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Abstract

In this study, we develop a search-based monetary growth model to analyze the effects of inflation on economic growth and social welfare by introducing endogenous economic growth via capital externality into a two-sector search model. We find that the channel through which inflation affects economic growth in the search model is different from the cash-in-advance model. In the quantitative analysis, we evaluate the welfare effect of inflation in the search-based endogenous growth model and compare it to a search-based exogenous growth model. We find that the welfare effect of inflation is nonlinear in the endogenous growth model whereas it is linear in the exogenous growth model. Furthermore, we find that the welfare cost of inflation under endogenous growth is about three times as large as the welfare cost under exogenous growth.

JEL classification: E41, O41, O42.
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1 Introduction

In this study, we analyze the effects of inflation and monetary policy on economic growth and social welfare. Although this important issue in monetary economics has received much attention and careful analysis in previous studies, our analysis provides novel elements and results. To highlight the novelty of this study, it is helpful to first discuss two related branches of literature in monetary economics. First, this study relates to the search-based literature on money and capital formation; see for example, Shi (1999), Menner (2006), Williamson and Wright (2010), Aruoba *et al.* (2011), Bencivenga and Camera (2011) and Waller (2011). This branch of literature analyzes the relationship between money and capital formation in a search-theoretic framework without considering economic growth as an endogenous process. Second, this study also relates to the branch of literature on inflation and economic growth; see for example, Wang and Yip (1992), Gomme (1993), Dotsey and Ireland (1996), Mino (1997) and more recently, Itaya and Mino (2003, 2007) and Lai and Chin (2010).[1] This branch of literature analyzes the growth and welfare effects of inflation by modeling money demand based on the classical approach, such as a cash-in-advance (CIA) constraint, money in utility and transaction costs, without considering search. In this study, we attempt to provide a bridge between these two branches of literature by analyzing the growth and welfare effects of inflation in a search-based monetary growth model. In summary, we introduce endogenous economic growth via capital externality as in Romer (1986) into a two-sector search model based on Lagos and Wright (2005), Aruoba *et al.* (2011) and Waller (2011).[2] In this search-based monetary growth model, we find the following results that have novel implications on the two branches of literature.

In a canonical monetary growth model with a CIA constraint on consumption, inflation affects economic growth via the consumption-leisure tradeoff; in other words, in the case of inelastic labor supply, inflation has no effect on economic growth in the CIA model. In contrast, in the search-based monetary growth model, the growth effects of inflation operate through a consumption effect in the decentralized market. Intuitively, a higher inflation rate increases the cost of holding money and reduces consumption in the decentralized market that requires the use of money for transactions. As a result of lower consumption in the decentralized market, capital demand decreases causing a reduction in capital accumulation and economic growth regardless of whether or not labor supply is elastic.

We also calibrate the model to quantitatively evaluate the welfare effects of inflation and compare the welfare effects of inflation under the search-based endogenous growth model to those from a search-based exogenous growth model similar to Aruoba *et al.* (2011) and Waller (2011) in order to highlight the importance of modelling economic growth as an endogenous process. We find that the welfare effect of inflation is nonlinear in the endogenous growth model,[3] whereas it is linear in the exogenous growth model. We discuss the intuition of this

[1] There is also a sub-branch of this literature that analyzes the relationship between inflation and innovation-driven economic growth; see for example, Marquis and Reffett (1994) and Chu and Lai (2012).

[2] A recent study by Chiu *et al.* (2011) also provides an interesting analysis on the effects of financial and search frictions on economic growth, but they do not consider money and inflation.

[3] Chiu and Molico (2010) also find that an extended version of the Lagos-Wright model with endogenous participation in the centralized market exhibits a nonlinear welfare function of inflation in contrast to the linear welfare function in the original Lagos-Wright model.

result in the main text.[4] Interestingly, we also find that the welfare cost of inflation under endogenous growth is about three times as large as the welfare cost under exogenous growth partly because reducing inflation increases the long-run growth rate in the endogenous growth model but not in the exogenous growth model. Furthermore, because the welfare effect of inflation is nonlinear in the endogenous growth model, the difference in the welfare costs of inflation across the two models is increasing in the size of the change in the money growth rate.

The rest of this study is organized as follows. Section 2 presents the search-based monetary growth model. Section 3 analyzes the effects of inflation on economic growth and social welfare. In Section 4, we calibrate the model to provide a quantitative analysis. The final section concludes.

2 A search-based monetary growth model

The two-sector search model is based on Lagos and Wright (2005). Aruoba *et al.* (2011) extend the Lagos-Wright model by introducing capital accumulation, whereas Waller (2011) further extends the model in Aruoba *et al.* (2011) by allowing for exogenous technological progress. Our model is based on Waller (2011), but we introduce capital externality into his model to generate endogenous growth. In what follows, we describe the basic features of the search-based monetary growth model.

2.1 Households

There is a unit measure of identical and infinitely-lived households in discrete time. In each period, households engage in economic activities first in the decentralized market (hereafter DM) and then in the centralized market (hereafter CM). The DM and the CM are distinguished as follows. In period t, households first enter the DM where they consume or produce special goods q_t. In this market, each meeting is random and anonymous so that money becomes essential.[5] Once the round of DM trade is completed, households proceed to the CM where they consume and produce general goods as in standard growth models. Following the common approach in the literature, we assume that there is no discounting between the DM and the CM within each period, and the discount factor between any two consecutive periods is $\beta \in (0, 1)$. In what follows, we first discuss households' optimization in the CM.

[4] See Section 4.2 for a discussion.

[5] Following the standard approach in the literature, we assume that capital cannot serve as a medium of exchange; see Williamson and Wright (2010) and Aruoba *et al.* (2011) for a useful discussion. Lagos and Rocheteau (2008) show that even when capital serves as a competing medium of exchange, fiat money can still be valued and used as a medium of exchange.

2.1.1 Households' optimization in the CM

In the CM, households have an instantaneous utility function $u_t = B \ln x_t - Ah_t$, which is increasing in the consumption of general goods x_t and decreasing in the supply of labor h_t. The parameters $A > 0$ and $B > 0$ determine respectively the disutility of labor supply and the importance of consumption.[6] Let $W(m_t, k_t)$ and $V(m_t, k_t)$ denote the period-t value functions for households in the CM and the DM respectively. m_t is the nominal money balance and k_t is the capital stock owned by households in period t. The maximization problem of households in the CM can be expressed as

$$W(m_t, k_t) = \max_{x_t, h_t, m_{t+1}, k_{t+1}} [B \ln x_t - Ah_t + \beta V(m_{t+1}, k_{t+1})] \tag{1}$$

subject to a sequence of budget constraints given by

$$k_{t+1} + \frac{m_{t+1}}{p_t} = w_t h_t + (1 + r_t - \delta)k_t + \frac{m_t}{p_t} + \tau_t - x_t. \tag{2}$$

p_t is the price of general goods. w_t is the real wage rate (denominated in the price of general goods). r_t is the rental price of capital. The parameter $\delta \in (0, 1)$ is the depreciation rate of capital. τ_t is a real lump-sum transfer from the government.

From standard optimization, the optimality condition for consumption in the CM is

$$\frac{B}{x_t} = \frac{A}{w_t}. \tag{3}$$

Equation (3) implies that all households consume the same amount of general goods x_t in the CM regardless of their holdings of capital and money. This useful property results from the quasi-linear utility function, which is a standard simplifying assumption in this branch of model to eliminate any dispersion in money holdings that arises from trades in the DM.[7] The standard intertemporal optimality conditions for the accumulation of capital and money are respectively

$$\frac{B}{x_t} = \beta V_k(m_{t+1}, k_{t+1}), \tag{4}$$

$$\frac{B}{p_t x_t} = \beta V_m(m_{t+1}, k_{t+1}). \tag{5}$$

Equations (3) to (5) imply that all households enter the DM in the next period with the same holdings of capital and money. In addition, the familiar envelope conditions are

$$W_k(m_t, k_t) = \frac{B(1 + r_t - \delta)}{x_t}, \tag{6}$$

$$W_m(m_t, k_t) = \frac{B}{p_t x_t}. \tag{7}$$

[6] Following Aruoba *et al.* (2011), we include the parameter B in order to match the money-consumption ratio in the calibration.

[7] See for example, Rocheteau and Wright (2005) and Aruoba *et al.* (2011) for a useful discussion.

2.1.2 Households' optimization in the DM

In the DM, a household either becomes (a) a buyer, (b) a seller or (c) a nontrader. The probability of becoming a buyer is $\sigma \in (0, 0.5)$, and the probability of becoming a seller is also $\sigma \in (0, 0.5)$. The probability of becoming a nontrader is $1 - 2\sigma > 0$. As $\sigma \to 0$, monetary policy would have no effects on economic growth and social welfare. This taste-and-technology-shock specification shows a random matching technology that buyers meet with sellers and is a standard feature of the Lagos-Wright model.[8] As a result of this taste-and-technology shock, the value of entering the DM is

$$V(m_t, k_t) = \sigma V^b(m_t, k_t) + \sigma V^s(m_t, k_t) + (1 - 2\sigma)W(m_t, k_t), \tag{8}$$

where $V^b(.)$ and $V^s(.)$ are the values of being a buyer and a seller respectively.

To analyze $V^b(.)$ and $V^s(.)$, we consider the following functional forms for the buyers' preference and the sellers' production technology. In the DM, each buyer's utility $\ln q_t^b$ is increasing and concave in the consumption of special goods. Each seller produces special goods q_t^s by combining her capital k_t and effort e_t subject to the following Cobb-Douglas production function.

$$q_t^s = z_t^{1-\alpha} k_t^{\alpha} e_t^{\eta}, \tag{9}$$

where z_t denotes aggregate technology. To achieve endogenous growth, we will follow Romer (1986) to assume that capital has a positive externality effect on aggregate technology such that $z_t = \overline{k}_t$, where $\overline{k}_t$ is the aggregate holding of capital in the economy.[9] The parameter $\alpha \in (0, 1)$ determines capital share. To ensure constant returns to scale, we will impose $\eta = 1 - \alpha$ on labor share; however, it would be useful for us to first present the analysis with η in order to isolate the effects of capital and labor shares.

Rewriting equation (9), we can express the utility cost of production as

$$e\left(\frac{q_t^s}{z_t}, \frac{k_t}{z_t}\right) = \left(\frac{q_t^s}{z_t}\right)^{1/\eta} \left(\frac{k_t}{z_t}\right)^{-\alpha/\eta}. \tag{10}$$

Buyers purchase special goods q_t^b by spending money d_t^b, whereas sellers earn money d_t^s by producing special goods q_t^s.[10] Given these terms of trade, the values of being a buyer and a seller are respectively[11]

$$V^b(m_t, k_t) = \ln q_t^b + W(m_t - d_t^b, k_t), \tag{11}$$

$$V^s(m_t, k_t) = -e\left(\frac{q_t^s}{z_t}, \frac{k_t}{z_t}\right) + W(m_t + d_t^s, k_t). \tag{12}$$

[8] See Corbae *et al.* (2002) for a model of endogenous matching as an alternative to random matching.

[9] It is useful to note that $k_t = \overline{k}_t$ in equilibrium.

[10] As a result of these different money holdings at the end of the DM, households supply different amounts of labor in the CM that eliminate any dispersion in money holdings.

[11] Adding a disutility parameter to the supply of effort in the DM would not change our qualitative and quantitative results. Therefore, we follow Aruoba *et al.* (2011) and Waller (2011) to normalize this parameter to unity.

Differentiating (11) and (12) and substituting them into (8), we can obtain the following envelope condition for m_t.

$$\begin{aligned}V_m(m_t, k_t) &= (1-2\sigma)W_m(m_t, k_t) + \sigma\left[\frac{1}{q_t^b}\frac{\partial q_t^b}{\partial m_t} + W_m(m_t - d_t^b, k_t)\left(1 - \frac{\partial d_t^b}{\partial m_t}\right)\right] \\ &\quad + \sigma\left[-e_1\left(\frac{q_t^s}{z_t}, \frac{k_t}{z_t}\right)\frac{1}{z_t}\frac{\partial q_t^s}{\partial m_t} + W_m(m_t + d_t^s, k_t)\left(1 + \frac{\partial d_t^s}{\partial m_t}\right)\right],\end{aligned} \quad (13)$$

where $W_m(m_t, k_t) = W_m(m_t - d_t^b, k_t) = W_m(m_t + d_t^s, k_t) = B/(p_t x_t)$ from (7). Similarly, we can obtain the following envelope condition for k_t.

$$\begin{aligned}V_k(m_t, k_t) &= (1-2\sigma)W_k(m_t, k_t) + \sigma\left[\frac{1}{q_t^b}\frac{\partial q_t^b}{\partial k_t} - W_m(m_t - d_t^b, k_t)\frac{\partial d_t^b}{\partial k_t} + W_k(m_t - d_t^b, k_t)\right] \\ &\quad + \sigma\left[-e_1\left(\frac{q_t^s}{z_t}, \frac{k_t}{z_t}\right)\frac{1}{z_t}\frac{\partial q_t^s}{\partial k_t} - e_2\left(\frac{q_t^s}{z_t}, \frac{k_t}{z_t}\right)\frac{1}{z_t} + W_m(m_t + d_t^s, k_t)\frac{\partial d_t^s}{\partial k_t} + W_k(m_t + d_t^s, k_t)\right],\end{aligned} \quad (14)$$

where $W_k(m_t, k_t) = W_k(m_t - d_t^b, k_t) = W_k(m_t + d_t^s, k_t) = B(1 + r_t - \delta)/x_t$ from (6).

To solve the marginal value of holding money (13) and capital (14), we consider a competitive equilibrium with price taking as in Aruoba *et al.* (2011) and Waller (2011).[12] Under price taking, once buyers and sellers are matched, they both act as price takers. Given the price $\widetilde{p}_t$ of special goods, buyers choose q_t^b to maximize

$$V^b(m_t, k_t) = \max_{q_t^b}[\ln q_t^b + W(m_t - \widetilde{p}_t q_t^b, k_t)] \quad (15)$$

subject to the budget constraint

$$d_t^b = \widetilde{p}_t q_t^b \leq m_t. \quad (16)$$

In the DM, buyers spend all their money,[13] so that the money constraint implies that

$$q_t^b = m_t/\widetilde{p}_t. \quad (17)$$

As for sellers' maximization problem in the DM, it is given by

$$V^s(m_t, k_t) = \max_{q_t^s}\left[-e\left(\frac{q_t^s}{z_t}, \frac{k_t}{z_t}\right) + W(m_t + \widetilde{p}_t q_t^s, k_t)\right]. \quad (18)$$

[12] In addition to the competitive equilibrium with price taking, Aruoba *et al.* (2011) and Waller (2011) also consider bargaining between buyers and sellers to determine the terms of trade. In this study, we focus on the competitive equilibrium with price taking because of economic growth. In the case of generalized Nash bargaining as in Aruoba *et al.* (2011) or proportional bargaining as in Waller (2011), the bargaining condition is incompatible with balanced growth because the buyers' utility, which determines their surplus, is increasing overtime due to economic growth whereas the sellers' disutility of effort is stationary on a balanced growth path. In Appendix A, we demonstrate this problem under proportional bargaining and show that only a special case in which buyers gain all surplus is consistent with balanced growth; however, under this special case, inflation has no effect on economic growth because sellers obtain zero surplus in the DM.

[13] See Appendix B for a proof. Intuitively, due to the opportunity cost of holding money and the possibility of not being a buyer in the DM, households do not carry a sufficient amount of money to the DM. Therefore, if a household turns out to be a buyer in the DM, it would be optimal to spend all the money on q_t^b.

Sellers' optimal supplies of special goods can be obtained from the following condition.

$$e_1\left(\frac{q_t^s}{z_t},\frac{k_t}{z_t}\right)\frac{1}{z_t}=\widetilde{p}_t W_m(m_t+\widetilde{p}_t q_t^s,k_t)\Leftrightarrow\frac{1}{\eta}e\left(\frac{q_t^s}{z_t},\frac{k_t}{z_t}\right)=B\frac{\widetilde{p}_t q_t^s}{p_t x_t},\tag{19}$$

where the second equality of (19) makes use of (7) and (10).

Using (17) and (19), we can obtain $\partial q_t^b/\partial m_t = 1/\widetilde{p}_t$, $\partial d_t^b/\partial m_t = 1$, and $\partial d_t^s/\partial k_t = \widetilde{p}_t\left(\partial q_t^s/\partial k_t\right)$, whereas the other partial derivatives, $\partial q_t^b/\partial k_t$, $\partial d_t^b/\partial k_t$, $\partial q_t^s/\partial m_t$ and $\partial d_t^s/\partial m_t$, in (13) and (14) are zero. Substituting these conditions, $q_t^b = q_t^s = q_t$ and (19) into (13) and (14), we can derive the following conditions.

$$V_m(m_t,k_t)=\frac{B\left(1-\sigma\right)}{p_t x_t}+\frac{\sigma}{\widetilde{p}_t q_t},\tag{20}$$

$$V_k(m_t,k_t)=\frac{B\left(1+r_t-\delta\right)}{x_t}-\frac{\sigma}{z_t}e_2\left(\frac{q_t}{z_t},\frac{k_t}{z_t}\right).\tag{21}$$

Intuitively, (20) states that the marginal benefit of holding money is the sum of the marginal utility from being able to consume special goods with probability σ (i.e., the household becomes a buyer in the DM) and the marginal utility from spending the money, which is also a valuable asset in the CM, on general goods with probability $1-\sigma$ (i.e., the household does not become a buyer in the DM). Equation (21) states that the marginal benefit of holding capital is the sum of the marginal utility from spending the capital return $1+r_t-\delta$ on general goods in the CM and the expected marginal utility from having to exert less effort (recall that $e_2 < 0$) in producing special goods in the DM with probability σ (i.e., the household becomes a seller in the DM).[14]

2.2 Firms in the CM

General goods are produced by using capital k_t and production labor h_t with the following Cobb-Douglas production function.

$$y_{x,t}=z_t^{1-\alpha}k_t^{\alpha}h_t^{\eta},\tag{22}$$

where aggregate technology is $z_t=\overline{k}_t$ as before. The producers act competitively by taking output and input prices as given. The conditional demand functions for capital and production labor are respectively

$$r_t=\alpha z_t^{1-\alpha}k_t^{\alpha-1}h_t^{\eta},\tag{23}$$

$$w_t=\eta z_t^{1-\alpha}k_t^{\alpha}h_t^{\eta-1}.\tag{24}$$

[14]Following Aruoba *et al.* (2011) and Waller (2011), we assume that the stock of capital does not depreciate within a period even upon usage in the DM. Capital depreciation only occurs at the end of a period after usage in the CM.

2.3 Monetary authority

Let $\mu_t = (m_{t+1} - m_t)/m_t$ denote the growth rate of money supply that is exogenously set by the monetary authority. Given the definition of real money balance m_t/p_t (denominated in the price of general goods), its evolution can then be expressed as

$$\frac{m_{t+1}}{p_{t+1}} = \left(\frac{1+\mu_t}{1+\pi_t}\right)\frac{m_t}{p_t}, \tag{25}$$

where π_t is the inflation rate that is endogenous and determines the cost of holding money. In each period, the monetary authority issues money to finance a lump-sum transfer that has a real value of $\tau_t = (m_{t+1} - m_t)/p_t = \mu_t m_t/p_t$.

2.4 Competitive equilibrium

The competitive equilibrium is a sequence of allocations $\{h_t, x_t, y_{x,t}, q_t, d_t, m_{t+1}, k_{t+1}\}_{t=0}^{\infty}$, a sequence of prices $\{w_t, r_t, p_t, \widetilde{p}_t, \pi_t\}_{t=0}^{\infty}$ and a sequence of policies $\{\mu_t, \tau_t\}_{t=0}^{\infty}$. Also, in each period, the following conditions hold.

- In the CM, households choose $\{h_t, x_t, m_{t+1}, k_{t+1}\}$ to maximize (1) subject to (2) taking $\{w_t, r_t, p_t, \tau_t\}$ as given;
- In the DM, buyers and sellers choose $\{q_t, d_t\}$ to maximize their value functions taking $\{\widetilde{p}_t\}$ as given;
- Competitive firms in the CM produce $\{y_{x,t}\}$ to maximize profit taking $\{w_t, r_t\}$ as given;
- The real value of aggregate consumption includes consumption in the CM and the DM such that $c_t \equiv (p_t x_t + \sigma \widetilde{p}_t q_t)/p_t$;
- The real value of aggregate output includes output in the CM and the DM such that $y_t \equiv (p_t y_{x,t} + \sigma \widetilde{p}_t q_t)/p_t$;
- The capital stock accumulates through investment from general goods such that $k_{t+1} = y_{x,t} - x_t + (1-\delta)k_t$;
- The monetary authority balances its budget such that $\tau_t = \mu_t m_t/p_t$.

2.5 Balanced growth path

In this subsection, we consider the dynamic properties of the model. Given that the monetary authority sets a stationary growth rate of money supply (i.e., $\mu_t = \mu$ for all t), Proposition 1 shows that the economy jumps to a unique and locally stable balanced growth path. The proof is relegated to Appendix C. Given this balanced growth behavior of the model, we analyze the effects of monetary policy on the balanced growth path in the next section.

Proposition 1 *Given a stationary sequence of monetary policy (i.e., $\mu_t = \mu$ for all t), the economy jumps to a unique and stable balanced growth path.*

3 Growth and welfare effects of monetary policy

In this section, we analyze the effects of monetary policy on the balanced growth path along which the supply of labor is stationary. Given the equilibrium condition $\overline{k}_t = k_t$, variables, such as output, consumption, capital and real money balance, exhibit a common growth rate g. Using (4), (10) and (21), we obtain

$$g \equiv \frac{x_{t+1}}{x_t} - 1 = \beta\left(1 + r - \delta + \frac{\sigma\alpha}{B\eta} f_c f_d^{1/\eta}\right) - 1, \tag{26}$$

where $r = \alpha h^\eta$ from (23). The variables $f_c \equiv x/k$ and $f_d \equiv q/k$ denote the consumption-capital ratios in the CM and the DM respectively.

We first make use of (5), (10), (19), (20) and (25) to derive the steady-state consumption-capital ratio in the DM. We obtain

$$f_d = \left[\frac{\sigma\eta}{(1+\mu)/\beta - (1-\sigma)}\right]^\eta. \tag{27}$$

f_d must be positive because $\mu > \beta - 1 > \beta(1-\sigma) - 1$.[15] Equation (27) shows that the consumption-capital ratio in the DM is decreasing in the growth rate of money supply, and this result can be shown as follows.

$$\frac{\partial f_d}{\partial \mu} = -\frac{\eta}{\beta}\frac{(\sigma\eta)^\eta}{[(1+\mu)/\beta - (1-\sigma)]^{1+\eta}} < 0. \tag{28}$$

Intuitively, a higher money growth rate increases inflation, which in turn increases the cost of consumption in the DM, where money is needed for transactions.

As for the steady-state consumption-capital ratio in the CM, we make use of (23), (24), (26) and the capital-accumulation equation $k_{t+1} = y_{x,t} - x_t + (1-\delta)k_t$ to derive

$$f_c = \frac{(1-\alpha\beta)h^\eta + (1-\beta)(1-\delta)}{1 + \sigma\alpha\beta f_d^{1/\eta}/(B\eta)}, \tag{29}$$

where aggregate labor h is still an endogenous variable and can be determined with the following condition.

$$Ah^{1-\eta} f_c = B\eta, \tag{30}$$

which uses (3) and (24). We use (30) to derive

$$\frac{\partial f_c}{\partial \mu} = -\frac{B\eta(1-\eta)}{Ah^{2-\eta}}\frac{\partial h}{\partial \mu}. \tag{31}$$

As for the derivative of h, we substitute (27) and (30) into (29) and then take the differentials of h with respect to μ to obtain

$$\frac{dh}{d\mu} = -\frac{\alpha(f_d^{1/\eta}/\eta)^2}{A(1-\alpha\beta) + (1-\eta)\left(B + \sigma\alpha\beta f_d^{1/\eta}/\eta\right)/h} < 0. \tag{32}$$

[15] It can be shown that as $\mu \to \beta - 1$, the nominal interest rate approaches the lower bound of zero. Here the nominal interest rate refers to the nominal rate of return on a conventional interest-bearing bond that pays interests in the CM (but not in the DM) of each period.

Substituting (32) into (31) shows that $\partial f_c/\partial\mu > 0$. In summary, a higher money growth rate induces households to increase leisure and shift consumption from the DM to the CM.

Substituting (29) into (26), we obtain

$$g = \alpha\beta h^{\eta} + \frac{(1-\alpha\beta)h^{\eta} + (1-\beta)(1-\delta)}{1 + B\eta/\left(\sigma\alpha\beta f_d^{1/\eta}\right)} + \beta(1-\delta) - 1. \tag{33}$$

From (33), it is easy to see that the growth rate g is decreasing in μ because $\partial h/\partial\mu < 0$ and $\partial f_d/\partial\mu < 0$. Intuitively, inflation affects consumption in the DM, which in turn has a direct effect and an indirect effect on economic growth. The direct channel is through the consumption-capital ratio f_d in the DM. Intuitively, a higher inflation increases the cost of holding money, thereby reducing the real money balance held by households and the value of goods traded in the DM. As a result, capital demand is depressed reducing the growth rate.

The indirect channel is endogenous labor supply, which is standard in traditional monetary growth models. Intuitively, a decrease in labor supply reduces the marginal product of capital thereby reducing capital accumulation. To separate the consumption and labor-supply effects, we consider the limiting case $\eta \to 0$.

$$\lim_{\eta\to 0} g = \alpha\beta + \frac{1-\alpha\beta + (1-\beta)(1-\delta)}{1 + B[(1+\mu)/\beta - (1-\sigma)]/(\sigma^2\alpha\beta)} + \beta(1-\delta) - 1, \tag{34}$$

where we have used $f_d^{1/\eta}/\eta = \sigma/[(1+\mu)/\beta - (1-\sigma)]$ from (27). Therefore, even when the search-based monetary growth model approaches the case with inelastic labor supply, inflation continues to have a detrimental effect on economic growth. This result stands in stark contrast to a canonical endogenous growth model with a CIA constraint on consumption, in which inflation has no effect on economic growth under inelastic labor supply.[16]

Proposition 2 *A higher money growth rate μ reduces economic growth through the consumption-capital ratio f_d in the DM.*

Before we proceed to analyze the welfare effects of inflation, it would be helpful to briefly present the equilibrium allocations under a special case of complete capital depreciation (i.e., $\delta = 1$). In this case, the DM consumption-capital ratio f_d in (27) remains unchanged; however, aggregate labor h in (30) nicely simplifies to

$$h = \frac{\eta}{A(1-\alpha\beta)}\left(B + \frac{\alpha\beta^2\sigma^2}{1+\mu-\beta+\sigma\beta}\right), \tag{30a}$$

which is increasing in σ because $1+\mu-\beta > 0$. Given (30a), we can simply express the CM consumption-capital ratio as $f_c = B\eta/(Ah^{1-\eta})$, which is decreasing in σ. Finally, the growth rate g in (33) simplifies to

$$g = \alpha\beta\left[1 + \frac{(1-\alpha\beta)\sigma^2\beta}{\alpha\sigma^2\beta^2 + B\,(1+\mu-\beta+\sigma\beta)}\right]h^{\eta} - 1, \tag{33a}$$

[16] In the previous version of this study, we provide an example of this model; see Chu *et al.* (2012).

which is increasing in σ. Intuitively, as the matching probability σ increases, households have more incentives to accumulate capital, which in turn increases the equilibrium growth rate.

Next, we examine the welfare effects of monetary policy under the general case of incomplete capital depreciation (i.e., $\delta \in (0,1)$). In this two-sector search model, households engage in two types of economic activities in the DM and the CM every period. On the balanced growth path, the lifetime utility U of households that includes the utility from the CM and the expected utility from the DM can be expressed as

$$(1-\beta)U = \underbrace{\sigma \ln q_0 - \sigma \left(\frac{q_0}{k_0}\right)^{1/\eta}}_{DM} + \underbrace{B \ln x_0 - Ah}_{CM} + \underbrace{\frac{\beta(B+\sigma)}{1-\beta} \ln(1+g)}_{growth}. \tag{35}$$

Substituting $q_0 = f_d k_0$ and $x_0 = f_c k_0$ into (35) and then normalizing initial k_0 to unity, (35) simplifies to

$$(1-\beta)U = \sigma \ln f_d - \sigma f_d^{1/\eta} + B \ln f_c - Ah + \frac{\beta(B+\sigma)}{1-\beta} \ln(1+g). \tag{36}$$

Differentiating (36) with respect to μ yields

$$(1-\beta)\frac{\partial U}{\partial \mu} = \frac{\sigma}{f_d}\left(1 - \frac{f_d^{1/\eta}}{\eta}\right)\underbrace{\frac{\partial f_d}{\partial \mu}}_{-} + \frac{B}{f_c}\underbrace{\frac{\partial f_c}{\partial \mu}}_{+} - A\underbrace{\frac{\partial h}{\partial \mu}}_{-} + \frac{\beta(B+\sigma)}{(1-\beta)(1+g)}\underbrace{\frac{\partial g}{\partial \mu}}_{-}, \tag{37}$$

where $f_d^{1/\eta}/\eta < 1$ from (27) because $\mu > \beta - 1$. A higher money growth rate (a) decreases the consumption-capital ratio f_d in the DM, (b) increases the consumption-capital ratio f_c in the CM, (c) decreases labor supply h in the CM, and (d) decreases economic growth g. Effects (a) and (d) hurt welfare, whereas effects (b) and (c) improve welfare. Although it appears that the overall effect of money growth on welfare is ambiguous, we show below that higher money growth is in fact detrimental to social welfare.

Comparing the equilibrium allocations and the first-best allocations, we find that (a) $f_d < f_d^*$, (b) $f_c > f_c^*$, (c) $h < h^*$, and (d) $g < g^*$, where the variables with superscript * denote first-best allocations.[17] In other words, there is too little consumption in the DM due to the cost of holding money. In the CM, there is too much consumption and too little labor supply due to capital externality. Finally, the equilibrium growth rate is also suboptimally low. Therefore, increasing the money growth rate that forces the equilibrium allocations to deviate further from the first-best allocations is detrimental to welfare. In other words, decreasing the money growth rate improves welfare, and the Friedman rule (given by $\mu \to \beta - 1$) is optimal in this model. However, although the Friedman rule is optimal, it does not achieve the first-best allocations due to the presence of capital externality.[18]

[17] In Appendix D, we derive the first-best allocations of the search model and prove these inequalities.

[18] It is useful to note that the Friedman rule is not always optimal under price taking in the search model. For example, Rocheteau and Wright (2005) show that the Friedman rule is not optimal when there exist search externalities.

Proposition 3 *A higher money growth rate μ reduces social welfare, and the Friedman rule is optimal but does not achieve the first-best allocations due to capital externality.*

4 Quantitative analysis

In this section, we calibrate the model in order to perform a numerical investigation on the effects of inflation on economic growth and social welfare. We consider two policy objectives (a) price stability (or equivalently, zero inflation) and (b) the Friedman rule. Both of these policy objectives are commonly analyzed in the literature; see for example, Dotsey and Ireland (1996), Lucas (2000), Lagos and Wright (2005) and Aruoba *et al.* (2011). Furthermore, we compare the welfare effects of inflation from the endogenous growth model to those from an exogenous growth model. To consider exogenous growth, we follow Waller (2011) to assume an exogenous technological progress; in other words, z_t in (9) increases according to an exogenous growth rate $g_z \equiv \dot{z}_t/z_t$.[19]

4.1 Calibration

We begin by characterizing a benchmark economy, in which each structural parameter is either set to a conventional value or matched to an empirical moment in the US economy. In the endogenous growth model, the discount factor β is set to 0.952 to match an annual discount rate ρ of 5%.[20] The capital-share parameter α is set to 0.3, which implies a labor share $\eta = 1 - \alpha$ of 0.7. We consider an initial money growth rate of 6.9%, so that the annual inflation rate is 3.6% (i.e., the average inflation rate in the US from 1959 to 2010) when the economy grows at an annual growth rate of 3.2% (i.e., the average output growth rate in the US from 1959 to 2010). We choose a value for the depreciation rate δ to match the investment-capital ratio of 0.07 (i.e., the average investment-capital ratio in the US from 1959 to 2010), and this value of δ is 0.038. The leisure parameter A is set to 5.275, so that the long-run growth rate is 3.2%. The consumption parameter B is set to 1.662 to match the average money-consumption ratio of 0.252 in the US using M1 as the measure of money as is standard in the literature. As for the probability σ, we use it to match the interest elasticity of money demand $\xi = -0.556$ estimated by Berentsen *et al.* (2011), and the calibrated value of σ is 0.092.

As for the exogenous growth model, we consider a similar set of parameter values. In this case, the exogenous growth model is also able to match the same set of empirical moments.[21]

[19] In an unpublished appendix (see Appendix E), we provide the derivations of equilibrium allocations under exogenous growth.

[20] We consider a relatively high discount rate in order to be conservative. If we consider a lower discount rate, the difference in the welfare cost of inflation between the endogenous-growth and exogenous-growth models would be even more dramatic because decreasing the discount rate magnifies the positive effect of a higher growth rate on social welfare.

[21] Because the growth rate of technology g_z is a parameter under exogenous growth, the exogenous growth model has a free parameter A for the calibration. We choose a value of A such that the calibrated value of h is the same as in the endogenous growth model.

Table 1a summarizes the parameter values. As for equilibrium values of the key variables, we report them in Table 1b. The consumption-capital ratio of 0.335 and the capital-output ratio of 2.467 from the models are in line with empirical moments in the US economy.

Table 1a: Benchmark parameter values

Parameter	α	η	β	μ	σ	B	δ	A
Targets	-	$1-\alpha$	ρ	π	ξ	$m/(pc)$	i/k	g
Target values	0.300	0.700	0.050	0.036	-0.556	0.252	0.070	0.032
Endogenous growth	0.300	0.700	0.952	0.069	0.092	1.662	0.038	5.275
Exogenous growth	0.300	0.700	0.952	0.069	0.093	1.669	0.038	5.297

Table 1b: Benchmark equilibrium values

Variable	g	π	h	f_d	f_c	i/k	c/k	k/y
Endogenous growth	0.032	0.036	0.268	0.431	0.328	0.070	0.335	2.467
Exogenous growth	0.032	0.036	0.268	0.432	0.327	0.070	0.335	2.467

4.2 Numerical results

Given the above parameter values, we consider the following policy experiments. First, we lower μ from 0.069 to a value that achieves zero inflation. This value is 0.033 in the endogenous growth model and 0.032 in the exogenous growth model. In this case, the inflation rate decreases from 3.6% to 0% in both models. As for the Friedman rule, we lower μ from 0.069 to -0.048, so that the nominal interest rate decreases and approaches zero. In Table 2, we report the results, which are expressed in percent changes, except for g and U. The changes in g are expressed in percentage point, and the changes in U are expressed in the usual equivalent variations in annual consumption.

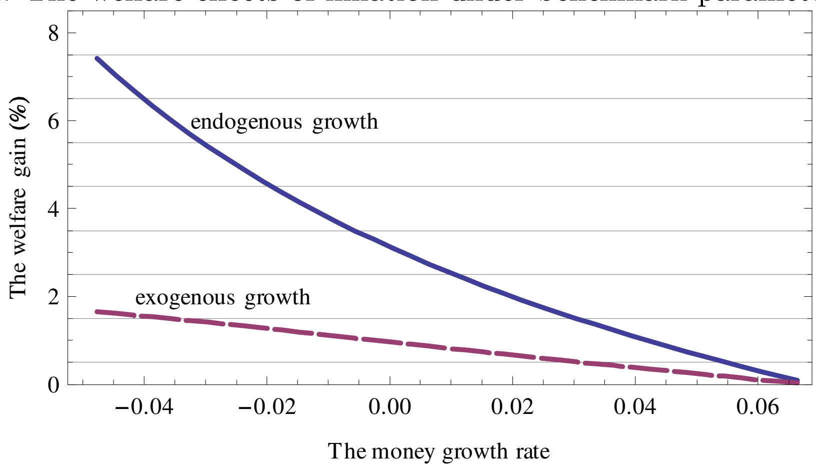
Table 2: Growth and welfare effects of a lower μ

	Δf_d (%)	Δf_c (%)	Δh (%)	Δg (%)	ΔU (%)
Zero inflation					
Endogenous growth	14.778	-0.049	0.162	0.061	1.398
Exogenous growth	14.454	-0.514	0.091	-	0.497
The Friedman rule					
Endogenous growth	80.877	-0.296	0.995	0.373	7.417
Exogenous growth	75.216	-2.997	0.544	-	1.662

In Table 2, we see that reducing the money growth rate has the following effects. First, it raises the supply of labor h in both models; however, the labor-supply effect is much larger in the endogenous growth model than in the exogenous growth model. Second, it also raises the consumption-capital ratio f_d in both models. The larger labor supply h and the higher consumption-capital ratio f_d serve to increase economic growth in the endogenous growth model, but they do not increase economic growth in the exogenous growth model.

Comparing Δf_d and Δh under zero inflation and the Friedman rule, we see that the increases in both f_d and h are disproportionately larger under Friedman rule than under zero inflation. From (27) and (30a), we see that both f_d and h are decreasing and convex functions in μ, and this property has the following implications on the growth and welfare effects of inflation. We find that the convexity carries over to the growth rate g, which is also a decreasing and convex function in μ. In Figure 1, we plot welfare changes against the money growth rate and see that the convexity in g in the endogenous growth model is also reflected in welfare U because the growth effect is numerically the most important component of the welfare effect in the endogenous growth model.

Figure 1: The welfare effects of inflation under benchmark parameter values



Given that the welfare effect of inflation is approximately linear in the exogenous growth model, the difference in the welfare costs of inflation across the two models is increasing in the size of the change in the money growth rate. In the case of zero inflation, we find that the welfare gain of 1.40% in the endogenous growth model is almost three times as large as the welfare gain of 0.50% in the exogenous growth model.[22] In the case of Friedman rule, the welfare gain of 7.42% in the endogenous growth model is 4.5 times as large as the welfare gain of 1.66% in the exogenous growth model.[23] The reasons why the endogenous growth model features a much larger welfare effect of inflation than the exogenous growth model are as follows. First, the equilibrium growth rate increases as the money growth rate decreases, whereas the balanced growth rate is constant in the exogenous growth model.[24] Second, Table 2 shows that as a result of a higher money growth rate, the consumption-capital ratio

[22] We focus on steady-state welfare in the exogenous growth model. Taking into account transition dynamics in the exogenous growth model would make its welfare effects even smaller.

[23] The welfare effect of inflation in the exogenous growth model is slightly smaller than the effect under competitive pricing in Aruoba *et al.* (2011) because we start from an inflation rate of 3.6% whereas they start from an inflation rate of 10%.

[24] In the exogenous growth model, it is the balanced growth level of capital that is affected by the money growth rate, but this level effect is not as strong as the growth effect under endogenous growth.

f_c in the CM falls by much more in the exogenous growth model than in the endogenous growth model. Finally, Table 2 also shows that labor supply h increases by much less under exogenous growth than under endogenous growth.

Given that the matching probability σ is the key parameter in determining the magnitude of the welfare effects of inflation, we perform a robustness check by varying the values of σ. We report the results for $\sigma \in \{0.05, 0.2\}$ in Figures 2 and 3, in which we see a similar pattern as before. In fact, considering a wide range of values of $\sigma \in (0, 0.5)$, we find that the endogenous growth model features larger welfare effects of inflation than the exogenous growth model in all cases. Furthermore, we find that the welfare effect of inflation is more sensitive to the value of σ under endogenous growth than under exogenous growth.

Figure 2: The welfare effects of inflation under $\sigma = 0.05$

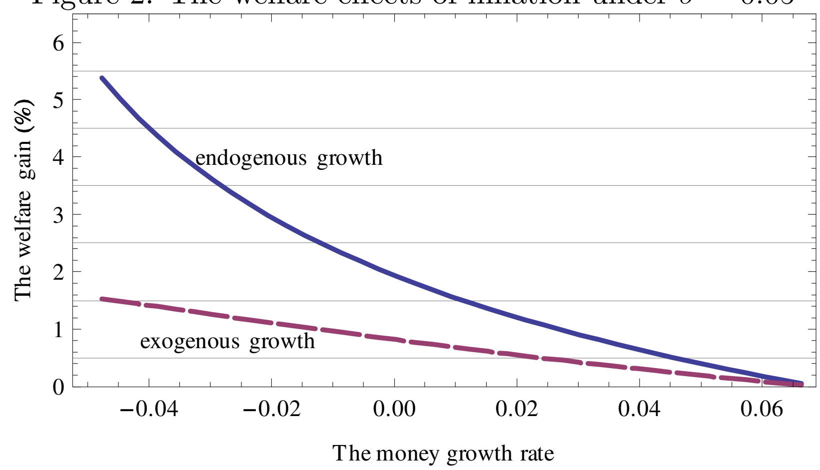
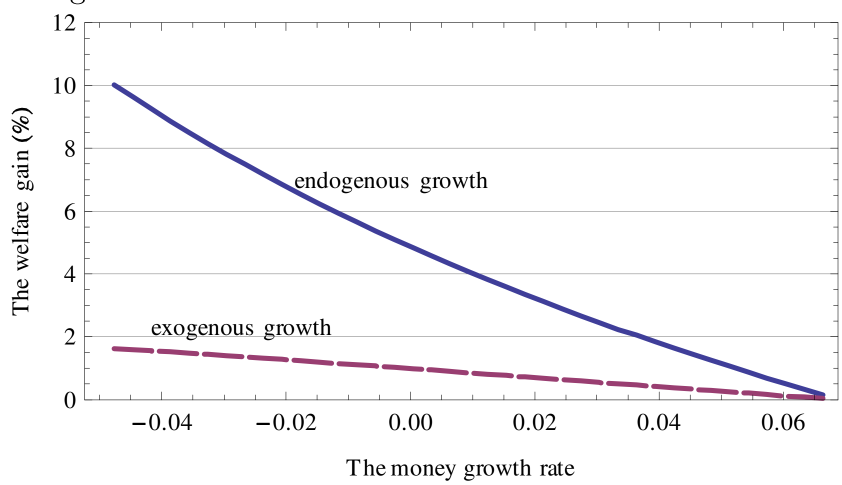


Figure 3: The welfare effects of inflation under $\sigma = 0.20$



5 Conclusion

In this study, we have analyzed the growth and welfare effects of inflation in a search-based monetary endogenous growth model. We find that modelling economic growth as an endogenous process increases the welfare cost of inflation. Furthermore, the channel through which inflation affects economic growth in a search-based monetary growth model is different from a CIA growth model. Therefore, given the relative tractability of recent vintages of search models, it would be a fruitful direction for future research to further revisit the interesting implications of monetary policy on economic growth and social welfare using variants of the search model. For example, it would be interesting to consider the welfare effects of inflation under different pricing mechanisms in a search-based monetary endogenous growth model.[25]

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Appendix A: Bargaining in the search model

In this appendix, we show that proportional bargaining is incompatible with balanced growth unless we focus on a special case in which buyers obtain all surplus.[26] If a buyer with state $\left(m_t^b, k_t^b\right)$ is matched with a seller with state (m_t^s, k_t^s), then a proportional bargaining problem, in which the buyer's gains from trade are a fixed share θ of the trade surplus can be expressed as

$$\underset{q_t}{Max} \underbrace{\ln q_t + W(m_t^b - d_t, k_t^b)}_{buyers' \ payoff} - \underbrace{W(m_t^b, k_t^b)}_{buyers' \ threat \ point} = \theta \underbrace{\left[\ln q_t - e\left(\frac{q_t}{z_t}, \frac{k_t^s}{z_t}\right)\right]}_{trade \ surplus}, \tag{A1}$$

subject to the budget constraint $d_t \leq m_t^b$. Using (7) and substituting $d_t = m_t^b$ into (A1), the bargaining condition for special goods is

$$(1-\theta)\ln q_t = \frac{Am_t^b}{p_t w_t} - \theta e\left(\frac{q_t}{z_t}, \frac{k_t^s}{z_t}\right). \tag{A2}$$

In (A2), q_t is increasing overtime due to economic growth whereas $m_t^b/\left(p_t w_t\right)$, q_t/z_t and k_t^s/z_t are stationary on a balanced growth path. As a result, the bargaining condition is incompatible with balanced growth unless the buyer obtains all surplus (i.e., $\theta = 1$); however, under this special case, inflation has no effect on economic growth because sellers obtain zero surplus in the DM.

Appendix B: Consumption in the DM

In this appendix, we show that it is optimal for the buyers to spend all their money to consume special goods in the DM. Differentiating (15) with respect to q_t^b yields

$$\frac{\partial V^b(m_t, k_t)}{\partial q_t^b} = \frac{1}{q_t^b} - \widetilde{p}_t W_m(m_t - \widetilde{p}_t q_t^b, k_t) = \frac{1}{q_t^b} - \frac{B\widetilde{p}_t}{p_t x_t}, \tag{B1}$$

where the second equality follows from (7). The second-order condition shows that $V^b(.)$ is globally concave in q_t^b and reaches a maximum at $q_t^b = p_t x_t/\left(B\widetilde{p}_t\right)$. In what follows, we show that $q_t^b = m_t/\widetilde{p}_t < p_t x_t/\left(B\widetilde{p}_t\right)$ implying that the money constraint must be binding because $q_t^b < p_t x_t/\left(B\widetilde{p}_t\right) \Leftrightarrow \partial V^b(.)/\partial q_t^b > 0$. Setting $q_t^b = q_t^s = q_t$ and $z_t = \overline{k}_t = k_t$ in (10) and (19), we have

$$B\frac{\widetilde{p}_t q_t}{p_t x_t} = \frac{1}{\eta}\left(\frac{q_t}{z_t}\right)^{1/\eta}, \tag{B2}$$

where $q_t/z_t = f_d$. From (27), we know that $f_d^{1/\eta}/\eta < 1$ because $\mu > \beta - 1$. Therefore, $\widetilde{p}_t q_t = m_t < p_t x_t/B$.

[26] The same result can be shown for the case of generalized Nash bargaining.

Appendix C: Dynamic properties of the search model

In this appendix, we show that the economy in the search model always jumps to a unique and locally stable balanced growth path given a stationary sequence of monetary policy (i.e., $\mu_t = \mu$ for all t). Combining (17) and (19) and using (10), we can obtain

$$B\frac{m_t}{p_t x_t} = \frac{1}{\eta}\left(\frac{q_t}{k_t}\right)^{1/\eta}. \tag{C1}$$

Here we define $f_{d,t} \equiv q_t/k_t$ as the ratio between consumption and capital in the DM and make use of (5), (17), (20), (25) and (C1) to derive

$$(1+\mu)\left(\frac{f_{d,t}}{f_{d,t+1}}\right)^{1/\eta} = \beta\left[1+\sigma+\frac{\sigma\eta}{f_{d,t+1}^{1/\eta}}\right]. \tag{C2}$$

Combining (3), (5), (21), (23) and (24) and using (10) to yield

$$\frac{k_{t+1}}{k_t}\left(\frac{h_{t+1}}{h_t}\right)^{\eta-1} = \beta\left[1+\alpha h_{t+1}^{\eta}-\delta+\left(\frac{\alpha\sigma}{A}\right)h_{t+1}^{\eta-1}f_{d,t+1}^{1/\eta}\right]. \tag{C3}$$

In addition, the capital-accumulation equation is $k_{t+1}/k_t = y_{x,t}/k_t - x_t/k_t + 1 - \delta = h_t^{\eta} - B\eta h_t^{\eta-1}/A + 1 - \delta$. Applying this equation to (C3) yields

$$\left(h_t^{\eta}-\frac{B\eta h_t^{\eta-1}}{A}+1-\delta\right)\left(\frac{h_{t+1}}{h_t}\right)^{\eta-1} = \beta\left[1+\alpha h_{t+1}^{\eta}-\delta+\left(\frac{\alpha\sigma}{A}\right)h_{t+1}^{\eta-1}f_{d,t+1}^{1/\eta}\right]. \tag{C4}$$

Log-linearizing (C2) and (C4) around the steady-state equilibrium yields the following deterministic system:

$$\begin{bmatrix}\log(h_{t+1}/h)\\ \log(f_{d,t+1}/f_d)\end{bmatrix} = \underbrace{\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}}_{Jacobian\ matrix}\begin{bmatrix}\log(h_t/h)\\ \log(f_{d,t}/f_d)\end{bmatrix}, \tag{C5}$$

where

$$a_{11} = \frac{\eta h^{\eta}+\beta(1-\eta)(1-\delta+\alpha h^{\eta})+(1-\eta)\left(B\eta+\alpha\beta\sigma h^{\eta-1}f_d^{1/\eta}\right)h^{\eta-1}/A}{(\alpha\beta\eta)h^{\eta}+\beta(1-\eta)(1-\delta+\alpha h^{\eta})} > 1,$$

$$a_{12} = -\frac{(\alpha\beta\sigma)h^{\eta-1}f_d^{1/\eta}/A\eta}{(\alpha\beta\eta)h^{\eta}+\beta(1-\eta)(1-\delta+\alpha h^{\eta})}\left[\frac{1+\mu}{\beta(1-\sigma)}\right] < 0,$$

$$a_{21} = 0,$$

$$a_{22} = \frac{1+\mu}{\beta(1-\sigma)} > 1,$$

where we have used $\mu > \beta(1-\sigma) - 1$. Let s_1 and s_2 be the two characteristic roots of the dynamic system. The trace and determinant of Jacobian are given by

$$\text{Tr} = s_1 + s_2 = a_{11} + a_{22} > 0, \tag{C6}$$

$$\text{Det} = s_1 s_2 = a_{11} a_{22} > 0. \tag{C7}$$

As indicated in (C6) and (C7), the dynamic system exists two unstable characteristic roots ($s_1 = a_{11} > 1$ and $s_2 = a_{22} > 1$). Given h and f_d are jump variables, two unstable characteristic roots imply that the economy jumps to a unique and locally stable balanced growth path.

Appendix D: First-best allocations of the search model

In this appendix, we derive the first-best allocations of the search model and compare them with the equilibrium allocations. The planner chooses all quantities directly, taking all relevant information into account. Here money is not essential. The planner's problem is

$$J(k_t) = \max_{q_t, x_t, h_t, k_{t+1}} \left\{ \sigma \ln q_t - \sigma \left(\frac{q_t}{k_t} \right)^{1/\eta} + B \ln x_t - A h_t + \beta J(k_{t+1}) \right\}, \tag{D1}$$

subject to the capital-accumulation equation

$$k_{t+1} = k_t h_t^{\eta} - x_t + (1-\delta) k_t. \tag{D2}$$

From standard dynamic optimization, the optimality conditions for q_t and h_t are respectively

$$\frac{q_t}{k_t} = \eta^{\eta}, \tag{D3}$$

$$A h_t^{1-\eta} \left(\frac{x_t}{k_t} \right) = B\eta. \tag{D4}$$

The intertemporal optimality condition for capital accumulation is

$$\frac{B}{x_t} = \beta J_k(k_{t+1}), \tag{D5}$$

and the envelope condition is

$$J_k(k_t) = \frac{\sigma}{\eta k_t} \left(\frac{q_t}{k_t} \right)^{1/\eta} + \frac{B}{x_t}(1 - \delta + h_t^{\eta}). \tag{D6}$$

Combining (D5) and (D6), we can derive the first-best balanced growth rate g^* given by

$$g^* \equiv \frac{x_{t+1}}{x_t} - 1 = \beta \left[1 + (h^*)^{\eta} - \delta + \frac{\sigma}{B\eta} f_c^* \left(f_d^* \right)^{1/\eta} \right] - 1, \tag{D7}$$

where $f_c^* \equiv (x/k)^*$ and $f_d^* \equiv (q/k)^*$ denote the first-best consumption-capital ratios for general goods and special goods respectively. From (D3), we can obtain

$$\frac{(f_d^*)^{1/\eta}}{\eta} = 1. \tag{D8}$$

As for f_c^*, combining (D7) and (D2) yields

$$f_c^* = \frac{(1-\beta)(h^*)^\eta + (1-\beta)(1-\delta)}{1 + \sigma\beta (f_d^*)^{1/\eta} / (B\eta)}. \tag{D9}$$

Rewriting (D4) yields

$$A(h^*)^{1-\eta} f_c^* = B\eta. \tag{D10}$$

Equations (D7), (D8), (D9) and (D10) determine the first-best allocations $\{g^*, f_d^*, f_c^*, h^*\}$.

Comparing (D8) and (27) shows that $f_d < f_d^*$ because $\mu > \beta - 1$. Substituting (D10) into (D9) yields

$$f_c^* = \frac{(1-\beta)[B\eta/(Af_c^*)]^{\eta/(1-\eta)} + (1-\beta)(1-\delta)}{1 + \sigma\beta (f_d^*)^{1/\eta} / (B\eta)}, \tag{D11}$$

where $(f_d^*)^{1/\eta}/\eta$ is determined by (D8). Substituting (30) into (29) yield

$$f_c = \frac{(1-\alpha\beta)[B\eta/(Af_c)]^{\eta/(1-\eta)} + (1-\beta)(1-\delta)}{1 + \sigma\alpha\beta f_d^{1/\eta} / (B\eta)}, \tag{D12}$$

where $f_d^{1/\eta}/\eta$ is determined by (27). Comparing (D11) and (D12) shows that $f_c > f_c^*$ because $f_d < f_d^*$ and $\alpha < 1$. Given $f_c > f_c^*$, (D10) and (30) imply that $h < h^*$. Rewriting (D2) yields

$$g = \frac{k_{t+1}}{k_t} - 1 = h^\eta - f_c - \delta. \tag{D13}$$

Given that $h < h^*$ and $f_c > f_c^*$, it must be the case that $g < g^*$.

Appendix E (not for publication): Exogenous technological progress

In this unpublished appendix, we provide the derivations of equilibrium allocations under exogenous growth. We follow Waller (2011) to assume an exogenous technological progress; in other words, z_t in (9) increases according to an exogenous growth rate $g_z \equiv (z_{t+1} - z_t) / z_t$. Based on (4), (10) and (21), the balanced growth rate of x_t can be expressed as

$$g_x \equiv \frac{x_{t+1}}{x_t} - 1 = \beta \left[1 + r - \delta + \frac{\sigma\alpha}{B\eta} f_c f_d^{1/\eta} s^{(1-\alpha)/\eta} \right] - 1, \tag{E1}$$

where $r = \alpha s^{\alpha-1} h^{\eta}$ from (23). On the balanced-growth path, variables such as output, consumption, capital and real money balance, grow at the same rate as z_t (i.e., $g_x = g_z$). The variable $s \equiv k/z$ denotes the capital-technology ratio and can be determined using the following condition

$$s^{\alpha-1} h^{\eta} = f_c + \delta + g_z, \tag{E2}$$

which uses the capital-accumulation equation $k_{t+1} = y_{x,t} - x_t + (1-\delta) k_t$.

As for f_d and f_c, we use analogous inference as in section 3 to obtain

$$f_d = s^{\alpha-1} \left(\frac{\sigma\eta}{R+\sigma} \right)^{\eta}, \tag{E3}$$

where R denotes the nominal interest rate and is given by $R = (1+\mu)/\beta - 1$ at the steady state. We make use of (E1), (E2) and (E3) to derive

$$f_c = \frac{B}{\alpha} \left(\frac{(1+g_z)/\beta - [1 - \delta + \alpha(\delta + g_z)]}{B + \sigma^2/(R+\sigma)} \right), \tag{E4}$$

which is positive because $(1+g_z)/\beta > 1 + g_z > 1 + \alpha g_z - (1-\alpha)\delta$.

Using (3) and (24), equation (E2) can be rearranged as

$$s^{\alpha-1} h^{\eta} = \frac{\delta + g_z}{1 - \eta B/(Ah)}. \tag{E5}$$

Combining (E2), (E4) and (E5), the equilibrium allocation of h is

$$h = \frac{\eta B}{A} \left\{ 1 + (\delta + g_z) \left(\frac{\alpha}{B} \right) \frac{B + \sigma^2/(R+\sigma)}{(1+g_z)/\beta - [1 - \delta + \alpha(\delta + g_z)]} \right\}, \tag{E6}$$

which is also positive. We substitute (E4) and (E6) into (E2) to obtain the equilibrium allocation of s as

$$\begin{aligned} s \quad = \quad & \left\{ \left(\frac{B}{\alpha} \right) \frac{(1+g_z)/\beta - [1 - \delta + \alpha(\delta + g_z)]}{B + \sigma^2/(R+\sigma)} + \delta + g_z \right\}^{1/(\alpha-1)} \\ & \times \left(\frac{\eta B}{A} \right)^{\eta/(1-\alpha)} \left\{ 1 + (\delta + g_z) \left(\frac{\alpha}{B} \right) \frac{B + \sigma^2/(R+\sigma)}{(1+g_z)/\beta - [1 - \delta + \alpha(\delta + g_z)]} \right\}^{\eta/(1-\alpha)}. \end{aligned} \tag{E7}$$

Substituting (E7) into (E3), the equilibrium allocation of f_d can be expressed as

$$\begin{aligned} f_d \;\; = \;\; & \left\{ \left(\frac{B}{\alpha} \right) \frac{(1+g_z)/\beta - [1-\delta+\alpha(\delta+g_z)]}{B+\sigma^2/(R+\sigma)} + \delta + g_z \right\} \\ & \times \left(\frac{\sigma A}{B} \right)^{\eta} \left\{ \frac{(1+g_z)/\beta - [1-\delta+\alpha(\delta+g_z)]}{(R+\sigma)[(1+g_z)/\beta-(1-\delta)]+\alpha(\delta+g_z)\sigma^2/B} \right\}^{\eta}. \end{aligned} \tag{E8}$$

As for social welfare, the lifetime utility of households on the balanced growth path is

$$(1-\beta)U = \sigma \ln q_0 - \sigma \left(\frac{q_0}{k_0} \right)^{1/\eta} \left(\frac{k_0}{z_0} \right)^{(1-\alpha)/\eta} + B \ln x_0 - Ah + \frac{\beta(B+\sigma)}{1-\beta} \ln(1+g_z). \tag{E9}$$

Substituting $q_0 = sf_d z_0$ and $x_0 = sf_c z_0$ into (E9) and then normalizing initial z_0 to unity, we simplify (E9) to

$$(1-\beta)U = (\sigma+B)\ln s + \sigma \ln f_d - \sigma f_d^{1/\eta} s^{(1-\alpha)/\eta} + B \ln f_c - Ah, \tag{E10}$$

where we have dropped the exogenous growth rate g_z.

In the rest of this appendix, we consider the first-best allocations under exogenous growth and compare them with the equilibrium allocations. The planner's problem is

$$J(k_t) = \max_{q_t, x_t, h_t, k_{t+1}} \left\{ \sigma \ln q_t - \sigma \left(\frac{q_t}{z_t} \right)^{1/\eta} \left(\frac{k_t}{z_t} \right)^{-\alpha/\eta} + B \ln x_t - Ah_t + \beta J(k_{t+1}) \right\}, \tag{E11}$$

subject to the capital-accumulation equation $k_{t+1} = z_t^{1-\alpha} k_t^{\alpha} h_t^{\eta} - x_t + (1-\delta)k_t$. From standard dynamic optimization, the optimality conditions for q_t and h_t are respectively

$$\left(\frac{q_t}{k_t} \right) \left(\frac{k_t}{z_t} \right)^{1-\alpha} = \eta^{\eta}, \tag{E12}$$

$$Ah_t^{1-\eta} \left(\frac{x_t}{k_t} \right) \left(\frac{k_t}{z_t} \right)^{1-\alpha} = B\eta. \tag{E13}$$

The intertemporal optimality condition for capital accumulation is

$$\frac{B}{x_t} = \beta J_k(k_{t+1}), \tag{E14}$$

and the envelope condition is

$$J_k(k_t) = \frac{\sigma\alpha}{\eta k_t} \left(\frac{q_t}{k_t} \right)^{1/\eta} \left(\frac{k_t}{z_t} \right)^{(1-\alpha)/\eta} + \frac{B}{x_t} \left[1 + \alpha \left(\frac{k_t}{z_t} \right)^{\alpha-1} h_t^{\eta} - \delta \right]. \tag{E15}$$

Combining (E14) and (E15), we can obtain the balanced growth rate of x_t given by

$$g_x = \frac{x_{t+1}}{x_t} - 1 = \beta \left[1 + \alpha (s^*)^{\alpha-1} (h^*)^{\eta} - \delta + \frac{\sigma\alpha}{B\eta} f_c^* (f_d^*)^{1/\eta} (s^*)^{(1-\alpha)/\eta} \right] - 1, \tag{E16}$$

where $s^* \equiv (k/z)^*$, $f_c^* \equiv (x/k)^*$ and $f_d^* \equiv (q/k)^*$ denote the first-best capital-technology ratio and the first-best consumption-capital ratios for general goods and special goods respectively. On the balanced growth path, the growth rate of x_t is equal to the growth rate of z_t (i.e., $g_x = g_z$). Using (E12), we obtain

$$(f_d^*)^{1/\eta} (s^*)^{(1-\alpha)/\eta} = \eta. \tag{E17}$$

In addition, the capital-accumulation equation can be rearranged as

$$(s^*)^{\alpha-1} (h^*)^{\eta} = f_c^* + \delta + g_z. \tag{E18}$$

Based on (E16), (E17) and (E18), the first-best consumption-capital for general goods can be expressed as

$$f_c^* = \frac{B}{\alpha} \left(\frac{(1+g_z)/\beta - [1-\delta+\alpha(\delta+g_z)]}{B+\sigma} \right). \tag{E19}$$

Combining (E13), (E18) and (E19) yields

$$h^* = \frac{\eta B}{A} \left\{ 1 + (\delta + g_z) \left(\frac{\alpha}{B} \right) \frac{B+\sigma}{(1+g_z)/\beta - [1-\delta+\alpha(\delta+g_z)]} \right\}. \tag{E20}$$

Substituting (E19) and (E20) into (E18), we obtain

$$\begin{aligned} s^* &= \left\{ \left(\frac{B}{\alpha} \right) \frac{(1+g_z)/\beta - [1-\delta+\alpha(\delta+g_z)]}{B+\sigma} + \delta + g_z \right\}^{1/(\alpha-1)} \\ &\times \left(\frac{\eta B}{A} \right)^{\eta/(1-\alpha)} \left\{ 1 + (\delta+g_z) \left(\frac{\alpha}{B} \right) \frac{B+\sigma}{(1+g_z)/\beta - [1-\delta+\alpha(\delta+g_z)]} \right\}^{\eta/(1-\alpha)}. \end{aligned} \tag{E21}$$

As for f_d^*, we make use of (E21) and (E17) to obtain

$$\begin{aligned} f_d^* &= \left\{ \frac{B}{\alpha} \left(\frac{(1+g_z)/\beta - [1-\delta+\alpha(\delta+g_z)]}{B+\sigma} \right) + \delta + g_z \right\} \\ &\times \left(\frac{A}{B} \right)^{\eta} \left\{ \frac{(1+g_z)/\beta - [1-\delta+\alpha(\delta+g_z)]}{[(1+g_z)/\beta - (1-\delta)] + \alpha(\delta+g_z)\sigma/B} \right\}^{\eta}. \end{aligned} \tag{E22}$$

Comparing the first-best allocations with equilibrium allocations, we find that $f_d = f_d^*$, $f_c = f_c^*$, $s = s^*$ and $h = h^*$ if the nominal interest rate R is equal to zero; therefore, the Friedman rule holds in the exogenous growth model.