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Market Expectations of the Short Rate and the Term Structure of Interest Rates: A New Perspective from the Classic Model

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Abstract

Based on the classic Gaussian dynamic term structure model $\mathbb{A}_0(3)$, I rotate the model to a special representation, the so called "Companion Form Realization", in which the state variables comprises the short rate and its related expectations. This unique feature makes the representation very useful in analyzing the response of the yield curve to the shocks in the short rate and its related expectations, and monitoring market expectations. Using the estimated model, I quantify a variety of yield responses to the changes in these important state variables; and also give an "unsurprising" pattern in which changes in state variables have little impact on the long end of the yield curve. Two case studies of recent unconventional monetary policies are also included.

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1 Introduction

How the yield curve responds to the changes in the market expectations of the short rate is a topic of great concerns to bond traders and monetary policy-makers alike. A good understanding on how the yield curve movement follows the changes in the market expectations can be the key to a successful trading strategy. Policy-makers monitor the market expectations constantly, a detailed knowledge about the impact of changes in the expectations is crucial for them to set up effective monetary policies.

There have been studies on how to extract the information about market expectations from the interest rate related instruments, for example: Söderlind and Svensson (1997); Brooke et al. (2000); Joyce et al. (2008). It seems that the market expectations of the short rates for the next 6 months to 1 year is most informative about the future monetary policy, and relatively free of term premia and liquidity premia. These expectations can be well extracted from the implied forward rate curve. They are closely related to the slope of the implied forward curve. Besides this, there is little attention has been paid to the difference between slopes at different maturities on the yield curve. Actually, this quantity measures the expectation of the change in the expectation. For most of the time, this quantity moves in the opposite direction as the expectation of the short rate due to the mean reverting feature of the economy system. Exceptions to this rule are very rare. However, once happen, they might be strong signs of substantial movements of the term structure.

There is no previous model coherently links these expectations with the dynamics of the term structure of interest rates. In this paper, I provide a model to fill in this gap. In this model, the market expectations are naturally constructed as the state variables which follow certain multivariate Gaussian dynamic. Thank to this unique feature, the model can be employed to effectively infer the market expectations from the term structure of interest rates for the purpose of monitoring market movement; more importantly, it can be used to precisely analyze the impacts of the changes in the market expectations, this is the key for successful monetary actions.

In the empirical analysis, using the estimated model, I provide a series of numerical examples to demonstrate how changes in the short rate and its related expectations impact the yield curve, and also show how the impacts evolve over time in the sense of impulse responses. I find that the market expectation variables are the crucial driver of the dynamics of the yield cuve long end. Based on these numerical examples, I give an intuitive explanation to a debatable puzzle (see, e.g., Ellingsen and Söderström, 2001) that why occasionally the long end of the yield curve moves in a direction opposite to the intent of the policy-makers.

As shown in Kuttner (2001), when monetary policies come in a "unsurprising" manner, the impact would be minor. I therefore quantify a "unsurprising" pattern using parameter estimates and filtered state variables. This pattern can serve as a benchmark to check if a monetary policy is anticipated or not and why so.

In the case studies, I also use the estimated model to analyze the impacts of the announcement effect of the large scale asset purchase program (LSAP) and the maturity extension program (MEP). Although the forward curve moved in line with the intents of the Fed, the market responded differently to these two announcements: the market responded clearly and positively to the LSAP by lowering the expected inflation at various horizons; however, it responded less so to the MEP by increasing the expected inflation at horizons shorter than 2 years and keeping those at longer horizons intact.

The rest of the paper is organized as follows. Section 2 describes the classic model and how the new model is built from the classic model. Section 3 contains the discussion of the empirical results. Section 4 concludes the paper. Appendix includes technical details.

2 A New Usage of the Classic Model

In this section, I apply the modeling framework recently developed by Li and Ye (2012) to set up the 3-factor Gaussian dynamic term structure model.

2.1 Instantaneous forward rate

The modeling framework proposed by Li and Ye (2012) (hereafter LY) is HJM-based, the starting point is the instantaneous forward rate. Following the Musiela parametrization (Brace et al., 1997), the time-t instantaneous forward rate for time-$t+x$ is written as:

$$\begin{aligned} r(t,x) &= r(0,t+x)+\Theta(t,x)+r_0(t,x) \\ \Theta(t,x) &= \int_0^t \sigma(x+t-s)\int_0^{x+t-s}\sigma(v)\,dvds \\ r_0(t,x) &= \int_0^t \sigma(x+t-s)\,dW_s \end{aligned} \tag{2.1}$$

where W_t is a 3-dimensional $\mathbb{Q}$-measure Brownian motion, $\sigma(\bullet)\in\mathbb{R}^3$ is the time-invariant volatility function defined in the following SDE:

$$dr(t,x)=\frac{\partial}{\partial x}r(t,x)+\sigma(x)\int_0^x \sigma(s)^{\intercal}\,ds+\sigma(x)\,dW_t$$

2.2 Volatility term structure and Markov representation

The key ingredient under LY's framework is the volatility function. Here I specify the volatility function to be consistent with the most classic 3-factor Gaussian dynamic model, $AM_0(3)$ due to Dai and Singleton (2000).

$$\sigma(x)_{1\times 3}=\begin{bmatrix} e^{-k_1x} & e^{-k_2x} & e^{-k_3x}\end{bmatrix}\begin{bmatrix}\Omega_1 & 0 & 0\\ \Omega_2 & \Omega_4 & 0\\ \Omega_3 & \Omega_5 & \Omega_6\end{bmatrix}.$$

Apparently, this volatility function guarantees a Markov representation for the model, as it is consistent with $AM_0(3)$. By the definition, if $\sigma(x)$ is factored as $\mathbf{C}\exp(\mathbf{A}x)\mathbf{B}$, then $r_0(t,x)$ has the following Markov representation:

$$\begin{aligned} r_0(t,x) &= \mathbf{C}(x)Z_t \\ dZ_t &= \mathbf{A}Z_t dt+\mathbf{B}dW_t, \quad Z_0=0, \end{aligned}$$

where $\mathbf{C}(x)=\mathbf{C}\exp(\mathbf{A}x)$. Following LY, I set $r(0,t+x)+\Theta(t,x)$ to its time-homogeneous counterpart $\varphi+\Theta^*(x)$, where

$$\Theta^*(x)=\mathbf{C}(x)\left(\mathbf{A}^{-1}\mathbf{B}\mathbf{B}^{\intercal}(\mathbf{A}^{\intercal})^{-1}\right)\mathbf{C}^{\intercal}-\frac{1}{2}\mathbf{C}(x)\left(\mathbf{A}^{-1}\mathbf{B}\mathbf{B}^{\intercal}(\mathbf{A}^{\intercal})^{-1}\right)\mathbf{C}(x)^{\intercal}.$$

There are infinitely many representations, as given one realization $\{\mathbf{A},\mathbf{B},\mathbf{C}(x)\}$, for any invertible matrix M, $\left\{M\mathbf{A}M^{-1}, M\mathbf{B}, \mathbf{C}(x)M^{-1}\right\}$ is another realization, since

$$\mathbf{C}\exp(\mathbf{A}x)\mathbf{B} = \mathbf{C}M^{-1}\exp\left(M\mathbf{A}M^{-1}x\right)M\mathbf{B} = \mathbf{C}(x)M^{-1}M\mathbf{B}.$$

The realization employed by Dai and Singleton (2000) is only one of them, the details are presented in LY. Here I present two realizations, the first one is the "Base realization", since this is a Jordan form realization, I use this realization to estimate parameters; the second one is the "Companion form realization" which has some unique features for analyzing the impacts of market expectations on the term structure of interest rates, and will be discussed more in the next section.

- Base realization

$$\begin{aligned}
\mathbf{A}_{\text{Base}} &= \begin{bmatrix} -k_1 & 0 & 0 \\ 0 & -k_2 & 0 \\ 0 & 0 & -k_3 \end{bmatrix} \\
\mathbf{B}_{\text{Base}} &= \begin{bmatrix} \Omega_1 & 0 & 0 \\ \Omega_2 & \Omega_4 & 0 \\ \Omega_3 & \Omega_5 & \Omega_6 \end{bmatrix} \\
\mathbf{C}(x)_{\text{Base}} &= \begin{bmatrix} e^{-k_1 x} & e^{-k_2 x} & e^{-k_3 x} \end{bmatrix}
\end{aligned}$$

- Companion form realization

$$\begin{aligned}
\mathbf{A}_{\text{CR}} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1k_2k_3 & -k_1k_2 - k_1k_3 - k_2k_3 & -k_1 - k_2 - k_3 \end{bmatrix} \\
\mathbf{B}_{\text{CR}} &= \begin{bmatrix} 1 & 1 & 1 \\ -k_1 & -k_2 & -k_3 \\ k_1^2 & k_2^2 & k_3^2 \end{bmatrix} \begin{bmatrix} \Omega_1 & 0 & 0 \\ \Omega_2 & \Omega_4 & 0 \\ \Omega_3 & \Omega_5 & \Omega_6 \end{bmatrix} \\
\mathbf{C}(x)_{\text{CR}} &= \begin{bmatrix} e^{-k_1 x} & e^{-k_2 x} & e^{-k_3 x} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -k_1 & -k_2 & -k_3 \\ k_1^2 & k_2^2 & k_3^2 \end{bmatrix}^{-1}
\end{aligned}$$

2.3 A closer look at the Companion form realization

The "Companion form realization" are firstly studied by Collin-Dufresne et al. (2008). The first important feature of the "Companion form realization" is that $Z_{1,t}$ is closely related to the short rate, $r(t,0)$. Specifically,

$$r(t,0) = \varphi + \frac{1}{2}\mathbf{C}\left(\mathbf{A}^{-1}\mathbf{B}\mathbf{B}^{\intercal}\left(\mathbf{A}^{\intercal}\right)^{-1}\right)\mathbf{C}^{\intercal} + Z_{1,t},$$

since

$$\mathbf{C}(0)_{\text{CR}} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -k_1 & -k_2 & -k_3 \\ k_1^2 & k_2^2 & k_3^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

The second important feature is that $Z_{2,t}$ is the risk neutrally expected instantaneous change of $Z_{1,t}$ or the short rate $r(t,0)$ per unit of time, and $Z_{3,t}$ is the risk neutrally expected instantaneous change of $Z_{2,t}$ per unit of time. This can be easily verified by looking at the conditional expectation of dZ_t

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}\left(\begin{bmatrix} dZ_{1,t} \\ dZ_{2,t} \\ dZ_{3,t} \end{bmatrix}\right) &= \mathbf{A}_{\text{CR}} \begin{bmatrix} Z_{1,t} \\ Z_{2,t} \\ Z_{3,t} \end{bmatrix} dt \\ &= \begin{bmatrix} Z_{2,t} \\ Z_{1,t} \\ -k_1k_2k_3Z_{1,t} - (k_1k_2 + k_1k_3 + k_2k_3)Z_{2,t} - (k_1 + k_2 + k_3)Z_{3,t} \end{bmatrix} dt. \end{aligned}$$

The expected instantaneous change of the short rate looks too abstract for the practical use, as mentioned in Söderlind and Svensson (1997) the expected change for a longer horizon, say 6-9 months, is actually more relevant. The last but not least important feature is that the model can be easily rotated to set per unit of time expectations of changes for a longer horizon as the state variables. Specifically, given a horizon h, a new set of state variables X_t^h can be defined as

$$\underbrace{X_t^h}_{4\times 1} = \begin{bmatrix} Z_{1,t} \\ \mathbb{E}_t^{\mathbb{Q}}\left(\begin{bmatrix}\Delta Z_{1,t}\\ \Delta Z_{2,t}\\ \Delta Z_{3,t}\end{bmatrix}\right)/h \end{bmatrix} = \begin{bmatrix} Z_{1,t} \\ \frac{\exp(\mathbf{A}_{\mathrm{CR}}h)-I}{h} Z_t \end{bmatrix}.$$

where $\Delta Z_t = Z_{t+h} - Z_t$. The first state in X_t^h is still $Z_{1,t}$ which is closely linked to the short rate, the second state is the risk neutrally expected $\Delta Z_{1,t}$ per unit of time, and so on. Therefore $r_0(t,x)$ can be re-written as

$$\begin{aligned} r_0(t,x) &= \underbrace{\begin{bmatrix}\mathbf{C}_1(x) & 0 & \mathbf{C}_2(x) & \mathbf{C}_3(x)\end{bmatrix}}_{4\times 1} M_1 M_2 M_{fun}(h)^{-1} X_t^h, \\ &\equiv \mathbf{D}(x,h) X_t^h \end{aligned} \tag{2.2}$$

where

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, M_{fun}(h) = \begin{bmatrix} 1 & 0_{1\times 3} \\ 0_{3\times 1} & \frac{\exp(\mathbf{A}_{\mathrm{CR}}h)-I}{h} \end{bmatrix}.$$

These features make the “Companion form realization” very useful in analyzing the impacts of the monetary policy on the term structure. For example, Kuttner (2001) argues that it is crucial to separate the expected and unexpected components in short rate changes when analyzing how the monetary policy impacts the term structure of interest rates. The “Companion form realization” gives us a natural separation on the expected and unexpected changes of the short rate in a consistent and non-arbitrage way. More insights on how the “Companion form realization” can be utilized to analyze the impacts of market expectations on the term structure of interest rates will be discussed in the empirical section.

Table 1: Parameter Estimates

$$\mathbf{A}_{\text{Base}} = \begin{bmatrix} \underset{(0.003)}{-0.1178} & 0 & 0 \\ 0 & \underset{(0.008)}{-0.4036} & 0 \\ 0 & 0 & \underset{(0.029)}{-1.3215} \end{bmatrix}, \mathbf{B}_{\text{Base}} = \begin{bmatrix} \underset{(0.001)}{0.0271} & 0 & 0 \\ \underset{(0.002)}{-0.0296} & \underset{(0.008)}{0.0176} & 0 \\ \underset{(0.001)}{0.0038} & \underset{(\text{N.A.})}{-0.0176} & \underset{(0.000)}{0.0071} \end{bmatrix}$$

$$\lambda = \begin{bmatrix} \underset{(0.252)}{0.0042} & \underset{(0.253)}{0.6864} & \underset{(0.255)}{1.4640} \end{bmatrix}^{\mathsf{T}}, \varphi = \underset{(0.000)}{0.0546}$$

This table reports parameter estimates using the Base realization. The standard errors are in parentheses. The constrain that $\omega_5 + \omega_4 = 0$ is imposed to identify ω_4. The sample is daily from October 1994 to December 2009

3 Empirical Results

3.1 Data, parameter estimates, and fitting performance

The data used in the empirical analysis are the U.S. Treasury constant maturities yield curve rates, downloaded from Federal Reserve Statistical Release, H.15, Selected Interest Rates (Daily). The data-set contains daily observations of the yields with maturities of 3, 6-month, 1, 2, 3, 5, 7, and 10-year. As the Treasury yield curve is considered as the par curve, the par curves are converted to the zero curves by first smoothing par rates then bootstrapping the zero rates from par rates.

Following Dai and Singleton (2000), I use the "completely affine" specification for the market price of risk setting. Specifically, the measure $\mathbb{P}$ dynamic of Z_t is assumed to be

$$dZ_t = \mathbf{A}\left(Z_t - \mathbf{A}^{-1}\mathbf{B}\lambda\right)dt + \mathbf{B}dW_t^{\mathbb{P}},$$

where λ is a 3-dimensional constant market price of risk vector, $W_t^{\mathbb{P}}$ is a 3-dimensional $\mathbb{P}$-measure Brownian motion. The parameters are estimated using Kalman filter in conjunction with QMLE, assuming IID normal measurement errors. The estimates are presented in Table 1.

In many traditional term structure estimations, yields at certain maturities are assumed to be priced without errors. By this assumption, even one uses a same set of data, the results are different when different maturities are assumed to be priced perfectly. Here I do not make this assumption, all yields are priced with errors. The summary statistics of the pricing errors are shown in Table 2. Unsurprisingly, this 3-factor Gaussian model does a good job in fitting the dynamic term structure of interest rates. The average MAE (mean absolute error) across maturities is only 4.2bp, and the average VR (variance ratio) across maturities is as high as 99.9%. This is consistent with the previous literature on the fitting performance of 3-factor Gaussian dynamic term structure models, e.g., Heidari and Wu (2009).

Table 2: Summary Statistics of Pricing Errors

Maturity	Mean(bp)	Medn(bp)	Std.(bp)	MAE(bp)	Max(%)	Min(%)	VR(%)
3m	-2.011	-1.321	6.695	4.575	0.225	-0.781	99.878
6m	2.638	2.232	5.299	4.397	0.496	-0.135	99.923
1y	-0.239	0.101	6.844	4.884	0.420	-0.239	99.865
2y	0.913	1.039	4.145	3.424	0.144	-0.230	99.946
3y	-1.853	-1.415	4.121	3.060	0.177	-0.306	99.937
5y	-1.472	-1.207	4.343	3.585	0.188	-0.171	99.902
7y	3.264	2.114	5.542	4.658	0.238	-0.121	99.801
10y	-1.342	-1.085	6.566	5.136	0.263	-0.196	99.617
Ave.	-0.013	0.057	5.444	4.215	0.269	-0.272	99.859

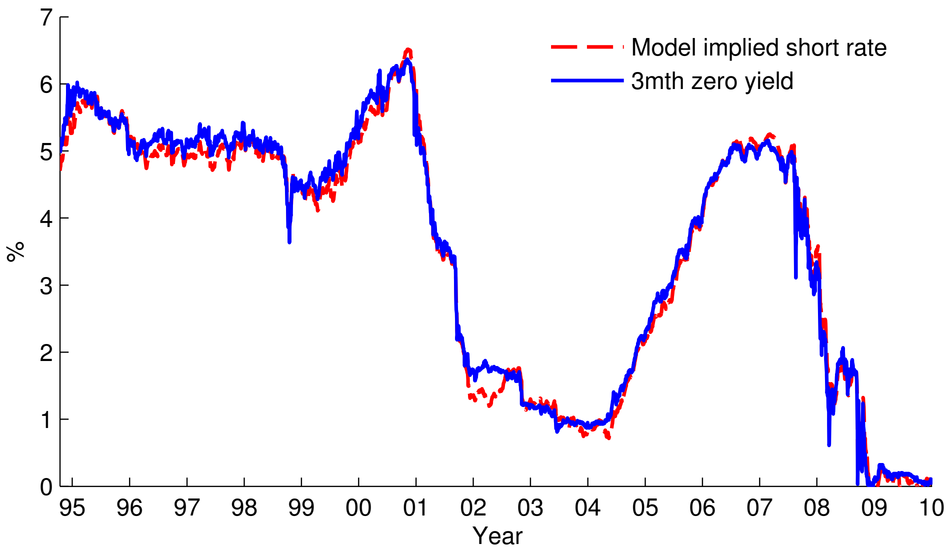
This table reports the summary statistics (sample mean, median, standard deviation, mean absolute error, maximum, and minimum) of the pricing errors over the daily sample from October 1994 to December 2009. The pricing error is defined as the difference between the market observation and the model implied value. The table also reports the variance ratio (VR) at each maturity, which is defined as one minus the ratio of the pricing error variance to the variance of the original series.

3.2 State Variables

In order to confirm that the first state variable in the "Companion form realization" represents the short rate, I compare it with the shortest yield, 3-mth zero yield, used in the estimation. Although the short rate is not the 3-mth zero yield as the short rate is with 0 time-to-maturity, they should look very alike since their maturities are close. The comparison is shown in Figure 3.1. By eyeballing, the model implied short rate is very

Figure 3.1: Model Implied Short Rate v.s. 3-mth Zero Yield

This figure compares the dynamics of the model implied short rate and 3mth zero yield from 1994 to 2009.

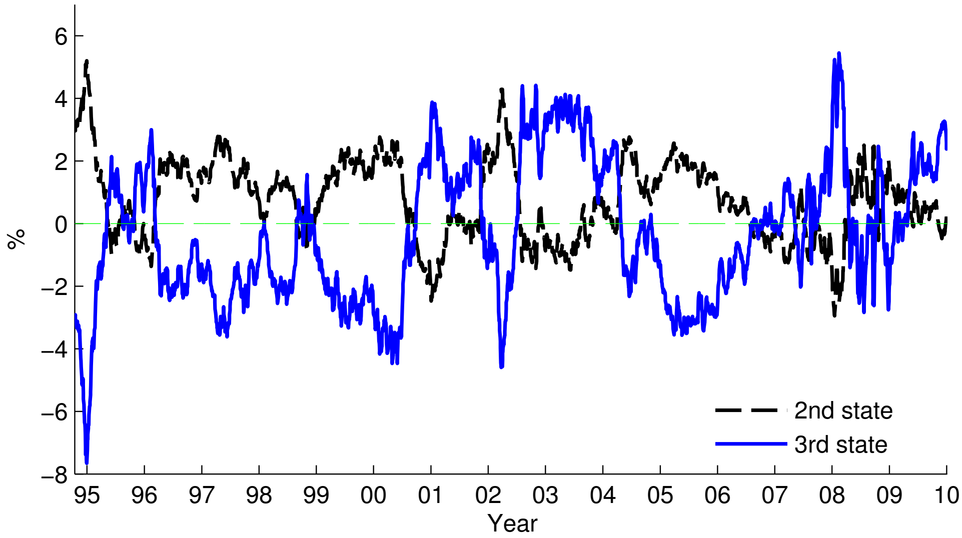


close to the 3-mth zero yield. Their correlation is as high as 99.7%. In Piazzesi (2005), the correlation between the short rate implied by her model and LIBOR rate is only 54%. This somewhat confirms the statement that the first state variable in the "Companion form realization" can represent the short rate and the model (the realization) used in this paper is much more better in capturing the short rate process than others.

The second and third states are shown in Figure 3.2. These two states are significantly negatively correlated. For most of the time, they almost mirror each other on the two sides of the zero line. In order to interpret this, let me borrow some results from Collin-Dufresne et al. (2008): other than the interpretations I mentioned before, these two states also have some other intuitive economic interpretations such as slope and curvature of the yield curve around a maturity of zero. Or in other words, the second and third states represent the first and second order derivatives, respectively, of the term structure function evaluated at the maturity zero. Bearing this understanding in mind, I find that the significant negative-correlation is actually a reflection of the stylized fact that for most of the time, the yield curve near the short end is concave when it is upward-sloping; convex when downward-sloping. Therefore, the shape of the yield curve in fact

Figure 3.2: The 2nd and 3rd State Variables in Z_t

This figure shows the dynamic of the 2nd and 3rd state variables in Z_t from 1994 to 2009.



carries important information about the market expectation of the short rate movements. Another explanation of the negative correlation lies in the mean-reverting feature of the short rate: increasing $Z_{2,t}$ means that the market is expecting the short rate to increase more or decrease less; at the same time, the market also expects a decrease in the speed of this adjustment due to the mean-reverting feature of the short rate. Therefore increasing (decreasing) $Z_{2,t}$ comes with decreasing (increasing) $Z_{3,t}$ in most of the time. The same observations can also be found among the 2nd, 3rd and 4th state variables in X_t^h, as shown in Figure 3.3.

3.3 Yields Responses

3.3.1 Contemporaneous Responses

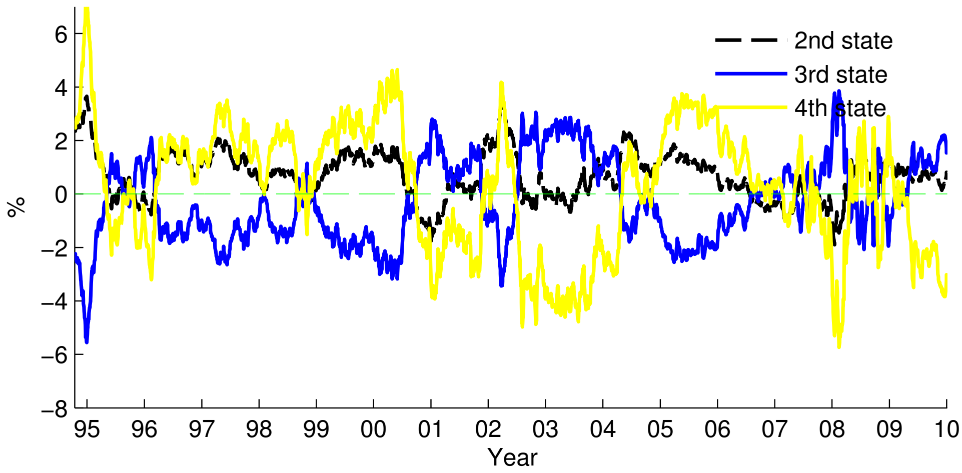
Given (2.1), the zero yield with time-to-maturity of x can be written as

$$y_t(x) = \varphi + \frac{\int_0^x \Theta^*(s)\,ds}{x} + \frac{\int_0^x \mathbf{C}_{\mathrm{CR}}(s)\,ds}{x} Z_t.$$

Figure 3.3: The 2nd, 3rd and 4th State Variables in X_t^h

This figure shows the dynamic of the 2nd, 3rd and 4th state variables in X_t from 1994 to 2009.

X_t^h, $h = 6$mths



X_t^h, $h = 12$mths

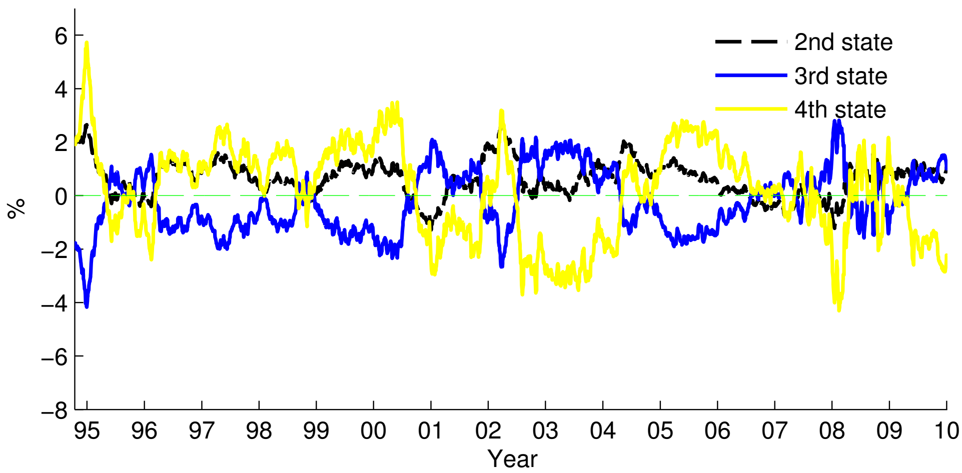
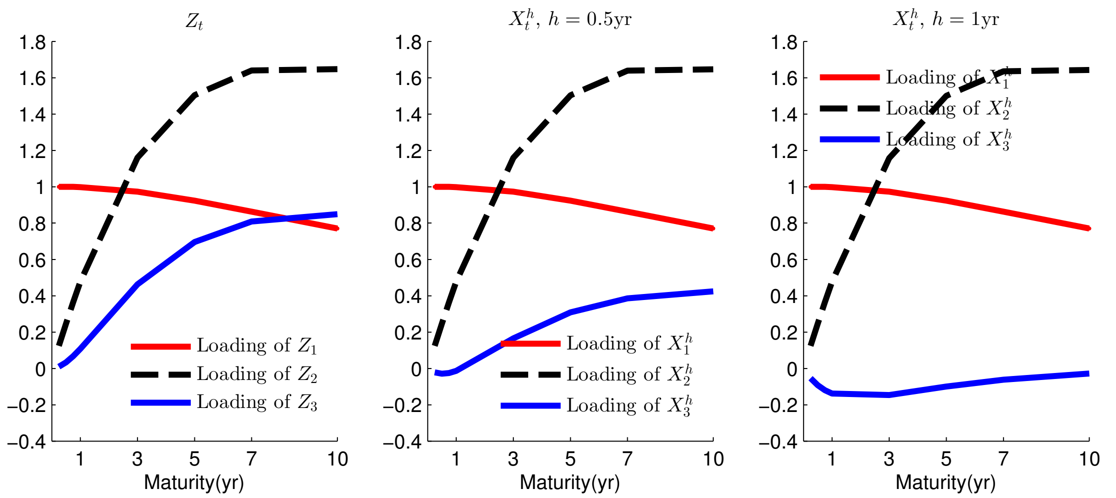


Figure 3.4: Contemporaneous Responses of the Zero Yields

This figure shows how the zero yields at different maturities respond to a change in 3 different sets of state variables (the left panel is for Z_t, the middel panel is for X_t^h when $h = 0.5$yr, the right panel is for X_t^h when $h = 1$yr).



It can also be written as an affine function of X_t^h based on (2.2)

$$y_t(x) = \varphi + \frac{\int_0^x \Theta^*(s)\,ds}{x} + \frac{\int_0^x \mathbf{D}(s,h)\,ds}{x} X_t^h.$$

The contemporaneous responses of the zero yield curve to the state variables are measured by the loading coefficients $\frac{\int_0^x \mathbf{C}(s)ds}{x}$ and $\frac{\int_0^x \mathbf{D}(s,h)ds}{x}$. Responses of the zero yields at different maturities are plotted in Figure 3.4.

From the figure, first of all we can see the short rate has a relatively flat loading across maturities. This means when other states keep constant, a change in the short rate can even significantly move the long end of the yield curve. Specifically, when there are no changes in other states, 10bps increase in the short rate can immediately induce 8bps increase in the 10yr zero yield. Secondly, I find that a change in the second state variable ($Z_{2,t}$ or $X_{2,t}^h$), which is the expectation of the short rate movement, has a small impact on the short end of the yield curve, but a much bigger impact on the long end: assuming all else being equal, 10bps increase in the second state can only increase the 3mth zero yield by 1.2bps, but increase the 10yr zero yield by 16bps. Notice that for horizons of 0, 6 months and 1 year, the contemporaneous impacts of changes in the expectation on the yield curve are almost the same, as we can see that the loadings of the second state variable in all three panels in Figure 3.4 have no noticeable difference. This means that even a change in the expectation of the short rate movement in 1 year can have an immediate impact on the long end of the yield curve. However, this is not the case for the expectation of the expectation movement. As we can see in Figure 3.4, the immediate impact of a change in $Z_{3,t}$ has a similar pattern as that of $Z_{2,t}$ but with a much smaller magnitude: almost zero impact on the short end, and a unit impact on the long end; when the horizon increases to 6 months, the impact is only half; when the horizon is 1 year, the impact is almost 0 at the long end, and becomes negative at the short end.

The findings for the first and the second state variables are consistent with the consensus in the literature (see, e.g.,Kuttner, 2001, Gürkaynak et al., 2005, Andersson et al., 2006, and Geiger, 2011) that only the "surprises" can impact the long end of the yield curve. Both state variables can significantly impact the long end in a condition of all else being equal. In other words, they can impact the long end when their changes come in surprise. However, their impacts are of different patterns: when a surprise happens in the short rate, the yield curve will be uniformly shifted; whereas, when a surprise happens in the expectation of the short rate movement, only the long end of the yield curve will be affected.

The finding for the third state variable is new. To my knowledge there are no previous studies focus on the expectation of the expectation change. The finding seems to tell that the immediate long-end impact of "surprises" in the expectation of the expectation movement has a short-term effect, meaning that as the horizon of the expectation increases (at least up to one year), the immediate impact on the long end decreases. Actually, if the horizon increases more the impact will become negative (this is not shown in the figure). This can be readily explained if we relate this expectation of the expectation change to the slopes of the forward curve at different maturities (see Appendix A): $X^h_{3,t}$ is proportional to the difference between the slopes of the forward curve at maturities h and 0. When $h = 0$, $X^h_{3,t}$ is just $Z_{3,t}$, its loading will be the same as that of $Z_{3,t}$; as h increases, the slope of the forward curve at h is approaching to zero as the part of forward curve at the maturity h is getting flat when h increases, therefore, $X^h_{3,t}$ will get close to $-Z_{2,t}$, and its loading will be the same as the negative loading of $Z_{2,t}$. The results shown in Figure 3.4 suggest that $h = 1$ is the "watershed", meaning when the horizon is small than one year, the change in $X_{3,t}^{h}$ has positive impacts on the yield curve, and negative impacts when the horizon is longer than one year.

Despite of the differences in the findings, one thing remains the same: the market expectation is one of the key determinants of the long term interest rates. There are more discussions on this in the next section.

3.3.2 Impulse Responses

The contemporaneous response analysis presented in the last section provides a good sense on how changes in the short rate and the market expectation of the short rate immediately impact the yield curve. This would be very useful for practitioners, as a good understanding on how the yield curve movement follows the changes in the state variables can be the key to a successful trading strategy. However, it is also of a great concern to measure the persistence effect of shocks on the yield curve. Especially from a policy maker's perspective, one would love to know what is the expected long-term effect of certain shocks (either endogenous or exogenous) on the dynamic path of the interest rates. When policy makers are armed with this knowledge, the short interest rates as an instrument can be better deployed to stimulate or restrain the economy. In this section, I conduct an impulse response analysis to further discuss the impact of changes in the short rate and expectations in terms of their persistent effects.

Following Koop et al. (1996), I define that impulse response function as

$$I(n,x,\Delta Z) \quad = \quad \mathbb{E}\left(y_{t+n}(x)\middle|Z_t = Z+\Delta Z\right) - \mathbb{E}\left(y_{t+n}(x)\middle|Z_t = Z\right),$$

where n denotes the concerned time horizon, ΔZ a vector of changes in state variables. Therefore for the model with Z_t, the impulse response function is

$$I(n,x,\Delta Z) = \frac{\int_0^x \mathbf{C}_{\mathrm{CR}}(s)\,ds}{x}\exp\left(\mathbf{A}_{\mathrm{CR}}n\right)\Delta Z;$$

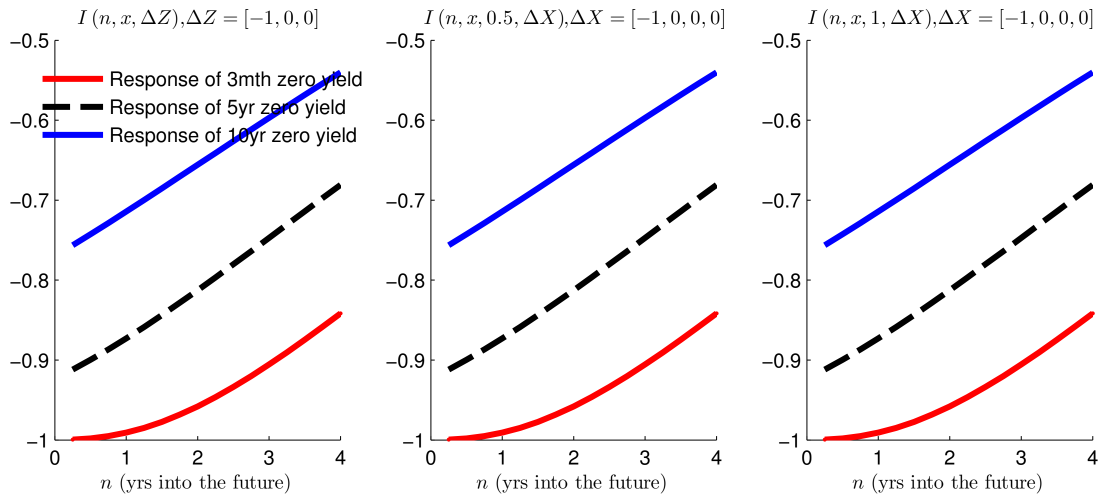
for the model with X_t^h,

$$I(n,x,h,\Delta X) = \begin{bmatrix} \frac{\int_0^x \mathbf{C}_1(s)ds}{x} & 0 & \frac{\int_0^x \mathbf{C}_2(s)ds}{x} & \frac{\int_0^x \mathbf{C}_2(s)ds}{x} \end{bmatrix} M_1 \exp\left(\mathbf{A}_{\mathrm{CR}}n\right) M_2 M_{fun}(h)^{-1}\Delta X.$$

Notice that the contemporaneous response presented before is a special case of the

Figure 3.5: Impulse Responses of the Zero Yields: Wicksell Effect

This figure shows how the 3mth, 5yr, and 10yr zero yields respond to a shock in the short rate due to the Wicksell effect in 4yr horizon.(the left panel is for $I(n,x,\Delta Z)$, the middel panel is for $I(n,x,h,\Delta X)$ when $h = 0.5$yr, the right panel is for $I(n,x,h,\Delta X)$ when $h = 1$yr).



impulse response when $n \to 0$. As mentioned in Koop et al. (1996), the impulse response function can be understood as a multiplier, it captures the properties of the model's propagation mechanism, and compares the value of a yield after the shock has occurred with its benchmark value where the economy has not been subject to any shocks.

Wicksell effect The "Wicksell Effect" used here refers to an "Interest Wicksell Effect" same as in Cwik (2005). When the monetary authority engages in a policy of monetary expansion, e.g., the Fed wants to stimulate the economy, the new money is injected into the monetary system by expanding the supply of investable funds to achieve the targeted Federal funds rate. This will lower the short rate immediately. However, it does not necessary affect the expected inflation immediately. In this section I show how the Wicksell effect impacts the yield curve dynamically.

I assume that the Wicksell effect only lowers the $Z_{1,t}$ without affecting other variables, i.e.,

$$\Delta Z = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}, \Delta X = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}.$$

Since the first state variable is the same for both Z_t and X_t^h, the yield curve responses to this state variable are the same for the three models. We can see from Figure 3.5, a shock in the short rate has a uniform impact on the yield curve even in 4 years of time. 1bp decrease in the short rate now is expected to lower the short rate by 0.86bp in 4 years; can lower the 10 year zero yield by 0.76bp now, and is expected to lower it by 0.55bp in 4 years.

Fisher effect When the expected inflation rises, interest rates will rise. This result has been named the *Fisher effect* (Mishkin, 2007). Ang et al. (2008) also find that nominal term spreads are primarily driven by changes in the expected inflation. Here I assume a scenario in which the expected inflation has changed and induced a shock on $Z_{2,t}$, $X_{2,t}^{0.5}$, and $X_{2,t}^{1}$ but no direct impact on the short rate and other state variables, i.e.,

$$\Delta Z = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \Delta X = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}.$$

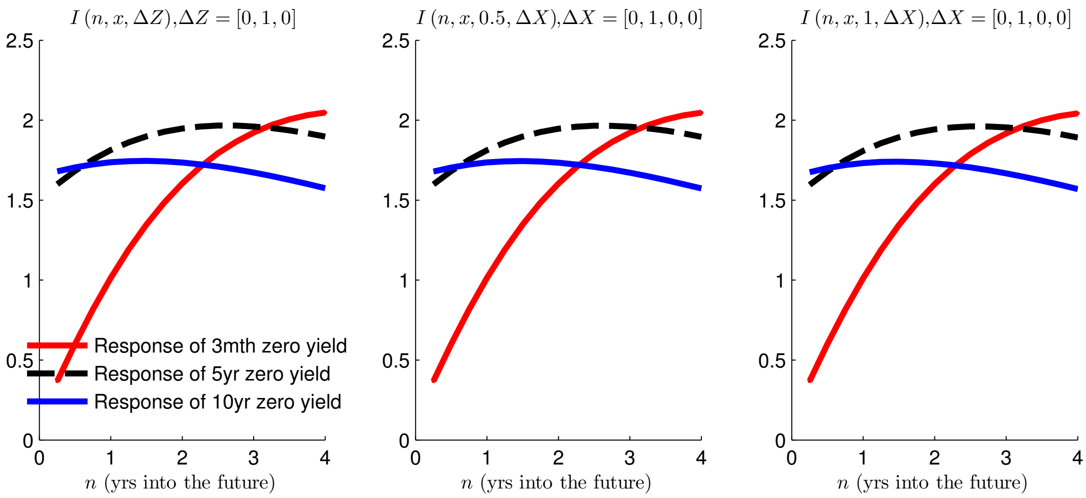
The impulse responses for a 4yr horizon are shown in Figure 3.6. I find that if the shock only happens in the expectation of the short rate change, the impact on the long end of the yield curve is significant, instant, and persistent. Specifically, 1bp increase in $Z_{2,t}$ can raise the 10yr zero yield by 1.6bp and the impact stays there for at least 4 years; the impact of an increase in $X_{2,t}$ has a similar pattern but with an even higher magnitude. The Fisher effect can also impact on the short rate, however, it comes in a lagged and slowly increasing manner (from 0.4bp to 2bps in 4 years of time). The impacts of shocks in $X_{2,t}^{0.5}$ and $X_{2,t}^{1}$ have no noticeable difference.

Therefore the Fisher effect in this scenario has indeed an impressive impact on the yield curve. However, I argue that this scenario is actually unreasonable and misleading. If we take a look at the dynamics of $Z_{2,t}$, $Z_{3,t}$ and $X_{2,t}^{h}$, $X_{3,t}^{h}$, $X_{4,t}^{h}$ in Figure 3.2 and Figure 3.3, we can quickly find that the absolute correlations between their changes are extremely high. The correlation coefficient matrix for $Z_{2,t}$ and $Z_{3,t}$ is

$$\begin{bmatrix} 1 & \\ -0.9708 & 1 \end{bmatrix};$$

Figure 3.6: Impulse Responses of the Zero Yields: Fisher Effect I

This figure shows how the 3mth, 5yr, and 10yr zero yields respond to a shock in the expectation of the short rate change in a 4yr horizon.(the left panel is for $I(n,x,\Delta Z)$, the middel panel is for $I(n,x,h,\Delta X)$ when $h = 0.5$yr, the right panel is for $I(n,x,h,\Delta X)$ when $h = 1$yr).



for $X_{2,t}^{0.5}$, $X_{3,t}^{0.5}$ and $X_{4,t}^{0.5}$ is

$$\begin{bmatrix} 1 & & \\ -0.9550 & 1 & \\ 0.9169 & -0.9939 & 1 \end{bmatrix};$$

and for $X_{2,t}^{1}$, $X_{3,t}^{1}$ and $X_{4,t}^{1}$ is

$$\begin{bmatrix} 1 & & \\ -0.9353 & 1 & \\ 0.8809 & -0.9912 & 1 \end{bmatrix}.$$

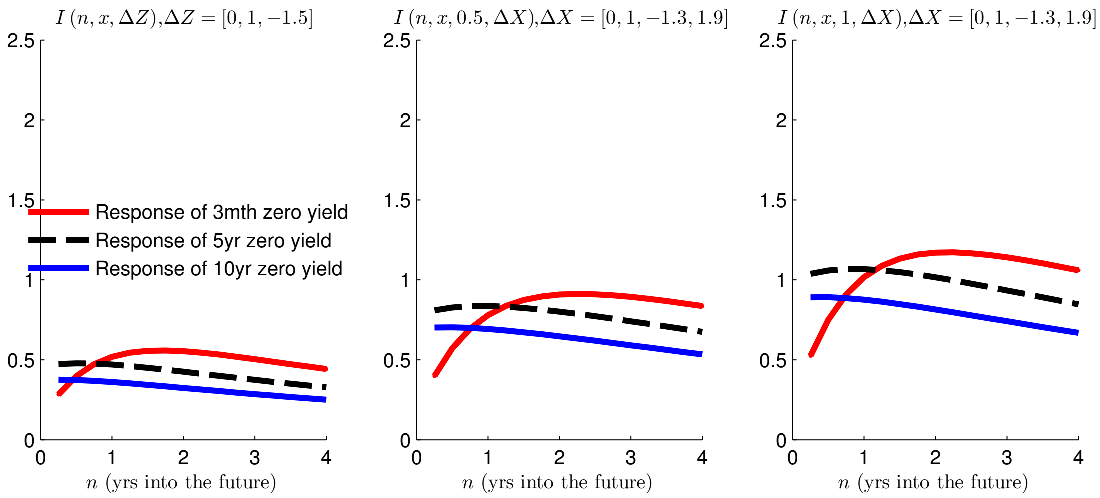
This means that it will be very rare that a shock only happens in $Z_{2,t}$ and $X_{2,t}^{h}$ without influencing $Z_{3,t}$ and $X_{3,t}^{h}$, $X_{4,t}^{h}$. Therefore, a more sensible scenario is a change in the expected inflation accompanies shocks on $Z_{2,t}$ $Z_{3,t}$ and $X_{2,t}^{h}$, $X_{3,t}^{h}$, $X_{4,t}^{h}$. I now set ΔZ and ΔX as

$$\Delta Z = \begin{bmatrix} 0 & 1 & -1.5 \end{bmatrix}, \Delta X = \begin{bmatrix} 0 & 1 & -1.3 & 1.9 \end{bmatrix},$$

where -1.5 is the sample average relative absolute size of $Z_{3,t}$ to $Z_{2,t}$; -1.3 and 1.9 are

Figure 3.7: Impulse Responses of the Zero Yields: Fisher Effect II

This figure shows how the 3mth, 5yr, and 10yr zero yields respond to shocks in the expectation related state variables in a 4yr horizon.(the left panel is for $I(n,x,\Delta Z)$, the middel panel is for $I(n,x,h,\Delta X)$ when $h=0.5$yr, the right panel is for $I(n,x,h,\Delta X)$ when $h=1$yr).



those for $X^h_{3,t}$ and $X^h_{4,t}$. The impulse response analysis of the Fisher effect in this scenario is presented in Figure 3.7. Surprisingly, I find that in this scenario, the impacts of the Fisher effect drop substantially. Now 1bp increase in $Z_{2,t}$ can only lift the long end by about 0.3bp in 4 years of time due to 1.5bp decrease in $Z_{3,t}$. Similarly, 1bp increase in $X^{0.5}_{2,t}$ ($X^1_{2,t}$) can only induce 0.6bp (0.9bp) increase in the long end.

Although this finding is somewhat surprising, it is quite consistent with the consensus that only the "surprises" can significant impact the long end of the yield curve. The shock in the second scenario are less surprising than that in the first one, because the first one is really rare given the historical observations. Therefore the Fisher effect in the first "unreasonable" scenario has a much more prominent impact on the long end. And this finding should also remind us not to ignore the expectation of the expectation changes.

Wicksell and Fisher effects combined In reality, the Wicksell and Fisher effects often come together. Intuitively, with a policy of monetary expansion, the Wicksell effect will lower the whole yield curve with a more significant impact on the short end; at the same time, the expected inflation will likely increase, so the Fisher effect will lift the long end to a certain extent. This combined effect is illustrated in Figure 3.8 which is a variant of Fig 1 in Cwik (2005). Although Figure 3.8 illustrates a policy easing, a similar analysis with everything in the opposite direction applies to a policy tightening. Asking how the long end of the yield curve will be affected by the combined effect leads to an open question which has been heavily discussed in the literature: Cook and Hahn (1989) report long-term interest rates move in the same direction as monetary policy actions based on a relatively old data-set; Gürkaynak et al. (2005) argue that the movement of the forward curve is the other way around based on an updated data set; and Ellingsen and Söderström (2001) build a theoretical model to prove that depending on whether the shocks are endogenous or exogenous both scenarios are possible (endogenous shocks lead to the gay line, exogenous ones lead to the dash line). Here I argue that which direction the long end will move is just a matter of domination. If the Wicksell effect dominates the Fisher effect, the long end of the yield curve will fall, and will rise the other way around. I quantitatively illustrate this point by presenting a few numerical examples.

For the sake of a clear comparison, I only consider $I(n,x,1,\Delta X)$, the results for three different scenarios are shown in Figure 3.9. The left panel in Figure 3.9 shows a scenario in which the Fisher effect dominates the Wicksell effect, i.e., if $X_{1,t}^{1}$ decreases by 1bp and $X_{2,t}^{1}$ increases by 1.3bp, the impact of the combined effect will tilt the yield curve counterclockwise in the short run and even lift the whole yield curve in the long run (say, in 4 years of time); the middle panel shows a scenario in which the combined effect only transitorily impact the short end and has no impact on the long end at all, i.e., the Fisher effect cancels out the Wicksell effect at the long end. For this to happen, when $X_{1,t}^{1}$ increases by 1bp, $X_{2,t}^{1}$ should decrease 0.83bp; the right panel shows a scenario where the Wicksell effect dominates the Fisher effect, so the combined effect shifts the whole yield curve downward. This happens when $X_{2,t}^{1}$ increases relatively less (about $1/3$) comparing to the amount by that $X_{1,t}^{1}$ decreases.

These observations convey an intuitive idea about why sometimes the monetary actions are ineffective and when they are so. If the central banks can convince the private sector that they will keep the future inflation low when conducting a monetary easing[1], then this easing will be considered as a sign of a positive economic outlook, so it will

[1]Here I assume the easing is conducted via a conventional monetary policy, e.g., setting the Fed funds rate's targets.

Figure 3.8: Wicksell and Fisher Effects Combined

This figure shows how the Wicksell and Fisher effects combined can move the yield curve. The black line is the initial curve, the gay line is the resulting curve when the Wicksell effect dominates the Fisher effect, the dash line is the resulting curve when the Fisher effect dominates the Wicksell effect, and the dot line is the resulting curve when the Wicksell effect and the Fisher effect cancel out at the long end of the curve.

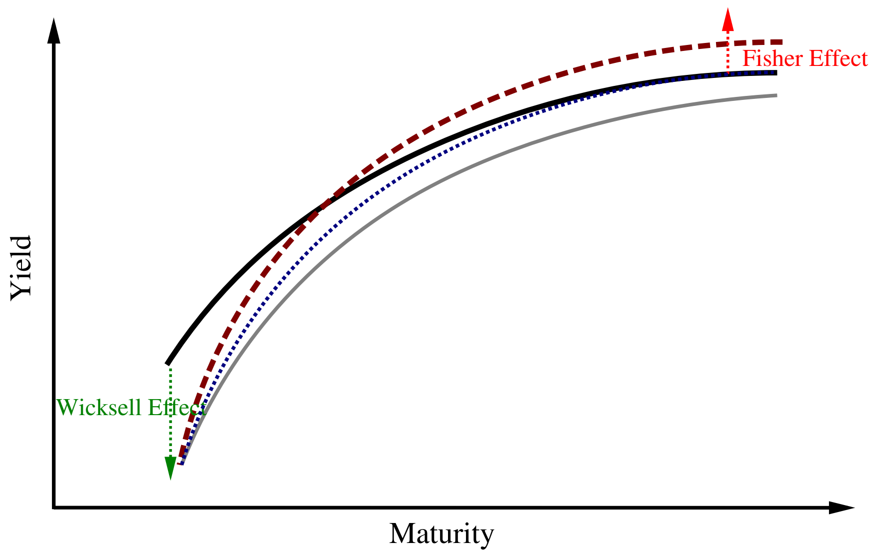
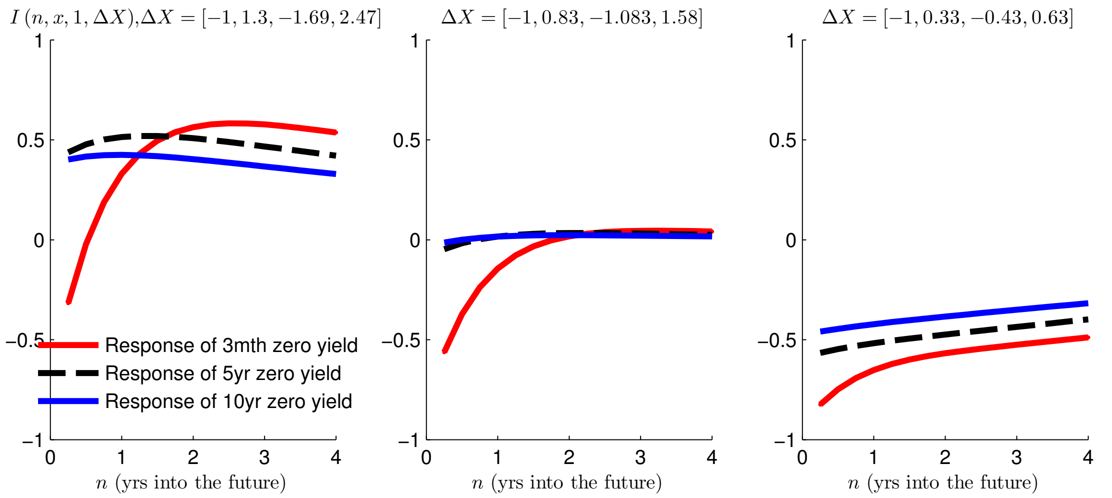


Figure 3.9: Responses of the Zero Yields: Wicksell and Fisher Effects Combined

This figure shows how the 3mth, 5yr, and 10yr zero yields respond to the combined Wicksell and Fisher effect in three different ways (in the left panel the Fisher effect dominates the Wicksell effect, in the middel panel the Wicksell effect and the Fisher effect cancel out, in the right panel the Wicksell effect dominates the Fisher effect).



typically take effect by inducing the Wicksell effect to dominate the Fisher effect, and lowering the whole yield curve; however, if they fails to do so, i.e., the private sector may doubt either the ability or the will of the central banks to achieve that future inflation, a monetary easing can only be interpreted as a further accelerant of the inflation, so it won't be effective as the Fisher effect dominates or cancels out the Wicksell effect, the long end of the yield will likely move in the opposite direction as the short end. Therefore, accurately measuring and monitoring the expected inflation, and the credibility of the central banks are the keys to build up the channels of monetary transmission.

Before finishing up this section, let's take a look at how the combined effect impacts the yield curve under an "unsurprising" circumstance. According to Kuttner (2001), anticipated monetary moves have little impact on the long end of the yield curve. Therefore, under an "unsurprising" circumstance the Wicksell effect and the Fisher effect should cancel out as the scenario in the middle panel of Figure 3.9. Now the question is: what would be a sensible "unsurprising" circumstance? A natural answer is a representative of the most often encountered circumstances. To find this representative, I estimate the following three regressions (again I take the model with X_t^1):

$$\begin{aligned}
\Delta X_{1,t}^1 &= \underset{[-2e-05,4e-06]}{0.000} + \beta_1 = \underset{[0.415,0.472]}{0.4439}\Delta X_{1,t-1}^1 + \varepsilon_{1,t}, R^2 = 0.20 \\
\Delta X_{2.t}^1 &= \underset{[-2e-05,9e-06]}{-0.000} + \beta_2 = \underset{[-0.4551-0.371]}{-0.4129}\Delta X_{1,t-1}^1 + \varepsilon_{2,t}, R^2 = 0.09 \\
\Delta X_{3,t}^1 &= \underset{[-9.5e-06,3.9e-05]}{0.000} + \beta_3 = \underset{[0.5315,0.6563]}{0.5939}\Delta X_{1,t-1}^1 + \varepsilon_{3,t}, R^2 = 0.08 \\
\Delta X_{4,t}^1 &= \underset{[-5.4e-05,1.2e-05]}{-0.000} + \beta_4 = \underset{[-0.888,-0.7163]}{-0.8022}\Delta X_{1,t-1}^1 + \varepsilon_{4,t}, R^2 = 0.08.
\end{aligned}$$

where the 95% confidence intervals for β_is ($i = 1,2,3,4$) are presented in the square brackets. So all estimates of β_is are significant at 5%. In an average sense 1pb decrease in $X_{1,t}^1$ accompanies $-\beta_2/\beta_1$bp increase in $X_{2,t}^1$, β_3/β_1bp decrease in $X_{3,t}^1$, and $-\beta_4/\beta_1$bp increase in $X_{4,t}^1$. Therefore, a representative ΔX_t^1 would be

$$\Delta X = \left[-1,\ \frac{\beta_2}{\beta_1},\ \frac{\beta_3}{\beta_1},\ \frac{\beta_4}{\beta_1}\right].$$

The response function $I(n,x,1,\Delta X)$ shown in Figure 3.10 is very similar to the middle panel in Figure 3.9. This just confirms that the model presented in the paper is able to infer useful information about the market expectations. By using the model to monitor the information, one might be able to easily isolate the impact of shocks by categorizing them as “surprises” and “unsurprises”.

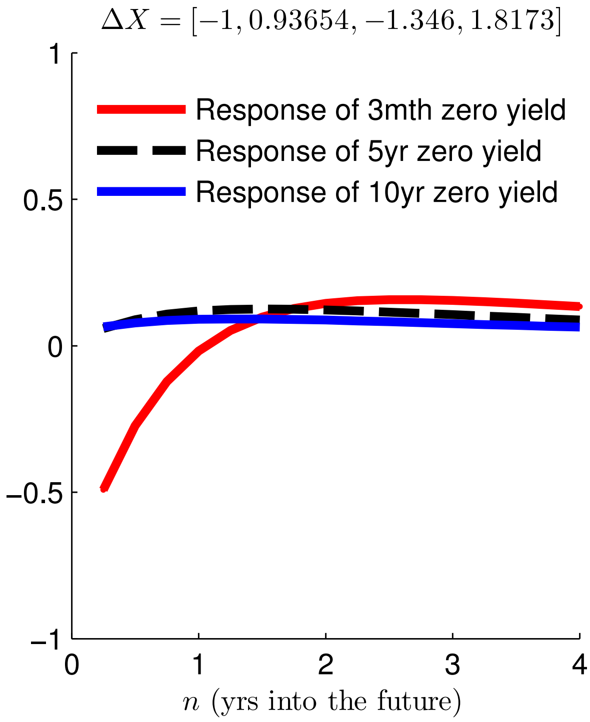
3.4 Case Studies: impacts of LSAP and MEP announcements

In December 2008, the Federal Reserve reduced its target for the federal funds rate—the traditional tool of U.S. monetary policy—to a range of 0 to 25 basis points, essentially the lower bound of zero. This means that despite the severity of the recession, the conventional option of reducing the Funds rate was no longer available. Concerned that economic conditions would deteriorate, the FOMC chose to pursue unconventional monetary policies since there was no scope for further cuts in short-term interest rates.

On November 25, 2008, the Federal Reserve announced that it would purchase up to \$100 billion in agency debt, and up to \$500 billion in agency MBS, and later on a series of large-scale asset purchase programs (LSAP) were implemented on those debts and

Figure 3.10: The Unsurprising Circumstance

This figure shows the responses of 3mth, 5yr, and 10yr zero yields to the combined Wicksell and Fisher effect in an unsurprising circumstance.



long-term Treasury securities as well. LSAP is also referred to as "Quantitative Easing" (QE). On September 21, 2011, the FOMC announced the Maturity Extension Program (MEP), under which the FOMC will purchase, by the end of June 2012, \$400 billion of Treasury securities with remaining maturities of 6 to 30 years while simultaneously selling an equal amount of Treasuries with remaining maturities of 3 years or less. MEP is also referred to as "operation twist".

On November 25, 2008, $\Delta X_t^1 = [-2.8613, -3.0635, -10.4201, 21.6822]$ bps; On September 21, 2011, $\Delta X_t^1 = [-0.58617, 3.9402, -13.0314, 20.9613]$ bps. By comparing these two ΔX_t^1s with the one presented in Figure 3.10, we can see LSAP and MEP are apparently surprises to the economy system. Now let's see how these surprises impact the yield curve.

From the implied forward curve changes shown in Figure 3.11a and Figure 3.13a, I find that the contemporaneous responses of market are consistent with the intents of the Fed: the LSAP is targeting the long-term yields, in Figure 3.11a it shows that the forward curve is tilted clockwise with the short end being intact; in Figure 3.11b the forward curve rises in the short end but falls in the long end, this is a response to the "twist" effect of MEP. I also find that MEP had a significantly smaller impact on the forward curve, this is somewhat expected, as MEP can actually be considered as a part of LSAP.

It is interesting to see how the expected change of the short rate responds to the LSAP announcement, this expected change is associated with the slope of the implied forward curve and often interpreted as the expected inflation. From the panels titled "X_2^h" and "ΔX_2^h" in Figure 3.11b, I find that the market seemed (at least on the day of the LSAP announcement) to have a strong believe that the LSAP would have a positive influence on the recovery of the economy instead of just boosting up the inflation without stimulating the recovery: upon the LSAP announcement, the expected inflation for horizons from 6 months to 6 years fell by different extents, only those at very short horizons (shorter than 6mth) rose. By looking at the shape change of the forward curves, I find that the LSAP announcement decreased the slope of the forward curve at middle and long horizons but increased it at short horizons, i.e., increasing the short end curvature of the forward curve. The reaction of the yield curve upon the LSAP announcement can be understood as a consequence of a new version of Wicksell and Fisher effects combined in which both effects push the yield curve to a same direction (downwards).

Figure 3.12 shows the expected impacts of the LSAP announcement on the yield curve 4 years into the future. Normally, X_2^1 and X_3^1 move in opposite directions, however, I find from the panels titled "X_2^h" and "X_3^h" in Figure 3.11b that X_3^h moved in a same direction as X_2^h for $h > 6$ months. This might indicate that the market was expecting a further decrease of X_2^h, and the LSAP announcement was indeed a big surprise to the market. Therefore the expected long run impacts of the LSAP announcement should be significant, and this is confirmed in Figure 3.12: the LSAP announcement had a persistent impact on the long end of the yield curve and a gradually increasing one on the short end, specifically, the LSAP announcement lowered the long end by 16bps instantly and the impact would stay there for at least 4 years; although it only contemporaneously lowered the short end by about 3bps, the impact would be gradually increasing, it is expected to lower the short end by 20bps in 4 years of time.[2] The impact of the LSAP announcement is mostly attributed to the changes in the expectation related state variables.

Although MEP had a much smaller impact on the yield curve, it impacted the yield curve quite differently. As we can see in the panel titled "ΔX_2^h" in Figure 3.13b, the announcement of MEP increased the X_2^h for $h < 2$yr, but had no impact for $h > 2$yr. This means that the announcement increased the short term expected inflation while kept the long term ones constant. Comparing the one day response of the forward curves upon the announcements of the LSAP and the MEP, it seems the market remained relatively calm upon the MEP announcement, and believed that the additional purchase of \$400 billion of long term Treasury securities financed by selling a same amount of the short term securities would actually boost up the inflation in the short run than having any further influence on the recovery. So under this scenario, the Fisher effect was slightly dominated by, if not completely canceled out, the Wicksell effect at the long end of the yield curve, and made the slope at the short end less negative. Unsurprisingly, the expected long run impact of the MEP announcement is much less notable. This is confirmed in Figure 3.14, specifically, the announcement is expected to lower the yield curve by about 4bps in 4 years of time, which is only $\frac{1}{4}$ of the impact of the LSAP announcement. Again the impact is mostly attributed to the changes in the expectation related state variables. However, one observation that needs to be mentioned is that,

[2]Of course the downward impact on the short end would be bounded by the zero lower bound.

unlike the LSAP, the MEP announcement is expected to increase the short rate in about 2 years before it eventually pushing down the short rate. This is consistent with the increase in the short term expected inflation.

4 Conclusions

In this paper, based on the classic $\mathbb{A}_0(3)$, I develop a new model in which the state variables comprises the short rate and its related expectations. This unique feature enable the model to be used in analyzing the responses of the yield curve to the market expectations, and monitoring the changes of the expectations. The model is estimated using about 16 years of daily zero yield data, then based on the estimated model, various numerical examples are provided to help understand the mechanism behind the monetary transmission.

Although in the paper, I assume the expected change of the short rate can be interpreted as the expected inflation based on results of many studies in the literature, e.g., Ang et al. (2008); Söderlind and Svensson (1997), this interpretation might need more rigorous empirical supports. This would be an interesting extension of the current paper. Also how the market expectations respond to the monetary policies should be analyzed not only numerically but also empirically. This deserves efforts in future work.

Appendices

A State Variables as the Forward Curve Characteristics

This section shows that the state variables Z_t and X_t^h can precisely be interpreted as some characteristics of the forward curve.

In what follows, Δt denotes the limit of a time interval which approaches to 0 infinitely, for the sake of convenience, the notation of lim is dropped.

$$\begin{aligned} Z_{2,t} &= \frac{\mathbb{E}_t^{\mathbb{Q}}\left(Z_{1,t+\Delta t}\right) - Z_{1,t}}{\Delta t} = \frac{r(t,\Delta t) - r(t,0)}{\Delta t}, \\ Z_{3,t} &= \frac{\mathbb{E}_t^{\mathbb{Q}}\left(Z_{2,t+\Delta t}\right) - Z_{2,t}}{\Delta t} = \frac{\frac{r(t,2\Delta t)-r(t,\Delta t)}{\Delta t} - \frac{r(t,\Delta t)-r(t,0)}{\Delta t}}{\Delta t}, \\ &= \frac{r(t,2\Delta t) - 2r(t,\Delta t) + r(t,0)}{(\Delta t)^2}. \end{aligned}$$

It is clear that $Z_{2,t}$ and $Z_{3,t}$ represent the slope and curvature of the forward curve $r(t,x)$ at $x = 0$, respectively.

$$\begin{aligned} X_{2,t}^h &= \frac{\mathbb{E}_t^{\mathbb{Q}}\left(Z_{1,t+h}\right) - Z_{1,t}}{h} = \frac{r(t,h) - r(t,0)}{h}, \\ X_{3,t}^h &= \frac{\mathbb{E}_t^{\mathbb{Q}}\left(Z_{2,t+h}\right) - Z_{2,t}}{h} = \frac{\mathbb{E}_t^{\mathbb{Q}}\left(\frac{r(t+h,\Delta t)-r(t+h,0)}{\Delta t}\right) - \frac{r(t,\Delta t)-r(t,0)}{\Delta t}}{h} \\ &= \frac{\frac{r(t,h+\Delta t)-r(t,h)}{\Delta t} - \frac{r(t,\Delta t)-r(t,0)}{\Delta t}}{h}, \\ X_{4,t}^h &= \frac{\mathbb{E}_t^{\mathbb{Q}}\left(Z_{3,t+h}\right) - Z_{3,t}}{h} = \frac{\mathbb{E}_t^{\mathbb{Q}}\left(\frac{r(t+h,2\Delta t)-2r(t+h,\Delta t)+r(t+h,0)}{(\Delta t)^2}\right) - \frac{r(t,2\Delta t)-2r(t,\Delta t)+r(t,0)}{(\Delta t)^2}}{h} \\ &= \frac{\frac{r(t,h+2\Delta t)-2r(t,h+\Delta t)+r(t,h)}{(\Delta t)^2} - \frac{r(t,2\Delta t)-2r(t,\Delta t)+r(t,0)}{(\Delta t)^2}}{h}. \end{aligned}$$

Therefore, $X_{2,t}^h$ represents the slope between $r(t,h)$ and $r(t,0)$; $X_{3,t}^h$ ($X_{4,t}^h$) represents the difference between slopes (curvatures) at $r(t,h)$ and $r(t,0)$.

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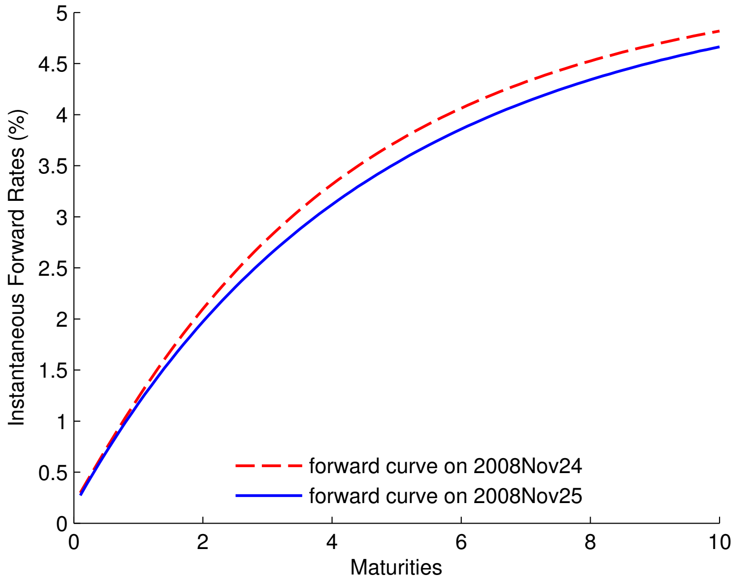
Figure 3.11: Changes upon the LSAP announcement

This figure shows changes of the forward curve and state variables happened on November 25, 2008 when the Fed annouced the LSAP.

(a) Changes in forward curves



(b) Changes in state variables

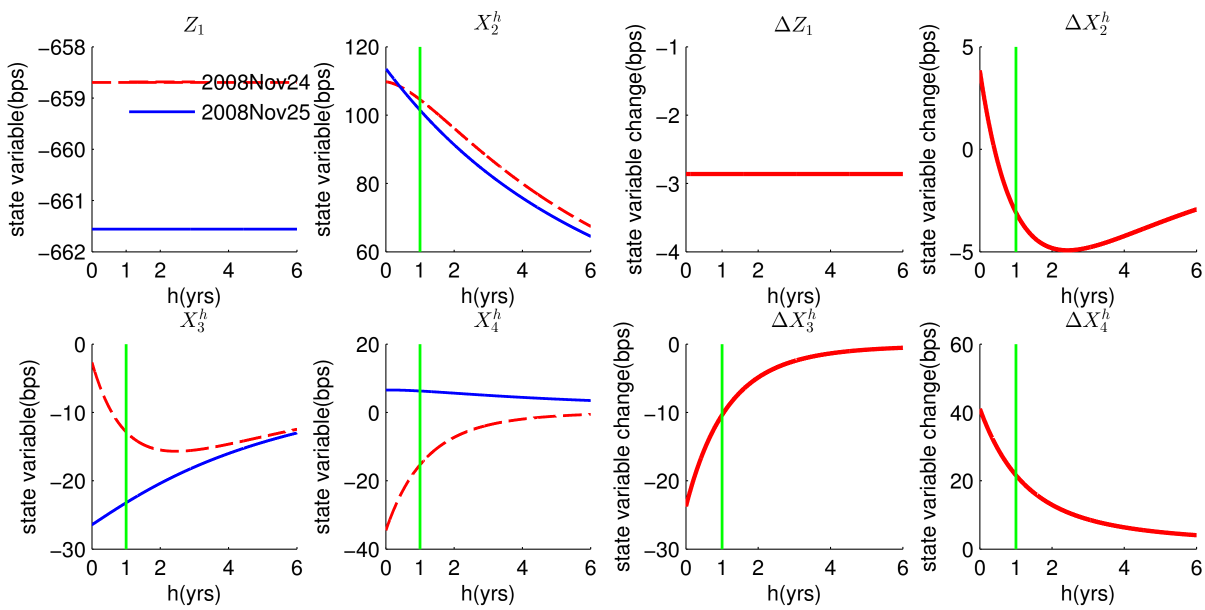


Figure 3.12: Responses of the Zero Yields upon the LSAP announcement

This figure shows the responses of the 3mth, 5yr, and 10yr zero yields to the shocks happened on November 25, 2008 when the Fed annouced the LSAP.

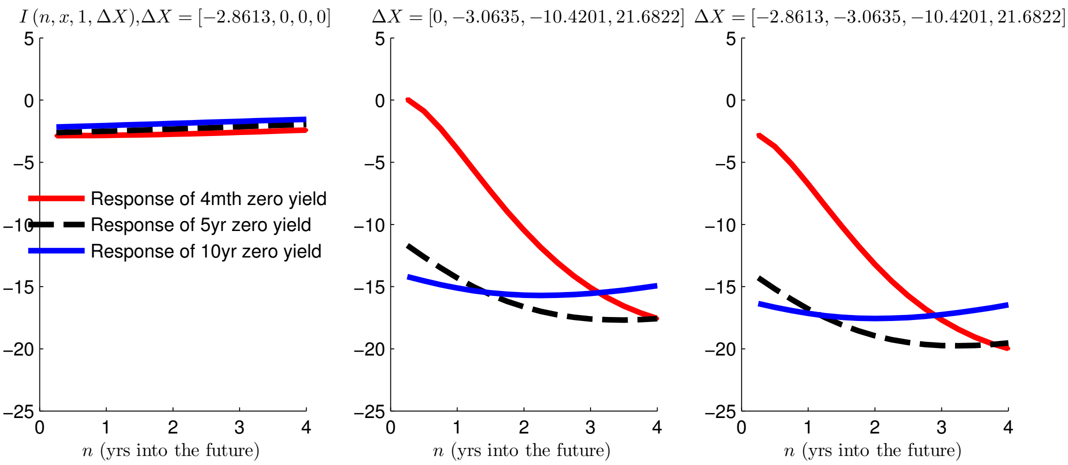
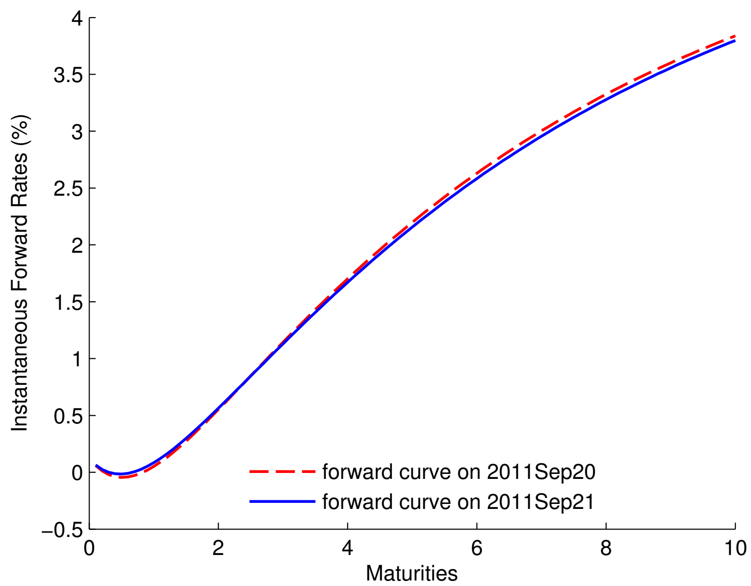


Figure 3.13: Changes upon the MEP announcement

This figure shows changes of the forward curve and state variables happened on September 21, 2011 when the Fed annouced the MEP.

(a) Changes in forward curves



(b) Changes in state variables

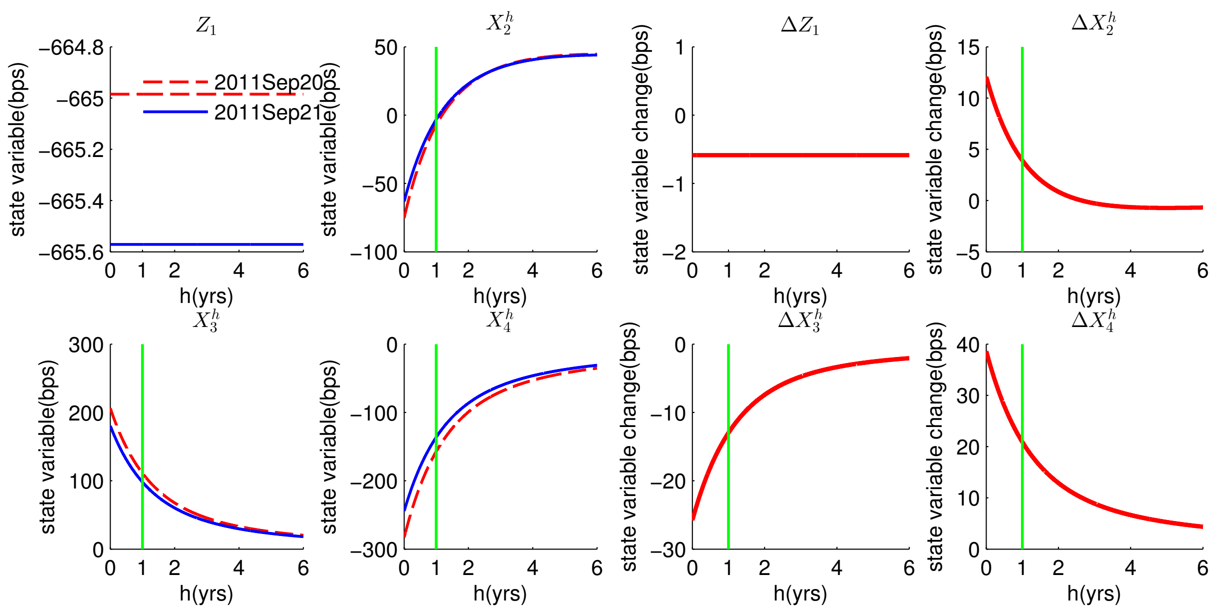


Figure 3.14: Responses of the Zero Yields upon the MEP announcement

This figure shows the responses of the 3mth, 5yr, and 10yr zero yields to the shocks happened September 21, 2011 when the Fed annouced the MEP.