Sustainable Heterogeneity as the Unique Socially Optimal Allocation for Almost All Social Welfare Functions

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Abstract

The socially optimal allocation has been regarded to be unspecifiable because of utility's interpersonal incomparability, Arrow's general possibility theorem, and other factors. This paper examines this problem by focusing not on the social welfare function but instead on the utility possibility frontier in dynamic models with a heterogeneous population. A unique balanced growth path was found on which all of the optimality conditions of all heterogeneous households are equally and indefinitely satisfied (sustainable heterogeneity). With appropriate government interventions, such a path is always achievable and is uniquely socially optimal for almost all generally usable (i.e., preferences are complete, transitive, and continuous) social welfare functions. The only exceptions are some variants in Nietzsche type social welfare functions, but those types of welfare functions will rarely be adopted in democratic societies. This result indicates that it is no longer necessary to specify the shape of the social welfare function to determine the socially optimal growth path in a heterogeneous population.

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1 INTRODUCTION

Problems of economic inequality, wealth disparity, and justice have long been central issues in economics and are again a hot topic in the midst of the great recession that began in 2008. The concerns of the Occupy Wall Street movement are a recent example. However, the criteria for socially optimal allocation have not been universally agreed upon because of utility's interpersonal incomparability, Arrow's general possibility theorem (Arrow, 1951), and other factors. Although the problem of utility's interpersonal incomparability was solved by Bergson (1938) and Samuelson (1947), their idea was fundamentally criticized by Arrow (1951). Arrow's criticism can be worked around if the assumptions in Arrow (1951) are modified, for example, the assumption that every individual has a single-peaked preference is added (see e.g., Black, 1958); thus, social welfare functions can be used for various analyses. Nevertheless, even if social welfare functions can be used, there is no consensus on their shape. Because of this limitation, it has been difficult to provide useful information for arguments of social optimality. Even though many people have protested that current levels of economic inequality and wealth disparity are too large, there is no theoretical basis on which to judge their arguments.

To shed light on the arguments, I take a different approach in this paper. I focus not on the nature of the social welfare function but instead on the nature of the utility possibility frontier, because if the shape of the utility possibility frontier has some special characteristics, particularly if it is very constrained by some factors, it may be able to narrow the opportunities for a socially optimal allocation, regardless of any differences in the social welfare functions.

In particular, this paper examines social optimality in dynamic models with a heterogeneous population and the condition for the state where all of the optimality conditions of all heterogeneous households are satisfied in these models. Intuitively, knowing whether the state where all of the optimality conditions of all heterogeneous households are satisfied is achieved seems to provide useful information for social optimality, but it is meaningless if we use static models because any competitive equilibrium naturally and always achieves this state even if the population is heterogeneous. It is also meaningless when dynamic models are used if the models use homogeneous populations, because such a state is naturally and always achieved and a homogeneous population generates no income inequality or wealth differential. Thus, the only remaining type of model to study is a dynamic model with a heterogeneous population. However, Becker (1980) showed that, in such models, the magnitudes of income inequality and wealth disparity eventually reach the limit; that is, the most patient household eventually will own all capital. All of the other households cannot satisfy their optimality conditions and will go bankrupt and, as it were, perish when even a very small negative shock occurs unless the authority intervenes. Consequently, examining social optimality in a heterogeneous population by using dynamic models has been regarded to be a meaningless task. As a result, little attention has been paid in the analyses of social optimality to the state where all of the optimality conditions of all heterogeneous households are satisfied.

This paper first shows that, in dynamic models with a heterogeneous population, there exists a state where all of the optimality conditions of all heterogeneous households are satisfied (i.e., "sustainable heterogeneity"), although this state is not guaranteed to be naturally and always achieved, and it is influenced by the behavior of the most advantaged household (see Harashima, 2010). Even though it is not naturally achievable, it can be always achieved with appropriate government intervention. The existence of this state is very important because, unlike the case with static and dynamic models with homogeneous populations, we can obtain additional meaningful and useful information about social optimality. Dynamic models with a heterogeneous population have another advantage—they describe the nature of economy far more realistically than static and dynamic models with homogeneous populations. Because little attention has been given to sustainable heterogeneity in analyses of social optimality, discoveries derived from such analyses add a new analytical tool and may help solve the previously discussed problem of the unspecifiability of social optimality.

A distinct feature of the model presented in this paper is that a common nature of utility across the population is assumed to exist as a result of human evolution. Although utility functions are different across a population, some common features have been assumed, for example, a diminishing marginal rate of substitution. In this paper, an additional common nature is assumed such that extreme disutility is generated if all of the optimality conditions are not satisfied. The reason for this assumption, as described in more detail in Section 4, is that only humans who have this nature could have survived the process of natural selection. This additional common nature of utility plays an important role in the analyses of social optimality presented in this paper.

The model shows that sustainable heterogeneity is the unique socially optimal allocation for almost all generally usable (i.e., preferences are complete, transitive, and continuous) social welfare functions. This result is very important because the socially optimal allocation is uniquely determined without having to specify the shape of the social welfare function. This result therefore implies that, with the additional information provided by sustainable heterogeneity in dynamic models with a heterogeneous population, the problem of unspecifiability of social optimality can be solved.

The paper is organized as follows. In Section 2, a multi-economy endogenous growth model with heterogeneous population is constructed, and sustainability of heterogeneity is examined by using it. The existence of a unique balanced growth path on which all optimality conditions of all heterogeneous households are satisfied is shown. Section 3 shows that sustainable heterogeneity is always achievable with appropriate government intervention even if the most advantaged household behaves unilaterally. In Section 4, extreme disutility to unsustainable heterogeneity is examined based on the gene theory of evolution. Section 5 introduces a utility possibility frontier and social welfare function modified to dynamic models and shows that sustainable heterogeneity represents the unique socially optimal allocation. Finally, some concluding remarks are offered in Section 6.

2 SUSTAINABLE HETEROGENEITY

2.1 The model

2.1.1 The base model

In this paper, sustainability of heterogeneity is examined in the framework of endogenous growth, but most endogenous growth models commonly have problems with scale effects or the influence of population growth (e.g., Jones, 1995a, b). Hence, this paper uses the model presented by Harashima (2004), which is free from both problems (see also Jones, 1995a; Aghion and Howitt, 1998; Peretto and Smulders, 2002). The production function is $Y_t = F(A_t, K_t, L_t)$, and the accumulation of capital is

$$\dot{K}_t = Y_t - C_t - v\dot{A}_t \quad , \tag{1}$$

where Y_t is outputs, A_t is technology, K_t is capital inputs, L_t is labor inputs, C_t is consumption, $v(>0)$ is a constant, and a unit of K_t and $\frac{1}{v}$ of a unit of A_t are equivalent: that is, they are produced using the same quantities of inputs. All firms are identical and have the same size, and for any period,

$$m = \frac{M_t}{L_t} \quad , \tag{2}$$

where M_t is the number of firms, and $m(>0)$ is a constant. In addition,

$$\frac{\partial Y_t}{\partial K_t} = \frac{\varpi}{M_t} \frac{\partial Y_t}{\partial (vA_t)} \quad ; \tag{3}$$

thus,

$$\frac{\partial y_t}{\partial k_t} = \frac{\varpi}{mv} \frac{\partial y_t}{\partial A_t} \tag{4}$$

is always kept, where y_t is output per capita, k_t is capital per capita, and $\varpi(>1)$ is a constant. For simplicity, the period of patent is assumed to be indefinite, and no capital depreciation is assumed. ϖ indicates the effect of patent protection. With patents, the income is distributed to not only capitals and labors but technologies. Equation (2) indicates that population and number of firms are positively correlated. Equations (3) and (4) indicate that returns on investing in K_t and in A_t are kept equal and that a firm that produces a new technology cannot obtain all the returns on an investment in A_t. This means that investing in A_t increases Y_t, but the investing firm's return on the investment in A_t is only a fraction of the increase of Y_t, such that $\frac{\varpi}{M_t} \frac{\partial Y_t}{\partial (vA_t)} = \frac{\varpi}{mL_t} \frac{\partial Y_t}{\partial (vA_t)}$ because of uncompensated knowledge spillovers to other firms and complementarity of technologies.

A part of the knowledge generated as a result of an investment made by a firm spills over to other firms. Researchers in firms as well as universities and research institutions could not effectively generate innovations if they were isolated from other researchers. They contact and stimulate each other. Probably, mutual partial knowledge spillovers among researchers and firms give each other reciprocal benefits. Researchers take hints on their researches in exchange for spilled knowledge. Therefore, even though the investing firm wishes to keep its knowledge secret, some parts of it will spill over. In addition, many uncompensated knowledge spillovers occur because many technologies are regarded as so minor that they are not applied for patents and left unprotected by patents. Nevertheless, even if a technology that was generated as a byproduct is completely useless for the investing firm, it may be a treasure for firms in a different industry. A_t includes all these technologies, and an investment in technology generates many technologies that the investing firm cannot protect by patents.

Broadly speaking, there are two types of uncompensated knowledge spillovers: intra-sectoral knowledge spillovers (i.e., Marshall-Arrow-Romer [MAR] externalities; Marshall, 1890; Arrow, 1962; Romer, 1986) and inter-sectoral knowledge spillovers (i.e., Jacobs externalities; Jacobs, 1969). MAR theory assumes that knowledge spillovers between homogenous firms work out most effectively and that spillovers will therefore primarily emerge within one sector. As a result, uncompensated knowledge spillovers will be more active if the number of firms within a sector is larger. On the other hand, Jacobs (1969) argues that knowledge spillovers are most effective among firms that practice different activities and that diversification (i.e., a variety of sectors) is important for spillovers. As a result, uncompensated knowledge spillovers will be more active if the number of sectors in the economy is larger. Nevertheless, if all sectors have the same number of firms, an increase in the number of firms in the economy results in more active knowledge spillovers in any case, owing to either MAR externalities or Jacobs externalities.

Furthermore, as the volume of uncompensated knowledge spillovers increases, the investing firm's returns on the investment in A_t decrease. $\frac{\partial Y_t}{\partial A_t}$ indicates the total increase in Y_t in the economy by an increase in A_t, which consists of increases in both outputs in the firm that invested in the new technologies and outputs in other firms that utilize the newly invented technologies, whether the firms obtained the technologies by compensating the originating firm or by using uncompensated knowledge spillovers. If the number of firms becomes larger and uncompensated knowledge spillovers occur more actively, the compensated fraction in $\frac{\partial Y_t}{\partial A_t}$ that the investing firm can obtain becomes smaller, and the investing firm's returns on the investment in A_t also become smaller.

Complementarity of technologies also reduces the fraction of $\frac{\partial Y_t}{\partial A_t}$ that the investing firm can obtain. If a new technology is effective only if it is combined with some particular technologies, the return on the investment in technology will belong not only to the investing firm but to the firms that hold these particular technologies. For example, an innovation in software technology generated by a software company increases the sales and profits of computer hardware companies. The economy's productivity increases because of the innovation but the increased incomes are attributed not only to the firm that generated the innovation but also to the firms that hold complementary technologies. A part of $\frac{\partial Y_t}{\partial A_t}$ leaks to these firms. For them, the leaked income is a kind of rent revenue unexpectedly become obtainable thanks to the innovation. Most new technologies will have complementary technologies. In addition, as the number of firms increases, the number of firms that holds complementary technologies will also increase, and thereby these leaks will also increase.

Because of the uncompensated knowledge spillovers and the complementarity of technologies, therefore, the fraction of $\frac{\partial Y_t}{\partial A_t}$ that an investing firm can obtain on average will be comparatively small, i.e., ϖ will be far smaller than M_t except that M_t is very small,[1] and the fraction will decrease as M_t increases.

The production function is specified as $Y_t = A_t^{\alpha} f(K_t, L_t)$ where $\alpha\,(0 < \alpha < 1)$ is a constant. Let $y_t = \frac{Y_t}{L_t}$, $k_t = \frac{K_t}{L_t}$, $c_t = \frac{C_t}{L_t}$, and $n_t = \frac{\dot{L}_t}{L_t}$, and assume that $f(K_t, L_t)$ is homogenous of degree one. Thus $y_t = A_t^{\alpha} f(k_t)$ and $\dot{k}_t = y_t - c_t - \frac{v\dot{A}_t}{L_t} - n_t k_t$. By equation (4), $A_t = \frac{\varpi \alpha f(k_t)}{m v f'(k_t)}$ because $\frac{\varpi \partial y_t}{m v \partial A_t} = \frac{\partial y_t}{\partial k_t} \Leftrightarrow \frac{\varpi \alpha}{m v} A_t^{\alpha - 1} f(k_t) = A_t^{\alpha} f'(k_t)$.

2.1.2 Models with heterogeneous households

[1] If M_t is very small, the value of ϖ will be far smaller than that for sufficiently large M_t, because the number of firms that can benefit from an innovation is constrained owing to very small M_t. The very small number of firms indicates that the economy is not sufficiently sophisticated, and thereby the benefit of an innovation cannot be fully realized in the economy. This constraint can be modeled as $\varpi = \tilde{\varpi}\left[1 - \left(1 - \tilde{\varpi}^{-1}\right)^{M_t}\right]$ where $\tilde{\varpi}(\geq 1)$ is a constant. Nevertheless, for sufficiently large M_t (i.e., in sufficiently sophisticated economies), the constraint is removed such that $\lim_{M_t \to \infty} \tilde{\varpi}\left[1 - \left(1 - \tilde{\varpi}^{-1}\right)^{M_t}\right] = \tilde{\varpi} = \varpi$.

Three heterogeneities—heterogeneous time preference, risk aversion, and productivity—are examined in endogenous growth models, which are modified versions of the model shown in Section 2.1.1. First, suppose that there are two economies— economy 1 and economy 2—that are identical except for time preference, risk aversion, or productivity. The population growth rate is zero (i.e., $n_t = 0$). The economies are fully open to each other, and goods, services, and capital are freely transacted between them, but labor is immobilized in each economy.

Each economy can be interpreted as representing either a country (the international interpretation) or a group of identical households in a country (the national interpretation). Because the economies are fully open, they are integrated through trade and form a combined economy. The combined economy is the world economy in the international interpretation and the national economy in the national interpretation. In the following discussion, a model based on the international interpretation is called an international model and that based on the national interpretation is called a national model. Usually, the concept of the balance of payments is used only for the international transactions. However, because both national and international interpretations are possible, this concept and terminology are also used for the national models in this paper.

2.1.2.1 Heterogeneous time preference model

First, a model in which the two economies are identical except for time preference is constructed.[2] The rate of time preference of the representative household in economy 1 is θ_1 and that in economy 2 is θ_2, and $\theta_1 < \theta_2$. The production function in economy 1 is $y_{1,t} = A_t^{\alpha} f(k_{1,t})$ and that in economy 2 is $y_{2,t} = A_t^{\alpha} f(k_{2,t})$, where $y_{i,t}$ and $k_{i,t}$ are, respectively, output and capital per capita in economy i in period t for i = 1, 2. The population of each economy is $\frac{L_t}{2}$; thus, the total for both is L_t, which is sufficiently large. Firms operate in both economies, and the number of firms is M_t. The current account balance in economy 1 is τ_t and that in economy 2 is - τ_t. Because a balanced growth path requires Harrod neutral technological progress, the production functions are further specified as

$$y_{i,t} = A_t^{\alpha} k_{i,t}^{1-\alpha} \quad ;$$

thus, $Y_{i,t} = K_{i,t}^{1-\alpha} (A_t L_t)^{\alpha} \ (i = 1,2)$.

Because both economies are fully open, returns on investments in each economy are kept equal through arbitration such that

$$\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\varpi}{2mv} \frac{\partial (y_{1,t} + y_{2,t})}{\partial A_t} = \frac{\partial y_{2,t}}{\partial k_{2,t}} \quad . \tag{5}$$

Equation (5) indicates that an increase in A_t enhances outputs in both economies such that $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\varpi}{M_t} \frac{\partial (Y_{1,t} + Y_{2,t})}{\partial (v A_t)}$, and because the population is equal ($\frac{L_t}{2}$), $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\partial y_{i,t}}{\partial k_{i,t}} =$ $\frac{\varpi}{M_t} \frac{\partial (Y_{1,t} + Y_{2,t})}{\partial (v A_t)} = \frac{\varpi}{m L_t} \frac{\partial (y_{1,t} + y_{2,t})}{\partial (v A_t)} \frac{L_t}{2} = \frac{\varpi}{2mv} \frac{\partial (y_{1,t} + y_{2,t})}{\partial A_t}$. Therefore,

[2] This type of endogenous growth model of heterogeneous time preference was originally shown by Harashima (2009b).

$$A_t = \frac{\varpi\alpha\left[f\left(k_{1,t}\right)+f\left(k_{2,t}\right)\right]}{2mvf'\left(k_{1,t}\right)} = \frac{\varpi\alpha\left[f\left(k_{1,t}\right)+f\left(k_{2,t}\right)\right]}{2mvf'\left(k_{2,t}\right)} \quad .$$

Because equation (5) is always held through arbitration, equations $k_{1,t} = k_{2,t}$, $\dot{k}_{1,t} = \dot{k}_{2,t}$, $y_{1,t} = y_{2,t}$ and $\dot{y}_{1,t} = \dot{y}_{2,t}$ are also held. Hence,

$$A_t = \frac{\varpi\alpha f\left(k_{1,t}\right)}{mvf'\left(k_{1,t}\right)} = \frac{\varpi\alpha f\left(k_{2,t}\right)}{mvf'\left(k_{2,t}\right)} \quad .$$

In addition, because $\frac{\partial\left(y_{1,t}+y_{2,t}\right)}{\partial A_{1,t}} = \frac{\partial\left(y_{1,t}+y_{2,t}\right)}{\partial A_{2,t}}$ through arbitration, then $\dot{A}_{1,t} = \dot{A}_{2,t}$ is held.

The accumulated current account balance $\int_0^t \tau_s ds$ mirrors capital flows between the two economies. The economy with current account surpluses invests them in the other economy. Since $\frac{\partial y_{1,t}}{\partial k_{1,t}} \left(= \frac{\partial y_{2,t}}{\partial k_{2,t}} \right)$ are returns on investments, $\frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds$ and $\frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s ds$ represent income receipts or payments on the assets that an economy owns in the other economy. Hence,

$$\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s ds$$

is the balance on goods and services of economy 1, and

$$\frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t$$

is that of economy 2. Because the current account balance mirrors capital flows between the economies, the balance is a function of capital in both economies such that

$$\tau_t = \kappa\left(k_{1,t}, k_{2,t}\right) \quad .$$

The government (or an international supranational organization) intervenes in activities of economies 1 and 2 by transferring money from economy 1 to economy 2. The amount of transfer in period t is g_t and it is assumed that g_t depends on capitals such that

$$g_t = \bar{g} k_{1,t}$$

where $\bar{g}$ is a constant. Because $k_{1,t} = k_{2,t}$ and $\dot{k}_{1,t} = \dot{k}_{2,t}$,

$$g_t = \bar{g} k_{1,t} = \bar{g} k_{2,t} \quad .$$

The representative household in economy 1 maximizes its expected utility

$$E\int_0^{\infty} u_1(c_{1,t})\exp(-\theta_1 t)dt \quad ,$$

subject to

$$\begin{aligned}\dot{k}_{1,t} &= y_{1,t} + \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s \, ds - \tau_t - c_{1,t} - g_t - v\dot{A}_{1,t}\left(\frac{L_t}{2}\right)^{-1} \\ &= y_{1,t} + \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s \, ds - \tau_t - c_{1,t} - \overline{g}k_{1,t} - v\dot{A}_{1,t}\left(\frac{L_t}{2}\right)^{-1} \quad ,\end{aligned} \tag{6}$$

and the representative household in economy 2 maximizes its expected utility

$$E\int_0^{\infty} u_2(c_{2,t})\exp(-\theta_2 t)dt \quad ,$$

subject to

$$\begin{aligned}\dot{k}_{2,t} &= y_{2,t} - \frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t \tau_s ds + \tau_t - c_{2,t} + g_t - v\dot{A}_{2,t}\left(\frac{L_t}{2}\right)^{-1} \\ &= y_{2,t} - \frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t \tau_s ds + \tau_t - c_{2,t} + \overline{g}k_{2,t} - v\dot{A}_{2,t}\left(\frac{L_t}{2}\right)^{-1} \quad ,\end{aligned} \tag{7}$$

where $u_{i,t}$, $c_{i,t}$, and $\dot{A}_{i,t}$, respectively, are the utility function, per capita consumption, and the increase in A_t by R&D activities in economy i in period t for i = 1, 2; E is the expectation operator; and $\dot{A}_t = \dot{A}_{1,t} + \dot{A}_{2,t}$. Equations (6) and (7) implicitly assume that each economy does not have foreign assets or debt in period t = 0.

Because the production function is Harrod neutral and because $A_t = \frac{\varpi\alpha f(k_{1,t})}{mvf'(k_{1,t})} = \frac{\varpi\alpha f(k_{2,t})}{mvf'(k_{2,t})}$ and $f = k_{i,t}^{1-\alpha}$, then

$$A_t = \frac{\varpi\alpha}{mv(1-\alpha)}k_{i,t}$$

and

$$\frac{\partial y_{i,t}}{\partial k_{i,t}} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} \quad .$$

Since $\dot{A}_{1,t} = \dot{A}_{2,t}$ and $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$, then

$$\dot{k}_{1,t} = y_{1,t} + \frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t - c_{1,t} - \bar{g} k_{1,t} - \frac{v \dot{A}_t}{2} \left(\frac{L_t}{2} \right)^{-1}$$

$$= \left(\frac{\varpi \alpha}{mv} \right)^{\alpha} (1-\alpha)^{-\alpha} k_{1,t} + \left(\frac{\varpi \alpha}{mv} \right)^{\alpha} (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds - \tau_t - c_{1,t} - \bar{g} k_{1,t} - \frac{\varpi \alpha}{m L_t (1-\alpha)} \dot{k}_{1,t}$$

and

$$\dot{k}_{1,t} = \frac{m L_t (1-\alpha)}{m L_t (1-\alpha) + \varpi \alpha} \left[\left(\frac{\varpi \alpha}{mv} \right)^{\alpha} (1-\alpha)^{-\alpha} k_{1,t} + \left(\frac{\varpi \alpha}{mv} \right)^{\alpha} (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds - \tau_t - c_{1,t} - \bar{g} k_{1,t} \right] \ .$$

Because L_t is sufficiently large and ϖ is far smaller than M_t, the problem of scale effects vanishes and thereby $\frac{m L_t (1-\alpha)}{m L_t (1-\alpha) + \varpi \alpha} = 1$.

Putting the above elements together, the optimization problem of economy 1 can be rewritten as

$$Max\ E \int_0^{\infty} u_1 (c_{1,t}) \exp(-\theta_1 t) dt \ ,$$

subject to

$$\dot{k}_{1,t} = \left(\frac{\varpi \alpha}{mv} \right)^{\alpha} (1-\alpha)^{-\alpha} k_{1,t} + \left(\frac{\varpi \alpha}{mv} \right)^{\alpha} (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds - \tau_t - c_{1,t} - \bar{g} k_{1,t} \ .$$

Similarly, that of economy 2 can be rewritten as

$$Max\ E \int_0^{\infty} u_2 (c_{2,t}) \exp(-\theta_2 t) dt \ ,$$

subject to

$$\dot{k}_{2,t} = \left(\frac{\varpi \alpha}{mv} \right)^{\alpha} (1-\alpha)^{-\alpha} k_{2,t} - \left(\frac{\varpi \alpha}{mv} \right)^{\alpha} (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds + \tau_t - c_{2,t} + \bar{g} k_{2,t} \ .$$

2.1.2.2 Heterogeneous risk aversion model

The basic structure of the model with heterogeneous risk aversion is the same as that of heterogeneous time preference. The two economies are identical except in regard to risk aversion.[3] The degree of relative risk aversion of economy 1 is $\varepsilon_1 = -\frac{c_{1,t} u_1''}{u_1'}$ and that of economy 2 is $\varepsilon_2 = -\frac{c_{2,t} u_2''}{u_2'}$, which are constant, and $\varepsilon_1 < \varepsilon_2$. The optimization problem of economy 1 is

[3] This type of endogenous growth model of heterogeneous risk aversion was originally shown by Harashima (2009c).

$$Max\ E\int_0^{\infty} u_1(c_{1,t})\exp(-\theta t)dt \quad ,$$

subject to

$$\dot{k}_{1,t} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} - \bar{g}k_{1,t} \quad ,$$

and that of economy 2 is

$$Max\ E\int_0^{\infty} u_2(c_{2,t})\exp(-\theta t)dt \quad ,$$

subject to

$$\dot{k}_{2,t} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_{2,t} - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t} + \bar{g}\,k_{2,t} \quad .$$

2.1.2.3 Heterogeneous productivity model

With heterogeneous productivity, the production function is heterogeneous, not the utility function. Because technology A_t is common to both economies, a heterogeneous production function requires heterogeneity in elements other than technology. Prescott (1998) argues that unknown factors other than technology have made total factor productivity (TFP) heterogeneous across countries. Harashima (2009a) argues that average workers' innovative activities are an essential element of productivity and make TFP heterogeneous across workers, firms, and economies. Since average workers are human and capable of creative intellectual activities, they can create innovations even if their innovations are minor. It is rational for firms to exploit all the opportunities that these ordinary workers' innovative activities offer. Furthermore, innovations created by ordinary workers are indispensable for efficient production. A production function incorporating average workers' innovations has been shown to have a Cobb-Douglas functional form with a labor share of about 70% (Harashima 2009a), such that

$$Y_t = \bar{\sigma}\omega_A\omega_L A_t^{\alpha}K_t^{1-\alpha}L_t^{\alpha} \quad , \tag{8}$$

where ω_A and ω_L are positive constant parameters with regard to average workers' creative activities, and $\bar{\sigma}$ is a parameter that represents a worker's accessibility limit to capital with regard to location. The parameters ω_A and ω_L are independent of A_t but are dependent on the creative activities of average workers. Thereby, unlike with technology A_t, these parameters can be heterogeneous across workers, firms, and economies.

In this model of heterogeneous productivity, it is assumed that workers whose households belong to different economies have different values of ω_A and ω_L. In addition, only productivity that is represented by $\bar{\sigma}\omega_A\omega_L A_t^{\alpha}$ in equation (8) is heterogeneous between the two economies. The production function of economy 1 is $y_{1,t} = \omega_1^{\alpha}A_t^{\alpha}f(k_{1,t})$ and that of economy 2 is $y_{2,t} = \omega_2^{\alpha}A_t^{\alpha}f(k_{2,t})$, where $\omega_1\ (0 < \omega_1 \leq 1)$ and $\omega_2\ (0 < \omega_2 \leq 1)$ are constants and $\omega_2 < \omega_1$. Since $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\partial y_{i,t}}{\partial k_{i,t}} = M_t^{-1}\frac{\partial(Y_{1,t}+Y_{2,t})}{\partial(vA_t)} = \frac{\varpi}{mL_t}\frac{\partial(y_{1,t}+y_{2,t})}{\partial(vA_t)}\frac{L_t}{2} = \frac{\varpi}{2mL_t}\frac{\partial(y_{1,t}+y_{2,t})}{\partial A_t}$ by equation (5), then

$$A_t = \frac{\varpi\alpha\left[\omega_1^\alpha f(k_{1,t}) + \omega_2^\alpha f(k_{2,t})\right]}{2mv\,\omega_1^\alpha f'(k_{1,t})} = \frac{\varpi\alpha\left[\omega_1^\alpha f(k_{1,t}) + \omega_2^\alpha f(k_{2,t})\right]}{2mv\,\omega_2^\alpha f'(k_{2,t})} \quad . \tag{9}$$

Because equation (5) is always held through arbitration, equations $k_{1,t} = \frac{\omega_1}{\omega_2} k_{2,t}$, $\dot{k}_{1,t} = \frac{\omega_1}{\omega_2} \dot{k}_{2,t}$, $y_{1,t} = \frac{\omega_1}{\omega_2} y_{2,t}$, and $\dot{y}_{1,t} = \frac{\omega_1}{\omega_2} \dot{y}_{2,t}$ are also held. In addition, since $\frac{\partial(y_{1,t} + y_{2,t})}{\partial A_{1,t}} = \frac{\partial(y_{1,t} + y_{2,t})}{\partial A_{2,t}}$ by arbitration, $\dot{A}_{1,t} = \frac{\omega_1}{\omega_2} \dot{A}_{2,t}$ is held. Because of equation (9) and $f = \omega_i^\alpha k_{i,t}^{1-\alpha}$, then $A_t = \frac{\varpi\alpha}{2mv(1-\alpha)\omega_1^\alpha}\left(\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha k_2^{1-\alpha}\right) = \frac{\varpi\alpha}{2mv(1-\alpha)\omega_2^\alpha}\left(\omega_1^\alpha k_1^{1-\alpha} k_2^\alpha + \omega_2^\alpha k_2\right)$, $\frac{\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha k_2^{1-\alpha}}{\omega_1^\alpha} = \frac{\omega_1^\alpha k_1^{1-\alpha} k_2^\alpha + \omega_2^\alpha k_2}{\omega_2^\alpha}$, and $\frac{\partial y_{i,t}}{\partial k_{i,t}} = \left(\frac{\varpi\alpha}{2mv}\right)^\alpha (1-\alpha)^{1-\alpha}\left(\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha k_2^{1-\alpha}\right)^\alpha k_1^{-\alpha} = \left(\frac{\varpi\alpha}{2mv}\right)^\alpha (1-\alpha)^{1-\alpha}\left(\omega_1^\alpha k_1^{1-\alpha} k_2^\alpha + \omega_2^\alpha k_2\right)^\alpha k_2^{-\alpha}$. Since $\frac{\omega_2}{\omega_1} k_{1,t} = k_{2,t}$, then $\frac{\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha k_2^{1-\alpha}}{\omega_1^\alpha} = \frac{\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha \left(\frac{\omega_2}{\omega_1}\right)^{1-\alpha} k_1^{1-\alpha}}{\omega_1^\alpha} = k_1\left(1 + \omega_1^{-1}\omega_2\right)$ and $\frac{\omega_1^\alpha k_1^{1-\alpha} k_2^\alpha + \omega_2^\alpha k_2}{\omega_2^\alpha} = \frac{\omega_1^\alpha k_1^{1-\alpha}\left(\frac{\omega_2}{\omega_1}\right)^\alpha k_1^\alpha + \omega_2^\alpha \frac{\omega_2}{\omega_1} k_1}{\omega_2^\alpha} = k_1 + \frac{\omega_2}{\omega_1} k_1 = k_1\left(1 + \omega_1^{-1}\omega_2\right) = k_2\left(1 + \omega_1\omega_2^{-1}\right)$. Hence,

$$A_t = k_1 \frac{\varpi\alpha\left(1 + \omega_1^{-1}\omega_2\right)}{2mv(1-\alpha)} = k_2 \frac{\varpi\alpha\left(1 + \omega_1\omega_2^{-1}\right)}{2mv(1-\alpha)} \quad ,$$

and

$$\frac{\partial y_{i,t}}{\partial k_{i,t}} = \left(\frac{\omega_1 + \omega_2}{2}\right)^\alpha \left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{1-\alpha}$$

for i = 1, 2. Because $\dot{A}_{1,t} = \left(\frac{\omega_2}{\omega_1}\right)^{-\frac{1}{\alpha}} \dot{A}_{2,t}$ (i.e., $\dot{A}_t = \dot{A}_{1,t} + \dot{A}_{2,t} = \left(1 + \omega_1^{-1}\omega_2\right)\dot{A}_{1,t}$) and $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$, then

$$\begin{aligned}
\dot{k}_{1,t} &= y_{1,t} + \frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t - c_{1,t} - g_t - v\dot{A}_{1,t}\left(\frac{L_t}{2}\right)^{-1} \\
&= y_{1,t} + \frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t - c_{1,t} - g_t - v\dot{A}_t\left(1 + \omega_1^{-1}\omega_2\right)^{-1}\left(\frac{L_t}{2}\right)^{-1} \\
&= \omega_1^\alpha\left[\frac{\left(1 + \omega_1^{-1}\omega_2\right)\varpi\alpha}{2mv(1-\alpha)}\right]^\alpha k_{1,t} + \left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv}\right]^\alpha (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds - \tau_t - c_{1,t} - g_t - \frac{\varpi\alpha}{mL_t(1-\alpha)} \dot{k}_{1,t} \quad ,
\end{aligned}$$

and

$$\dot{k}_{1,t} = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} k_{1,t} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} - g_t\right\} .$$

Because L_t is sufficiently large and ϖ is far smaller than M_t and thus $\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha} = 1$, the optimization problem of economy 1 is

$$Max\ E\int_0^{\infty} u_1(c_{1,t})\exp(-\theta t)dt \quad ,$$

subject to

$$\dot{k}_{1,t} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} k_{1,t} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} - g_t \quad ,$$

and similarly, that of economy 2 is

$$Max\ E\int_0^{\infty} u_2(c_{2,t})\exp(-\theta t)dt \quad ,$$

subject to

$$\dot{k}_{2,t} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} k_{2,t} - \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t} + g_t \quad .$$

2.2 *The multilateral path*

Heterogeneity is defined as being sustainable if all the optimality conditions of all heterogeneous households are satisfied indefinitely. Although the previously discussed state of Becker (1980) is Pareto efficient, by this definition, the heterogeneity is not sustainable because only the most patient household can achieve optimality. Sustainability is therefore the stricter criterion for welfare than Pareto efficiency.

In this section, the growth path that makes heterogeneity sustainable is examined. First, the basic natures of the models presented in Section 2.1 when the government does not intervene, i.e., $\bar{g} = 0$ are examined.

2.1.1 The consumption growth rate

2.2.3 Sustainability

Because balanced growth is the focal point for the growth path analysis, the following analyses focus on the steady state such that $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$, $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, $\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}$, $\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$, and $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}$ are constants.

2.2.3.1 Heterogeneous time preference model

The balanced growth path in the heterogeneous time preference model has the following properties.

Lemma 2-1: In the model of heterogeneous time preference, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} = \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} .$$

Proof: See Harashima (2010)

Proposition 1-1: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$ $= \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, all the optimality conditions of both economies are satisfied at steady state.

The path on which $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant has the following properties.

Proof: See Harashima (2010)

Corollary 1-1: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} =$ $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} = \lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}} = \lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t} = \text{constant.}$$

Proof: See Harashima (2010)

Note that the limit of the growth rate on this path is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right] . \tag{10}$$

[4]

Corollary 2-1: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} =$ $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant,

[4] See Harashima (2010)

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\int_0^t \tau_s ds}{dt}}{\int_0^t \tau_s ds}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$$

$$=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\text{constant.}$$

Proof: See Harashima (2010)

Because current account imbalances eventually grow at the same rate as output, consumption, and capital on the multilateral path, the ratios of the current account balance to output, consumption, and capital do not explode, but they stabilize as shown in the proof of Proposition 1-1; that is, $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}=\lim_{t\to\infty}\frac{\tau_t}{k_{2,t}}=\Xi$.

On the balanced growth path satisfying Proposition 1-1 and Corollaries 1-1 and 2-1, heterogeneity in time preference is sustainable by definition because all the optimality conditions of the two economies are indefinitely satisfied. The balanced growth path satisfying Proposition 1-1 and Corollaries 1-1 and 2-1 is called the "multilateral balanced growth path" or (more briefly) the "multilateral path" in the following discussion. The term "multilateral" is used even though there are only two economies, because the two-economy models shown can easily be extended to the multi-economy models shown in Section 2.2.6.

Because technology will not decrease persistently (i.e., $\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}>0$), only the case such that $\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}>0$ (i.e., $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}>0$ on the multilateral path by Corollary 1-1) is examined in the following discussion.

2.2.3.2 Heterogeneous risk aversion model

On the multilateral path in the heterogeneous risk aversion model, the same Proposition, Lemmas, and Corollaries are proved by arguments similar to those shown in Section 2.2.3.1.

Lemma 2-2: In the model of heterogeneous risk aversion, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}\ .$$

Proposition 1-2: In the model of heterogeneous risk aversion, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant, all the optimality conditions of both economies are satisfied at steady state.

Corollary 1-2: In the model of heterogeneous risk aversion, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\text{constant.}$$

Note that the limit of the growth rate on this path is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta\right] \text{.[5]} \tag{11}$$

Corollary 2-2: In the model of heterogeneous risk aversion, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\int_0^t\tau_s ds}{dt}}{\int_0^t\tau_s ds}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$$
$$=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\text{constant.}$$

On the balanced growth path satisfying Proposition 1-2 and Corollaries 1-2 and 2-2, heterogeneity in risk aversion is also sustainable by definition because all the optimality conditions of the two economies are indefinitely satisfied, and this path is the multilateral path.

2.2.3.3 Heterogeneous productivity model

Similar Proposition, Lemmas, and Corollaries also hold in the heterogeneous productivity model. However, unlike heterogeneous preferences, $\lim_{t\to\infty}\tau_t=0$ and $\lim_{t\to\infty}\int_0^t\tau_s ds=0$ are possible even if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$ as equations (24) and (25) indicate. Therefore, the case of $\lim_{t\to\infty}\tau_t=0$ and $\lim_{t\to\infty}\int_0^t\tau_s ds=0$ will be dealt with separately from the case of $\lim_{t\to\infty}\tau_t\neq 0$ and $\lim_{t\to\infty}\int_0^t\tau_s ds\neq 0$ if necessary.

Lemma 2-3: In the model of heterogeneous productivity, if $\lim_{t\to\infty}\tau_t=0$ and $\lim_{t\to\infty}\int_0^t\tau_s ds=0$,

[5] See Harashima (2010)

then if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} \quad ,$$

and if $\lim_{t\to\infty}\tau_t\neq 0$ and $\lim_{t\to\infty}\int_0^t\tau_s ds\neq 0$, then if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$$

and

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t\tau_s ds\right)}{dt}}{\int_0^t\tau_s ds}=\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} \quad .$$

By Lemma 2-3, if all the optimality conditions of both economies are satisfied, either

$$\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}=\lim_{t\to\infty}\frac{\int_0^t\tau_s ds}{k_{1,t}}=0 \tag{12}$$

or

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t\tau_s ds\right)}{dt}}{\int_0^t\tau_s ds}=\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} \quad . \tag{13}$$

Proposition 1-3: If and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant, all the optimality conditions of both economies are satisfied at steady state.

Corollary 1-3: In the model of heterogeneous productivity, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=$ $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\text{constant.}$$

Corollary 2-3: In the model of heterogeneous productivity, if $\lim_{t\to\infty}\tau_t \neq 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds \neq 0$, then if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \text{constant}$,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} = \lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}} = \lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t} = \text{constant}$$

and

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} .$$

On the two balanced growth paths satisfying Proposition 1-3 and Corollaries 1-3 and 2-3, heterogeneity in productivity is sustainable by definition because all the optimality conditions of the two economies are indefinitely satisfied.

By equations (24) and (25), the limit of the growth rate on these sustainable paths is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \theta\right\} .$$

2.2.4 The balance of payments

2.2.4.1 Heterogeneous time preference model

As shown in the proof of Proposition 1-1, $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}} = \lim_{t\to\infty}\frac{\tau_t}{k_{2,t}} = \Xi$ and $\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} = \lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{2,t}} = \Xi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}$ on the multilateral path. Because $k_{i,t}$ is positive, if the sign of Ξ is negative, the current account of economy 1 will eventually show permanent deficits and vice versa.

Lemma 3-1: In the model of heterogeneous time preference,

$$\Xi = \frac{\theta_1-\theta_2}{2}\left\{\varepsilon\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right]^{-1} - 1\right\}^{-1} .$$

Proof: See Harashima (2010)

Lemma 3-1 indicates that the value of Ξ is uniquely determined on the multilateral path, and the sign of Ξ is also therefore uniquely determined.

Proposition 2-1: In the model of heterogeneous time preference, $\Xi < 0$ if $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}[1-(1-\alpha)\varepsilon] < \frac{\theta_1+\theta_2}{2}$.

Proof: See Harashima (2010)

Proposition 2-1 indicates that the current account deficit of economy 1 and the current account surplus of economy 2 continue indefinitely on the multilateral path. The condition $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}[1-(1-\alpha)\varepsilon] < \frac{\theta_1+\theta_2}{2}$ is generally satisfied for reasonable parameter values.

Conversely, the opposite is true for the trade balance.

Corollary 3-1: In the model of heterogeneous time preference, $\lim_{t\to\infty}\left(\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds\right) > 0$ if $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}[1-(1-\alpha)\varepsilon] < \frac{\theta_1+\theta_2}{2}$.

Proof: See Harashima (2010)

Corollary 3-1 indicates that, on the multilateral path, the trade surpluses of economy 1 continue indefinitely and vice versa. That is, goods and services are transferred from economy 1 to economy 2 in each period indefinitely in exchange for the returns on the accumulated current account deficits (i.e., debts) of economy 1.

Nevertheless, the trade balance of economy 1 is not a surplus from the beginning. Before Corollary 3-1 is satisfied, negative $\int_0^t \tau_s ds$ should be accumulated. In the early periods, when $\int_0^t \tau_s ds$ is small, the balance on goods and services of economy 1 ($\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds$) continues to be a deficit. After a sufficient negative amount of $\int_0^t \tau_s ds$ is accumulated, the trade balances of economy 1 shift to surpluses.

Current account deficit of economy 1 means for example that a firm that is owned by economy 1 borrows money from a bank in which economy 2 deposits money. Economy 1 indirectly borrows money from economy 2. This situation can be easily understood if you see the current account deficit of the United States.

2.2.4.2 Heterogeneous risk aversion model

Similarly, the value of Ξ in the heterogeneous risk aversion model is uniquely determined on the multilateral path.

Lemma 3-2: In the model of heterogeneous risk aversion,

$$\Xi = \frac{(\varepsilon_1-\varepsilon_2)\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta\right]}{(\varepsilon_1+\varepsilon_2)\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta\right]^{-1}-1\right\}} .$$

Proposition 2-2: In the model of heterogeneous risk aversion, $\Xi < 0$ if $1-\theta\left(\frac{\varpi\alpha}{mv}\right)^{-\alpha}(1-\alpha)^{-1+\alpha} < \frac{\varepsilon_1+\varepsilon_2}{2}$.

The condition $1-\theta\left(\frac{\varpi\alpha}{mv}\right)^{-\alpha}(1-\alpha)^{-1+\alpha} < \frac{\varepsilon_1+\varepsilon_2}{2}$ is generally satisfied for reasonable parameter values.

Corollary 3-2: In the model of heterogeneous risk aversion, $\lim_{t\to\infty}\left(\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds\right) > 0$.

By Lemma 3-2 and equations (21) and (22), the limit of the growth rate on the multilateral path is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta\right] .$$

2.2.4.3 Heterogeneous productivity model

As Lemma 2-3 shows, on the multilateral path, either $\lim_{t\to\infty}\tau_t = 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds = 0$ or $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}$. On the former path, $\Xi = 0$ and heterogeneous productivity does not result in permanent trade imbalances. However, on the latter path, trade imbalances usually grow at a higher rate than consumption, because usually $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} > \varepsilon^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}-\theta\right\} = \lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$; thus, Ξ explodes to infinity. Hence, the latter path will generally not be selected. The question of which path is selected is examined in detail in the Section 2.3.3.

2.2.5 A model with heterogeneities in multiple elements

The three heterogeneities are not exclusive. It is particularly likely that heterogeneities in time preference and productivity coexist. Many empirical studies conclude that the rate of time preference is negatively correlated with income (e.g., Lawrance, 1991; Samwick, 1998; Ventura, 2003); this indicates that the economy with the higher productivity has a lower rate of time preference and vice versa. In this section, the models are extended to include heterogeneity in multiple elements.

Suppose that economies 1 and 2 are identical except for time preference, risk aversion, and productivity. The Hamiltonian for economy 1 is

$$H_1 = u_1(c_{1,t})\exp(-\theta_1 t) + \lambda_{1t}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} k_{1,t} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t}\right\} ,$$

and that for economy 2 is

$$H_2 = u_2(c_{2,t})\exp(-\theta_2 t) + \lambda_{2,t}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} k_{2,t} - \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t}\right\} .$$

The growth rates are

$$\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon_1^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}} - \frac{\partial \tau_t}{\partial k_{1,t}} - \theta_1\right\} ,$$

and

$$\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon_2^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{2,t}} + \frac{\partial \tau_t}{\partial k_{2,t}} - \theta_2\right\} .$$

Here, $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}} = \Xi$, $\lim_{t\to\infty}\frac{\tau_t}{k_{2,t}} = \frac{\omega_1}{\omega_2}\Xi$, $\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} = \Xi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}$, and $\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{2,t}} = \frac{\omega_1}{\omega_2}\Xi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}$ at steady state. Thus, $\Xi = \dfrac{\frac{\varepsilon_2}{\varepsilon_1}\theta_1 - \theta_2 - \left(\frac{\varepsilon_2}{\varepsilon_1} - 1\right)\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}}{\left(\frac{\varepsilon_2}{\varepsilon_1} + \frac{\omega_1}{\omega_2}\right)\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1} - 1\right\}}$,
and the limit of the growth rate on the multilateral path is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\varepsilon_1\omega_1 + \varepsilon_2\omega_2}{\omega_1+\omega_2}\right)^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\,\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \frac{\theta_1\omega_1+\theta_2\omega_2}{\omega_1+\omega_2}\right\} . \tag{14}$$

Clearly, if $\varepsilon_1 = \varepsilon_2$ and $\omega_1 = \omega_2 = 1$, then $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon_1^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right]$; if $\theta_1 = \theta_2$ and $\omega_1 = \omega_2 = 1$, then $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta_1\right]$; and if $\theta_1 = \theta_2$ and $\varepsilon_1 = \varepsilon_2$, then $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon_1^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \theta_1\right\}$ as shown in Sections 2.2.3 and 2.2.4.

The sign of Ξ on the multilateral path depends on the relative values between θ_1 and θ_2,

ε_1 and ε_2, and ω_1 and ω_2. Nevertheless, if the rate of time preference and productivity are negatively correlated, as argued above (i.e., if $\theta_1 < \theta_2$ and $\omega_1 > \omega_2$ while $\varepsilon_1 = \varepsilon_2$), then by similar proofs as those presented for Proposition 2-1 and Corollary 3-1, if $\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}\left[1-(1-\alpha)^{1-\alpha}\varepsilon_1\right] < \frac{\omega_1\theta_1+\omega_2\theta_2}{\omega_1+\omega_2}$, then $\Xi < 0$ and $\lim_{t\to\infty}\left(\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds\right) > 0$ on the multilateral path; that is, the current account deficits and trade surpluses of economy 1 continue indefinitely. The condition $\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}\left[1-(1-\alpha)^{1-\alpha}\varepsilon_1\right] < \frac{\omega_1\theta_1+\omega_2\theta_2}{\omega_1+\omega_2}$ is generally satisfied for reasonable parameter values.

2.2.6 Multi-economy models

The two-economy models can be extended to include numerous economies that have differing degrees of heterogeneity.

2.2.6.1 Heterogeneous time preference model

Suppose that there are H economies that are identical except for time preference. Let θ_i be the rate of time preference of economy i and $\tau_{i,j,t}$ be the current account balance of economy i with economy j, where $i = 1, 2, \dots, H$, $j = 1, 2, \dots, H$, and $i \neq j$. Because the total population is L_t, the population in each economy is $\frac{L_t}{H}$. The representative household of economy i maximizes its expected utility

$$E\int_0^{\infty} u_i(c_{i,t})\exp(-\theta_i t)dt \quad ,$$

subject to

$$\dot{k}_{i,t} = y_{i,t} + \sum_{j=1}^{H}\frac{\partial y_{j,t}}{\partial k_{j,t}}\int_0^t \tau_{i,j,s}ds - \sum_{j=1}^{H}\tau_{i,j,t} - c_{i,t} - v\dot{A}_{i,t}\left(\frac{L_t}{H}\right)^{-1}$$

for $i \neq j$.

Proposition 3-1: In the multi-economy model of heterogeneous time preference, if and only if

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\sum_{q=1}^{H}\theta_q}{H}\right] \tag{15}$$

for any i, all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}}=\lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}}=\lim_{t\to\infty}\frac{\frac{d\int_0^t\tau_{i,j,s}ds}{dt}}{\int_0^t\tau_{i,j,s}ds}$$

for any i and j ($i \neq j$).
Proof: See Harashima (2010)

2.2.6.2 Heterogeneous risk aversion model

The heterogeneous risk aversion model can be extended to the multi-economy model by a proof similar to that for Proposition 3-1. Suppose that H economies are identical except for risk aversion, and their degrees of risk aversion are ε_i (i = 1, 2, … , H).

Proposition 3-2: In the multi-economy model of heterogeneous risk aversion, if and only if

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\left(\frac{\sum_{q=1}^{H}\varepsilon_q}{H}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta\right] \tag{16}$$

for any i, all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}}=\lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}}=\lim_{t\to\infty}\frac{\frac{d\int_0^t\tau_{i,j,s}ds}{dt}}{\int_0^t\tau_{i,j,s}ds}$$

for any i and j ($i \neq j$).

2.2.6.3 Heterogeneous productivity model

The heterogeneous productivity model can also be extended by a proof similar to that for Proposition 3-1. Suppose that H economies are identical except for productivity, and their productivities are ω_i (i = 1, 2, … , H). Note that, because $k_{1+2,t}=k_{1,t}+k_{2,t}=k_{2,t}\left[\frac{\omega_1}{\omega_2}+1\right]$, the productivity of economy 1+2 is $y_{1+2,t}=A_t^{\alpha}\left(\omega_1^{\alpha}k_{1,t}^{1-\alpha}+\omega_2^{\alpha}k_{2,t}^{1-\alpha}\right)=(\omega_1+\omega_2)^{\alpha}A_t^{\alpha}k_{1+2,t}^{1-\alpha}$.

Proposition 3-3: In the multi-economy model of heterogeneous productivity, if and only if

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\varepsilon^{-1}\left\{\left[\frac{\left(\sum_{q=1}^{H}\omega_q\right)\varpi\alpha}{Hmv(1-\alpha)}\right]^{\alpha}-\theta\right\}$$

for any i, all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}}=\lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}$$

for any i and j ($i \neq j$).

2.2.6.4 Heterogeneity in multiple elements

Similarly, the multi-economy model can be extended to heterogeneity in multiple elements, as follows.

Proposition 3-4: In the multi-economy model of heterogeneity in multiple elements, if and only if

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\left(\frac{\sum_{q=1}^{H}\varepsilon_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right)^{-1}\left\{\left[\frac{\varpi\alpha\sum_{q=1}^{H}\omega_q}{Hmv(1-\alpha)}\right]^{\alpha}-\frac{\sum_{q=1}^{H}\theta_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right\} \tag{17}$$

for any i (= 1, 2, … , H), all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}}=\lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}}=\lim_{t\to\infty}\frac{\frac{d\int_0^t\tau_{i,j,s}ds}{dt}}{\int_0^t\tau_{i,j,s}ds}$$

for any i and j ($i \neq j$).

Proposition 3-4 implies that the concept of the representative household in a heterogeneous population implicitly assumes that all households are on the multilateral path.

2.3 The unilateral path

The multilateral path satisfies all the optimality conditions, but that does not mean that the two economies naturally select the multilateral path. Ghiglino (2002) predicts that it is likely that, under appropriate assumptions, the results of Becker (1980) still hold in endogenous growth models. Farmer and Lahiri (2005) show that balanced growth equilibria do not exist in a multi-agent economy in general, except in the special case that all agents have the same constant rate of time preference. How the economies behave in the environments described in Sections 2 and 3 when the government does not intervene, i.e., $\bar{g}=0$. is examined in this section.

2.3.1 Heterogeneous time preference model

The multilateral path is not the only path on which all the optimality conditions of economy 1 are satisfied. Even if economy 1 behaves unilaterally, it can achieve optimality, but economy 2 cannot.

Lemma 4-1: In the heterogeneous time preference model, if each economy sets τ_t without regarding the other economy's optimality conditions, then it is not possible to satisfy all the optimality conditions of both economies.
Proof: See Harashima (2010)

Since $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\varepsilon^{-1}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}+\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}\left(\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}\right)^{-1}-\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}-\theta_1\right\}$

at steady state, all the optimality conditions of economy 1 can be satisfied only if either

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t\tau_s ds\right)}{dt}}{\int_0^t\tau_s ds}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} \tag{18}$$

or

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t\tau_s ds\right)}{dt}}{\int_0^t\tau_s ds}=\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} \quad . \tag{19}$$

That is, $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$ can be constant only when either equation (18) or (19) is satisfied. Conversely, economy 1 has two paths on which all its optimality conditions are satisfied. Equation (18) indicates that $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}=$ constant, and equation (19) indicates that $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left(\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}\right)^{-1}-1=0$ for any $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}$. Equation (18) corresponds to the multilateral path. On the path satisfying equation (19), $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\neq\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t\tau_s ds\right)}{dt}}{\int_0^t\tau_s ds}$, and $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}>\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$. Here, by equations (6) and (7),

$$c_{1,t}-c_{2,t}=2\left(\frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t\tau_s ds-\tau_t\right)=2\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t\tau_s ds-\tau_t\right] \quad ,$$

and

$$\lim_{t\to\infty}(c_{1,t}-c_{2,t})=0$$

is required because $\lim_{t\to\infty}\frac{\tau_t}{\int_0^t \tau_s ds}=\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}$. However, because $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}>\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, economy 2 must initially set consumption such that $c_{2,0}=\infty$, which violates the optimality condition of economy 2. Therefore, unlike with the multilateral path, all the optimality conditions of economy 2 cannot be satisfied on the path satisfying equation (19) even though those of economy 1 can. Hence, economy 2 has only one path on which all its optimality conditions can be satisfied—the multilateral path. The path satisfying equation (19) is called the "unilateral balanced growth path" or the "unilateral path" in the following discussion. Clearly, heterogeneity in time preference is not sustainable on the unilateral path.

How should economy 2 respond to the unilateral behavior of economy 1? Possibly, both economies negotiate for the trade between them, and some agreements may be reached. If no agreement is reached, however, and economy 1 never regards economy 2's optimality conditions, economy 2 generally will fall into the following unfavorable situation.

Remark 1-1: In the model of heterogeneous time preference, if economy 1 does not regard the optimality conditions of economy 2, the ratio of economy 2's debts (owed to economy 1) to its consumption explodes to infinity while all the optimality conditions of economy 1 are satisfied.

The reasoning behind Remark 1-1 is as follows. When economy 1 selects the unilateral path and sets $c_{1,0}$ so as to achieve this path, there are two options for economy 2. The first option is for economy 2 to also pursue its own optimality without regarding economy 1: that is, to select its own unilateral path. The second option is to adapt to the behavior of economy 1 as a follower. If economy 2 takes the first option, it sets $c_{2,0}$ without regarding $c_{1,0}$. As the proof of Lemma 4-1 indicates, unilaterally optimal growth rates are different between the two economies and $\frac{\dot{c}_{1,t}}{c_{1,t}}>\frac{\dot{c}_{2,t}}{c_{2,t}}$; thus, the initial consumption should be set as $c_{1,0}<c_{2,0}$. Because $\frac{\partial y_{1,t}}{\partial k_{1,t}}=\frac{\varpi}{2mv}\frac{\partial(y_{1,t}+y_{2,t})}{\partial A_t}=\frac{\partial y_{2,t}}{\partial k_{2,t}}$ and $k_{1,t}=k_{2,t}$ must be kept, capital and technology are equal and grow at the same rate in both economies. Hence, because $c_{1,0}<c_{2,0}$, more capital is initially produced in economy 1 than in economy 2 and some of it will need to be exported to economy 2. As a result, $\frac{\dot{c}_{1,t}}{c_{1,t}}>\frac{\dot{k}_{1,t}}{k_{1,t}}=\frac{\dot{k}_{2,t}}{k_{2,t}}>\frac{\dot{c}_{2,t}}{c_{2,t}}$, which means that all the optimality conditions of both economies cannot be satisfied. Since $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}>\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}>\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, capital soon becomes abundant in economy 2, and excess goods and services are produced in that economy. These excess products are exported to and utilized in economy 1. This process escalates as time passes because $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}>\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}>\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, and eventually almost all consumer goods and services produced in economy 2 are consumed by households in economy 1. These consequences will be unfavorable for economy 2.

If economy 2 takes the second option, it should set $c_{2,0}=\infty$ to satisfy all its optimality conditions, as the proof of Lemma 4-1 indicates. Setting $c_{2,0}=\infty$ is impossible, but economy 2 as the follower will initially set $c_{2,t}$ as large as possible. This action gives economy 2 a higher expected utility than that of the first option, because consumption in economy 2 in the second case is always higher. As a result, economy 2 imports as many goods and services as possible from economy 1, and the trade deficit of economy 2 continues until $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds = \tau_t$ is achieved; this is, $\frac{\dot{\tau}_t}{\tau_t} = \frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$ is achieved. The current account deficits and the accumulated debts of economy 2 will continue to increase indefinitely. Furthermore, they will increase more rapidly than the growth rate of outputs ($\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}$) because, in general, $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} < \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}$; that is, $(1-\varepsilon)\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} < \theta_1(<\theta_2)$. If no disturbance occurs, the expansion of debts may be sustained forever, but economy 2 becomes extremely vulnerable to even a very tiny negative disturbance. If such a disturbance occurs, economy 2 will lose all its capital and will no longer be able to repay its debts. This result corresponds to the state shown by Becker (1980), and it will also be unfavorable for economy 2. Because $\lim_{t\to\infty}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t\right] = 0$, inequality (27) holds, and the transversality condition for economy 1 is satisfied by Lemma 1-1. Thus, all the optimality conditions of economy 1 are satisfied if economy 2 takes the second option.

As a result, all the optimality conditions of economy 2 cannot be satisfied in any case if economy 1 takes the unilateral path. Both options to counter the unilateral behavior of economy 1 are unfavorable for economy 2. However, the expected utility of economy 2 is higher if it takes the second option rather than the first, and economy 2 will choose the second option. Hence, if economy 1 does not regard economy 2's optimality conditions, the debts owed by economy 2 to economy 1 increase indefinitely at a higher rate than consumption.

2.3.2 Heterogeneous risk aversion model

The same consequences are observed in this model.

Lemma 4-2: In the model of heterogeneous risk aversion, if each economy sets τ_t without regard for the other economy's optimality conditions, then all the optimality conditions of both economies cannot be satisfied.

Therefore, heterogeneity in risk aversion is not sustainable on the unilateral path.

Remark 1-2: In the model of heterogeneous risk aversion, if economy 1 does not regard economy 2's optimality conditions, the ratio of economy 2's debts (owed to economy 1) to its consumption explodes to infinity while all the optimality conditions of economy 1 are satisfied.

2.3.3 Heterogeneous productivity model

Unlike the heterogeneous preferences shown in Sections 2.3.1 and 2.3.2, heterogeneity in productivity can be sustainable even on the unilateral path.

Lemma 4-3: In the heterogeneous productivity model, even if each economy sets τ_t without regard for the other economy's optimality conditions, it is possible that all the optimality conditions of both economies are satisfied if

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\dfrac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}=\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} \quad .$$

Proof: See Harashima (2010)

All the optimality conditions of economy 1 can be satisfied only if either equation (12) or (13) holds, because $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$ can be constant only when equation (12) or (13) holds. Equation (12) corresponds to the multilateral path, and equation (13) corresponds to the unilateral path. Unlike the heterogeneity in preferences, Lemma 4-3 shows that, even on the unilateral path, all the optimality conditions of both economies are satisfied because the limit of both economies' growth rates is identical on the path of either equation (12) or (13), such that $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\varepsilon^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}-\theta\right\}$. Therefore, heterogeneity in productivity is sustainable even on the unilateral path.

Nevertheless, on the unilateral path, current account imbalances generally grow steadily at a higher rate than consumption; this is not the case on the multilateral path. How does economy 1 set τ? If economy 1 imports as many goods and services as possible before reaching the steady state at which $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}=\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}$ (i.e., if it initially sets τ_i as $\tau_t < 0$ and $\tau_t-\frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds<0$), the expected utility of economy 1 will be higher than it is in either case where $\tau_t > 0$ or in the multilateral path. However, the debts economy 1 owes to economy 2 will grow indefinitely at a higher rate than consumption, and the ratio of debt to consumption explodes to infinity. If there is no disturbance, this situation will be sustained forever, but economy 1 will become extremely vulnerable to even a very tiny negative disturbance. Hence, the unilateral path will not necessarily be favorable for economy 1 although all its optimality conditions are satisfied on this path, and economy 1 will prefer the multilateral path.

Remark 1-3: In the heterogeneous productivity model, even though economy 1 does not regard economy 2's optimality conditions, the multilateral balanced growth path will be selected.

Hence, the state shown by Becker (1980) will not be observed in the case of heterogeneous productivity.

2.3.4 Doom of the less advantaged economies

Remarks 1-1 and 1-2 indicate that economy 2's ratio of debt to consumption continues to increase indefinitely on the unilateral path. Such an indefinitely increasing ratio may not matter if there is no shock or disturbance. However, if even a very tribunal negative shock occurs, economy 2 will be ruined because the huge amount of accumulated debts cannot be refinanced. In this case, "ruin" means that economy 2 will go bankrupt or be exterminated because its consumption has to be zero unless the authority intervenes to some extent (e.g., debt relief after personal bankruptcy). Even if economy 2 continues to exist by the mercy of economy 1, it will fall into a slave-like state indefinitely without the authority's intervention.

3 SUSTAINABLE HETEROGENEITY WITH GOVERNMENT INTERVENTION

Sustainable heterogeneity, as described in this paper, is a very different state from what Becker (1980) described. The difference emerges because, on a multilateral path, economy 1 behaves fully considering economy 2's situation. The multilateral path therefore will not be naturally selected by economy 1, and the path selection may have to be decided politically (Harashima, 2010). On the other hand, when economy 1 behaves unilaterally, the government may intervene in economic activities so as to achieve, for example, social justice.

In this section, I show that even if economy 1 behaves unilaterally, sustainable heterogeneity can always be achieved with appropriate government intervention.

3.1 Heterogeneous time preference model

Government intervention was first considered in the two-economy model constructed in Section 2. If the government intervenes (i.e., $\bar{g} > 0$),

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} \neq \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$$

and

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta_1 - \bar{g}\right\}$$

on the path satisfying equation (19). At the same time,

$$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta_2 + \bar{g}\right\} .$$

Therefore, if

$$\bar{g} = \frac{\theta_2 - \theta_1}{2} ,$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right] . \tag{20}$$

Equation (20) is identical to equation (10). The government's appropriate redistribution from economy 1 to economy 2 by $\bar{g}$ leads to the same consequence with a multilateral path.

Note that if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} > \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$ or $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} < \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, neither economy can achieve optimality. In this sense, the only appropriate amount of government intervention is $\bar{g} = \frac{\theta_2 - \theta_1}{2}$.

3.2 Heterogeneous risk aversion model

Similarly, all of the optimality conditions of economy 1 can be satisfied only if either equation (18) or (19) is satisfied, and if

$$\bar{g} = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right] ,$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right]. \tag{21}$$

Equation (21) is identical to equation (11). Similar to the case with a heterogeneous time preference, the government's appropriate redistribution by $\bar{g} = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right]$ leads to the same consequence with a multilateral path.

3.3 Heterogeneous productivity model

Heterogeneity in productivity can be sustainable even on the unilateral path. Hence, government intervention is not necessary; that is, even if $\bar{g} = 0$, the unilateral path is sustainable.

3.4 A model with heterogeneities in multiple elements

By similar procedures as those used in Sections 3.1 and 3.2, if

$$\bar{g} = \left\{(\varepsilon_2 - \varepsilon_1)\left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} + \varepsilon_1\theta_2 - \varepsilon_2\theta_1\right\}\left(\varepsilon_1\frac{\omega_1}{\omega_2} + \varepsilon_2\right)^{-1} ,$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\omega_1\varepsilon_1 + \omega_2\varepsilon_2}{\omega_1 + \omega_2}\right)^{-1}\left\{\left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \frac{\omega_1\theta_1 + \omega_2\theta_2}{\omega_1 + \omega_2}\right\} , \tag{22}$$

and equation (22) is identical to equation (14).

3.5 Multi-economy models

3.5.1 Heterogeneous time preference model

If H = 2, when sustainable heterogeneity is achieved, economies 1 and 2 consist of a combined economy (economy 1+2) with twice the population and a rate of time preference of $\frac{\theta_1+\theta_2}{2}$. Suppose there is a third economy with a time preference of θ_3. Because economy 1+2 has twice the population of economy 3, if

$$\bar{g} = \frac{\theta_3 - \frac{\theta_1+\theta_2}{2}}{3} \quad ,$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{c}_{3,t}}{c_{3,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\sum_{q=1}^{3}\theta_q}{3}\right] \ .$$

By iterating similar procedures, if the government's transfers between economy H and economy 1+2+ · · · + (H – 1) is such that

$$\bar{g} = \frac{\theta_H - \frac{\sum_{q=1}^{H-1}\theta_q}{H-1}}{H} \quad ,$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\sum_{q=1}^{H}\theta_q}{H}\right] \tag{23}$$

for any i (= 1, 2, · · ·, H). Equation (23) is the same as equation (15).

3.5.2 Heterogeneous risk aversion model

By a similar procedure as that used for heterogeneous time preference, if the sum of the government transfers between economy H and economy 1+2+ · · · + (H – 1) is such that

$$\bar{g} = \frac{\varepsilon_H - \frac{\sum_{q=1}^{H-1}\varepsilon_q}{H-1}}{\sum_{q=1}^{H}\varepsilon_q}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right] \quad ,$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\left(\frac{\sum_{q=1}^{H}\varepsilon_q}{H}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta\right] \tag{24}$$

for any i (= 1, 2, · · ·, H). Equation (24) is the same as equation (16).

3.5.3 Heterogeneous productivity model

As discussed in Section 3.3, even if government transfers between economy H and economy 1+2+ · · · + (H – 1) is nil (i.e., $\bar{g}=0$), the unilateral path is sustainable.

3.5.4 Heterogeneity in multiple elements

By combining the procedures and results presented in Section 3.4, 3.5.1 and 3.5.2, it can be shown that, if the sum of government transfers between economy H and economy 1+2+ · · · + (H – 1) is such that

$$\bar{g}=\left[\frac{\sum_{q=1}^{H}\varepsilon_q\omega_q}{\omega_H}\right]^{-1}\left\{\frac{\left(\varepsilon_H\sum_{q=1}^{H}\omega_q-\sum_{q=1}^{H}\varepsilon_q\omega_q\right)}{\sum_{q=1}^{H-1}\omega_q}\left[\frac{\varpi\alpha\sum_{q=1}^{H}\omega_q}{Hmv(1-\alpha)}\right]^{\alpha}-\frac{\varepsilon_H\sum_{q=1}^{H}\theta_q\omega_q-\theta_H\sum_{q=1}^{H}\varepsilon_q\omega_q}{\sum_{q=1}^{H-1}\omega_q}\right\}, \tag{25}$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\left(\frac{\sum_{q=1}^{H}\varepsilon_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right)^{-1}\left\{\left[\frac{\varpi\alpha\sum_{q=1}^{H}\omega_q}{Hmv(1-\alpha)}\right]^{\alpha}-\frac{\sum_{q=1}^{H}\theta_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right\}. \tag{26}$$

for any i (= 1, 2, · · ·, H). Equation (26) is the same as equation (17).

3.6 Models of partially unilateral behaviors

Here, suppose that economy 1 undertakes partly unilateral and partly multilateral behaviors such that ψ of $k_{1,0}$ is allocated to the unilateral path and (1 − ψ) of $k_{1,0}$ is allocated to the multilateral path, where 0 < ψ < 1. In this case, if an appropriate value of $\bar{g}$ is set, both the unilateral and multilateral parts of $k_{1,t}$ achieve sustainable heterogeneity because the unilateral part of $k_{1,t}$ is forced on a path of sustainable heterogeneity by appropriate government intervention, whereas the multilateral part of $k_{1,t}$ naturally takes a path of sustainable heterogeneity. Therefore, even though economy 1 behaves partly unilaterally and partly multilaterally, if an appropriate value of $\bar{g}$ is set, the combined path can be sustainable. I call this a "sustainable partly unilateral path."

Corresponding to different values of ψ, sustainable partly unilateral paths are different and will fall somewhere between the multilateral path and the fully unilateral path with appropriate government transfers described by equation (26). In addition, sustainable partly unilateral paths will move continuously as the value of ψ continuously moves.

Note that this paper assumes that government intervention can be represented only by $\bar{g}$, but many other types of interventions are actually possible. For example, debt relief after personal bankruptcy would work as a measure to achieve sustainable heterogeneity, but the paths of the less advantaged economies may not be continuous in that case.

4 EVOLUTIONARY ORIGIN OF UTILITY

4.1 Genes and utility

The gene-centered view of evolution indicates that evolution is the result of the differential survival of competing genes (see, e.g., Hamilton 1964a; b, Williams, 1966), and the gene is the unit of selection. Genes compete to survive, and only genes that "won" the competition have survived the evolutionary process by fully utilizing their phenotypic effects. The gene-centered view implies that species are governed by an extremely strong desire for the indefinite continuation of their genes. Although some mutations may have existed that made an individual lack such a desire, such mutations must eventually be exterminated through natural selection. A strong desire to survive as a phenotypic effect indicates that humans are extremely motivated to avoid of being exterminated.

Altruistic behaviors of individuals in a group that shares a common pool of genes may be observed, but the gene-centered view implies that the group as a whole will demonstrate an extremely strong desire to escape the possibility of being exterminated. Some individuals may even die to save the group, but the group will never willingly choose to be destroyed because the common pool of genes would be lost.

The concept of utility should be consistent with the theory of evolution, and the above arguments indicate that the prospect of being exterminated should produce extreme fear (i.e., extreme disutility) in human beings. As Becker (1980) showed, unless sustainable heterogeneity is achieved, less advantaged households will perish when even a very small negative shock occurs, so the possibility of extinction does occur in dynamic models with a heterogeneous population. The possibility of extinction should result in a situation of extreme disutility for households in an economy, and human beings are "programmed" to take extreme actions to try to escape this result, thereby enabling the common pool of genes to survive. The gene-centered view of evolution indicates that the extreme disutility experienced in this situation is a natural outcome of evolution.

4.2 Extreme disutility to unsustainable heterogeneity

Section 2.3 indicates that, on a unilateral path without government intervention, less advantaged economies are exterminated or, at best, fall into a slave-like state. The slave-like state can be seen as equivalent to being exterminated in the sense that the members of those economies are treated more like disposable materials. As discussed in Section 4.1, either extermination or living in a slave-like state should generate extreme fear and disutility in residents of these economies. Hence, the unilateral path without government intervention will generate extreme disutility in the less advantaged economies.

Note that households are assumed to live infinitely long in this paper; thus, extermination does not mean the death of an individual with a finite lifespan. It is the extinction of a dynasty, and in biological terms, indicates that all group members who share a common pool of genes perish.

It could be argued that being forced to live in a slave-like state does not generate extreme disutility because the common pool of genes is preserved. However, the members of these economies can be exterminated at will at any time by the most advantaged economy. Therefore, such states merely mean that extermination is postponed, and the expectations of either being exterminated or falling into in a slave-like state will equally generate extreme disutility.

4.3 The utility of being exterminated

The utility function $u_i(c_{i,t})$ is modified to

$$u_i(\sigma_{i,t}, c_{i,t}) \quad ,$$

where $\sigma_{i,t}$ takes two values, 1 and 0. $\sigma_{i,t} = 0$ if economy i is exterminated, and $\sigma_{i,t} = 1$ if economy i is not exterminated (extermination includes falling in a slave-like state). The utility function allows negative values of utility. Being exterminated (i.e., $\sigma_{i,t} = 0$) generates extreme disutility such that

$$u_i(0, c_{i,t}) = -\infty$$

for any $c_{i,t}$; that is, extreme disutility is expressed as infinite disutility. If economy i expects to be exterminated in some future period t' such that $E(\sigma_{i,t}) = 0$ for $t > t'$, then

$$Eu_i(\sigma_t, c_{i,t}) = -\infty$$

for $t > t'$. If economy i does not expect to be exterminated in the future such that $E(\sigma_{i,t}) = 1$ for any t, then

$$Eu_i(\sigma_t, c_{i,t}) = Eu_i(1, c_{i,t}) \quad .$$

Note that infinite disutility may indicate that utility is cardinal. Nevertheless, the infinite disutility of $u_i(0, c_{i,t})$ expressed here as $u_i(0, c_{i,t}) = -\infty$ can be defined by an ordinal expression such that $u_i(0, c_{i,t})$ is identical for any $c_{i,t}$, and

$$u_i(1,0) \succ u_i(0, c_{i,t})$$

for any $c_{i,t}$ [e.g., $u_i(1,0) \succ u_i(0,\infty)$], where $u_i(1, c_{i,t,1}) \succ u_i(1, c_{i,t,2})$ when $c_{i,t,1} > c_{i,t,2}$.

5 SUSTAINABLE HETEROGENEITY AS THE UNIQUE SOCIALLY OPTIMAL ALLOCATION

5.1 The utility possibility frontier

A modified utility possibility frontier is needed for analyses using dynamic models with a heterogeneous population.

5.1.1 The utility possibility frontier for an endogenous growth model

Because the model used in this paper is a dynamic one, streams of utilities have to be compared. The utility possibility frontier, therefore, does not consist of period-utilities but of discounted sums of expected utilities. For simplicity, the two-economy model is again used where economy 1 has a lower rate of time preference and a lower degree of risk aversion than economy 2. Let

$$\tilde{U}\left[E\int_{t=0}^{\infty} u_1(\sigma_{1,t}, c_{1,t}) \exp(-\theta_1 t) dt \,,\; E\int_{t=0}^{\infty} u_2(\sigma_{2,t}, c_{2,t}) \exp(-\theta_2 t) dt \right] = 0$$

be the utility possibility frontier of economies 1 and 2, where $\sigma_{i,t}$ is σ of economy i (= 1, 2) in period t and $\widetilde{U}(\bullet)$ is a two-dimensional function.

The summation of expected period-utilities indicates that period-utilities are cardinal over time in an economy. Nevertheless, the discounted sums of expected utilities derived from different future paths are not required to be cardinal. They merely express ordinal rankings; for example, a higher value of $E\int_{t=0}^{\infty} u_i(1, c_{i,t}) \exp(-\theta_i t) dt$ simply means that economy i prefers the path that leads to the higher value over another path with a lower value, and

$$E\int_{t=0}^{\infty} u_i(1,0) \exp(-\theta_i t) dt \succ E\int_{t=0}^{\infty} u_i(\sigma_{i,t}, c_{i,t}) \exp(-\theta_i t) dt$$

for any $c_{i,t}$ if $E(\sigma_{i,t}) = 0$ for $t > t'$ ($E\int_{t=0}^{\infty} u_i(\sigma_{i,t}, c_{i,t}) \exp(-\theta_i t) dt$ is expressed here as $-\infty$ in this case). In addition, comparability of utilities among different economies is not required; that is, the utilities of economies 1 and 2 do need not to be comparable in this model. Note however that although an ordinal expression is possible, a cardinal expression is used for simplicity in the following examinations.

Let $\overline{c}_{i,\psi,t}$ be the consumption of economy i (= 1, 2) with fully appropriate government transfers corresponding to a given degree of unilateral behavior of economy 1 (ψ). The points on the utility possibility frontier that achieve sustainable heterogeneity are expressed by

$$\left[E\int_{t=0}^{\infty} u_1(1, \overline{c}_{1,\psi,t}) \exp(-\theta_1 t) dt \ , \ E\int_{t=0}^{\infty} u_2(1, \overline{c}_{2,\psi,t}) \exp(-\theta_2 t) dt \right] .$$

5.1.2 The shape of the utility possibility frontier

The analyses in Sections 2 and 3 indicate that the points on the utility possibility frontier that achieve sustainable heterogeneity consist only of the curve segment *AB* in Figure 1. Point *A* indicates the multilateral path, and point *B* indicates the fully unilateral path with appropriate government intervention. Because, as shown in Section 3.6, sustainable partly unilateral paths continuously move as the degree of unilateral behavior of economy 1 (ψ) continuously moves, curve segment *AB* is continuous.

Curve segment *AB* will slope downward to the right; that is, the discounted sum of expected utilities of economy 1 will decrease and that of economy 2 will increase as economy 1 engages in more unilateral behavior (i.e., ψ becomes larger) and the scale of government intervention increases. The reason for this correlation is that part of economy 1's accumulated capital is being consumed by economy 2. Therefore, future consumption of economy 1 is smaller than it would be without government intervention and that of economy 2 is larger. In addition, in the early periods of the consumption path, consumption in economy 1 on a unilateral path is smaller than that on the multilateral path because capital is accumulated more quickly on a unilateral path. Because the government's intervention reduces accumulated capital and future utilities are discounted by the rate of time preference, these early smaller utilities will not be compensated fully by future increases in consumption that would result from the quicker capital accumulation. As a result, the discounted sums of expected utilities of economy 1 on sustainable partly unilateral paths will be smaller than that on the multilateral path and the curve segment will slope downward to the right. Whether the curve slopes downward or upward is not important for the results shown below, however. The results depend not on the direction but on the monotonicity of the curve segment, that is, the monotonous relationship between ψ and $E\int_{t=0}^{\infty} u_i(1, \overline{c}_{i,\psi,t}) \exp(-\theta_i t) dt$.

The government's responses to the unilateral behaviors of economy 1, by which sustainable heterogeneity is achieved, are very limited—only responses corresponding to sustainable partly unilateral paths are chosen. Given a degree of unilateral behavior of economy 1 (i.e., given a value of ψ), only one government response, which is indicated by equation (25), correspondingly can successfully achieve sustainable heterogeneity. Therefore only one point on curve segment *AB* consists of the utility possibility frontier for any given value of ψ.

For simplicity, the possibility of too much government intervention is not considered, and $\bar{g}$ never exceeds the value for sustainable heterogeneity. Hence, all other responses result in a disutility of $-\infty$ for economy 2 because it expects to be exterminated in future such that

$$Eu_2(0, c_{2,t}) = -\infty$$

after a finite period of time; thus,

$$E\int_{t=0}^{\infty} u_2(\sigma_{2,t}, c_{2,t}) \exp(-\theta_2 t) dt = -\infty \quad .$$

The utility possibilities of such unsustainable heterogeneity for all values of ψ are depicted by the line *CD* in Figure 1. Given a value of ψ, a part of the line *CD* correspondingly consists of the utility possibility frontier of unsustainable heterogeneity. Let such part of the line *CD* be "the line $C(\psi)D(\psi)$," where point $C(\psi)$ indicates the insufficient intervention that gives the smallest discounted sum of expected utility of economy 1 and point $D(\psi)$ indicates the insufficient intervention that gives the largest. Each point on the line $C(\psi)D(\psi)$ has a corresponding value of $\bar{g}$, all of which are insufficient to achieve sustainable heterogeneity for the given ψ.

As a result, given a degree of unilateral behavior of economy 1 (i.e., given a value of ψ), the utility possibility frontier is composed of the two parts: a point on the curve segment *AB* and the line $C(\psi)D(\psi)$.

5.2 The social welfare function

Here, a social welfare function is assumed to be adopted by the society consisting of the all economies. The assumptions in Arrow (1951) are modified (e.g., the assumption that every individual has a single-peaked preference is added). The social welfare function that is defined on the same space as the utility possibility frontier is

$$\tilde{W}\left[E\int_{t=0}^{\infty} u_1(\sigma_{1,t}, c_{1,t}) \exp(-\theta_1 t) dt \, , \; E\int_{t=0}^{\infty} u_2(\sigma_{2,t}, c_{2,t}) \exp(-\theta_2 t) dt\right] = W \quad ,$$

where $\tilde{W}(\bullet)$ is a two-dimensional function and W is a variable. Its shape is not specified but it at least satisfies the following typical features: completeness, transitivity, and continuity. Thus, its indifference curves do not cross and are sloping downward to the right. The social welfare function's indifference curves are either convex or concave to the origin. In addition, on any indifference curve, as $c_{i,t} \rightarrow 0$ for any t, $c_{i,t} \rightarrow \infty$ for any t ($i \neq j$). I call this type of social welfare function a "general type social welfare function."

Next, suppose a continuous function such that

$$\tilde{V}\left[E\int_{t=0}^{\infty} u_1(\sigma_{1,t}, c_{1,t}) \exp(-\theta_1 t) dt \, , \; E\int_{t=0}^{\infty} u_2(\sigma_{2,t}, c_{2,t}) \exp(-\theta_2 t) dt\right] = 0$$

defined on the same space as the utility possibility frontier is. Points satisfying this function are indicated by $\left(v_1, v_2\right)$, where $\dfrac{dv_2}{dv_1} > 0$ and $v_2 = 0$ when $v_1 = 0$, as shown as the dotted line in Figure 2. The indifference curve that crosses the function $\tilde{V}(\bullet) = 0$ at point $\left(v_1, v_2\right)$ is

$$\tilde{W}\left(v_1, v_2\right) = \tilde{W}\left[E\int_{t=0}^{\infty} u_1\left(\sigma_{1,t}, c_{1,t}\right)\exp\left(-\theta_1 t\right)dt \ , \ E\int_{t=0}^{\infty} u_2\left(\sigma_{2,t}, c_{2,t}\right)\exp\left(-\theta_2 t\right)dt\right] \ .$$

Suppose another type of social welfare function such that, for any point $\left(v_1, v_2\right)$, $\tilde{W}\left(v_1, v_2\right) = \tilde{W}\left[v_1 \ , \ E\int_{t=0}^{\infty} u_2\left(\sigma_{2,t}, c_{2,t}\right)\exp\left(-\theta_2 t\right)dt\right]$ for any $E\int_{t=0}^{\infty} u_2\left(\sigma_{2,t}, c_{2,t}\right)\exp\left(-\theta_2 t\right)dt \le v_2$. That is, the indifference curves are vertical if $E\int_{t=0}^{\infty} u_2\left(\sigma_{2,t}, c_{2,t}\right)\exp\left(-\theta_2 t\right)dt \le v_2$, as shown as the solid lines in Figure 2. I call this type of social welfare function a "Nietzsche type social welfare function." This type of social welfare function is completely different from the general type social welfare function because it does not possess the nature that as $c_{i,t} \to 0$ for any t, $c_{i,t} \to \infty$ for any t $(i \neq j)$ on any indifference curve. The Nietzsche type social welfare function may be loathed by many people because it indicates that a society should not care about its members being exterminated and does not exclude the social preference that only the strongest should prevail. Although a few people may support the Nietzsche type social welfare function, the probability of violent political conflicts will become extremely high if a society adopts it (see Harashima, 2010).

5.3 The almost unique socially optimal allocation

The socially optimal state is given by the point where the utility possibility frontier and an indifference curve of the social welfare function come in contact with each other. As shown in Section 5.1, however, the utility possibility frontier's shape is not simple. Given a degree of unilateral behavior of economy 1, it is composed of a point on the curve segment *AB* and the line $C(\psi)D(\psi)$.

Given a value of ψ, let the corresponding point on the curve segment *AB* be indicated by ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$). Let also $W(\varsigma)$ be W of the indifference curve that crosses the point ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$), and ($\gamma_{1,W(\varsigma)}$, $\gamma_{2,W(\varsigma)}$) indicate points on the indifference curve $W(\varsigma)$. In addition, let the point $D(\psi)$ be indicated by (δ_1, δ_2), $W(\delta)$ be W of the indifference curve that crosses the point $D(\psi)$, and ($\gamma_{1,W(\delta)}$, $\gamma_{2,W(\delta)}$) indicate points on the indifference curve $W(\delta)$. As argued in Sections 4.3 and 5.1.1, δ_2 is expressed as $-\infty$.

Because of the nature of the point of sustainable heterogeneity ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$), the following proposition is self-evident.

Proposition 4: If the social welfare function is a general type and its indifference curves are convex to the origin, then only the point ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$) is optimal.

Because it is highly likely that social welfare functions in most societies are general type functions and their indifferent curves are convex to the origin, Proposition 4 indicates that generally the point of sustainable heterogeneity ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$) is uniquely socially optimal.

I next examine social optimality when the social welfare function's indifference curves are concave to the origin.

Lemma 5: If the social welfare function is a general type and its indifference curves are concave to the origin, then only the point ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$) is optimal.

Proof: Because the social welfare function is a general type and its indifference curves are concave to the origin, then $\gamma_{1,W(\varsigma)} > \varsigma_{1,\psi}$ if $\gamma_{2,W(\varsigma)} < \varsigma_{2,\psi}$, and as $\gamma_{1,W(\varsigma)}$ becomes larger, $\gamma_{2,W(\varsigma)}$

becomes smaller. Let $\gamma_{2,W(\varsigma),D}$ be $\gamma_{2,W(\varsigma)}$ when $\gamma_{1,W(\varsigma)} = \delta_1$. Because the social welfare function is not a Nietzsche type, then $\gamma_{2,W(\varsigma),D} > \delta_2 = -\infty$. Therefore, $W(\varsigma) > W(\delta)$. Because the values of W of the indifference curves that cross any other point on the line $C(\psi)D(\psi)$ than the point $D(\psi)$ are less than $W(\delta)$, then only the point $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is optimal. ■

Lemma 5 shows that even though the social welfare function's indifference curves are concave to the origin, the point of sustainable heterogeneity $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is uniquely determined to be socially optimal if the social welfare function is a general type.

Next, I examine social optimality when the social welfare function is a Nietzsche type. Let $(v_{1,W(\varsigma)}, v_{2,W(\varsigma)})$ be (v_1, v_2) on the indifference curve $W(\varsigma)$. When the social welfare function is Nietzsche type, then $\gamma_{1,W(\varsigma)} \leq v_{1,W(\varsigma)}$, where $\gamma_{1,W(\varsigma)} < v_{1,W(\varsigma)}$ if $\gamma_{2,W(\varsigma)} > v_{2,W(\varsigma)}$ and $\gamma_{1,W(\varsigma)} = v_{1,W(\varsigma)}$ if $\gamma_{2,W(\varsigma)} \leq v_{2,W(\varsigma)}$.

Lemma 6: If the social welfare function is a Nietzsche type, and
(a) if $v_{1,W(\varsigma)} < \delta_1$, then only the point (δ_1, δ_2) is optimal,
(b) if $v_{1,W(\varsigma)} > \delta_1$, then only the point $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is optimal, and
(c) if $v_{1,W(\varsigma)} = \delta_1$, then only the points $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ and (δ_1, δ_2) are optimal.
Proof: Because the social welfare function is a Nietzsche type and thus its indifference curves are concave to the origin, then $\delta_1 = \gamma_{1,W(\delta)}$ and if $\gamma_{2,W(\varsigma)} = \delta_2 = -\infty$, then $v_{1,W(\varsigma)} = \gamma_{1,W(\varsigma)}$. Hence, the following statements apply.
(a) If $v_{1,W(\varsigma)} < \delta_1$, then $\gamma_{1,W(\varsigma)} < \gamma_{1,W(\delta)}$ for $\gamma_{2,W(\varsigma)} = \delta_2 = -\infty$, and thus $W(\varsigma) < W(\delta)$. Because the values of W of the indifference curves that cross any other point on the line $C(\psi)D(\psi)$ than the point $D(\psi)$ are less than $W(\delta)$, then only the point (δ_1, δ_2) is optimal.
(b) If $v_{1,W(\varsigma)} > \delta_1$, then $\gamma_{1,W(\varsigma)} > \gamma_{1,W(\delta)}$ for $\gamma_{2,W(\varsigma)} = \delta_2 = -\infty$, and thus $W(\varsigma) > W(\delta)$. By the same reason as the latter part of (a), only the point $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is optimal.
(c) If $v_{1,W(\varsigma)} = \delta_1$, then $\gamma_{1,W(\varsigma)} = \gamma_{1,W(\delta)}$ for $\gamma_{2,W(\varsigma)} = \delta_2 = -\infty$, and thus $W(\varsigma) = W(\delta)$. Again, by the same reason as the latter part of (a), only points $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ and (δ_1, δ_2) are optimal. ■

Lemma 6 indicates that Nietzsche type social welfare functions are distinguished into the following three categories.
Category (i): only point (δ_1, δ_2) is socially optimal (corresponding to the case $v_{1,W(\varsigma)} < \delta_1$).
Category (ii): only point $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is only socially optimal (corresponding to the case $v_{1,W(\varsigma)} > \delta_1$).
Category (iii): only points $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ and (δ_1, δ_2) are socially optimal (corresponding to the case $v_{1,W(\varsigma)} = \delta_1$).

Proposition 5: If the social welfare function is either a general or Nietzsche type, the point $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is only socially optimal allocation for any social welfare function except categories (i) and (iii) Nietzsche type social welfare functions,
Proof: First, by Proposition 4, if the social welfare function is a general type and its indifferent curves are convex to the origin, the point $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is optimal. Second, by Lemma 5, if the social welfare function is a general type and its indifferent curves are concave to the origin, the point $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is optimal. Finally, by Lemma 6, if the social welfare function is a category (ii) Nietzsche type, the point $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is optimal, whereas if it is either a category (i) or (iii) Nietzsche type, the point (δ_1, δ_2) can be socially optimal. ■

Proposition 5 is important because it indicates that, for almost all generally usable (i.e., preferences are complete, transitive, and continuous) social welfare functions, the point of sustainable heterogeneity $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is the only socially optimal allocation. In addition, it is highly likely that very few people actually support category (i) or (iii) Nietzsche type social welfare functions because they will generate violent political conflicts (see Harashima, 2010), and they will almost certainly always be in the minority. Hence these types of welfare functions will be rarely adopted in democratic societies where policies are decided by majority.[6] In other words, category (i) or (iii) Nietzsche type social welfare functions would only be adopted by a democratic society when its economic and social situations were extraordinary abnormal. If the situation is not extraordinarily abnormal, category (i) and (iii) Nietzsche type social welfare functions can be excluded, and we can assert that for any generally usable social welfare function, the point of sustainable heterogeneity ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$) is uniquely socially optimal.

Proposition 5 provides a clue to solve an important problem in studies of social welfare, that is, the unspecifiability of socially optimal allocation resulting from the difficulty in specifying the shape of the social welfare function. Proposition 5 escapes this problem because the socially optimal allocation is uniquely determined no matter the shape of the social welfare function. Therefore, it is no longer necessary to form a specific social ordering to determine the socially optimal growth path in a heterogeneous population.

6 CONCLUDING REMARKS

Historically, it has been difficult to universally agree upon a criterion for socially optimal allocation because of utility's interpersonal incomparability, Arrow's general possibility theorem, and other factors. This paper examined social optimality in dynamic models with a heterogeneous population and showed that a state exists in which all of the optimality conditions of a heterogeneous population are satisfied. The existence of such a state provides us with additional meaningful information for studying social optimality.

The model in this paper shows that sustainable heterogeneity, which is defined as the state at which all optimality conditions of all heterogeneous households are satisfied, is uniquely determined to be the socially optimal allocation for almost all generally usable social welfare functions. The only exceptions are some variants of a Nietzsche type social welfare function, which will rarely be adopted in democratic societies unless the economic and social situations are extraordinarily abnormal. Sustainable heterogeneity is achievable even if the most advantaged household behaves unilaterally if the government appropriately intervenes. The uniquely determined socially optimal allocation in a heterogeneous population can be accomplished without specifying the shape of the social welfare function, and therefore, the problem of unspecifiability of social optimality can be solved.

Sustainable heterogeneity as the unique socially optimal allocation will have important implications to currently passionately disputed issues such as the Occupy Wall Street movement, anti-globalization (e.g., Klein, 2000; Stiglitz, 2002), anti-market fundamentalism (e.g., Gray, 1998; Stiglitz, 2002, 2009; Soros, 2008), and true measures of happiness (e.g., Sen, 1976; Arrow et al., 1995). In addition, sustainable heterogeneity will provide additional theoretical foundations for debt relief, wealth taxes, progressive taxation, and international aid. On the other hand, sustainable heterogeneity also indicates that there is a unique sustainable level of inequality in consumption.

[6] As shown in Section 5.2, it is assumed that the assumptions in Arrow (1951) are modified.

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Figure 1 ***The utility possibility frontiers of sustainable and unsustainable heterogeneity***

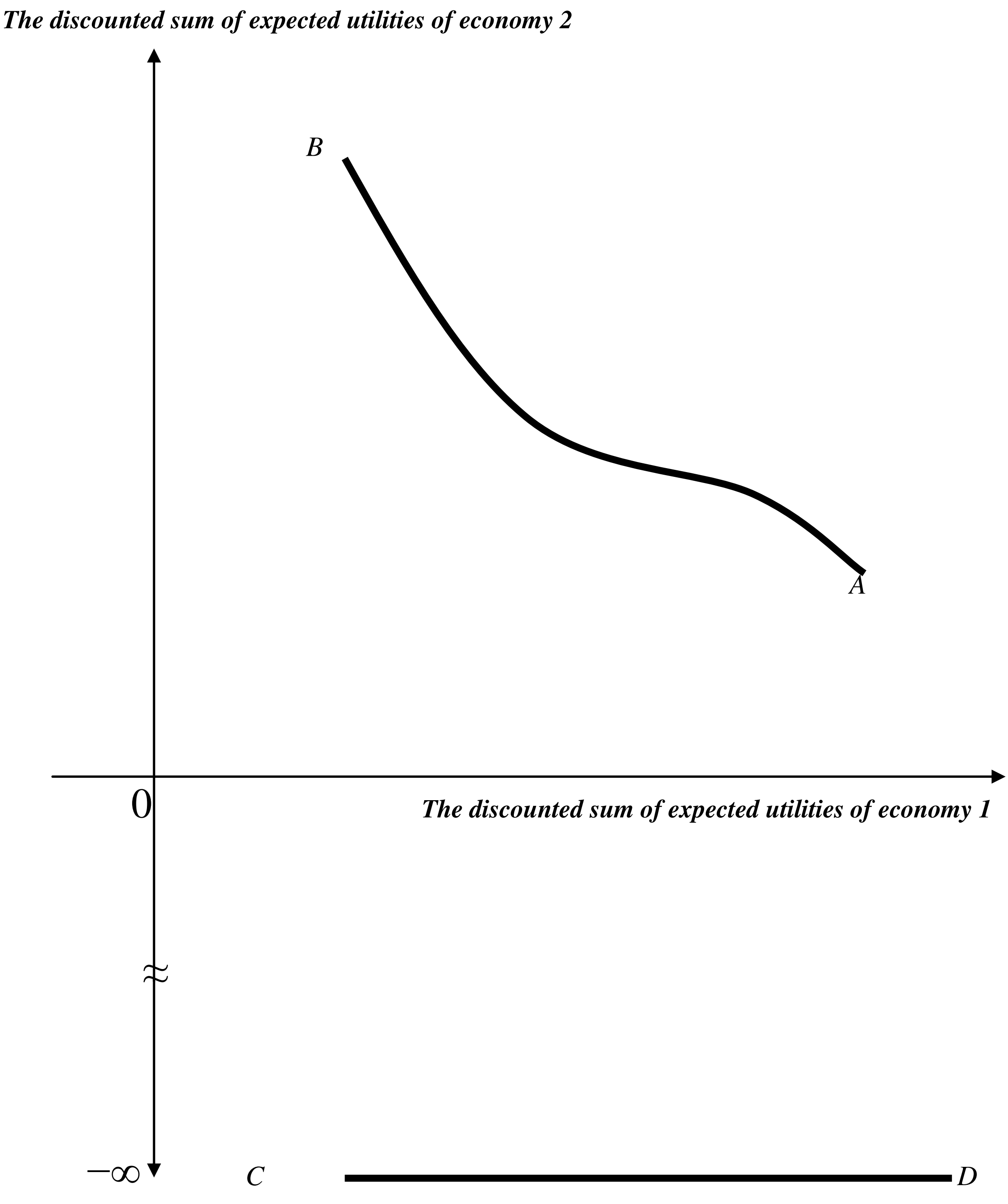

Figure 2 ***Indifference curves of a Nietzsche type social welfare function***