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Abstract

In the present paper, a new approach to equilibrium selection for very general normal form games has been constructed by introducing stochastic optimal stopping theory into classical evolutionary game theory. That is, the new game equilibrium is induced by both stochastic group evolution and decentralized rational individual choice. Moreover, stability of the game equilibrium is confirmed from both time and space dimensions.

Keywords: Stochastic replicator dynamics; Rational choice; Normal-form game; Stability.

JEL classification: C62; C70.

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1. Introduction

The major goal of the present paper is to construct a new approach to equilibrium selection for very general normal form game situations, especially, those games consist of two groups of populations. The existence and uniqueness of the new game equilibrium induced by stochastic group evolution and rational individual choice have been proved, and also the stability of the game equilibrium is confirmed from both time (i.e., in the sense of stochastic stopping time) and space (i.e., from the viewpoint of invariant probability measure) dimensions, which is different from the classical approach of Kohlberg and Mertens (1986). Furthermore, Theorem 1 in section 2 not only provides us with the explicit form of the new game equilibrium but also provides us with the explicit time length needed so that the game equilibrium can be achieved by decentralized players. And this would be regarded as an important characteristic of the new game equilibrium relative to traditional approach (see, Samuelson and Zhang, 1992; Kandori et al., 1993; Young, 1993; Matsui and Matsuyama, 1995; Foster and Young, 2003; Binmore et al., 2003).

Noting that both evolutionary game theory and rational choice theory have very important economic implications, the current paper introduces a new game equilibrium by combining both of the above, i.e., evolutionary game theory (see, Friedman, 1991; Ritzberger and Weibull, 1995; Hofbauer and Sigmund, 2003; Benaïm and Weibull, 2003, and among others) corresponds to group-level deterministic or stochastic evolution via both the well-known random-matching rule (e.g., Ellison, 1994; Zhou, 1999; Bogomolnaia and Moulin, 2004; Duffie and Sun, 2007; Aliprantis et al., 2007, and among others) and deterministic or stochastic replicator dynamics (e.g., Foster and Young, 1990; Fudenberg and Harris, 1992; Binmore et al., 1995; Cabrales, 2000; Corradi and Sarin, 2000; Imhof, 2005; Benaïm et al., 2008, and among others) while rational choice theory (see, Harsanyi, 1966; Bernheim, 1984; Aumann, 1987, and among others) corresponds to individual-level and decentralized rational decision. Most importantly, the present paper successfully shows that optimal stopping theory that has been widely applied in mathematical finance (see, Myneni, 1992; Shepp and Shiryaev, 1993; Hobson, 1998; Guo and Shepp, 2001; Avram et al., 2004; Choi et al., 2004; Alili and Kyprianou, 2005) plays a crucial role in characterizing and finally demonstrating the existence and uniqueness of the new game equilibrium.

The rest of the paper is organized as follows: section 2 introduces the model where the formal definition of the new game equilibrium is given and the existence and uniqueness of the game equilibrium are proved; section 3 demonstrates the stability of the game equilibrium from both time and space dimensions; section 4 concludes and the Appendix provides the main mathematical derivations.

2. The Model

Let $A_{I_1 \times I_2}$ be the payoff matrix for row-players and $B_{I_1 \times I_2}$ be the payoff matrix for column-players with $A_{I_1 \times I_2}$, $B_{I_1 \times I_2} \in \mathbb{R}^{I_1 \times I_2}$, and I_1, $I_2 \geq 1$. Here, and throughout the current paper, we study the replicator dynamics of $I_1 \times I_2$ normal form games between two groups of populations. Put $\sum_{i_1=1}^{I_1} M^{i_1}(t) \triangleq M$, where $M^{i_1}(t)$ denotes the number of strategy-i_1 players at period t. Similarly, let $\sum_{i_2=1}^{I_2} N^{i_2}(t) \triangleq N$, where $N^{i_2}(t)$ denotes the number of strategy-i_2 players at period t. And, we introduce the following technical assumption,

ASSUMPTION 1: *Throughout the current paper, both* M *and* N*, sufficiently large, are assumed to be finite constants.*

REMARK 2.1: Some of existing literatures (see, Fudenberg et al., 2004; Nowak et al., 2004; Imhof and Nowak, 2006, and among others) have confirmed that Assumption 1 has very important implications. That is to say, on the one hand, Assumption 1 is used to make things much easier from the viewpoint of pure mathematics; and also, Assumption 1 is indeed without loss of any generality in the sense of economic and biological intuitions on the other hand.

We let $X^{i_1}(t) \triangleq M^{i_1}(t)/M$, $Y^{i_2}(t) \triangleq N^{i_2}(t)/N$ denote the frequencies of strategies i_1 and i_2, respectively, with $i_1 = 1,2,...,I_1$ and $i_2 = 1,2,...,I_2$. Therefore, the average payoffs of strategy i_1 and strategy i_2 are given by $u\left(i_1, Y(t)\right) \triangleq \bar{e}_{i_1}^T A Y(t)$ and $u\left(i_2, X(t)\right) \triangleq \tilde{e}_{i_2}^T B^T X(t)$, respectively, with the superscript " T " denoting transpose, and $X(t) \triangleq \left(X^1(t),...,X^{i_1}(t), ..., X^{I_1}(t)\right)^T$, $Y(t) \triangleq \left(Y^1(t),...,Y^{i_2}(t), ...,Y^{I_2}(t)\right)^T$, and also $\bar{e}_{i_1} = (0,...,1,...,0)^T$, $\tilde{e}_{i_2} = (0,...,1,...,0)^T$, where the i_1-th entry and i_2-entry are ones, respectively, for $i_1 = 1,2,...,I_1$ and $i_2 = 1,2,...,I_2$.

We now denote by $\left(\Omega^{(W^{i_\beta})}, \mathcal{F}^{(W^{i_\beta})}, \left\{\mathcal{F}_t^{(W^{i_\beta})}\right\}_{0\le t\le \tau^{i_\beta}(\omega)}, \mathbb{P}^{(W^{i_\beta})}\right)$ the filtered probability space with $\mathbb{F}^{(W^{i_\beta})} \triangleq \left\{\mathcal{F}_t^{(W^{i_\beta})}\right\}_{0\le t\le \tau^{i_\beta}(\omega)}$ the $\mathbb{P}^{(W^{i_\beta})}-$augmented filtration generated by $d_\beta-$dimensional standard Brownian motion $\left(W^{i_\beta}(t), 0 \le t \le \tau^{i_\beta}(\omega)\right)$ with $\mathcal{F}^{(W^{i_\beta})} \triangleq \mathcal{F}_{\tau^{i_\beta}(\omega)}^{(W^{i_\beta})}$, $\omega \in \Omega^{(W^{i_\beta})}$ and $\tau^{i_\beta}(\omega)$ a stopping time, to be endogenously determined. Moreover, we define $\tilde{N}^{i_\beta}\left(dt, dz^{i_\beta}\right) \triangleq \left(\tilde{N}_1^{i_\beta}\left(dt, dz_1^{i_\beta}\right), ..., \tilde{N}_{n_\beta}^{i_\beta}\left(dt, dz_{n_\beta}^{i_\beta}\right)\right)^T$ $\triangleq \left(N_1^{i_\beta}\left(dt, dz_1^{i_\beta}\right) - \nu_1^{i_\beta}\left(dz_1^{i_\beta}\right)dt, ..., N_{n_\beta}^{i_\beta}\left(dt, dz_{n_\beta}^{i_\beta}\right) - \nu_{n_\beta}^{i_\beta}\left(dz_{n_\beta}^{i_\beta}\right)dt\right)^T$, in which $\left\{N_{l_\beta}^{i_\beta}\right\}_{l_\beta=1}^{n_\beta}$ are independent Poisson random measures with Lévy measures $\nu_{l_\beta}^{i_\beta}$ coming from n_β independent (1-dimensional) Lévy processes $\eta_1^{i_\beta}(t) \triangleq \int_0^t \int_{\mathbb{R}_0} z_1^{i_\beta} \tilde{N}_1^{i_\beta}\left(ds, dz_1^{i_\beta}\right)$, ..., $\eta_{n_\beta}^{i_\beta}(t) \triangleq \int_0^t \int_{\mathbb{R}_0} z_{n_\beta}^{i_\beta} \tilde{N}_{n_\beta}^{i_\beta}\left(ds, dz_{n_\beta}^{i_\beta}\right)$ with $\mathbb{R}_0 \triangleq \mathbb{R} - \{0\}$, and then the corresponding stochastic basis is given by $\left(\Omega^{(\tilde{N}^{i_\beta})}, \mathcal{F}^{(\tilde{N}^{i_\beta})}, \left\{\mathcal{F}_t^{(\tilde{N}^{i_\beta})}\right\}_{0\le t\le \tau^{i_\beta}(\omega)}, \mathbb{P}^{(\tilde{N}^{i_\beta})}\right)$ with $\mathbb{F}^{(\tilde{N}^{i_\beta})} \triangleq \left\{\mathcal{F}_t^{(\tilde{N}^{i_\beta})}\right\}_{0\le t\le \tau^{i_\beta}(\omega)}$ the $\mathbb{P}^{(\tilde{N}^{i_\beta})}-$augmented filtration and $\mathcal{F}^{(\tilde{N}^{i_\beta})} \triangleq \mathcal{F}_{\tau^{i_\beta}(\omega)}^{(\tilde{N}^{i_\beta})}$, $\omega \in \Omega^{(\tilde{N}^{i_\beta})}$ and $\tau^{i_\beta}(\omega)$ a stopping time, to be endogenously determined. Thus, we are provided with a new stochastic basis $\left(\Omega^{i_\beta}, \mathcal{F}^{i_\beta}, \left\{\mathcal{F}_t^{i_\beta}\right\}_{0\le t\le \tau^{i_\beta}(\omega)}, \mathbb{P}^{i_\beta}\right)$, where $\Omega^{i_\beta} \triangleq \Omega^{(W^{i_\beta})} \times \Omega^{(\tilde{N}^{i_\beta})}$, $\mathcal{F}^{i_\beta} \triangleq \mathcal{F}^{(W^{i_\beta})} \otimes \mathcal{F}^{(\tilde{N}^{i_\beta})}$, $\mathcal{F}_t^{i_\beta} \triangleq \mathcal{F}_t^{(W^{i_\beta})} \otimes \mathcal{F}_t^{(\tilde{N}^{i_\beta})}$, $\mathbb{P}^{i_\beta} \triangleq \mathbb{P}^{(W^{i_\beta})} \otimes \mathbb{P}^{(\tilde{N}^{i_\beta})}$ and $\mathbb{F}^{i_\beta} \triangleq \left\{\mathcal{F}_t^{i_\beta}\right\}_{0\le t\le \tau^{i_\beta}(\omega)}$ denotes the corresponding filtration satisfying the well-known "usual conditions". Here, and throughout the current paper, $\mathbb{E}^{i_\beta}$ is used to denote the expectation operator with respect to (w. r. t.) the probability law $\mathbb{P}^{i_\beta}$ for $\forall\ i_\beta = 1,2,...,I_\beta$ and for $\beta = 1,2$. Naturally, we have stochastic basis $\left(\Omega^\beta, \mathcal{F}^\beta, \left\{\mathcal{F}_t^\beta\right\}_{0 \le t \le \tau^\beta(\omega)}, \mathbb{P}^\beta\right)$ with $\Omega^\beta \triangleq \Omega^1 \times ... \times \Omega^{I_\beta}$, $\mathcal{F}^\beta \triangleq \mathcal{F}^1 \otimes ... \otimes \mathcal{F}^{I_\beta}$, $\mathcal{F}_t^\beta \triangleq \mathcal{F}_t^1 \otimes ... \otimes \mathcal{F}_t^{I_\beta}$, $\mathbb{P}^\beta \triangleq \mathbb{P}^1 \otimes ... \otimes \mathbb{P}^{I_\beta}$, $\tau^\beta(\omega) \triangleq \tau^1(\omega) \vee ... \vee \tau^{I_\beta}(\omega) \triangleq \overline{\tau}^1(\omega) \vee ... \vee \overline{\tau}^{I_\beta}(\omega)$ if $\beta = 1$, and $\tau^\beta(\omega) \triangleq \tau^1(\omega) \vee ... \vee \tau^{I_\beta}(\omega) \triangleq \tilde{\tau}^1(\omega) \vee ... \vee \tilde{\tau}^{I_\beta}(\omega)$ if $\beta = 2$ with $\omega \in \Omega^\beta$, $\mathbb{F}^\beta \triangleq \left\{\mathcal{F}_t^\beta\right\}_{0 \le t \le \tau^\beta(\omega)}$ denoting the corresponding filtration satisfying the usual conditions, and $\mathbb{E}^\beta$ is used to denote the expectation operator with respect to (w. r. t.) the probability law $\mathbb{P}^\beta$ for $\beta = 1,2$.

We now define the canonical Lebesgue measure μ on measure space $\left(\mathbb{R}_+, \mathfrak{B}\left(\mathbb{R}_+\right)\right)$ with $\mathbb{R}_+ \triangleq \left[0,\infty\right)$, $\mathbb{R}_{++} \triangleq (0,\infty)$ and $\mathfrak{B}\left(\mathbb{R}_+\right)$ the Borel sigma-algebra, and also the corresponding regular properties about Lebesgue measure are supposed to be fulfilled. Thus, we can define the following product measure spaces $\left(\Omega^{i_\beta} \times \mathbb{R}_+, \mathbb{F}^{i_\beta} \otimes \mathfrak{B}\left(\mathbb{R}_+\right)\right)$ and $\left(\Omega^\beta \times \mathbb{R}_+, \mathbb{F}^\beta \otimes \mathfrak{B}\left(\mathbb{R}_+\right)\right)$ with corresponding product measures $\mu \otimes \mathbb{P}^{i_\beta}$ and $\mu \otimes \mathbb{P}^\beta$, respectively, for $\forall\ i_\beta = 1,2,...,I_\beta$ and for $\beta = 1,2$.

Now, based upon the probability space $\left(\Omega^{i_\beta}, \mathcal{F}^{i_\beta}, \mathbb{F}^{i_\beta}, \mathbb{P}^{i_\beta}\right)$ for $\beta = 1,2$, and following Fudenberg and Harris (1992), Cabrales (2000), Imhof (2005), Benaïm et al (2008), Hofbauer and Imhof (2009), the stochastic replicator dynamics of the two groups of populations can be respectively given as follows,

$$dM^{i_1}(t) = M^{i_1}(t)\left[\overline{e}_{i_1}^T AY(t)dt + \sum_{k_1=1}^{d_1} \overline{\sigma}_{i_1k_1}(t)dW_{k_1}^{i_1}(t) + \sum_{l_1=1}^{n_1} \int_{\mathbb{R}_0} \overline{\gamma}_{i_1l_1}\left(t, z_{l_1}^{i_1}\right)\tilde{N}_{l_1}^{i_1}\left(dt, dz_{l_1}^{i_1}\right)\right],$$

$$dN^{i_2}(t) = N^{i_2}(t)\left[\tilde{e}_{i_2}^T B^T X(t)dt + \sum_{k_2=1}^{d_2} \tilde{\sigma}_{i_2k_2}(t)dW_{k_2}^{i_2}(t) + \sum_{l_2=1}^{n_2} \int_{\mathbb{R}_0} \tilde{\gamma}_{i_2l_2}\left(t, z_{l_2}^{i_2}\right)\tilde{N}_{l_2}^{i_2}\left(dt, dz_{l_2}^{i_2}\right)\right]$$

where $M^{i_1}(t)$ is $\mathbb{F}^{i_1} \otimes \mathfrak{B}\left(\mathbb{R}_+\right)-$adapted, $N^{i_2}(t)$ is $\mathbb{F}^{i_2} \otimes \mathfrak{B}\left(\mathbb{R}_+\right)-$adapted, $Y(t)$ is $\mathbb{F}^2 \otimes \mathfrak{B}\left(\mathbb{R}_+\right)-$adapted, $X(t)$ is $\mathbb{F}^1 \otimes \mathfrak{B}\left(\mathbb{R}_+\right)-$adapted, $\overline{\sigma}_{i_1k_1}(t)$ and $\overline{\gamma}_{i_1l_1}\left(t, z_{l_1}^{i_1}\right)$ are

$\mathbb{F}^{i_1} \otimes \mathfrak{B}(\mathbb{R}_+)$ – progressively measurable, and $\tilde{\sigma}_{i_2 k_2}(t)$ and $\tilde{\gamma}_{i_2 l_2}\left(t, z_{l_2}^{i_2}\right)$ are $\mathbb{F}^{i_2} \otimes \mathfrak{B}(\mathbb{R}_+)$ – progressively measurable, for $\forall\ i_1 = 1,2,...,I_1$, $\forall\ i_2 = 1,2,...,I_2$, $\forall\ k_1 = 1,2,...,d_1$, $\forall\ k_2 = 1,2,...,d_2$, $\forall\ l_1 = 1,2,...,n_1$ and $\forall\ l_2 = 1,2,...,n_2$. Notice from Assumption 1 that the sizes of the two populations are finite constants, based on Itô's rule one can easily find,

$$\begin{aligned} dX^{i_1}(t) &= X^{i_1}(t)\left[\bar{e}_{i_1}^T AY(t)dt + \sum_{k_1=1}^{d_1} \bar{\sigma}_{i_1 k_1}(t) dW_{k_1}^{i_1}(t) + \sum_{l_1=1}^{n_1} \int_{\mathbb{R}_0} \bar{\gamma}_{i_1 l_1}\left(t, z_{l_1}^{i_1}\right) \tilde{N}_{l_1}^{i_1}\left(dt, dz_{l_1}^{i_1}\right)\right] \\ &\triangleq X^{i_1}(t)\left[\bar{e}_{i_1}^T AY(t)dt + \bar{\sigma}^{i_1}(t) dW^{i_1}(t) + \int_{\mathbb{R}_0^{n_1}} \bar{\gamma}^{i_1}\left(t, z^{i_1}\right) \tilde{N}^{i_1}\left(dt, dz^{i_1}\right)\right] \\ &\triangleq f^{i_1}\left(X^{i_1}(t)\right)dt + g^{i_1}\left(X^{i_1}(t)\right)dW^{i_1}(t) + \int_{\mathbb{R}_0^{n_1}} h^{i_1}\left(X^{i_1}(t), z^{i_1}\right) \tilde{N}^{i_1}\left(dt, dz^{i_1}\right), \end{aligned}$$

$$\begin{aligned} dY^{i_2}(t) &= Y^{i_2}(t)\left[\tilde{e}_{i_2}^T B^T X(t)dt + \sum_{k_2=1}^{d_2} \tilde{\sigma}_{i_2 k_2}(t) dW_{k_2}^{i_2}(t) + \sum_{l_2=1}^{n_2} \int_{\mathbb{R}_0} \tilde{\gamma}_{i_2 l_2}\left(t, z_{l_2}^{i_2}\right) \tilde{N}_{l_2}^{i_2}\left(dt, dz_{l_2}^{i_2}\right)\right] \\ &\triangleq Y^{i_2}(t)\left[\tilde{e}_{i_2}^T B^T X(t)dt + \tilde{\sigma}^{i_2}(t) dW^{i_2}(t) + \int_{\mathbb{R}_0^{n_2}} \tilde{\gamma}^{i_2}\left(t, z^{i_2}\right) \tilde{N}^{i_2}\left(dt, dz^{i_2}\right)\right] \\ &\triangleq f^{i_2}\left(Y^{i_2}(t)\right)dt + g^{i_2}\left(Y^{i_2}(t)\right)dW^{i_2}(t) + \int_{\mathbb{R}_0^{n_2}} h^{i_2}\left(Y^{i_2}(t), z^{i_2}\right) \tilde{N}^{i_2}\left(dt, dz^{i_2}\right), \quad (1) \end{aligned}$$

subject to $W^{i_1}(0) = (0,...,0)^T$ $\mathbb{P}^{i_1}$ – a.s., $W^{i_2}(0) = (0,...,0)^T$ $\mathbb{P}^{i_2}$ – a.s., $X(0) = \left(X^1(0), ..., X^{i_1}(0), ..., X^{I_1}(0)\right)^T \triangleq \left(x^1, ..., x^{i_1}, ..., x^{I_1}\right)^T \triangleq x > 0$ $\mathbb{P}^1$ – a.s., $Y(0) = \left(Y^1(0), ..., Y^{i_2}(0), ..., Y^{I_2}(0)\right)^T \triangleq \left(y^1, ..., y^{i_2}, ..., y^{I_2}\right)^T \triangleq y > 0$ $\mathbb{P}^2$ – a.s., $X^{i_1}(t)$ is assumed to be $\mathbb{F}^{i_1} \otimes \mathfrak{B}(\mathbb{R}_+)$ – adapted, and $Y^{i_2}(t)$ is assumed to be $\mathbb{F}^{i_2} \otimes \mathfrak{B}(\mathbb{R}_+)$ – adapted, for $\forall\ i_1 = 1,2,...,I_1$ and $\forall\ i_2 = 1,2,...,I_2$. Moreover, with a little abuse of notations, we put $\bar{\sigma}^{i_1}(0) = \left(\bar{\sigma}_{i_1 1}(0), ..., \bar{\sigma}_{i_1 k_1}(0), ..., \bar{\sigma}_{i_1 d_1}(0)\right)^T \triangleq \left(\bar{\sigma}_{i_1 1}, ..., \bar{\sigma}_{i_1 k_1}, ..., \bar{\sigma}_{i_1 d_1}\right)^T \triangleq \bar{\sigma}^{i_1}$, $\bar{\gamma}^{i_1}\left(0, z^{i_1}\right) = \left(\bar{\gamma}_{i_1 1}\left(0, z_1^{i_1}\right), ..., \bar{\gamma}_{i_1 l_1}\left(0, z_{l_1}^{i_1}\right), ..., \bar{\gamma}_{i_1 n_1}\left(0, z_{n_1}^{i_1}\right)\right)^T \triangleq \left(\bar{\gamma}_{i_1 1}\left(z_1^{i_1}\right), ..., \bar{\gamma}_{i_1 l_1}\left(z_{l_1}^{i_1}\right), ..., \bar{\gamma}_{i_1 n_1}\left(z_{n_1}^{i_1}\right)\right)^T \triangleq \bar{\gamma}^{i_1}\left(z^{i_1}\right)$, $\tilde{\sigma}^{i_2}(0) = \left(\tilde{\sigma}_{i_2 1}(0), ..., \tilde{\sigma}_{i_2 k_2}(0), ..., \tilde{\sigma}_{i_2 d_2}(0)\right)^T \triangleq \left(\tilde{\sigma}_{i_2 1}, ..., \tilde{\sigma}_{i_2 k_2}, ..., \tilde{\sigma}_{i_2 d_2}\right)^T \triangleq \tilde{\sigma}^{i_2}$, and $\tilde{\gamma}^{i_2}\left(0, z^{i_2}\right) = \left(\tilde{\gamma}_{i_2 1}\left(0, z_1^{i_2}\right), ..., \tilde{\gamma}_{i_2 l_2}\left(0, z_{l_2}^{i_2}\right), ..., \tilde{\gamma}_{i_2 n_2}\left(0, z_{n_2}^{i_2}\right)\right)^T \triangleq \left(\tilde{\gamma}_{i_2 1}\left(z_1^{i_2}\right), ..., \tilde{\gamma}_{i_2 l_2}\left(z_{l_2}^{i_2}\right), ..., \tilde{\gamma}_{i_2 n_2}\left(z_{n_2}^{i_2}\right)\right)^T \triangleq \tilde{\gamma}^{i_2}\left(z^{i_2}\right)$, for $\forall\ i_1 = 1,2,...,I_1$ and $\forall\ i_2 = 1,2,...,I_2$. Moreover, we have,

$$dX(t) = f^1\left(X(t)\right)dt + g^1\left(X(t)\right)dW^1(t) + \int_{\mathbb{R}_0^{I_1 n_1}} h^1\left(X(t), z^1\right)\tilde{N}^1\left(dt, dz^1\right),$$

$$dY(t) = f^2\left(Y(t)\right)dt + g^2\left(Y(t)\right)dW^2(t) + \int_{\mathbb{R}_0^{I_2 n_2}} h^2\left(Y(t), z^2\right)\tilde{N}^2\left(dt, dz^2\right), \tag{2}$$

with $X(t) \triangleq \left(X^1(t),..., X^{i_1}(t),..., X^{I_1}(t)\right)^T$ and $Y(t) \triangleq \left(Y^1(t),..., Y^{i_2}(t),..., Y^{I_2}(t)\right)^T$.

Next, we introduce some necessary assumptions,

ASSUMPTION 2: *The initial conditions* $X^{i_1}(0) = x^{i_1} > 0$, $Y^{i_2}(0) = y^{i_2} > 0$, $X(0) = x > 0$ *and* $Y(0) = y > 0$ *are all supposed to be deterministic and bounded for* $\forall\ i_1 = 1, 2, ..., I_1$ *and* $\forall\ i_2 = 1, 2, ..., I_2$. *Furthermore,* $\bar{\sigma}^{i_1} \neq 0$ $\mathbb{P}^{i_1} - a.s.$, $\tilde{\sigma}^{i_2} \neq 0$ $\mathbb{P}^{i_2} - a.s.$, $\bar{\gamma}_{i_1 l_1}\left(t, z_{l_1}^{i_1}\right) > -1 + \varepsilon_{l_1}^{i_1}$ $\mu \otimes \mathbb{P}^{i_1} - a.e.$, *and* $\tilde{\gamma}_{i_2 l_2}\left(t, z_{l_2}^{i_2}\right) > -1 + \varepsilon_{l_2}^{i_2}$ $\mu \otimes \mathbb{P}^{i_2} -$ *a.e., for* $\forall\ \varepsilon_{l_1}^{i_1} > 0$, $\varepsilon_{l_2}^{i_2} > 0$ *and for* $\forall\ i_1 = 1, 2, ..., I_1$; $i_2 = 1, 2, ..., I_2$; $l_1 = 1, 2, ..., n_1$ *and* $l_2 = 1, 2, ..., n_2$.

ASSUMPTION 3: *The following linear growth and local Lipschitz continuity conditions are fulfilled, respectively,*

$$\left|f^{i_1}\left(x^{i_1}\right)\right|^2 + \left\|g^{i_1}\left(x^{i_1}\right)\right\|_2^2 + \int_{\mathbb{R}_0} \sum_{l_1=1}^{n_1} \left\|h^{i_1}\left(x^{i_1}, z_{l_1}^{i_1}\right)\right\|_2^2 \nu_{l_1}^{i_1}\left(dz_{l_1}^{i_1}\right) \leq C^{i_1}\left(1 + \left|x^{i_1}\right|^2\right),$$

$$\left|f^{i_2}\left(y^{i_2}\right)\right|^2 + \left\|g^{i_2}\left(y^{i_2}\right)\right\|_2^2 + \int_{\mathbb{R}_0} \sum_{l_2=1}^{n_2} \left\|h^{i_2}\left(y^{i_2}, z_{l_2}^{i_2}\right)\right\|_2^2 \nu_{l_2}^{i_2}\left(dz_{l_2}^{i_2}\right) \leq C^{i_2}\left(1 + \left|y^{i_2}\right|^2\right),$$

$$\left\|f^1(x)\right\|_2^2 + \left\|g^1(x)\right\|_2^2 + \int_{\mathbb{R}_0} \sum_{l_1=1}^{I_1 n_1} \left\|h^1\left(x, z_{l_1}^1\right)\right\|_2^2 \nu_{l_1}^1\left(dz_{l_1}^1\right) \leq C^1\left(1 + \|x\|_2^2\right),$$

$$\left\|f^2(y)\right\|_2^2 + \left\|g^2(y)\right\|_2^2 + \int_{\mathbb{R}_0} \sum_{l_2=1}^{I_2 n_2} \left\|h^2\left(y, z_{l_2}^2\right)\right\|_2^2 \nu_{l_2}^2\left(dz_{l_2}^2\right) \leq C^2\left(1 + \|y\|_2^2\right),$$

for some constants C^{i_1}, C^{i_2}, C^1, $C^2 < \infty$. *And,*

$$\left|f^{i_1}\left(x^{i_1}\right) - f^{i_1}\left(\hat{x}^{i_1}\right)\right|^2 \vee \left\|g^{i_1}\left(x^{i_1}\right) - g^{i_1}\left(\hat{x}^{i_1}\right)\right\|_2^2 \vee$$

$$\int_{\mathbb{R}_0} \sum_{l_1=1}^{n_1} \left\|h^{i_1}\left(x^{i_1}, z_{l_1}^{i_1}\right) - h^{i_1}\left(\hat{x}^{i_1}, z_{l_1}^{i_1}\right)\right\|_2^2 \nu_{l_1}^{i_1}\left(dz_{l_1}^{i_1}\right) \leq L_{R^{i_1}}^{i_1}\left|x^{i_1} - \hat{x}^{i_1}\right|^2,$$

$$\left|f^{i_2}\left(y^{i_2}\right) - f^{i_2}\left(\hat{y}^{i_2}\right)\right|^2 \vee \left\|g^{i_2}\left(y^{i_2}\right) - g^{i_2}\left(\hat{y}^{i_2}\right)\right\|_2^2 \vee$$

$$\int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\|h^{i_2}\left(y^{i_2},z_{l_2}^{i_2}\right)-h^{i_2}\left(\hat{y}^{i_2},z_{l_2}^{i_2}\right)\right\|_2^2 \nu_{l_2}^{i_2}\left(dz_{l_2}^{i_2}\right)\leq L_{R^{i_2}}^{i_2}\left|y^{i_2}-\hat{y}^{i_2}\right|^2,$$

$$\left\|f^1(x)-f^1(\hat{x})\right\|_2^2 \vee \left\|g^1(x)-g^1(\hat{x})\right\|_2^2 \vee$$

$$\int_{\mathbb{R}_0}\sum_{l_1=1}^{I_1 n_1}\left\|h^{1(l_1)}\left(x,z^1\right)-h^{1(l_1)}\left(\hat{x},z^1\right)\right\|_2^2 \nu_{l_1}^1\left(dz_{l_1}^1\right)\leq L_{R^1}^1\left\|x-\hat{x}\right\|_2^2,$$

$$\left\|f^2(y)-f^2(\hat{y})\right\|_2^2 \vee \left\|g^2(y)-g^2(\hat{y})\right\|_2^2 \vee$$

$$\int_{\mathbb{R}_0}\sum_{l_2=1}^{I_2 n_2}\left\|h^{2(l_2)}\left(y,z^2\right)-h^{2(l_2)}\left(\hat{y},z^2\right)\right\|_2^2 \nu_{l_2}^2\left(dz_{l_2}^2\right)\leq L_{R^2}^2\left\|y-\hat{y}\right\|_2^2,$$

for any given constants R^{i_1}, R^{i_2}, R^1, $R^2>0$ *with* $\left|x^{i_1}\right|\vee\left|\hat{x}^{i_1}\right|\leq R^{i_1}$, $\left|y^{i_2}\right|\vee\left|\hat{y}^{i_2}\right|\leq R^{i_2}$, $\|x\|_2\vee\|\hat{x}\|_2\leq R^1$, $\|y\|_2\vee\|\hat{y}\|_2\leq R^2$, *and constants* $L_{R^{i_1}}^{i_1}$, $L_{R^{i_2}}^{i_2}$, $L_{R^1}^1$, $L_{R^2}^2<\infty$ *that depend only on* R^{i_1}, R^{i_2}, R^1 *and* R^2, *respectively, for all* x^{i_1}, $\hat{x}^{i_1}$, y^{i_2}, $\hat{y}^{i_2}\in\mathbb{R}_{++}$; x, $\hat{x}\in\mathbb{R}_{++}^{I_1}$ *and* y, $\hat{y}\in\mathbb{R}_{++}^{I_2}$ *with* $h^{1(l_1)}\left(x,z^1\right)$, $h^{1(l_1)}\left(\hat{x},z^1\right)$ *representing the* l_1*-th columns of matrixes* $h^1\left(x,z^1\right)$, $h^1\left(\hat{x},z^1\right)$, *respectively, and* $h^{2(l_2)}\left(y,z^2\right)$, $h^{2(l_2)}\left(\hat{y},z^2\right)$ *denoting the* l_2*-th columns of matrixes* $h^2\left(y,z^2\right)$, $h^2\left(\hat{y},z^2\right)$, *respectively, for* $\forall\ l_1=1,2,\ldots,n_1$, $l_2=1,2,\ldots,n_2$, *and also* $\forall\ i_1=1,2,\ldots,I_1$, $i_2=1,2,\ldots,I_2$.

REMARK 2.2: (i) Provided Assumption 3, the existence and uniqueness of strong solutions of the Lévy SDEs given in (1) and (2) are ensured, respectively.

(ii) Assumption 3 is indeed weak in the following sense, local Lipschitz continuity conditions can be naturally satisfied for any C^1 functions or correspondences thanks to the Mean Value Theorem.

(iii) Here, and throughout the current paper, $|\cdot|$ is used to denote absolute value, $\|\cdot\|_2$ is used to represent both Euclidean vector norm and the Frobenius (or trace) matrix norm, and $<\cdot>$ is used to denote the scalar product.

Now, as in the model of Fudenberg et al (2004), Nowak et al (2004), and Imhof and Nowak (2006), we define the following expected discounted fitness functions,

$$\bar{f}_{i_1}\left(t, Y(t)\right) \triangleq \mathbb{E}^2_{(s,y)}\left[\exp\left(-\bar{\theta}^{i_1} t\right)\left\{1-\bar{w}^{i_1}+\bar{w}^{i_1}\left(\bar{e}^T_{i_1} AY(t)\right)\right\}\right],$$

$$\tilde{f}_{i_2}\left(t, X(t)\right) \triangleq \mathbb{E}^1_{(s,x)}\left[\exp\left(-\tilde{\theta}^{i_2} t\right)\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left(\tilde{e}^T_{i_2} B^T X(t)\right)\right\}\right].$$

with $\bar{\theta}^{i_1}$, $\tilde{\theta}^{i_2} \in [0,1]$ ($\forall\ i_1 = 1, 2, ..., I_1$; $i_2 = 1, 2, ..., I_2$) denoting the discounted factors, $\bar{w}^{i_1}$, $\tilde{w}^{i_2} \in [0,1]$ ($\forall\ i_1 = 1, 2, ..., I_1$; $i_2 = 1, 2, ..., I_2$) the parameters that measure the contributions of the matrix payoffs of the game to the fitness of the corresponding strategies, and $\mathbb{E}^2_{(s,y)}$, $\mathbb{E}^1_{(s,x)}$ representing the expectation operators w. r. t. the complete probability law $\mathbb{P}^2$, $\mathbb{P}^1$ with depending on initial conditions (s, y) $\in \mathbb{R}_+ \times [0,1]^{I_2}$ and $(s, x) \in \mathbb{R}_+ \times [0,1]^{I_1}$, respectively. Thus, the problem facing us is,

PROBLEM 1: We need to demonstrate that there exist two vectors of $\mathcal{F}^1-$ stopping times $\bar{\tau}^*(\omega) \triangleq \left(\bar{\tau}^{1*}(\omega), ..., \bar{\tau}^{i_1*}(\omega), ..., \bar{\tau}^{I_1*}(\omega)\right)^T$ with $\omega \in \Omega^1$ and $\mathcal{F}^2-$ stopping times $\tilde{\tau}^*(\omega) \triangleq \left(\tilde{\tau}^{1*}(\omega), ..., \tilde{\tau}^{i_2*}(\omega), ..., \tilde{\tau}^{I_2*}(\omega)\right)^T$ with $\omega \in \Omega^2$ such that,

$$\bar{f}^*_{i_1}\left(\bar{\tau}^{i_1*}(\omega), Y\left(\bar{\tau}^{i_1*}(\omega)\right)\right)$$

$$\triangleq \sup_{\bar{\tau}^{i_1}(\omega)\leq\infty} \mathbb{E}^1\left[\mathbb{E}^2_{(s,y)}\left[\exp\left(-\bar{\theta}^{i_1}\bar{\tau}^{i_1}(\omega)\right)\left\{1-\bar{w}^{i_1}+\bar{w}^{i_1}\left(\bar{e}^T_{i_1} AY\left(\bar{\tau}^{i_1}(\omega)\right)\right)\right\}\right]\right],$$

$$\triangleq \mathbb{E}^1\left[\mathbb{E}^2_{(s,y)}\left[\exp\left(-\bar{\theta}^{i_1}\bar{\tau}^{i_1*}(\omega)\right)\left\{1-\bar{w}^{i_1}+\bar{w}^{i_1}\left(\bar{e}^T_{i_1} AY\left(\bar{\tau}^{i_1*}(\omega)\right)\right)\right\}\right]\right].$$

And simultaneously,

$$\tilde{f}^*_{i_2}\left(\tilde{\tau}^{i_2*}(\omega), X\left(\tilde{\tau}^{i_2*}(\omega)\right)\right)$$

$$\triangleq \sup_{\tilde{\tau}^{i_2}(\omega)\leq\infty} \mathbb{E}^2\left[\mathbb{E}^1_{(s,x)}\left[\exp\left(-\tilde{\theta}^{i_2}\tilde{\tau}^{i_2}(\omega)\right)\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left(\tilde{e}^T_{i_2} B^T X\left(\tilde{\tau}^{i_2}(\omega)\right)\right)\right\}\right]\right],$$

$$\triangleq \mathbb{E}^2\left[\mathbb{E}^1_{(s,x)}\left[\exp\left(-\tilde{\theta}^{i_2}\tilde{\tau}^{i_2*}(\omega)\right)\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left(\tilde{e}^T_{i_2} B^T X\left(\tilde{\tau}^{i_2*}(\omega)\right)\right)\right\}\right]\right].$$

with $\bar{\tau}^{i_1*}(\omega) = \bar{\tau}^{k_1*}(\omega)$ ($\forall\ i_1 \neq k_1, i_1, k_1 = 1, 2, ..., I_1$) $\mathbb{P}^1-$a.s. and $\tilde{\tau}^{i_2*}(\omega) = \tilde{\tau}^{k_2*}(\omega)$ ($\forall\ i_2 \neq k_2, i_2, k_2 = 1, 2, ..., I_2$) $\mathbb{P}^2-$a.s..

REMARK 2.3: Indeed, $\bar{w}^{i_1}$, $\tilde{w}^{i_2} \in [0,1]$ ($\forall\ i_1 = 1, 2, ..., I_1$; $i_2 = 1, 2, ..., I_2$) can be regarded as objective parameters that measure the intensity of evolutionary selection (see, Ohtsuki et al., 2007), and also, this specification reflects the idea that, in reality, individuals or players inclined to use different strategies may feel different importance of the game payoff to their fitness, thereby determining different degrees of participation which in turn will greatly affect the strategy choice of the players.

DEFINITION 1 (Game Equilibrium): *The solution, if it exists, to Problem* 1 *defines a* game equilibrium *induced by stochastic group evolution and rational individual choice corresponding to the very general normal form game situations*.

REMARK 2.4: Here, and throughout the current paper, we study the game equilibrium by employing evolutionary game theory under uncertainty, which implies that the game equilibrium is characterized from the viewpoint of group level, thereby leading to a case where classical optimal control theory is not suitable for rational individual choice while stochastic optimal stopping theory is powerful and hence plays a crucial role in proving and characterizing the existence and uniqueness of the game equilibrium. Nevertheless, on the other hand, it is specifically worth emphasizing that the game equilibrium is achieved through decentralized rational individual choice of many players in the corresponding game although the game equilibrium is characterized in the sense of group level based upon the classical evolutionary game theory.

We now define $\tilde{Z}(t) \triangleq \left(s+t, X(t)\right)$ for $\forall\ t \in \mathbb{R}_+$ with $\tilde{Z}(0) \triangleq (s,x) \in \mathbb{R}_+ \times \left[0,1\right]^{I_1}$, and $\bar{Z}(t) \triangleq \left(s+t, Y(t)\right)$ for $\forall\ t \in \mathbb{R}_+$ with $\bar{Z}(0) \triangleq (s,y) \in \mathbb{R}_+ \times \left[0,1\right]^{I_2}$. And also we let $\nabla\tilde{f}(s,x) \triangleq \left(\frac{\partial\tilde{f}}{\partial x^1}(s,x),\ldots,\frac{\partial\tilde{f}}{\partial x^{i_1}}(s,x),\ldots,\frac{\partial\tilde{f}}{\partial x^{I_1}}(s,x)\right)^T$, $\bar{\gamma}_{l_1}(x) \triangleq \left(x^1\bar{\gamma}_{1l_1}\left(z_{l_1}^1\right),\ldots,x^{i_1}\bar{\gamma}_{i_1l_1}\left(z_{l_1}^{i_1}\right),\ldots,x^{I_1}\bar{\gamma}_{I_1l_1}\left(z_{l_1}^{I_1}\right)\right)^T$, $\nabla\bar{f}(s,y) \triangleq \left(\frac{\partial\bar{f}}{\partial y^1}(s,y),\ldots,\frac{\partial\bar{f}}{\partial y^{i_2}}(s,y),\ldots,\frac{\partial\bar{f}}{\partial y^{I_2}}(s,y)\right)^T$ and $\tilde{\gamma}_{l_2}(y) \triangleq \left(y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right),\ldots,y^{i_2}\tilde{\gamma}_{i_2l_2}\left(z_{l_2}^{i_2}\right),\ldots,y^{I_2}\tilde{\gamma}_{I_2l_2}\left(z_{l_2}^{I_2}\right)\right)^T$. Then the characteristic operators of $\tilde{Z}(t)$ and $\bar{Z}(t)$ can be respectively given by,

$$\mathcal{A}\tilde{f}(s,x) = \frac{\partial\tilde{f}}{\partial s}(s,x) + \sum_{i_1=1}^{I_1} x^{i_1}\left(\bar{e}_{i_1}^T Ay\right)\frac{\partial\tilde{f}}{\partial x^{i_1}}(s,x)$$

$$
+\frac{1}{2}\sum_{i_1=1}^{I_1}\left(x^{i_1}\right)^2\left(\bar{\sigma}^{i_1}\right)^T\bar{\sigma}^{i_1}\frac{\partial^2\tilde{f}}{\partial(x^{i_1})^2}(s,x)
$$

$$
+\sum_{k_1=1}^{I_1}\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\tilde{f}\left(s,x+\bar{\gamma}_{l_1}(x)\right)-\tilde{f}(s,x)-\left\langle\nabla\tilde{f}(s,x),\bar{\gamma}_{l_1}(x)\right\rangle\right\}\nu_{l_1}^{k_1}\left(dz_{l_1}^{k_1}\right),
$$

$$
\forall\ \tilde{f}\in C^2\left(\mathbb{R}^{I_1+1}\right).
$$

And,

$$
\mathcal{A}\bar{f}(s,y)=\frac{\partial\bar{f}}{\partial s}(s,y)+\sum_{i_2=1}^{I_2}y^{i_2}\left(\tilde{e}_{i_2}^T B^T x\right)\frac{\partial\bar{f}}{\partial y^{i_2}}(s,y)
$$

$$
+\frac{1}{2}\sum_{i_2=1}^{I_2}\left(y^{i_2}\right)^2\left(\tilde{\sigma}^{i_2}\right)^T\tilde{\sigma}^{i_2}\frac{\partial^2\bar{f}}{\partial(y^{i_2})^2}(s,y)
$$

$$
+\sum_{k_2=1}^{I_2}\int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\{\bar{f}\left(s,y+\tilde{\gamma}_{l_2}(y)\right)-\bar{f}(s,y)-\left\langle\nabla\bar{f}(s,y),\tilde{\gamma}_{l_2}(y)\right\rangle\right\}\nu_{l_2}^{k_2}\left(dz_{l_2}^{k_2}\right),
$$

$$
\forall\ \bar{f}\in C^2\left(\mathbb{R}^{I_2+1}\right).
$$

Furthermore, we let $\sum_{i_1=1}^{I_1-1}x^{i_1}=\bar{\delta}_1$, then $x^{I_1}=1-\bar{\delta}_1$ with $0\le\bar{\delta}_1\le 1$ by noting that $\sum_{i_1=1}^{I_1}x^{i_1}=1$. Let $\sum_{i_1=1}^{I_1-2}x^{i_1}=\bar{\delta}_2$, then we get $x^{I_1-1}=\bar{\delta}_1-\bar{\delta}_2$ with $0\le\bar{\delta}_2\le\bar{\delta}_1\le 1$. Inductively, let $\sum_{i_1=1}^{I_1-(I_1-2)}x^{i_1}=\bar{\delta}_{I_1-2}$, then we have $x^3=x^{I_1-(I_1-3)}=\bar{\delta}_{I_1-3}-\bar{\delta}_{I_1-2}$ with $0\le\bar{\delta}_{I_1-2}\le\bar{\delta}_{I_1-3}\le\ldots\le\bar{\delta}_1\le 1$; let $\sum_{i_1=1}^{I_1-(I_1-1)}x^{i_1}=\bar{\delta}_{I_1-1}$, i.e., $x^1=\bar{\delta}_{I_1-1}$, then we get $x^2=x^{I_1-(I_1-2)}=\bar{\delta}_{I_1-2}-\bar{\delta}_{I_1-1}$ with $0\le\bar{\delta}_{I_1-1}\le\bar{\delta}_{I_1-2}\le\bar{\delta}_{I_1-3}\le\ldots\le\bar{\delta}_1\le 1$. And without loss of any generality, we put $\bar{\delta}_0\equiv 1$. Then we obtain,

$$
u\left(i_2,x\right)=\tilde{e}_{i_2}^T B^T x=\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\bar{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1}-\bar{\delta}_{I_1-i_1+1}\right).
$$

Similarly, notice that $\sum_{i_2=1}^{I_2}y^{i_2}\equiv 1$ and let $\sum_{i_2=1}^{I_2-1}y^{i_2}=\tilde{\delta}_1$, then we have $y^{I_2}=1-\tilde{\delta}_1$ with $0\le\tilde{\delta}_1\le 1$. Let $\sum_{i_2=1}^{I_2-2}y^{i_2}=\tilde{\delta}_2$, then we see that $y^{I_2-1}=\tilde{\delta}_1-\tilde{\delta}_2$ with $0\le\tilde{\delta}_2\le\tilde{\delta}_1\le 1$. Inductively, let $\sum_{i_2=1}^{I_2-(I_2-2)}y^{i_2}=\tilde{\delta}_{I_2-2}$, then we have $y^3=y^{I_2-(I_2-3)}=\tilde{\delta}_{I_2-3}-\tilde{\delta}_{I_2-2}$ with $0\le\tilde{\delta}_{I_2-2}\le\tilde{\delta}_{I_2-3}\le\ldots\le\tilde{\delta}_1\le 1$; let $\sum_{i_2=1}^{I_2-(I_2-1)}y^{i_2}=\tilde{\delta}_{I_2-1}$, i.e., $y^1=\tilde{\delta}_{I_2-1}$, then it follows that $y^2 = y^{I_2-(I_2-2)} = \tilde{\delta}_{I_2-2} - \tilde{\delta}_{I_2-1}$ with $0 \le \tilde{\delta}_{I_2-1} \le \tilde{\delta}_{I_2-2} \le \tilde{\delta}_{I_2-3} \le \ldots \le \tilde{\delta}_1 \le 1$. And we, without loss of any generality, put $\tilde{\delta}_0 \equiv 1$. Then we get,

$$u\left(i_1, y\right) = \overline{e}_{i_1}^T A y = \left(a_{i_1 1} - a_{i_1 2}\right) y^1 + a_{i_1 2}\tilde{\delta}_{I_2-2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2} - \tilde{\delta}_{I_2-i_2+1}\right).$$

Therefore, the fitness functions become,

$$\overline{f}_{i_1}\left(s, y^1\right) = \exp\left(-\overline{\theta}^{i_1} s\right)$$

$$\times\left\{1 - \overline{w}^{i_1} + \overline{w}^{i_1}\left[\left(a_{i_1 1} - a_{i_1 2}\right) y^1 + a_{i_1 2}\tilde{\delta}_{I_2-2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2} - \tilde{\delta}_{I_2-i_2+1}\right)\right]\right\},$$

$$\forall\ i_1 = 1, 2, \ldots, I_1.$$

$$\tilde{f}_{i_2}\left(s, x^1\right) = \exp\left(-\tilde{\theta}^{i_2} s\right)$$

$$\times\left\{1 - \tilde{w}^{i_2} + \tilde{w}^{i_2}\left[\left(b_{i_2 1} - b_{i_2 2}\right) x^1 + b_{i_2 2}\overline{\delta}_{I_1-2} + \sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1} - \overline{\delta}_{I_1-i_1+1}\right)\right]\right\},$$

$$\forall\ i_2 = 1, 2, \ldots, I_2. \quad (3)$$

with $0 \le \tilde{\delta}_{I_2-1} \le \tilde{\delta}_{I_2-2} \le \tilde{\delta}_{I_2-3} \le \ldots \le \tilde{\delta}_1 \le \tilde{\delta}_0 \equiv 1$ and $0 \le \overline{\delta}_{I_1-1} \le \overline{\delta}_{I_1-2} \le \overline{\delta}_{I_1-3} \le \ldots \le \overline{\delta}_1 \le \overline{\delta}_0 \equiv 1$. And inspection of the fitness functions given in (3) reveals that one can just define $\tilde{Z}(t) \triangleq \left(s+t, X^1(t)\right)$ for $\forall\ t \in \mathbb{R}_+$ with $\tilde{Z}(0) \triangleq \left(s, x^1\right) \in \mathbb{R}_+ \times [0,1]$, and $\overline{Z}(t) \triangleq \left(s+t, Y^1(t)\right)$ for $\forall\ t \in \mathbb{R}_+$ with $\overline{Z}(0) \triangleq \left(s, y^1\right) \in \mathbb{R}_+ \times [0,1]$. And hence the corresponding characteristic operators of $\tilde{Z}(t)$ and $\overline{Z}(t)$ are respectively given by,

$$\mathcal{A}\tilde{f}\left(s, x^1\right) = \frac{\partial \tilde{f}}{\partial s}\left(s, x^1\right) + x^1\left(\overline{e}_1^T A y\right)\frac{\partial \tilde{f}}{\partial x^1}\left(s, x^1\right) + \frac{1}{2}\left(x^1\right)^2\left(\overline{\sigma}^1\right)^T \overline{\sigma}^1 \frac{\partial^2 \tilde{f}}{\partial (x^1)^2}\left(s, x^1\right)$$

$$+\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\tilde{f}\left(s, x^1 + x^1\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\right) - \tilde{f}\left(s, x^1\right) - x^1\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\tfrac{\partial \tilde{f}}{\partial x^1}\left(s, x^1\right)\right\}\nu_{l_1}^1\left(dz_{l_1}^1\right),$$

$$\forall\ \tilde{f} \in C^2\left(\mathbb{R}^2\right).$$

And,

$$\mathcal{A}\overline{f}\left(s, y^1\right) = \frac{\partial \overline{f}}{\partial s}\left(s, y^1\right) + y^1\left(\tilde{e}_1^T B^T x\right)\frac{\partial \overline{f}}{\partial y^1}\left(s, y^1\right) + \frac{1}{2}\left(y^1\right)^2\left(\tilde{\sigma}^1\right)^T \tilde{\sigma}^1 \frac{\partial^2 \overline{f}}{\partial (y^1)^2}\left(s, y^1\right)$$

$$+\int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\{\overline{f}\left(s,y^1+y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\right)-\overline{f}\left(s,y^1\right)-y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\tfrac{\partial\overline{f}}{\partial y^1}\left(s,y^1\right)\right\}\nu_{l_2}^1\left(dz_{l_2}^1\right)$$

$$\forall\ \overline{f}\in C^2\left(\mathbb{R}^2\right).$$

Therefore, based upon the above assumptions and specifications, the following theorem is derived,

THEOREM 1: *There exists a unique solution to Problem* 1 *under very weak conditions, and accordingly the existence and uniqueness of the game equilibrium are confirmed*.

PROOF: See Appendix A. ▌

REMARK 2.5: It is especially worth noting that Theorem 1 not only shows the existence and uniqueness of the game equilibrium given by Definition 1 but also provides us with the explicit time length needed so that the game equilibrium can be achieved by decentralized players. And hence this would be regarded as one major characteristic of the corresponding game equilibrium relative to the traditional approach and hence Nash equilibrium concept (see, Nash, 1950, 1951).

3. Stability of the Game Equilibrium

By solving Problem 1 defined in the previous section, we get the optimal stopping times as follows,

$$\overline{\tau}^{i_1 *}(\omega)\triangleq\inf\left\{t\geq 0;Y^1(t)=y_{i_1}^{1*}\right\},\ \ \forall\ i_1=1,2,\ldots,I_1. \tag{4}$$

with $y_{i_1}^{1*}$ determined by (A.5). And,

$$\tilde{\tau}^{i_2 *}(\omega)\triangleq\inf\left\{t\geq 0;X^1(t)=x_{i_2}^{1*}\right\},\ \ \forall\ i_2=1,2,\ldots,I_2. \tag{5}$$

where $x_{i_2}^{1*}$ is given by (A.13). Moreover, it follows from Theorem 1 that the new game equilibrium is characterized via letting $y_1^{1*}=y_2^{1*}=\ldots=y_{i_1}^{1*}=\ldots=y_{I_1}^{1*}$, $x_1^{1*}=x_2^{1*}$ $=\ldots=x_{i_2}^{1*}=\ldots=x_{I_2}^{1*}$, and meanwhile $\tilde{\delta}=\overline{\Sigma}^{+}\overline{\Gamma}$ in (A.15) and $\overline{\delta}=\tilde{\Sigma}^{+}\tilde{\Gamma}$ in (A.16).That is to say, we get from Theorem 1 that $y^{2*}=\tilde{\delta}_{I_2-2}-y^{1*}$, $y^{3*}=\tilde{\delta}_{I_2-3}-\tilde{\delta}_{I_2-2}$, $\ldots$, $y^{I_2 *}=$

$1-\tilde{\delta}_1$ and $x^{2*}=\bar{\delta}_{I_1-2}-x^{1*}$, $x^{3*}=\bar{\delta}_{I_1-3}-\bar{\delta}_{I_1-2}$, … , $x^{I_1*}=1-\bar{\delta}_1$ with $y^{1*}\equiv y_1^{1*}=y_2^{1*}$ $=\ldots=y_{i_1}^{1*}=\ldots=y_{I_1}^{1*}$ and $x^{1*}\equiv x_1^{1*}=x_2^{1*}=\ldots=x_{i_2}^{1*}=\ldots=x_{I_2}^{1*}$. In what follows, we are encouraged to show the stability of the game equilibrium. And we do so by first giving some necessary definitions and assumptions,

DEFINITION 2 (Simplex of Evolutionary Dynamics): *Here, and throughout the present paper, we put* $\bar{\Delta}\triangleq\left\{y\in(0,1)^{I_2}; y^1+y^2+\ldots+y^{i_2}+\ldots+y^{I_2}=1\right\}$ *and* $\tilde{\Delta}\triangleq\left\{x\in(0,1)^{I_1}; x^1+x^2+\ldots+x^{i_1}+\ldots+x^{I_1}=1\right\}$ *as the* simplexes of evolutionary dynamics. *If we set up* $y^*\triangleq\left(y^{1*},\ldots,y^{i_2*},\ldots,y^{I_2*}\right)^T$ *and* $x^*\triangleq\left(x^{1*},\ldots,x^{i_1*},\ldots,x^{I_1*}\right)^T$, *then we always get* $y^*\in cl\bar{\Delta}$ *and* $x^*\in cl\tilde{\Delta}$ *with* $cl\bar{\Delta}$ *and* $cl\tilde{\Delta}$ *denoting the closures of* $\bar{\Delta}$ *and* $\tilde{\Delta}$, *respectively.*

ASSUMPTION 4: *We, without loss of any generality, assume that* $y^*\in\bar{\Delta}$, $x^*\in\tilde{\Delta}$, *there exists a unique invariant probability measure* $\bar{\pi}$ *on* $\bar{\Delta}$, *and there exists a unique invariant probability measure* $\tilde{\pi}$ *on* $\tilde{\Delta}$.

REMARK 3.1: It is especially worth noting that we employ Assumption 4 is just for the sake of simplicity. Indeed, the existence and uniqueness of the probability measure can be ensured under certain weak conditions (see, Garay and Hofbauer, 2003), and one can also refer to Theorem 2.1 of Imhof (2005) and Theorem 3.1 of Benaïm et al (2008) for much more details.

ASSUMPTION 5: *There exist constants* $\tilde{L}<\infty$ *and* $\bar{L}<\infty$ *such that,*

$$\left\langle x-x^*,f^1(x)\right\rangle\vee\left\|g^1(x)\right\|_2^2\vee\sum_{l_1=1}^{I_1n_1}\int_{\mathbb{R}_0}\left\|h^{1(l_1)}\left(x,z^1\right)\right\|_2^2\nu_{l_1}^1\left(dz_{l_1}^1\right)\vee\left\|h^{1(l_1^*)}\left(x,z^1\right)\right\|_2^2\le\tilde{L}\|x\|_2^2,$$

$$\left\langle y-y^*,f^2(y)\right\rangle\vee\left\|g^2(y)\right\|_2^2\vee\sum_{l_2=1}^{I_2n_2}\int_{\mathbb{R}_0}\left\|h^{2(l_2)}\left(y,z^2\right)\right\|_2^2\nu_{l_2}^2\left(dz_{l_2}^2\right)\vee\left\|h^{2(l_2^*)}\left(y,z^2\right)\right\|_2^2\le\bar{L}\|y\|_2^2$$

where $h^{1(l_1^*)}\left(x,z^1\right)\triangleq\max_{l_1\in\{1,2,\ldots,I_1n_1\}}h^{1(l_1)}\left(x,z^1\right)$ *and* $h^{2(l_2^*)}\left(y,z^2\right)\triangleq\max_{l_2\in\{1,2,\ldots,I_2n_2\}}h^{2(l_2)}\left(y,z^2\right)$ *for* $\forall\ x\in\tilde{\Delta}$, $\forall\ y\in\bar{\Delta}$, *and* x^* *and* y^* *are given by Definition* 2.

Now, the following lemma can be derived, LEMMA 1: *Based upon the above definitions and assumptions, then there exist two constants* $\tilde{\zeta}\left(p,\tilde{\tau}^*(\omega),x,x^*\right)<\infty$ *and* $\bar{\zeta}\left(p,\bar{\tau}^*(\omega),y,y^*\right)<\infty$ *with* $\tilde{\tau}^*(\omega)\equiv\tilde{\tau}^{1*}(\omega)=\ldots=\tilde{\tau}^{i_2*}(\omega)=\ldots=\tilde{\tau}^{I_2*}(\omega)$ *and* $\bar{\tau}^*(\omega)\equiv\bar{\tau}^{1*}(\omega)=\ldots=\bar{\tau}^{i_1*}(\omega)=\ldots=\bar{\tau}^{I_1*}(\omega)$ *such that,*

$$\mathbb{E}^1\left[\sup_{0\le t\le\tilde{\tau}^*(\omega)}\left\|X(t)-x^*\right\|_2^p\right]\le\tilde{\zeta}\left(p,\tilde{\tau}^*(\omega),x,x^*\right).$$

And,

$$\mathbb{E}^2\left[\sup_{0\le t\le\bar{\tau}^*(\omega)}\left\|Y(t)-y^*\right\|_2^p\right]\le\bar{\zeta}\left(p,\bar{\tau}^*(\omega),y,y^*\right).$$

for $\forall\ p\in\mathbb{N}$ *and* $p\ge 2$.

PROOF: See Appendix B. ▌

Then, the stability theorem of the game equilibrium can be established and expressed as follows,

THEOREM 2 (Stability of the Game Equilibrium): *Based upon Lemma* 1 *and the above assumptions, then there exist constants* $\tilde{\psi}<\infty$ *and* $\bar{\psi}<\infty$ *such that,*

(a.1) $\mathbb{E}^1\left[\tilde{\tau}_{\bar{\tilde{B}}_{\tilde{\alpha}}(x^*)}(\omega)\right]\le\dfrac{dist\left(x,x^*\right)}{\tilde{\alpha}^p-\tilde{\psi}}$,

(a.2) $\tilde{\pi}\left[\bar{\tilde{B}}_{\tilde{\alpha}}\left(x^*\right)\right]\ge 1-\dfrac{\tilde{\psi}}{\tilde{\alpha}^p}\triangleq 1-\tilde{\varepsilon}$,

And,

(b.1) $\mathbb{E}^2\left[\bar{\tau}_{\bar{\bar{B}}_{\bar{\alpha}}(y^*)}(\omega)\right]\le\dfrac{dist\left(y,y^*\right)}{\bar{\alpha}^p-\bar{\psi}}$,

(b.2) $\bar{\pi}\left[\bar{\bar{B}}_{\bar{\alpha}}\left(y^*\right)\right]\ge 1-\dfrac{\bar{\psi}}{\bar{\alpha}^p}\triangleq 1-\bar{\varepsilon}$,

where,

$$\tilde{B}_{\tilde{\alpha}}\left(x^*\right)\triangleq\left\{X(t)\in cl\tilde{\Delta};\left\|X(t)-x^*\right\|_2<\tilde{\alpha},t\ge 0\right\},$$

$$\tilde{\tau}_{\bar{\tilde{B}}_{\tilde{\alpha}}(x^*)}(\omega)\triangleq\inf\left\{t\ge 0;X(t,\omega)\in\bar{\tilde{B}}_{\tilde{\alpha}}\left(x^*\right)\triangleq cl\tilde{B}_{\tilde{\alpha}}\left(x^*\right)\right\},$$

$$\bar{B}_{\bar{\alpha}}\left(y^*\right)\triangleq\left\{Y(t)\in cl\bar{\Delta};\left\|Y(t)-y^*\right\|_2<\bar{\alpha},t\ge 0\right\},$$

$$\overline{\tau}_{\overline{\overline{B}}_{\overline{\alpha}}(y^*)}(\omega) \triangleq \inf\left\{t \geq 0; Y(t,\omega) \in \overline{\overline{B}}_{\overline{\alpha}}\left(y^*\right) \triangleq cl\overline{B}_{\overline{\alpha}}\left(y^*\right)\right\},$$

and also $dist\left(x,x^*\right) \triangleq \sum_{i_1=1}^{I_1} x^{i_1*}\log\left(\frac{x^{i_1*}}{x^{i_1}}\right)$, $dist\left(y,y^*\right) \triangleq \sum_{i_2=1}^{I_2} y^{i_2*}\log\left(\frac{y^{i_2*}}{y^{i_2}}\right)$ *denote the Kullback-Leibler distances between* $x \triangleq X(0)$ *and* x^*, *and* $y \triangleq Y(0)$ *and* y^*, *respectively, with* $\tilde{\psi} < \tilde{\alpha}^p$, $\overline{\psi} < \overline{\alpha}^p$ *and for* $\forall\ \tilde{\alpha} > 0$, $\overline{\alpha} > 0$, $\forall\ p \in \mathbb{N}$ *and* $p \geq 2$.

PROOF: See Appendix C. ▌

REMARK 3.2: Theorem 2 brings the idea from Theorem 2.1 of Imhof (2005) and also one can refer to the similar technique employed by Dai (2012). It follows from Theorem 1 and Definition 2 that x^* and y^* equivalently characterize the game equilibrium given by Definition 1, hence Theorem 2 confirms the stability of the game equilibrium from both time dimension (i.e., stochastic stopping time) and space dimension (i.e., invariant probability measure).

4. Concluding Remarks

In this paper, we have studied a new approach to equilibrium selection for very general normal form games. The basic economic intuition behind the approach is very simple, i.e., classical evolutionary game theory emphasizes the deterministic or stochastic evolution of the populations as a group or many groups while rational individual choice theory also has very important economic implications, that is, the present paper chooses the way that reasonably combines both group-level stochastic evolution and individual-level rational choice. By noting that optimal control theory (i.e., dynamic optimization theory) is not suitable for the present case, we do so by introducing optimal stopping theory into classical evolutionary game theory. And hence the existence and uniqueness of the new game equilibrium have been demonstrated. Moreover, the stability of the new game equilibrium is confirmed from both time and space dimensions.

Noting that our approach provides us with a general framework, the corresponding applications will be very rich, e.g., inducing cooperative equilibrium in PD games and Pareto optimal equilibrium in coordination games. Finally, our approach can be easily extended to include multiple priors (see, Riedel, 2009, for instance) and also study evolutionary dynamics and corresponding equilibria in complex networks (see, Pacheco et al., 2006, for example) and on graphs (see, Ohtsuki and Nowak, 2006; Ohtsuki et al., 2007; Ohtsuki and Nowak, 2008, and among others).

APPENDIX

A. Proof of Theorem 1.

STEP 1: For strategy i_1, $\forall\ i_1 = 1,2,...,I_1$. Notice that,

$$\begin{aligned}
\mathcal{A}\overline{f}_{i_1}\left(s, y^1\right) &= -\overline{\theta}^{i_1} \exp\left(-\overline{\theta}^{i_1} s\right) \\
&\quad \times \left\{1 - \overline{w}^{i_1} + \overline{w}^{i_1}\left[\left(a_{i_1 1} - a_{i_1 2}\right) y^1 + a_{i_1 2}\tilde{\delta}_{I_2 - 2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2 - i_2} - \tilde{\delta}_{I_2 - i_2 + 1}\right)\right]\right\} \\
&\quad + y^1\left(\tilde{e}_1^T B^T x\right)\exp\left(-\overline{\theta}^{i_1} s\right)\overline{w}^{i_1}\left(a_{i_1 1} - a_{i_1 2}\right) \geq 0 \\
&\Leftrightarrow \left(\tilde{e}_1^T B^T x - \overline{\theta}^{i_1}\right)\overline{w}^{i_1}\left(a_{i_1 1} - a_{i_1 2}\right) y^1 \\
&\quad \geq \overline{\theta}^{i_1}\left(1 - \overline{w}^{i_1}\right) + \overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2 - 2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2 - i_2} - \tilde{\delta}_{I_2 - i_2 + 1}\right)\right].
\end{aligned}$$

Case 1.1: $\begin{cases} \overline{\theta}^{i_1}\left(1 - \overline{w}^{i_1}\right) + \overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2 - 2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2 - i_2} - \tilde{\delta}_{I_2 - i_2 + 1}\right)\right] < 0 \\ \operatorname{sgn}\left(\tilde{e}_1^T B^T x - \overline{\theta}^{i_1}\right) = \operatorname{sgn}\left(a_{i_1 2} - a_{i_1 1}\right) \end{cases}$

Then,

$$\begin{aligned}
&\mathcal{A}\overline{f}_{i_1}\left(s, y^1\right) \geq 0 \\
&\Leftrightarrow y^1 \leq \frac{\overline{\theta}^{i_1}\left(1 - \overline{w}^{i_1}\right) + \overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2 - 2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2 - i_2} - \tilde{\delta}_{I_2 - i_2 + 1}\right)\right]}{\left(\tilde{e}_1^T B^T x - \overline{\theta}^{i_1}\right)\overline{w}^{i_1}\left(a_{i_1 1} - a_{i_1 2}\right)}.
\end{aligned}$$

Hence, we have,

$$U^{i_1} = \left\{ \left(s, y^1\right); y^1 \le \frac{\overline{\theta}^{i_1}\left(1-\overline{w}^{i_1}\right)+\overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)\right]}{\left(\tilde{e}_1^T B^T x-\overline{\theta}^{i_1}\right)\overline{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)} \right\}. \quad \text{(A.1)}$$

And it is natural to guess that the continuation region D^{i_1} has the following form,

$$D^{i_1}\left(y_{i_1}^{1*}\right) = \left\{\left(s, y^1\right); 0 \le y^1 \le y_{i_1}^{1*}\right\}.$$

where,

$$y_{i_1}^{1*} \ge \frac{\overline{\theta}^{i_1}\left(1-\overline{w}^{i_1}\right)+\overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)\right]}{\left(\tilde{e}_1^T B^T x-\overline{\theta}^{i_1}\right)\overline{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)}. \quad \text{(A.2)}$$

Notice that the generator of $\overline{Z}(t)$ is given by,

$$\begin{aligned}\mathcal{A}\overline{\phi}_{i_1}\left(s, y^1\right) &= \frac{\partial \overline{\phi}_{i_1}}{\partial s} + y^1\left(\tilde{e}_1^T B^T x\right)\frac{\partial \overline{\phi}_{i_1}}{\partial y^1} + \frac{1}{2}\left(y^1\right)^2\left(\tilde{\sigma}^1\right)^T\tilde{\sigma}^1\frac{\partial^2 \overline{\phi}_{i_1}}{\partial (y^1)^2} \\ &\quad + \int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\{\overline{\phi}_{i_1}\left(s, y^1+y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\right)-\overline{\phi}_{i_1}\left(s, y^1\right)-y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\tfrac{\partial \overline{\phi}_{i_1}}{\partial y^1}\left(s, y^1\right)\right\}\nu_{l_2}^1\left(dz_{l_2}^1\right)\end{aligned}$$

for $\forall \overline{\phi}_{i_1}\left(s, y^1\right) \in C^2\left(\mathbb{R}^2\right)$. If we try a function $\overline{\phi}_{i_1}$ of the following form,

$$\overline{\phi}_{i_1}\left(s, y^1\right) = \exp\left(-\overline{\theta}^{i_1}s\right)\left(y^1\right)^{\overline{\lambda}^{i_1}} \quad \text{for some constant } \overline{\lambda}^{i_1} \in \mathbb{R}.$$

We then get,

$$\begin{aligned}\mathcal{A}\overline{\phi}_{i_1}\left(s, y^1\right) &= \exp\left(-\overline{\theta}^{i_1}s\right)\Big[-\overline{\theta}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}} + \left(\tilde{e}_1^T B^T x\right)y^1\overline{\lambda}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}-1} \\ &\quad + \frac{1}{2}\left(\tilde{\sigma}^1\right)^T\tilde{\sigma}^1\left(y^1\right)^2\overline{\lambda}^{i_1}\left(\overline{\lambda}^{i_1}-1\right)\left(y^1\right)^{\overline{\lambda}^{i_1}-2} \\ &\quad + \int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\{\left[y^1+y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\right]^{\overline{\lambda}^{i_1}} - \left(y^1\right)^{\overline{\lambda}^{i_1}} - \tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)y^1\overline{\lambda}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}-1}\right\}\nu_{l_2}^1\left(dz_{l_2}^1\right)\Big] \\ &= \exp\left(-\overline{\theta}^{i_1}s\right)\left(y^1\right)^{\overline{\lambda}^{i_1}}\overline{h}_{i_1}\left(\overline{\lambda}^{i_1}\right).\end{aligned}$$

where,

$$\begin{aligned}\overline{h}_{i_1}\left(\overline{\lambda}^{i_1}\right) &\triangleq -\overline{\theta}^{i_1} + \left(\tilde{e}_1^T B^T x\right)\overline{\lambda}^{i_1} + \frac{1}{2}\left(\tilde{\sigma}^1\right)^T\tilde{\sigma}^1\overline{\lambda}^{i_1}\left(\overline{\lambda}^{i_1}-1\right) \\ &\quad + \int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\{\left[1+\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\right]^{\overline{\lambda}^{i_1}} - 1 - \tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\overline{\lambda}^{i_1}\right\}\nu_{l_2}^1\left(dz_{l_2}^1\right).\end{aligned}$$

Note that,

$$\overline{h}_{i_1}(1) = \tilde{e}_1^T B^T x - \overline{\theta}^{i_1} \quad \text{and} \quad \lim_{\overline{\lambda}^{i_1} \to \infty} \overline{h}_{i_1}\left(\overline{\lambda}^{i_1}\right) = \infty .$$

Therefore, if we assume that,

$$\tilde{e}_1^T B^T x < \overline{\theta}^{i_1}, \tag{A.3}$$

Then we find that there exists $\overline{\lambda}^{i_1} > 1$ such that,

$$\overline{h}_{i_1}\left(\overline{\lambda}^{i_1}\right) = 0 . \tag{A.4}$$

with this value of $\overline{\lambda}^{i_1}$ we put,

$$\overline{\phi}_{i_1}\left(s, y^1\right) = \begin{cases} \exp\left(-\overline{\theta}^{i_1} s\right) \overline{C}^{i_1} \left(y^1\right)^{\overline{\lambda}^{i_1}}, 0 \le y^1 \le y_{i_1}^{1*} \\ \exp\left(-\overline{\theta}^{i_1} s\right) \left\{ 1 - \overline{w}^{i_1} + \overline{w}^{i_1} \left[\left(a_{i_1 1} - a_{i_1 2}\right) y^1 + a_{i_1 2} \tilde{\delta}_{I_2 - 2} + \sum_{i_2 = 3}^{I_2} a_{i_1 i_2} \left(\tilde{\delta}_{I_2 - i_2} - \tilde{\delta}_{I_2 - i_2 + 1} \right) \right] \right\}, y_{i_1}^{1*} \le y^1 \le 1 \end{cases}$$

for some constant $\overline{C}^{i_1} > 0$, to be determined. We, without loss of any generality, guess that the value function is C^1 at $y^1 = y_{i_1}^{1*}$ and this leads us to the following "high contact" conditions,

$$\overline{C}^{i_1} \left(y_{i_1}^{1*}\right)^{\overline{\lambda}^{i_1}} = 1 - \overline{w}^{i_1} + \overline{w}^{i_1} \left[\left(a_{i_1 1} - a_{i_1 2}\right) y_{i_1}^{1*} + a_{i_1 2} \tilde{\delta}_{I_2 - 2} + \sum_{i_2 = 3}^{I_2} a_{i_1 i_2} \left(\tilde{\delta}_{I_2 - i_2} - \tilde{\delta}_{I_2 - i_2 + 1} \right) \right]$$

(continuity at $y^1 = y_{i_1}^{1*}$)

$$\overline{C}^{i_1} \overline{\lambda}^{i_1} \left(y_{i_1}^{1*}\right)^{\overline{\lambda}^{i_1} - 1} = \overline{w}^{i_1} \left(a_{i_1 1} - a_{i_1 2}\right) \quad \text{(differentiability at } y^1 = y_{i_1}^{1*}\text{)}$$

Combining the above equations shows that,

$$\frac{\overline{C}^{i_1} \left(y_{i_1}^{1*}\right)^{\overline{\lambda}^{i_1}}}{\overline{C}^{i_1} \overline{\lambda}^{i_1} \left(y_{i_1}^{1*}\right)^{\overline{\lambda}^{i_1} - 1}} = \frac{1 - \overline{w}^{i_1} + \overline{w}^{i_1} \left[\left(a_{i_1 1} - a_{i_1 2}\right) y_{i_1}^{1*} + a_{i_1 2} \tilde{\delta}_{I_2 - 2} + \sum_{i_2 = 3}^{I_2} a_{i_1 i_2} \left(\tilde{\delta}_{I_2 - i_2} - \tilde{\delta}_{I_2 - i_2 + 1} \right) \right]}{\overline{w}^{i_1} \left(a_{i_1 1} - a_{i_1 2}\right)}$$

$$\Leftrightarrow y_{i_1}^{1*} = \frac{\overline{\lambda}^{i_1} \left\{ 1 - \overline{w}^{i_1} + \overline{w}^{i_1} \left[a_{i_1 2} \tilde{\delta}_{I_2 - 2} + \sum_{i_2 = 3}^{I_2} a_{i_1 i_2} \left(\tilde{\delta}_{I_2 - i_2} - \tilde{\delta}_{I_2 - i_2 + 1} \right) \right] \right\}}{\left(1 - \overline{\lambda}^{i_1}\right) \overline{w}^{i_1} \left(a_{i_1 1} - a_{i_1 2}\right)} . \tag{A.5}$$

And this gives,

$$\overline{C}^{i_1} = \frac{\overline{w}^{i_1} \left(a_{i_1 1} - a_{i_1 2}\right)}{\overline{\lambda}^{i_1} \left(y_{i_1}^{1*}\right)^{\overline{\lambda}^{i_1} - 1}} . \tag{A.6}$$

Hence, by (A.4), (A.5) and (A.6), we can define,

$$\overline{\overline{f}}_{i_1}\left(s, y^1\right) \triangleq \exp\left(-\overline{\theta}^{i_1} s\right) \overline{C}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}}.$$

And then we are in the position to prove that,

$$\overline{\overline{f}}_{i_1}\left(s, y^1\right) \triangleq \exp\left(-\overline{\theta}^{i_1} s\right) \overline{C}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}} = \overline{f}_{i_1}^{*}\left(s, y^1\right).$$

in which $\overline{f}_{i_1}^{*}\left(s, y^1\right)$ is a supermeanvalued majorant of $\overline{f}_{i_1}\left(s, y^1\right)$. Firstly, noting that,

$$\begin{aligned}
\mathcal{A}\overline{f}_{i_1}\left(s, y^1\right) &= -\overline{\theta}^{i_1} \exp\left(-\overline{\theta}^{i_1} s\right) \\
&\quad \times \left\{1 - \overline{w}^{i_1} + \overline{w}^{i_1}\left[\left(a_{i_1 1} - a_{i_1 2}\right) y^1 + a_{i_1 2}\tilde{\delta}_{I_2-2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2} - \tilde{\delta}_{I_2-i_2+1}\right)\right]\right\} \\
&\quad + y^1\left(\tilde{e}_1^T B^T x\right)\exp\left(-\overline{\theta}^{i_1} s\right)\overline{w}^{i_1}\left(a_{i_1 1} - a_{i_1 2}\right) \le 0, \quad \forall y^1 \ge y_{i_1}^{1*}.
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \left(\tilde{e}_1^T B^T x - \overline{\theta}^{i_1}\right)\overline{w}^{i_1}\left(a_{i_1 1} - a_{i_1 2}\right) y^1 \\
&\quad \le \overline{\theta}^{i_1}\left(1 - \overline{w}^{i_1}\right) + \overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2} - \tilde{\delta}_{I_2-i_2+1}\right)\right], \quad \forall y^1 \ge y_{i_1}^{1*}.
\end{aligned}$$

$$\Leftrightarrow \; y_{i_1}^{1*} \ge \frac{\overline{\theta}^{i_1}\left(1 - \overline{w}^{i_1}\right) + \overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2} - \tilde{\delta}_{I_2-i_2+1}\right)\right]}{\left(\tilde{e}_1^T B^T x - \overline{\theta}^{i_1}\right)\overline{w}^{i_1}\left(a_{i_1 1} - a_{i_1 2}\right)}.$$

which holds by (A.2). Secondly, to prove,

$$\overline{C}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}} \ge 1 - \overline{w}^{i_1} + \overline{w}^{i_1}\left[\left(a_{i_1 1} - a_{i_1 2}\right) y^1 + a_{i_1 2}\tilde{\delta}_{I_2-2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2} - \tilde{\delta}_{I_2-i_2+1}\right)\right],$$

$$\text{for } \forall 0 \le y^1 \le y_{i_1}^{1*}.$$

Define

$$\overline{\xi}^{i_1}\left(y^1\right) \triangleq \overline{C}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}} - 1 + \overline{w}^{i_1} - \overline{w}^{i_1}\left[\left(a_{i_1 1} - a_{i_1 2}\right) y^1 + a_{i_1 2}\tilde{\delta}_{I_2-2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2} - \tilde{\delta}_{I_2-i_2+1}\right)\right].$$

Then with our chosen values of $\overline{C}^{i_1}$ and $\overline{\lambda}^{i_1}$, we see that $\overline{\xi}^{i_1}\left(y_{i_1}^{1*}\right) = \overline{\xi}^{i_1\prime}\left(y_{i_1}^{1*}\right) = 0$. Furthermore, noting that $\overline{\xi}^{i_1\prime\prime}\left(y^1\right) = \overline{C}^{i_1}\overline{\lambda}^{i_1}\left(\overline{\lambda}^{i_1} - 1\right)\left(y^1\right)^{\overline{\lambda}^{i_1}-2}$, and hence $\overline{\xi}^{i_1\prime\prime}\left(y^1\right) > 0$ holds for $\forall 0 \le y^1 \le y_{i_1}^{1*}$ given $\overline{\lambda}^{i_1} > 1$ in (A.4), that is, $\overline{\xi}^{i_1}\left(y^1\right) > 0$ follows for $\forall\ 0 \le y^1 \le y_{i_1}^{1*}$. And this completes the short proof.

Case 1.2: $\begin{cases} \bar{\theta}^{i_1}\left(1-\bar{w}^{i_1}\right)+\bar{\theta}^{i_1}\bar{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)\right]>0 \\ \operatorname{sgn}\left(\tilde{e}_1^T B^T x-\bar{\theta}^{i_1}\right)=\operatorname{sgn}\left(a_{i_1 1}-a_{i_1 2}\right) \end{cases}$

It is easy to see that the proof is quite similar to that of case 1.1, so we take it omitted.

STEP 2: For strategy i_2, $\forall\ i_2=1,2,...,I_2$. Notice that,

$$\begin{aligned}
\mathcal{A}\tilde{f}_{i_2}\left(s,x^1\right) &= -\tilde{\theta}^{i_2}\exp\left(-\tilde{\theta}^{i_2}s\right) \\
&\times\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\bar{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1}-\bar{\delta}_{I_1-i_1+1}\right)\right]\right\} \\
&+x^1\left(\bar{e}_1^T Ay\right)\exp\left(-\tilde{\theta}^{i_2}s\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)\geq 0 \\
\Leftrightarrow &\left(\bar{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)x^1 \\
&\geq \tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\bar{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1}-\bar{\delta}_{I_1-i_1+1}\right)\right].
\end{aligned}$$

Case 2.1: $\begin{cases} \tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\bar{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1}-\bar{\delta}_{I_1-i_1+1}\right)\right]<0 \\ \operatorname{sgn}\left(\bar{e}_1^T Ay-\tilde{\theta}^{i_2}\right)=\operatorname{sgn}\left(b_{i_2 2}-b_{i_2 1}\right) \end{cases}$

Hence,

$$\begin{aligned}
&\mathcal{A}\tilde{f}_{i_2}\left(s,x^1\right)\geq 0 \\
&\Leftrightarrow x^1\leq\frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\bar{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1}-\bar{\delta}_{I_1-i_1+1}\right)\right]}{\left(\bar{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}.
\end{aligned}$$

Then, we have,

$$U^{i_2}=\left\{\left(s,x^1\right);x^1\leq\frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\bar{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1}-\bar{\delta}_{I_1-i_1+1}\right)\right]}{\left(\bar{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}\right\}. \tag{A.7}$$

So it is natural to guess that the continuation region D^{i_2} has the following form,

$$D^{i_2}\left(x_{i_2}^{1*}\right)=\left\{\left(s,x^1\right);0\le x^1\le x_{i_2}^{1*}\right\}.$$

where,

$$x_{i_2}^{1*}\ge\frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)\right]}{\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}. \tag{A.8}$$

Notice that the generator of $\tilde{Z}(t)$ is given by,

$$\mathcal{A}\tilde{\phi}_{i_2}\left(s,x^1\right)=\frac{\partial\tilde{\phi}_{i_2}}{\partial s}+x^1\left(\overline{e}_1^T Ay\right)\frac{\partial\tilde{\phi}_{i_2}}{\partial x^1}+\frac{1}{2}\left(x^1\right)^2\left(\overline{\sigma}^1\right)^T\overline{\sigma}^1\frac{\partial^2\tilde{\phi}_{i_2}}{\partial(x^1)^2}$$

$$+\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\tilde{\phi}_{i_2}\left(s,x^1+x^1\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\right)-\tilde{\phi}_{i_2}\left(s,x^1\right)-x^1\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\tfrac{\partial\tilde{\phi}_{i_2}}{\partial x^1}\left(s,x^1\right)\right\}\nu_{l_1}^1\left(dz_{l_1}^1\right)$$

for $\forall\tilde{\phi}_{i_2}\left(s,x^1\right)\in C^2\left(\mathbb{R}^2\right)$. If we choose $\tilde{\phi}_{i_2}\left(s,x^1\right)=\exp\left(-\tilde{\theta}^{i_2}s\right)\left(x^1\right)^{\tilde{\lambda}^{i_2}}$ for some constant $\tilde{\lambda}^{i_2}\in\mathbb{R}$. Then we get,

$$\mathcal{A}\tilde{\phi}_{i_2}\left(s,x^1\right)=\exp\left(-\tilde{\theta}^{i_2}s\right)\Big[-\tilde{\theta}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}}+\left(\overline{e}_1^T Ay\right)x^1\tilde{\lambda}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}-1}$$

$$+\frac{1}{2}\left(\overline{\sigma}^1\right)^T\overline{\sigma}^1\left(x^1\right)^2\tilde{\lambda}^{i_2}\left(\tilde{\lambda}^{i_2}-1\right)\left(x^1\right)^{\tilde{\lambda}^{i_2}-2}$$

$$+\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\left[x^1+x^1\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\right]^{\tilde{\lambda}^{i_2}}-\left(x^1\right)^{\tilde{\lambda}^{i_2}}-\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)x^1\tilde{\lambda}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}-1}\right\}\nu_{l_1}^1\left(dz_{l_1}^1\right)\Big]$$

$$=\exp\left(-\tilde{\theta}^{i_2}s\right)\left(x^1\right)^{\tilde{\lambda}^{i_2}}\tilde{h}_{i_2}\left(\tilde{\lambda}^{i_2}\right).$$

where,

$$\tilde{h}_{i_2}\left(\tilde{\lambda}^{i_2}\right)\triangleq-\tilde{\theta}^{i_2}+\left(\overline{e}_1^T Ay\right)\tilde{\lambda}^{i_2}+\frac{1}{2}\left(\overline{\sigma}^1\right)^T\overline{\sigma}^1\tilde{\lambda}^{i_2}\left(\tilde{\lambda}^{i_2}-1\right)$$

$$+\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\left[1+\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\right]^{\tilde{\lambda}^{i_2}}-1-\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\tilde{\lambda}^{i_2}\right\}\nu_{l_1}^1\left(dz_{l_1}^1\right).$$

Noting that,

$$\tilde{h}_{i_2}(1)=\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\quad\text{and}\quad\lim_{\tilde{\lambda}^{i_2}\to\infty}\tilde{h}_{i_2}\left(\tilde{\lambda}^{i_2}\right)=\infty.$$

Consequently, if we suppose that,

$$\overline{e}_1^T Ay<\tilde{\theta}^{i_2}, \tag{A.9}$$

Thus, it is easily seen that there exists $\tilde{\lambda}^{i_2}>1$ such that,

$$\tilde{h}_{i_2}\left(\tilde{\lambda}^{i_2}\right)=0. \tag{A.10}$$

with this value of $\tilde{\lambda}^{i_2}$ we put,

$$\tilde{\phi}_{i_2}\left(s,x^1\right)=\begin{cases}\exp\left(-\tilde{\theta}^{i_2}s\right)\tilde{C}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}}, 0\le x^1\le x_{i_2}^{1*}\\ \exp\left(-\tilde{\theta}^{i_2}s\right)\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)\right]\right\}, x_{i_2}^{1*}\le x^1\le 1\end{cases}$$

in which $\tilde{C}^{i_2}>0$ is some constant that remains to be determined. If we require that $\tilde{\phi}_{i_2}$ is continuous at $x^1=x_{i_2}^{1*}$ we get the following equation,

$$\tilde{C}^{i_2}\left(x_{i_2}^{1*}\right)^{\tilde{\lambda}^{i_2}}=1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x_{i_2}^{1*}+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)\right], \tag{A.11}$$

If we require that $\tilde{\phi}_{i_2}$ is differentiable at $x^1=x_{i_2}^{1*}$ we get the additional equation,

$$\tilde{C}^{i_2}\tilde{\lambda}^{i_2}\left(x_{i_2}^{1*}\right)^{\tilde{\lambda}^{i_2}-1}=\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right). \tag{A.12}$$

So, combining equation (A.11) and equation (A.12) yields,

$$\frac{\tilde{C}^{i_2}\left(x_{i_2}^{1*}\right)^{\tilde{\lambda}^{i_2}}}{\tilde{C}^{i_2}\tilde{\lambda}^{i_2}\left(x_{i_2}^{1*}\right)^{\tilde{\lambda}^{i_2}-1}}=\frac{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x_{i_2}^{1*}+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)\right]}{\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}$$

$$\Leftrightarrow x_{i_2}^{1*}=\frac{\tilde{\lambda}^{i_2}\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)\right]\right\}}{\left(1-\tilde{\lambda}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}. \tag{A.13}$$

And this produces,

$$\tilde{C}^{i_2}=\frac{\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}{\tilde{\lambda}^{i_2}\left(x_{i_2}^{1*}\right)^{\tilde{\lambda}^{i_2}-1}}. \tag{A.14}$$

Then, by applying equation (A.10), equation (A.13) and equation (A.14), we are in the position to prove that $\tilde{f}_{i_2}^{*}\left(s,x^1\right)=\exp\left(-\tilde{\theta}^{i_2}s\right)\tilde{C}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}}$ is a supermeanvalued majorant of $\tilde{f}_{i_2}\left(s,x^1\right)$. Firstly, noting that,

$$\begin{aligned}\mathcal{A}\tilde{f}_{i_2}\left(s,x^1\right)&=-\tilde{\theta}^{i_2}\exp\left(-\tilde{\theta}^{i_2}s\right)\\&\times\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)\right]\right\}\end{aligned}$$

$$+x^1\left(\overline{e}_1^T Ay\right)\exp\left(-\tilde{\theta}^{i_2}s\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)\le 0,\ \ \forall x^1\ge x_{i_2}^{1*}$$

$$\Leftrightarrow\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)x^1$$

$$\le\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)\right],\ \ \forall x^1\ge x_{i_2}^{1*}$$

$$\Leftrightarrow x^1\ge\frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)\right]}{\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)},$$

$$\text{for}\ \ \forall x^1\ge x_{i_2}^{1*}$$

$$\Leftrightarrow x_{i_2}^{1*}\ge\frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)\right]}{\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}$$

which holds by (A.8). Secondly, to show that,

$$\tilde{C}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}}\ge 1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)\right],$$

$$\text{for}\ \ \forall 0\le x^1\le x_{i_2}^{1*}.$$

Define

$$\tilde{\xi}^{i_2}\left(x^1\right)\triangleq\tilde{C}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}}-1+\tilde{w}^{i_2}-\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)\right].$$

Then with our chosen values of $\tilde{C}^{i_2}$ and $\tilde{\lambda}^{i_2}$, we see that $\tilde{\xi}^{i_2}\left(x_{i_2}^{1*}\right)=\tilde{\xi}^{i_2\prime}\left(x_{i_2}^{1*}\right)=0$. Furthermore, noting that $\tilde{\xi}^{i_2\prime\prime}\left(x^1\right)=\tilde{C}^{i_2}\tilde{\lambda}^{i_2}\left(\tilde{\lambda}^{i_2}-1\right)\left(x^1\right)^{\tilde{\lambda}^{i_2}-2}$, and hence $\tilde{\xi}^{i_2\prime\prime}\left(x^1\right)>0$ holds for $\forall 0\le x^1\le x_{i_2}^{1*}$ given $\tilde{\lambda}^{i_2}>1$ in (A.10), that is, $\tilde{\xi}^{i_2}\left(x^1\right)>0$ follows for $\forall 0\le x^1\le x_{i_2}^{1*}$. And hence the desired result is established.

Case 2.2: $\begin{cases}\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)\right]>0\\ \operatorname{sgn}\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)=\operatorname{sgn}\left(b_{i_2 1}-b_{i_2 2}\right)\end{cases}$

Similar to case 1.2 and we take the proof of case 2.2, which is quite similar to that of case 2.1, omitted.

STEP 3: The existence and uniqueness of the game equilibrium.

It follows from the requirements of Problem 1 that $y_1^{1*} = y_2^{1*} = \ldots = y_{i_1}^{1*} = \ldots = y_{I_1}^{1*}$ with $y_{i_1}^{1*}$ defined in (A.5). Let $y_{i_1}^{1*} = y_{k_1}^{1*}$ ($\forall i_1 \neq k_1, i_1, k_1 = 1,2,\ldots,I_1$), then one can easily see that,

$$\overline{\Sigma}_{i_1k_1,23}\tilde{\delta}_{I_2-2} + \overline{\Sigma}_{i_1k_1,34}\tilde{\delta}_{I_2-3} + \ldots + \overline{\Sigma}_{i_1k_1,I_2-1,I_2}\tilde{\delta}_1 = \overline{\Gamma}_{i_1k_1}.$$

where,

$$\overline{\Sigma}_{i_1k_1,j_2,j_2+1} \triangleq \frac{\overline{\lambda}^{i_1}\left(a_{i_1j_2} - a_{i_1,j_2+1}\right)}{\left(1-\overline{\lambda}^{i_1}\right)\left(a_{i_11} - a_{i_12}\right)} - \frac{\overline{\lambda}^{k_1}\left(a_{k_1j_2} - a_{k_1,j_2+1}\right)}{\left(1-\overline{\lambda}^{k_1}\right)\left(a_{k_11} - a_{k_12}\right)}.$$

$$\overline{\Gamma}_{i_1k_1} \triangleq \frac{\overline{\lambda}^{k_1}\left[\left(1-\overline{w}^{k_1}\right) + \overline{w}^{k_1}a_{k_1I_2}\right]}{\left(1-\overline{\lambda}^{k_1}\right)\left(a_{k_11} - a_{k_12}\right)\overline{w}^{k_1}} - \frac{\overline{\lambda}^{i_1}\left[\left(1-\overline{w}^{i_1}\right) + \overline{w}^{i_1}a_{i_1I_2}\right]}{\left(1-\overline{\lambda}^{i_1}\right)\left(a_{i_11} - a_{i_12}\right)\overline{w}^{i_1}}.$$

$$\forall i_1 \neq k_1,\ i_1, k_1 = 1,2,\ldots,I_1;\ j_2 = 2,3,\ldots,I_2-1.$$

Accordingly, we have,

$$\begin{bmatrix} \overline{\Sigma}_{12,23} & \overline{\Sigma}_{12,34} & \cdots & \overline{\Sigma}_{12,I_2-1,I_2} \\ \overline{\Sigma}_{23,23} & \overline{\Sigma}_{23,34} & \cdots & \overline{\Sigma}_{23,I_2-1,I_2} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\Sigma}_{I_1-1,I_1,23} & \overline{\Sigma}_{I_1-1,I_1,34} & \cdots & \overline{\Sigma}_{I_1-1,I_1,I_2-1,I_2} \end{bmatrix}_{(I_1-1)\times(I_2-2)} \begin{bmatrix} \tilde{\delta}_{I_2-2} \\ \tilde{\delta}_{I_2-3} \\ \vdots \\ \tilde{\delta}_1 \end{bmatrix}_{(I_2-2)\times 1} = \begin{bmatrix} \overline{\Gamma}_{12} \\ \overline{\Gamma}_{23} \\ \vdots \\ \overline{\Gamma}_{I_1-1,I_1} \end{bmatrix}_{(I_1-1)\times 1}.$$

which implies that,

$$\tilde{\delta} = \overline{\Sigma}^{+}\overline{\Gamma}. \tag{A.15}$$

where "$+$" denotes Moore-Penrose generalized inverse.

Similarly, we obtain $x_1^{1*} = x_2^{1*} = \ldots = x_{i_2}^{1*} = \ldots = x_{I_2}^{1*}$ with $x_{i_2}^{1*}$ defined in (A.13) according to Problem 1. Now, let $x_{i_2}^{1*} = x_{k_2}^{1*}$ ($\forall i_2 \neq k_2, i_2, k_2 = 1,2,\ldots,I_2$), then we get,

$$\tilde{\Sigma}_{i_2k_2,23}\overline{\delta}_{I_1-2} + \tilde{\Sigma}_{i_2k_2,34}\overline{\delta}_{I_1-3} + \ldots + \tilde{\Sigma}_{i_2k_2,I_1-1,I_1}\overline{\delta}_1 = \tilde{\Gamma}_{i_2k_2}.$$

where,

$$\tilde{\Sigma}_{i_2k_2,j_1,j_1+1} \triangleq \frac{\tilde{\lambda}^{i_2}\left(b_{i_2j_1} - b_{i_2,j_1+1}\right)}{\left(1-\tilde{\lambda}^{i_2}\right)\left(b_{i_21} - b_{i_22}\right)} - \frac{\tilde{\lambda}^{k_2}\left(b_{k_2j_1} - b_{k_2,j_1+1}\right)}{\left(1-\tilde{\lambda}^{k_2}\right)\left(b_{k_21} - b_{k_22}\right)}.$$

$$\tilde{\Gamma}_{i_2k_2} \triangleq \frac{\tilde{\lambda}^{k_2}\left[\left(1-\tilde{w}^{k_2}\right)+\tilde{w}^{k_2}b_{k_2I_1}\right]}{\left(1-\tilde{\lambda}^{k_2}\right)\left(b_{k_21}-b_{k_22}\right)\tilde{w}^{k_2}}-\frac{\tilde{\lambda}^{i_2}\left[\left(1-\tilde{w}^{i_2}\right)+\tilde{w}^{i_2}b_{i_2I_1}\right]}{\left(1-\tilde{\lambda}^{i_2}\right)\left(b_{i_21}-b_{i_22}\right)\tilde{w}^{i_2}}.$$

$$\forall i_2 \neq k_2,\ \ i_2,k_2=1,2,...,I_2;\ \ j_1=2,3,...,I_1-1.$$

Consequently, we obtain,

$$\begin{bmatrix} \tilde{\Sigma}_{12,23} & \tilde{\Sigma}_{12,34} & \cdots & \tilde{\Sigma}_{12,I_1-1,I_1} \\ \tilde{\Sigma}_{23,23} & \tilde{\Sigma}_{23,34} & \cdots & \tilde{\Sigma}_{23,I_1-1,I_1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\Sigma}_{I_2-1,I_2,23} & \tilde{\Sigma}_{I_2-1,I_2,34} & \cdots & \tilde{\Sigma}_{I_2-1,I_2,I_1-1,I_1} \end{bmatrix}_{(I_2-1)\times(I_1-2)} \begin{bmatrix} \overline{\delta}_{I_1-2} \\ \overline{\delta}_{I_1-3} \\ \vdots \\ \overline{\delta}_1 \end{bmatrix}_{(I_1-2)\times 1} = \begin{bmatrix} \tilde{\Gamma}_{12} \\ \tilde{\Gamma}_{23} \\ \vdots \\ \tilde{\Gamma}_{I_2-1,I_2} \end{bmatrix}_{(I_2-1)\times 1}.$$

which leads us to the following equation,

$$\overline{\delta}=\tilde{\Sigma}^{+}\tilde{\Gamma}. \tag{A.16}$$

where " $+$ " stands for the Moore-Penrose generalized inverse.

To summarize, we get the following theorem,

THEOREM 1': *If we are provided that* $\tilde{e}_1^T B^T x < \overline{\theta}^{i_1}$ *in* (A.3) *and* $\overline{e}_1^T A y < \tilde{\theta}^{i_2}$ *in* (A.9)*, then Problem* 1 *is solved as long as* $\tilde{\delta}=\overline{\Sigma}^{+}\overline{\Gamma}$ *in* (A.15) *and* $\overline{\delta}=\tilde{\Sigma}^{+}\tilde{\Gamma}$ *in* (A.16)*. That is, the existence and uniqueness of the game equilibrium are confirmed.*

Therefore, Theorem 1 is established thanks to Theorem 1'. ▌

B. Proof of Lemma 1.

By Itô's rule, we obtain from (2) that,

$$\begin{aligned}\left\|X(t)-x^*\right\|_2^2 &= \left\|X(0)-x^*\right\|_2^2 + 2\int_0^t \left\langle X(s)-x^*, f^1\left(X(s)\right)\right\rangle ds \\ &+2\int_0^t \left\langle X(s)-x^*, g^1\left(X(s)\right)dW^1(s)\right\rangle + \int_0^t \left\|g^1\left(X(s)\right)\right\|_2^2 ds \\ &+\sum_{l_1=1}^{I_1n_1}\int_0^t\int_{\mathbb{R}_0}\left\|h^{1(l_1)}\left(X(s),z^1\right)\right\|_2^2 \nu_{l_1}^1\left(dz_{l_1}^1\right)ds \\ &+\sum_{l_1=1}^{I_1n_1}\int_0^t\int_{\mathbb{R}_0}\left\langle h^{1(l_1)}\left(X(s),z^1\right), h^{1(l_1)}\left(X(s),z^1\right)\tilde{N}_{l_1}^1\left(ds,dz_{l_1}^1\right)\right\rangle \\ &+2\sum_{l_1=1}^{I_1n_1}\int_0^t\int_{\mathbb{R}_0}\left\langle X(s)-x^*, h^{1(l_1)}\left(X(s),z^1\right)\tilde{N}_{l_1}^1\left(ds,dz_{l_1}^1\right)\right\rangle,\end{aligned}$$

Then for $\forall t_1 \in \left[0, \tilde{\tau}^*(\omega)\right]$ with $\tilde{\tau}^*(\omega) \equiv \tilde{\tau}^{1*}(\omega) = ... = \tilde{\tau}^{i_2 *}(\omega) = ... = \tilde{\tau}^{I_2 *}(\omega)$, and $\tilde{\zeta} = \tilde{\zeta}(p)$ ($\forall p \geq 2$), which may be different from line to line throughout the current proof, we get,

$$
\begin{aligned}
\sup_{0\le t\le t_1}\left\|X(t)-x^*\right\|_2^p \le \tilde{\zeta}\Bigg\{&\left\|X(0)-x^*\right\|_2^p + 3\left[\int_0^{t_1}\tilde{L}\left\|X(t)\right\|_2^2 dt\right]^{\frac{p}{2}} \\
&+\sup_{0\le t\le t_1}\left|\int_0^t\left\langle X(s)-x^*, g^1\left(X(s)\right)dW^1(s)\right\rangle\right|^{\frac{p}{2}} \\
&+\sup_{0\le t\le t_1}\left|\sum_{l_1=1}^{I_1 n_1}\int_0^t\int_{\mathbb{R}_0}\left\langle h^{1(l_1)}\left(X(s),z^1\right), h^{1(l_1)}\left(X(s),z^1\right)\tilde{N}_{l_1}^1\left(ds,dz_{l_1}^1\right)\right\rangle\right|^{\frac{p}{2}} \\
&+\sup_{0\le t\le t_1}\left|\sum_{l_1=1}^{I_1 n_1}\int_0^t\int_{\mathbb{R}_0}\left\langle X(s)-x^*, h^{1(l_1)}\left(X(s),z^1\right)\tilde{N}_{l_1}^1\left(ds,dz_{l_1}^1\right)\right\rangle\right|^{\frac{p}{2}}\Bigg\},
\end{aligned}
$$

in which we have used Assumption 5. It follows from Cauchy-Schwartz Inequality that,

$$
\begin{aligned}
\sup_{0\le t\le t_1}\left\|X(t)-x^*\right\|_2^p \le \tilde{\zeta}\Bigg\{&\left\|X(0)-x^*\right\|_2^p + \int_0^{t_1}\left\|X(t)\right\|_2^p dt \\
&+\sup_{0\le t\le t_1}\left|\int_0^t\left\langle X(s)-x^*, g^1\left(X(s)\right)dW^1(s)\right\rangle\right|^{\frac{p}{2}} \\
&+\sup_{0\le t\le t_1}\left|\sum_{l_1=1}^{I_1 n_1}\int_0^t\int_{\mathbb{R}_0}\left\langle h^{1(l_1)}\left(X(s),z^1\right), h^{1(l_1)}\left(X(s),z^1\right)\tilde{N}_{l_1}^1\left(ds,dz_{l_1}^1\right)\right\rangle\right|^{\frac{p}{2}} \\
&+\sup_{0\le t\le t_1}\left|\sum_{l_1=1}^{I_1 n_1}\int_0^t\int_{\mathbb{R}_0}\left\langle X(s)-x^*, h^{1(l_1)}\left(X(s),z^1\right)\tilde{N}_{l_1}^1\left(ds,dz_{l_1}^1\right)\right\rangle\right|^{\frac{p}{2}}\Bigg\} \\
\le \tilde{\zeta}\Bigg\{&\left\|X(0)-x^*\right\|_2^p + \int_0^{t_1}\left\|X(t)\right\|_2^p dt \\
&+\sup_{0\le t\le t_1}\left|\int_0^t\left\langle X(s)-x^*, g^1\left(X(s)\right)dW^1(s)\right\rangle\right|^{\frac{p}{2}} \\
&+\sup_{0\le t\le t_1}\left|I_1 n_1 \sup_{1\le l_1\le I_1 n_1}\int_0^t\int_{\mathbb{R}_0}\left\langle h^{1(l_1)}\left(X(s),z^1\right), h^{1(l_1)}\left(X(s),z^1\right)\tilde{N}_{l_1}^1\left(ds,dz_{l_1}^1\right)\right\rangle\right|^{\frac{p}{2}}
\end{aligned}
$$

$$+\sup_{0\le t\le t_1}\left|I_1 n_1 \sup_{1\le l_1\le I_1 n_1}\int_0^t\int_{\mathbb{R}_0}\left\langle X(s)-x^*, h^{1(l_1)}\left(X(s),z^1\right)\tilde{N}^1_{l_1}\left(ds,dz^1_{l_1}\right)\right\rangle\right|^{\frac{p}{2}}\Bigg\}$$

$$\le\tilde{\zeta}\Bigg\{\left\|X(0)-x^*\right\|_2^p+\int_0^{t_1}\left\|X(t)\right\|_2^p dt$$

$$+\sup_{0\le t\le t_1}\left|\int_0^t\left\langle X(s)-x^*, g^1\left(X(s)\right)dW^1(s)\right\rangle\right|^{\frac{p}{2}}$$

$$+\sup_{0\le t\le t_1}\left|\int_0^t\int_{\mathbb{R}_0}\left\langle h^{1(l_1^*)}\left(X(s),z^1\right), h^{1(l_1^*)}\left(X(s),z^1\right)\tilde{N}^1_{l_1^*}\left(ds,dz^1_{l_1^*}\right)\right\rangle\right|^{\frac{p}{2}}$$

$$+\sup_{0\le t\le t_1}\left|\int_0^t\int_{\mathbb{R}_0}\left\langle X(s)-x^*, h^{1(l_1^*)}\left(X(s),z^1\right)\tilde{N}^1_{l_1^*}\left(ds,dz^1_{l_1^*}\right)\right\rangle\right|^{\frac{p}{2}}\Bigg\},$$

for some $l_1^*\in\{1,2,...,l_1,...,I_1n_1\}$. Now, taking expectations on both sides and applying the well-known Burkholder-Davis-Gundy Inequality (see, Karatzas and Shreve, 1991, pp.166) produces,

$$\mathbb{E}^1\left[\sup_{0\le t\le t_1}\left\|X(t)-x^*\right\|_2^p\right]\le\tilde{\zeta}\Bigg\{\mathbb{E}^1\left[\left\|X(0)-x^*\right\|_2^p\right]+\mathbb{E}^1\left[\int_0^{t_1}\left\|X(t)\right\|_2^p dt\right]$$

$$+\mathbb{E}^1\left[\int_0^{t_1}\left\|X(t)-x^*\right\|_2^2\left\|g^1\left(X(t)\right)\right\|_2^2 dt\right]^{\frac{p}{4}}$$

$$+\mathbb{E}^1\left[\int_0^{t_1}\left\|h^{1(l_1^*)}\left(X(t),z^1\right)\right\|_2^2\left\|h^{1(l_1^*)}\left(X(t),z^1\right)\right\|_2^2 dt\right]^{\frac{p}{4}}$$

$$+\mathbb{E}^1\left[\int_0^{t_1}\left\|X(t)-x^*\right\|_2^2\left\|h^{1(l_1^*)}\left(X(t),z^1\right)\right\|_2^2 dt\right]^{\frac{p}{4}}\Bigg\}, \qquad \text{(B.1)}$$

Now, employing the Young Inequality (see, Higham et al, 2003), Hölder Inequality and Assumption 5 leads us to,

$$\mathbb{E}^1\left[\int_0^{t_1}\left\|X(t)-x^*\right\|_2^2\left\|g^1\left(X(t)\right)\right\|_2^2 dt\right]^{\frac{p}{4}}$$

$$\le\mathbb{E}^1\left[\sup_{0\le t\le t_1}\left\|X(t)-x^*\right\|_2^{\frac{p}{2}}\left(\int_0^{t_1}\left\|g^1\left(X(t)\right)\right\|_2^2 dt\right)^{\frac{p}{4}}\right]$$

$$\le\frac{1}{2\left(2\tilde{\zeta}\right)}\mathbb{E}^1\left[\sup_{0\le t\le t_1}\left\|X(t)-x^*\right\|_2^p\right]+\frac{2\tilde{\zeta}}{2}\mathbb{E}^1\left[\int_0^{t_1}\left\|g^1\left(X(t)\right)\right\|_2^2 dt\right]^{\frac{p}{2}}$$

$$\le \frac{1}{4\tilde{\zeta}}\mathbb{E}^1\left[\sup_{0\le t\le t_1}\left\|X(t)-x^*\right\|_2^p\right]+\tilde{\zeta}\tilde{L}^{\frac{p}{2}}\mathbb{E}^1\left[\int_0^{t_1}\left\|X(t)\right\|_2^2 dt\right]^{\frac{p}{2}}$$

$$\le \frac{1}{4\tilde{\zeta}}\mathbb{E}^1\left[\sup_{0\le t\le t_1}\left\|X(t)-x^*\right\|_2^p\right]+\tilde{\zeta}\tilde{L}^{\frac{p}{2}}t_1^{\frac{p-2}{2}}\mathbb{E}^1\left[\int_0^{t_1}\left\|X(t)\right\|_2^p dt\right], \tag{B.2}$$

for that $\tilde{\zeta}$ appears in (B.1). Similarly, we obtain,

$$\mathbb{E}^1\left[\int_0^{t_1}\left\|X(t)-x^*\right\|_2^2\left\|h^{1(l_1^*)}\left(X(t),z^1\right)\right\|_2^2 dt\right]^{\frac{p}{4}}$$

$$\le \frac{1}{4\tilde{\zeta}}\mathbb{E}^1\left[\sup_{0\le t\le t_1}\left\|X(t)-x^*\right\|_2^p\right]+\tilde{\zeta}\tilde{L}^{\frac{p}{2}}t_1^{\frac{p-2}{2}}\mathbb{E}^1\left[\int_0^{t_1}\left\|X(t)\right\|_2^p dt\right], \tag{B.3}$$

And,

$$\mathbb{E}^1\left[\int_0^{t_1}\left\|h^{1(l_1^*)}\left(X(t),z^1\right)\right\|_2^2\left\|h^{1(l_1^*)}\left(X(t),z^1\right)\right\|_2^2 dt\right]^{\frac{p}{4}}$$

$$\le \frac{1}{2\left(\frac{1}{2}\tilde{\zeta}\right)}\mathbb{E}^1\left[\sup_{0\le t\le t_1}\left\|h^{1(l_1^*)}\left(X(t),z^1\right)\right\|_2^p\right]+\frac{\frac{1}{2}\tilde{\zeta}}{2}\tilde{L}^{\frac{p}{2}}t_1^{\frac{p-2}{2}}\mathbb{E}^1\left[\int_0^{t_1}\left\|X(t)\right\|_2^p dt\right]$$

$$\le \frac{1}{\tilde{\zeta}}\tilde{L}^{\frac{p}{2}}\mathbb{E}^1\left[\left\|X(t)\right\|_2^p\right]+\frac{\tilde{\zeta}}{4}\tilde{L}^{\frac{p}{2}}t_1^{\frac{p-2}{2}}\mathbb{E}^1\left[\int_0^{t_1}\left\|X(t)\right\|_2^p dt\right], \tag{B.4}$$

Substituting (B.2)-(B.4) into (B.1) yields,

$$\mathbb{E}^1\left[\sup_{0\le t\le t_1}\left\|X(t)-x^*\right\|_2^p\right]\le\tilde{\zeta}\left\{\mathbb{E}^1\left[\left\|X(0)-x^*\right\|_2^p\right]+\mathbb{E}^1\left[\int_0^{t_1}\left\|X(t)\right\|_2^p dt\right]+\mathbb{E}^1\left[\left\|X(t)\right\|_2^p\right]\right\},$$

Thus, applying Assumption 2 and the following fact (see, Higham et al, 2003),

$$\mathbb{E}^1\left[\left\|X(t)\right\|_2^p\right]\le\tilde{\zeta}\left(p,\tilde{\tau}^*(\omega)\right)\left[1+\mathbb{E}^1\left\|X(0)\right\|_2^p\right]$$

$$=\tilde{\zeta}\left(p,\tilde{\tau}^*(\omega)\right)\left[1+\left\|x\right\|_2^p\right],$$

We obtain,

$$\mathbb{E}^1\left[\sup_{0\le t\le\tilde{\tau}^*(\omega)}\left\|X(t)-x^*\right\|_2^p\right]$$

$$\le\tilde{\zeta}\left(p,\tilde{\tau}^*(\omega)\right)\left\{1+\left\|x-x^*\right\|_2^p+\left\|x\right\|_2^p+\mathbb{E}^1\left[\int_0^{\tilde{\tau}^*(\omega)}\left\|X(t)\right\|_2^p dt\right]\right\}$$

$$<\infty,$$

which implies that there exists a constant $\tilde{\zeta} \triangleq \tilde{\zeta}\left(p, \tilde{\tau}^*(\omega), x, x^*\right) < \infty$ such that,

$$\mathbb{E}^1\left[\sup_{0\le t\le \tilde{\tau}^*(\omega)} \left\|X(t)-x^*\right\|_2^p\right] \le \tilde{\zeta}\left(p, \tilde{\tau}^*(\omega), x, x^*\right).$$

Similarly, one can also get,

$$\mathbb{E}^2\left[\sup_{0\le t\le \overline{\tau}^*(\omega)} \left\|Y(t)-y^*\right\|_2^p\right] \le \overline{\zeta}\left(p, \overline{\tau}^*(\omega), y, y^*\right) < \infty .$$

where $\overline{\tau}^*(\omega) \equiv \overline{\tau}^{1*}(\omega) = \ldots = \overline{\tau}^{i_1*}(\omega) = \ldots = \overline{\tau}^{I_1*}(\omega)$. And the required assertions follow and this completes the whole proof. ▌

C. Proof of Theorem 2.

We define the characteristic operator of $X(t)$ as follows,

$$\mathcal{A}\tilde{\varphi}(x) = \sum_{i_1=1}^{I_1} x^{i_1}\left(\overline{e}_{i_1}^T Ay\right)\frac{\partial\tilde{\varphi}}{\partial x^{i_1}}(x) + \frac{1}{2}\sum_{i_1=1}^{I_1}\left(x^{i_1}\right)^2\left(\overline{\sigma}^{i_1}\right)^T\overline{\sigma}^{i_1}\frac{\partial^2\tilde{\varphi}}{\partial (x^{i_1})^2}(x)$$

$$+\sum_{k_1=1}^{I_1}\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\tilde{\varphi}\left(x+\overline{\gamma}_{l_1}(x)\right)-\tilde{\varphi}(x)-\left\langle\nabla\tilde{\varphi}(x), \overline{\gamma}_{l_1}(x)\right\rangle\right\}\nu_{l_1}^{k_1}\left(dz_{l_1}^{k_1}\right),$$

where,

$$\overline{\gamma}_{l_1}(x) \triangleq \left(x^1\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right), \ldots, x^{i_1}\overline{\gamma}_{i_1l_1}\left(z_{l_1}^{i_1}\right), \ldots, x^{I_1}\overline{\gamma}_{I_1l_1}\left(z_{l_1}^{I_1}\right)\right)^T,$$

$$x \triangleq \left(x^1, \ldots, x^{i_1}, \ldots, x^{I_1}\right)^T = X(0) \in \tilde{\Delta}.$$

And we define the Kullback-Leibler distance (see, Bomze, 1991; Imhof, 2005) between x and x^* as follows,

$$\tilde{\varphi}(x) = dist\left(x, x^*\right) \triangleq \sum_{i_1=1}^{I_1} x^{i_1*}\log\left(\tfrac{x^{i_1*}}{x^{i_1}}\right).$$

Thus, we have,

$$\mathcal{A}\tilde{\varphi}(x) = -\sum_{i_1=1}^{I_1}\left(\overline{e}_{i_1}^T Ay\right)x^{i_1*} + \frac{1}{2}\sum_{i_1=1}^{I_1}\left(\overline{\sigma}^{i_1}\right)^T\overline{\sigma}^{i_1}x^{i_1*}$$

$$+\sum_{k_1=1}^{I_1}\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\sum_{i_1=1}^{I_1}x^{i_1*}\log\frac{1}{1+\overline{\gamma}_{i_1l_1}\left(z_{l_1}^{i_1}\right)} + \sum_{i_1=1}^{I_1}x^{i_1*}\overline{\gamma}_{i_1l_1}\left(z_{l_1}^{i_1}\right)\right\}\nu_{l_1}^{k_1}\left(dz_{l_1}^{k_1}\right),$$

Then by Lemma 1, we get,

$$
\begin{aligned}
\mathcal{A}\tilde{\varphi}(x) &\leq -\sum_{i_1=1}^{I_1}\left(\overline{e}_{i_1}^{T} Ay\right) x^{i_1 *} + \frac{1}{2}\sum_{i_1=1}^{I_1}\left(\overline{\sigma}^{i_1}\right)^{T} \overline{\sigma}^{i_1} x^{i_1 *} \\
&\quad + \sum_{k_1=1}^{I_1}\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\sum_{i_1=1}^{I_1} x^{i_1 *}\log\frac{1}{1+\overline{\gamma}_{i_1 l_1}\left(z_{l_1}^{i_1}\right)} + \sum_{i_1=1}^{I_1} x^{i_1 *}\overline{\gamma}_{i_1 l_1}\left(z_{l_1}^{i_1}\right)\right\}\nu_{l_1}^{k_1}\left(dz_{l_1}^{k_1}\right) \\
&\quad + \tilde{\zeta}\left(p, \tilde{\tau}^{*}(\omega), x, x^{*}\right) - \left\|x - x^{*}\right\|_2^{p} \\
&= -\left\|x - x^{*}\right\|_2^{p} + \tilde{\psi}, \qquad \text{(C.1)}
\end{aligned}
$$

where,

$$
\begin{aligned}
\tilde{\psi} &\triangleq \tilde{\zeta}\left(p, \tilde{\tau}^{*}(\omega), x, x^{*}\right) - \sum_{i_1=1}^{I_1}\left(\overline{e}_{i_1}^{T} Ay\right) x^{i_1 *} + \frac{1}{2}\sum_{i_1=1}^{I_1}\left(\overline{\sigma}^{i_1}\right)^{T} \overline{\sigma}^{i_1} x^{i_1 *} \\
&\quad + \sum_{k_1=1}^{I_1}\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\sum_{i_1=1}^{I_1} x^{i_1 *}\log\frac{1}{1+\overline{\gamma}_{i_1 l_1}\left(z_{l_1}^{i_1}\right)} + \sum_{i_1=1}^{I_1} x^{i_1 *}\overline{\gamma}_{i_1 l_1}\left(z_{l_1}^{i_1}\right)\right\}\nu_{l_1}^{k_1}\left(dz_{l_1}^{k_1}\right),
\end{aligned}
$$

is some positive constant. Now, define,

$$
\tilde{B}_{\tilde{\alpha}}\left(x^{*}\right) \triangleq \left\{X(t) \in cl\tilde{\Delta}; \left\|X(t) - x^{*}\right\|_2 < \tilde{\alpha}, t \geq 0\right\},
$$

$$
\hat{\tilde{\tau}}(\omega) \triangleq \tilde{\tau}_{\bar{\tilde{B}}_{\tilde{\alpha}}(x^{*})}(\omega) \triangleq \inf\left\{t \geq 0; X(t,\omega) \in \bar{\tilde{B}}_{\tilde{\alpha}}\left(x^{*}\right) \triangleq cl\tilde{B}_{\tilde{\alpha}}\left(x^{*}\right)\right\},
$$

where $\bar{\tilde{B}}_{\tilde{\alpha}}\left(x^{*}\right)$ denotes the closure of $\tilde{B}_{\tilde{\alpha}}\left(x^{*}\right)$. Suppose that $\tilde{\alpha}^{p} > \tilde{\psi}$, for any $x \notin \bar{\tilde{B}}_{\tilde{\alpha}}\left(x^{*}\right)$, i.e., $x \in \bar{\tilde{B}}_{\tilde{\alpha}}^{C}\left(x^{*}\right)$, we have,

$$
\mathcal{A}\tilde{\varphi}(x) \leq -\tilde{\alpha} + \tilde{\psi},
$$

by (C.1). Then, applying the well-known Dynkin's formula yields,

$$
\begin{aligned}
&0 \\
&\leq \mathbb{E}^{1}\left[\tilde{\varphi}\left\{X\left(t \wedge \hat{\tilde{\tau}}(\omega)\right)\right\}\right] \\
&= \tilde{\varphi}(x) + \mathbb{E}^{1}\left[\int_0^{t \wedge \hat{\tilde{\tau}}(\omega)} \mathcal{A}\tilde{\varphi}\left(X(s)\right) ds\right] \\
&\leq \tilde{\varphi}(x) + \left(\tilde{\psi} - \tilde{\alpha}^{p}\right)\mathbb{E}^{1}\left[t \wedge \hat{\tilde{\tau}}(\omega)\right],
\end{aligned}
$$

Notice that $t \wedge \hat{\tilde{\tau}}(\omega) \nearrow \hat{\tilde{\tau}}(\omega)$ as $t \to \infty$, thus by using Lebesgue Monotone Convergence Theorem, we get,

$$
0 \leq \tilde{\varphi}(x) + \left(\tilde{\psi} - \tilde{\alpha}^{p}\right)\mathbb{E}^{1}\left[\hat{\tilde{\tau}}(\omega)\right],
$$

which implies that,

$$\mathbb{E}^1\left[\tilde{\tau}_{\bar{\tilde{B}}_{\tilde{\alpha}}(x^*)}(\omega)\right]=\mathbb{E}^1\left[\hat{\tilde{\tau}}(\omega)\right]\le\frac{\tilde{\varphi}(x)}{\tilde{\alpha}^p-\tilde{\psi}}=\frac{dist\left(x,x^*\right)}{\tilde{\alpha}^p-\tilde{\psi}},$$

as required in (a.1). Moreover, for some constant $\tilde{Q}>\tilde{\varphi}(x)$, we set up,

$$\tilde{\tau}_{\tilde{Q}}(\omega)\triangleq\inf\left\{t\ge 0;\tilde{\varphi}\left(X(t,\omega)\right)=\tilde{Q}\right\}.$$

Then, by employing Dynkin's formula and inequality in (C.1),

$$\begin{aligned}
&0\\
&\le\mathbb{E}^1\left[\tilde{\varphi}\left\{X\left(t\wedge\tilde{\tau}_{\tilde{Q}}(\omega)\right)\right\}\right]\\
&=\tilde{\varphi}(x)+\mathbb{E}^1\left[\int_0^{t\wedge\tilde{\tau}_{\tilde{Q}}(\omega)}\mathcal{A}\tilde{\varphi}\left(X(s)\right)ds\right]\\
&\le\tilde{\varphi}(x)-\mathbb{E}^1\left[\int_0^{t\wedge\tilde{\tau}_{\tilde{Q}}(\omega)}\left\|X(s)-x^*\right\|_2^p ds\right]+\tilde{\psi}\mathbb{E}^1\left[t\wedge\tilde{\tau}_{\tilde{Q}}(\omega)\right],
\end{aligned}$$

If $\tilde{Q}\to\infty$, then $t\wedge\tilde{\tau}_{\tilde{Q}}(\omega)\to t$, and application of Lebesgue Dominated Convergence Theorem reveals that,

$$0\le\tilde{\varphi}(x)-\mathbb{E}^1\left[\int_0^t\left\|X(s)-x^*\right\|_2^p ds\right]+\tilde{\psi}t,$$

which implies that,

$$\mathbb{E}^1\left[\frac{1}{t}\int_0^t\left\|X(s)-x^*\right\|_2^p ds\right]\le\frac{\tilde{\varphi}(x)}{t}+\tilde{\psi},$$

Consequently,

$$\limsup_{t\to\infty}\mathbb{E}^1\left[\frac{1}{t}\int_0^t\left\|X(s)-x^*\right\|_2^p ds\right]\le\tilde{\psi},$$

which combines with Assumption 4 leads us to,

$$\begin{aligned}
\tilde{\pi}\left[\bar{\tilde{B}}_{\tilde{\alpha}}^C\left(x^*\right)\right]&=\limsup_{t\to\infty}\mathbb{E}^1\left[\frac{1}{t}\int_0^t\chi_{\bar{\tilde{B}}_{\tilde{\alpha}}^C(x^*)}\left(X(s,\omega)\right)ds\right]\\
&\le\limsup_{t\to\infty}\mathbb{E}^1\left[\frac{1}{t}\int_0^t\frac{\left\|X(s)-x^*\right\|_2^p}{\tilde{\alpha}^p}ds\right]\\
&\le\frac{\tilde{\psi}}{\tilde{\alpha}^p}\triangleq\tilde{\varepsilon},
\end{aligned}$$

where $\chi_{\{\}}$ stands for the characteristic function of the set $\{\cdot\}$. Accordingly, one may obtain,

$$\tilde{\pi}\left[\bar{\tilde{B}}_{\tilde{\alpha}}\left(x^*\right)\right]\geq 1-\frac{\tilde{\psi}}{\tilde{\alpha}^p}\triangleq 1-\tilde{\varepsilon}\ ,$$

which gives the desired result in (a.2). Noting that the proof of (b.1) and (b.2) is quite similar to that of (a.1) and (a.2) shown above, so we take it omitted. And hence the whole proof of Theorem 2 is completed. ▌

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