup_{0\le t\le T}\left\|X(t)\right\|_2^p\right]\le\frac{1}{E^p}.$$

Similarly, $\mathbb{P}^1\left(\hat{\tau}_E\le T\right)\le 1/E^p$. So,

$$\mathbb{P}^1\left(\tau_E\le T,or\hat{\tau}_E\le T\right)\le\mathbb{P}^1\left(\tau_E\le T\right)+\mathbb{P}^1\left(\hat{\tau}_E\le T\right)\le\frac{2}{E^p}.$$

Moreover, we obtain by (D.1),

$$\mathbb{E}^1\left[\sup_{0\le t\le T}\left\|X(t)-\hat{X}(t)\right\|_2^p\right]\le 2^{p-1}\mathbb{E}^1\left[\sup_{0\le t\le T}\left(\left\|X(t)\right\|_2^p+\left\|\hat{X}(t)\right\|_2^p\right)\right]\le 2^p.$$

Hence, (D.2) becomes,

$$\mathbb{E}^1\left[\sup_{0\le t\le T}\left\|X(t)-\hat{X}(t)\right\|_2^2\right]\le\mathbb{E}^1\left[\sup_{0\le t\le T}\left\|X\left(t\wedge\tau_E^0\right)-\hat{X}\left(t\wedge\tau_E^0\right)\right\|_2^2\right]+\frac{2^{p+1}S}{p}+\frac{2(p-2)}{pS^{\frac{2}{p-2}}E^p}.\tag{D.3}$$

By the Cauchy-Bunyakovsky-Schwarz Inequality, we get,

$$\left\|X\left(t\wedge\tau_E^0\right)-\hat{X}\left(t\wedge\tau_E^0\right)\right\|_2^2$$

$$=\left\|\int_0^{t\wedge\tau_E^0}\left[X(s)\circ F^1\left(Y(s)\right)-\hat{X}(s)\circ\hat{F}^1\left(Y(s)\right)\right]ds+\int_0^{t\wedge\tau_E^0}\left[X(s)\circ G^1(s)-\hat{X}(s)\circ\hat{G}^1(s)\right]dW^1(s)\right.$$

$$\left.+\int_0^{t\wedge\tau_E^0}\int_{\mathbb{R}_0^{l_1m_1}}\left[X(s)\circ H^1\left(s,z^1\right)-\hat{X}(s)\circ\hat{H}^1\left(s,z^1\right)\right]\tilde{N}^1\left(ds,dz^1\right)\right\|_2^2$$

$$\leq 2\left\{T\int_0^{t\wedge\tau_E^0}\left\|X(s)\circ F^1\left(Y(s)\right)-\hat{X}(s)\circ\hat{F}^1\left(Y(s)\right)\right\|_2^2 ds+\left\|\int_0^{t\wedge\tau_E^0}\left[X(s)\circ G^1(s)-\hat{X}(s)\circ\hat{G}^1(s)\right]dW^1(s)\right\|_2^2\right.$$

$$\left.+\left\|\int_0^{t\wedge\tau_E^0}\int_{\mathbb{R}_0^{l_1n_1}}\left[X(s)\circ H^1\left(s,z^1\right)-\hat{X}(s)\circ\hat{H}^1\left(s,z^1\right)\right]\tilde{N}^1\left(ds,dz^1\right)\right\|_2^2\right\}$$

Taking expectations on both sides, and using Itô's Isometry and the stochastic Fubini Theorem, we have for any $\tau\leq T$,

$$\mathbb{E}^1\left[\sup_{0\leq t\leq\tau}\left\|X\left(t\wedge\tau_E^0\right)-\hat{X}\left(t\wedge\tau_E^0\right)\right\|_2^2\right]$$

$$\leq 4\left\{T\mathbb{E}^1\left[\int_0^{t\wedge\tau_E^0}\left\|X(s)\circ F^1\left(Y(s)\right)-\hat{X}(s)\circ F^1\left(Y(s)\right)\right\|_2^2 ds\right]\right.$$

$$+T\mathbb{E}^1\left[\int_0^{t\wedge\tau_E^0}\left\|\hat{X}(s)\circ F^1\left(Y(s)\right)-\hat{X}(s)\circ\hat{F}^1\left(Y(s)\right)\right\|_2^2 ds\right]$$

$$+\mathbb{E}^1\left[\int_0^{t\wedge\tau_E^0}\left\|X(s)\circ G^1(s)-\hat{X}(s)\circ G^1(s)\right\|_2^2 ds\right]+\mathbb{E}^1\left[\int_0^{t\wedge\tau_E^0}\left\|\hat{X}(s)\circ G^1(s)-\hat{X}(s)\circ\hat{G}^1(s)\right\|_2^2 ds\right]$$

$$+\mathbb{E}^1\left[\int_0^{t\wedge\tau_E^0}\int_{\mathbb{R}_0}\sum_{l_1=1}^{l_1n_1}\left\|X(s)\circ H^1\left(s,z^1\right)-\hat{X}(s)\circ H^1\left(s,z^1\right)\right\|_2^2\nu_{l_1}^1\left(dz_{l_1}^1\right)ds\right]$$

$$\left.+\mathbb{E}^1\left[\int_0^{t\wedge\tau_E^0}\int_{\mathbb{R}_0}\sum_{l_1=1}^{l_1n_1}\left\|\hat{X}(s)\circ H^1\left(s,z^1\right)-\hat{X}(s)\circ\hat{H}^1\left(s,z^1\right)\right\|_2^2\nu_{l_1}^1\left(dz_{l_1}^1\right)ds\right]\right\}$$

$$\leq 4\left\{TK^1\mathbb{E}^1\left[\int_0^{t\wedge\tau_E^0}\left\|X(s)-\hat{X}(s)\right\|_2^2 ds\right]+T\left(\varsigma^1\right)^2\mathbb{E}^1\left[\int_0^{t\wedge\tau_E^0}\left\|\hat{X}(s)\right\|_2^2 ds\right]\right.$$

$$+K^1\mathbb{E}^1\left[\int_0^{t\wedge\tau_E^0}\left\|X(s)-\hat{X}(s)\right\|_2^2 ds\right]+K^1\mathbb{E}^1\left[\int_0^{t\wedge\tau_E^0}\int_{\mathbb{R}_0}\sum_{l_1=1}^{l_1n_1}\left\|X(s)-\hat{X}(s)\right\|_2^2\nu_{l_1}^1\left(dz_{l_1}^1\right)ds\right]$$

$$\left.+\left(\varsigma^1\right)^2\mathbb{E}^1\left[\int_0^{t\wedge\tau_E^0}\left\|\hat{X}(s)\right\|_2^2 ds\right]+\left(\varsigma^1\right)^2\mathbb{E}^1\left[\int_0^{t\wedge\tau_E^0}\int_{\mathbb{R}_0}\sum_{l_1=1}^{l_1n_1}\left\|\hat{X}(s)\right\|_2^2\nu_{l_1}^1\left(dz_{l_1}^1\right)ds\right]\right\}$$

$$\leq 4\left\{(T+1)K^1\mathbb{E}^1\left[\int_0^{t\wedge\tau_E^0}\left\|X(s)-\hat{X}(s)\right\|_2^2 ds\right]+T(T+1)\left(\varsigma^1\right)^2\right.$$

$$+K^1\mathbb{E}^1\left[\sum_{l_1=1}^{l_1 n_1}\int_{\mathbb{R}_0}\nu_{l_1}^1\left(dz_{l_1}^1\right)\int_0^{t\wedge\tau_E^0}\left\|X(s)-\hat{X}(s)\right\|_2^2 ds\right]+\left(\varsigma^1\right)^2\mathbb{E}^1\left[\sum_{l_1=1}^{l_1 n_1}\int_{\mathbb{R}_0}\nu_{l_1}^1\left(dz_{l_1}^1\right)\int_0^{t\wedge\tau_E^0}\left\|\hat{X}(s)\right\|_2^2 ds\right]\Bigg\}$$

$$\leq 4\left\{\left[(T+1)+\hat{K}^1\right]K^1\mathbb{E}^1\left[\int_0^{t\wedge\tau_E^0}\left\|X(s)-\hat{X}(s)\right\|_2^2 ds\right]+T\left[(T+1)+\hat{K}^1\right]\left(\varsigma^1\right)^2\right\}$$

$$\leq 4\left[(T+1)+\hat{K}^1\right]K^1\int_0^\tau\mathbb{E}^1\left[\sup_{0\leq t_0\leq s}\left\|X\left(t_0\wedge\tau_E^0\right)-\hat{X}\left(t_0\wedge\tau_E^0\right)\right\|_2^2\right]ds+4T\left[(T+1)+\hat{K}^1\right]\left(\varsigma^1\right)^2$$

where we have used Assumption 7 and 8. Hence, applying Gronwall's Inequality (see, Higham et al., 2003) implies that,

$$\mathbb{E}^1\left[\sup_{0\leq t\leq\tau}\left\|X\left(t\wedge\tau_E^0\right)-\hat{X}\left(t\wedge\tau_E^0\right)\right\|_2^2\right]\leq 4T\left[(T+1)+\hat{K}^1\right]\exp\left\{4\left[(T+1)+\hat{K}^1\right]K^1\right\}\left(\varsigma^1\right)^2$$

Inserting this into (D.3) leads us to,

$$\mathbb{E}^1\left[\sup_{0\leq t\leq T}\left\|X(t)-\hat{X}(t)\right\|_2^2\right]$$

$$\leq 4T\left[(T+1)+\hat{K}^1\right]\exp\left\{4\left[(T+1)+\hat{K}^1\right]K^1\right\}\left(\varsigma^1\right)^2+\frac{2^{p+1}S}{p}+\frac{2(p-2)}{pS^{\frac{2}{p-2}}E^p}$$

Hence, for $\forall\varepsilon>0$, we can shoose S and E such that,

$$\frac{2^{p+1}S}{p}\leq\frac{\varepsilon}{3}\quad\text{and}\quad\frac{2(p-2)}{pS^{\frac{2}{p-2}}E^p}\leq\frac{\varepsilon}{3}$$

And for any given $T>0$, we put ς^1 such that,

$$4T\left[(T+1)+\hat{K}^1\right]\exp\left\{4\left[(T+1)+\hat{K}^1\right]K^1\right\}\left(\varsigma^1\right)^2\leq\frac{\varepsilon}{3}.$$

Thus, for $\forall\varepsilon>0$, we obtain,

$$\mathbb{E}^1\left[\sup_{0\leq t\leq T}\left\|X(t)-\hat{X}(t)\right\|_2^2\right]\leq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

Notice the arbitrariness of ε, and employ the well-known Levi Lemma gives the desired result in (i). One can easily check that the proof of (ii) is quite similar to that of (i), so we omit it. And this completes the whole proof. ▌