Second-Price Auctions with Different Participation Costs

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Abstract

This paper studies equilibria of second price auctions in independent private value environments with different participation costs. Two types of equilibria are identified: monotonic equilibria in which a bidder with a lower participation cost results in a lower cutoff for submitting a bid, and non-monotonic equilibria in which a lower participation cost results in a higher cutoff. We show that there always exists a monotonic equilibrium, and further, that the monotonic equilibrium is unique for either concave distribution functions or strictly convex distribution functions with non-increasing reverse hazard rates. There exist non-monotonic equilibria when the distribution functions are strictly convex and the difference of the participation costs is sufficiently small. We also provide comparative static analysis and study the limiting properties of equilibria when the difference in bidders' participation costs approaches zero.

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Key Words: Private Values, Differentiated Participation Costs, Second Price Auctions, Non-monotonic Equilibrium, Existence and Uniqueness of Equilibrium.

1 Introduction

Auctions are efficient ways to allocate resources by increasing the competition among potential buyers. However, not all bidders can participate in an auction freely. The existence of bidders' participation costs can substantially change the outcome of an auction.

Generally an auction with participation cost is the one in which an indivisible object is allocated to one of potential buyers via a second price auction, and in order to participate, bidders must incur a non-refundable cost that may be the costs of traveling to an auction site, to pay for the process of learning the rules of auction, to acquire information (Persico (2000), Cremer, Spiegel and Zheng (2009)), or more generally the opportunity cost of attending an auction (Lu (2009), Lu and Sun (2007)), etc. Hence the question of whether to participate in auctions may be more crucial than the standard question of how to bid, suggesting that such decisions should be modeled and included as part of an equilibrium. This paper studies (Bayesian-Nash) equilibria of sealed-bid second price auctions with private values and different participation costs. The entry behavior of potential bidders, in turn, provides a solid foundation for further analyzing the impacts of participation costs on revenue and welfare.

Different participation costs are important in practice. For example, in China's leasehold auctions, city officials provide hidden help to favored bidders, which has been attributed to corruption, see Cai et al (2010). Bidders from different cities have different transportation costs to show up on an auction spot. One bidder may have more advantage in acquiring the value of the object being auctioned than others. All these give rise to different participation costs.

Study of auctions with participation costs is mainly focused on the second price auction due to its simplicity of bidding behavior.[1] In second price auctions, if a bidder finds participating in this second price auction optimal, he cannot do better than bid his true valuation.

Green and Laffont (1984) studied second price auctions with participation costs in a general framework where bidders' valuations and participation costs are both private information. However, their study is incomplete, having additionally imposed a restrictive assumption of uniform distributions for both values and participation costs. The difficulty lies in the two-dimensional random framework. Some recent work studied second price auctions with participation costs in simplified versions, where either only valuations or participation costs are private while the other is assumed to be common knowledge.

There are a number of studies of auctions with equal participation costs. Campbell (1998) considered the equilibria in an independent private value environment with equal participation

[1] Cao and Tian (2010) studied the equilibria in first price auctions with equal participation costs.

costs when bidders' values are private information and participation costs are common knowledge. He focused on the coordination of equilibrium choice when multiple equilibria exist. Tan and Yilankaya (2006) considered the same problem as in Campbell (1998) by assuming bidders are asymmetric in the sense that they have different valuation distribution functions while maintaining identical participation costs. Some others, such as Samuelson (1985), McAfee and McMillan (1987), Harstad, Kagel and Levin (1990), Levin and Smith (1994), Stageman (1996), and Menezes and Monterio (2000) studied auctions with participation costs assuming bidders' participation costs are the same. Kaplan and Sela (2006) studied equilibria of the second price auction with participation costs when bidders' participation costs are private information while valuations are common knowledge.

However, the equal participation costs assumptions are stringent and unrealistic in many real world situations as we have mentioned above. Another advantage of considering different participation costs is that it can include the equal participation costs as a special case, which is interesting from the theoretical point.

Economic environments where bidders have private valuations for the object and different participation costs that are common knowledge are studied in this paper. Bidders submit bids if and only if their values are greater than the corresponding cutoffs. We identify two types of equilibria: monotonic equilibria in which a lower participation cost results in a lower cutoff to participate in an auction, and non-monotonic equilibria in which a lower participation cost results in a higher cutoff. We show that there always exists a monotonic equilibrium, and further that, the monotonic equilibrium is unique for concave distribution functions and strictly convex distribution functions with non-increasing reverse hazard rate. When bidders' distribution functions are strictly convex and the differences among the bidders' participation costs are sufficiently small, there is a non-monotonic equilibrium. There is no non-monotonic equilibrium for either strictly convex distributions when the difference is sufficiently large or for concave distributions, which implies that in the land leasehold auctions, the corrupt city officials may give a bidder sufficient help to make sure the targeted bidder is more likely to participate in the auction.

Our study on auction with different participation costs is not only more realistic, but also provides a deeper insight that would help us understand the existence or non-existence of asymmetric equilibria well in auctions with equal participation costs. This can be seen by investigating the limit behavior of the monotonic and non-monotonic equilibria when bidders' participation costs converge to the same value. We show that, when the distribution function of valuation is concave, the monotonic equilibrium converges to the symmetric equilibrium when bidders have the same participation costs. However, when the distribution is strictly convex, the monotonic equilibrium converges to the asymmetric equilibrium. In this case one non-monotonic equilibrium converges to a symmetric equilibrium, and another non-monotonic equilibrium converges to an asymmetric equilibrium.

We also provide comparative static analysis. It is shown that the cutoff is increasing in one's own participation costs, but decreasing in his opponents' participation costs, and further, as the number of bidders increases, the cutoffs of all bidders will increase.

The organization of the paper is as follows. In Section 2, we describe the economic environments. In Section 3, we focus on two bidders with the same distribution functions and different participation costs to study the existence, uniqueness, and limit properties of the equilibria and to make a comparative analysis. In Section 4, we extend our basic results to more general economic environments. Concluding remarks are provided in Section 5. All the proofs are presented in the appendix.

2 The Setup

We consider an independent private value economic environment with one seller and $n \geq 2$ potential buyers. The seller is risk-neutral and has an indivisible object to sell to one of the buyers via a sealed-bid second price auction (see Vickrey, 1961). The seller values the object as 0. However, in order to submit a bid, bidder i must incur a participation cost c_i. Bidder i's valuation v_i is private information, which is independently distributed with a cumulative distribution function $F_i(v)$ that has continuously differentiable density $f_i(v)$ with full support $[0, 1]$.[2] The participation costs $c_i \in (0, 1]$ for all i are common knowledge.

Each bidder knows his value and the distributions of the others' valuations. If participating in the auction, he incurs a non-refundable participation fee. The bidder with the highest bid wins the object and pays the second-highest bid. If there is only one bidder in the auction, he wins the object and pays 0. If the highest bids are equal for more than one bidder, he pays his own bid and gains nothing.

In this second price auction mechanism with participation cost, the individually rational action set for any type of bidder is $\{No\} \cup [0, 1]$, where "$\{No\}$" denotes not participating in the auction. Bidder i incurs the participation cost if and only if his action is different from "$\{No\}$".

[2]Here "0" denotes the value is zero while "1" is a normalization of the highest possible valuation among all bidders.

Let $b_i(v_i, \mathbf{c})$ denote bidder i's strategy where $\mathbf{c} = (c_1, ..., c_n)$.

If a bidder finds participating in this second price auction optimal, he cannot do better than bid his true value.[3] All of our results about the uniqueness or multiplicity of the equilibria should be interpreted accordingly.

Given the equilibrium strategies of all others, a bidder's expected payoff from participating in the auction is a non-decreasing function of his valuation. Therefore, we can focus on Bayesian-Nash equilibria in which each bidder uses a cutoff strategy[4] denoted by $v_i^*(\mathbf{c})$, i.e., he bids his valuation if it is greater than or equal to the cutoff[5] and does not enter otherwise. Thus the bidding decision function of each bidder is characterized by

$$b_i(v_i, \mathbf{c}) = \begin{cases} v_i & \text{if } v_i^*(\mathbf{c}) \leq v_i \leq 1 \\ \text{No} & \text{otherwise.} \end{cases}$$

For notational convenience, we simply denote $v_i^*(\mathbf{c}) = v_i^*$.

Remark 1 When $v_i^* \leq 1$, bidder i will participate in the auction whenever his true value v_i satisfies $v_i^* \leq v_i \leq 1$. However, when bidder i's expected payoff is always less than his participation cost c_i for any $v_i \in [0, 1]$, he will never participate in the auction. In this case, his equilibrium strategy (action) is "$\{No\}$". For notational convenience and simplicity of discussion, we use $v_i^* > 1$ to denote the equilibrium strategy of "$\{No\}$". This allows us to use a unified notation v_i^* to denote an equilibrium strategy of bidder i, including the strategy of "$\{No\}$".

For the game described above, each bidder's action is to choose a cutoff and decide how to bid when he participates. Thus, a (Bayesian-Nash) equilibrium of the second price auctions with participation costs is composed of bidders' cutoff strategies and participants' bidding strategies. However, note that, once the cutoffs are determined, the game is reduced to the standard second price auction and each bidder bids his true value. From now on we focus exclusively on cutoffs, since they are sufficient to describe equilibria.

Definition 1 An equilibrium is a cutoff vector $(v_1^*, v_2^*, ..., v_n^*) \in \mathbb{R}_+^n$ such that each bidder i's action is optimal, given others' cutoff strategies.

[3] There may exist an equilibrium in which bidders do not bid their true value when they participate. See the example given in Remark 4.7 below.

[4] Lu and Sun (2007) showed that for any auction mechanism with participation costs, the participating and nonparticipating types of any bidder are divided by a nondecreasing and equicontinuous shutdown curve. Thus in our framework, when participation cost is given, the participating and nonparticipating types of any bidder can be divided by a cutoff value and the threshold form is the only form of equilibria.

[5] In Milgrom and Weber (1982), the term of "screening level" is used instead of "cutoff."

We then immediately have the following result:

Lemma 1 $v_i^* \leq 1$ for at least some i.

One may come to the intuitive conclusion that the bidder with smaller participation cost is always more likely to participate in the auction by choosing a smaller cutoff. However, as we will show in the paper, it is possible that a bidder with a higher participation cost may actually have a lower cutoff. For illustration simplicity, we distinguish two types of equilibria: monotonic equilibria and non-monotonic equilibria which are defined formally as below.

Definition 2 An equilibrium $(v_1^*, v_2^*, \ldots, v_n^*) \in \mathbb{R}_+^n$ is called a *monotonic equilibrium* (*resp. non-monotonic equilibrium*) if, for any two bidders i and j, $c_i < c_j$ implies $v_i^* < v_j^*$ (resp. there exist two bidders i and j, $c_i < c_j$ implies $v_i^* \geq v_j^*$).[6]

Remark 2 The term "monotonic" used here means that two variables c_i and v_i^* vary in the same direction: a higher participation cost results in a higher cutoff. When bidders' distribution functions are the same, as one will see in Section 3, $v_1^* = v_2^*$ cannot be an equilibrium, provided bidders' participation costs are different. Thus, $c_i < c_j$ implies $v_i^* > v_j^*$ for every non-monotonic equilibrium, and $c_i < c_j$ implies $v_i^* < v_j^*$ for every monotonic equilibrium. However, when bidders' distribution functions are different, $v_1^* = v_2^*$ may be an equilibrium although bidders' participation costs are different. That is, we have a special non-monotonic equilibrium with $v_i^* = v_j^*$ even when $c_i < c_j$.

Example 1 We give an example to illustrate the notion of monotonic and non-monotonic equilibria. Suppose there is one object for sale to two bidders. Valuation distributions are $F_1(v) = F_2(v) = \frac{v+v^3}{2}$ and participation costs are $c_1 = 0.3$ and $c_2 = 0.32$. The equilibrium cutoff vectors are $(0.3753, 0.8911)$, $(0.6995, 0.6142)$ and $(0.8301, 0.4564)$[7]. The first one is monotonic and the latter two are non-monotonic.

As usual, when bidders' distribution functions and participation costs are the same; i.e., $F_1(\cdot) = F_2(\cdot) = \ldots = F_n(\cdot) = F(\cdot)$ and $c_1 = c_2 = \ldots = c_n$, we define the usual symmetric and asymmetric equilibria.

[6]The 'intuitive equilibrium' in Tan and Yilankaya (2006) is defined based on different valuation distributions (one distribution first order stochastically dominates another distribution) when participation costs are assumed to be the same, while our 'monotonic equilibrium' is defined based on different participation costs. Therefore, they are two different equilibrium concepts.

[7]In the next section, we will show how these equilibria are found.

Definition 3 An equilibrium $(v_1^*, v_2^*, \ldots, v_n^*) \in \mathbb{R}_+^n$ is called a *symmetric equilibrium* (*resp. asymmetric equilibrium*) if $v_1^* = v_2^* = \ldots = v_n^*$ (resp. there exist two bidders i and j such that $v_i^* \neq v_j^*$).

3 Two Bidders with Different Participation Costs

In this section we consider an economy with two bidders who have different participation costs c_1 and c_2 with $c_1 < c_2$, and the same distribution function $F(.)$.

We first assume, provisionally, that a monotonic equilibrium (v_1^*, v_2^*) exists, i.e., $v_1^* < v_2^*$. By Lemma 1, it must be $v_1^* \leq 1$. When bidder 1's valuation is $v_1 = v_1^*$, his expected payoff from participating is given by $v_1^* F(v_2^*) + 0(1 - F(v_2^*))$, where $F(v_2^*)$ is the probability that bidder 2 does not participate in the auction. Indeed, when bidder 1 participates and bidder 2 does not participate, bidder 1's revenue is v_1^*. When bidder 2 participates in the auction, it must be the case that $v_2 \geq v_2^*$. Then bidder 1 cannot win the object since $v_2 \geq v_2^* > v_1^* = v_1$, and thus his revenue is zero. Therefore, his expected payoff from the auction is $v_1^* F(v_2^*)$. Zero net-payoff (equilibrium) condition requires that

$$c_1 = v_1^* F(v_2^*). \tag{1}$$

When bidder 2's participation cost is too large, he may never participate in the auction, whatever his valuation is. In this case, bidder 1 uses $v_1^* = c_1$ as his cutoff, and bidder 2's expected payoff must satisfy

$$F(c_1) + \int_{c_1}^{1} (1 - v) dF(v) = c_1 F(c_1) + \int_{c_1}^{1} F(v) dv < c_2;$$

i.e., the expected payoff he obtains from participating even when his value is 1 is less than his participation cost c_2, given bidder 1 uses c_1 as his cutoff. In this case, we have a monotonic equilibrium with $v_1^* = c_1$ and $v_2^* > 1$.

Now suppose $v_2^* \leq 1$. Then, when bidder 2's valuation is $v_2 = v_2^*$, his expected payoff is

$$v_2^* F(v_1^*) + \int_{v_1^*}^{v_2^*} (v_2^* - v) dF(v),$$

where the first term is the expected payoff when bidder 1 does not participate, and the second term is the expected payoff when both bidders participate in the auction. Note that bidder 2 will lose the object if $v_1 > v_2^*$. The zero expected net-payoff (equilibrium) condition requires that

$$v_2^* F(v_1^*) + \int_{v_1^*}^{v_2^*} (v_2^* - v) dF(v) = c_2. \tag{2}$$

Integrating by parts in the left side of (2), we have

$$v_1^* F(v_1^*) + \int_{v_1^*}^{v_2^*} F(v)dv = c_2. \tag{3}$$

Remark 3 Note that, from (1) and (3), one can see the claim in Remark 2 is true. It is impossible for both bidders to use the same cutoff $v_1^* = v_2^* = v^*$ when their participation costs are different. Indeed, suppose not. Then we must have $c_1 = v^* F(v^*)$ by (1) and $c_2 = v^* F(v^*)$ by (2). Thus $c_1 = c_2$, which contradicts the fact that $c_2 > c_1$.

Let v_1^s be the symmetric equilibrium defined by

$v_1^s F(v_1^s) = c_1$ if both bidders have the same participation cost c_1. Similarly define v_2^s as the symmetric equilibrium if both bidders have the same participation cost c_2.

The following lemma shows the relationship between a monotonic equilibrium and symmetric equilibria.

Lemma 2 *Suppose (v_1^*, v_2^*) is a monotonic equilibrium, (v_1^s, v_1^s) and (v_2^s, v_2^s) are symmetric equilibria associated with participation costs $c_1 < c_2$, respectively. Then, we have $v_1^* < v_1^s < v_2^s < v_2^*$.*

Lemma 2 shows that, when bidders have different participation costs, at a monotonic equilibrium, the cutoff for the bidder with lower participation cost is lower than the cutoff at the symmetric equilibrium when both bidders have the same lower participation cost c_1 and the cutoff for the bidder with higher participation cost is higher than the cutoff at the symmetric equilibrium when both bidders have the same higher participation cost c_2.

To find a monotonic equilibrium, we define the following two cutoff reaction function equations.

$$xF(y) = c_1 \tag{4}$$

$$xF(x) + \int_x^y F(v)dv = c_2 \tag{5}$$

with $x < y$, where x corresponds to v_1^*, and y corresponds to v_2^*. It can be easily seen that we have $x \geq c_1$ and $y \geq c_2$. They can be regarded as cutoff reaction functions because (4) shows how bidder 1 will choose a cutoff x, given bidder 2's action y. Equation (5) shows how bidder 2 will choose a cutoff y, given bidder 1's action x. A monotonic equilibrium $(v_1^*, v_2^*) \in [c_1, 1] \times [c_2, 1]$ is obtained when x and y satisfy these two equations simultaneously.

From (4), we have $x = x(y) = \frac{c_1}{F(y)}$. Then $\frac{dx}{dy} = -\frac{c_1 f(y)}{F^2(y)} < 0$. This implicitly defines y as a decreasing function of x, denoted by $y = y(x)$. Substitute $y = y(x)$ into the left side of (5) and let

$$h(x) = xF(x) + \int_x^{y(x)} F(v)dv - c_2.$$

Substitute $x = x(y)$ into the left side of (5) and let

$$\lambda(y) = \frac{c_1}{F(y)} F(\frac{c_1}{F(y)}) + \int_{\frac{c_1}{F(y)}}^{y} F(v) dv.$$

To consider the existence of non-monotonic equilibria in which $v_2^* < v_1^*$ whenever $c_1 < c_2$, we follow the above process similarly and at equilibrium get

$$c_2 = v_2^* F(v_1^*), \tag{6}$$

and

$$v_1^* F(v_2^*) + \int_{v_2^*}^{v_1^*} (v_1^* - v) dF(v) - c_1 \leq 0, \tag{7}$$

where the equality holds whenever $v_1^* \leq 1$.

Integrating by parts, we get

$$c_1 \geq v_2^* F(v_2^*) + \int_{v_2^*}^{v_1^*} F(v) dv. \tag{8}$$

To find a non-monotonic equilibrium, through (6) and (8), we define the two cutoff reaction functions

$$\begin{aligned} y(x) &= c_2/F(x) \\ \phi(x) &= \frac{c_2}{F(x)} F(\frac{c_2}{F(x)}) + \int_{\frac{c_2}{F(x)}}^{x} F(v) dv. \end{aligned}$$

Again, we use x to correspond to v_1^* and y to correspond to v_2^*. Note that we have $x \geq y \geq c_2$.

When two bidders have the same participation cost c_2 and $F(.)$ is strictly convex, there exists a unique symmetric equilibrium $x = y = v_2^s$ that satisfies $y = x = c_2/F(x)$ and an asymmetric equilibrium (x_0, y_0) with $x_0 > v_2^s$ and $y_0 < v_2^s$ (cf. Campbell (1998) and Tan and Yilankaya (2006)), indicating that $\phi(x)$ intersects with c_2 when $x = v_2^s$ and $x = x_0$. Also, by the uniqueness of symmetric equilibrium, $v_1^* \geq v_2^s$ if it exists. Let c_m be the minimum of $\phi(x) = \frac{c_2}{F(x)} F(\frac{c_2}{F(x)}) + \int_{\frac{c_2}{F(x)}}^{x} F(v) dv$ in the interval $[v_2^s, 1]$.

We then have the following proposition on the existence and uniqueness of equilibrium:

Proposition 1 (Existence and Uniqueness Theorem) *For the independent private values economic environment with two bidders who have different participation costs $c_2 > c_1$, we have the following conclusions:*

(1) There always exists a monotonic equilibrium.

(2) Suppose $F(.)$ is concave. Then the equilibrium is unique and monotonic.

(3) Suppose $F(.)$ is strictly convex. Then

(3.i) the monotonic equilibrium is unique when the reverse hazard rate of $F(.)$, i.e., when $\frac{f(.)}{F(.)}$ is non-increasing,

(3.ii) the non-monotonic equilibrium is unique when $c_1 = c_m$,

(3.iii) there is no non-monotonic equilibrium when $c_1 < c_m$, and

(3.iv) there are at least two non-monotonic equilibria when $c_m < c_1 < c_2$.

We provide the brief idea about the proof in the appendix. To investigate the existence and uniqueness of the equilibria, we examine how functions $\lambda(y)$ and $\phi(x)$ intersect with c_2 and c_1, respectively, keeping in mind that there may be an equilibrium in which one bidder never participates. The existence of a monotonic equilibrium can be established by the intermediate value theorem. The uniqueness of the monotonic (resp. non-monotonic) equilibrium comes from the fact that $\lambda(y)$ (resp. $\phi(x)$) intersects with c_2 (resp. c_1) at most once on the interval $y \in [v_1^s, 1]$ (resp. $[v_2^s, 1]$). When $F(.)$ is concave, $\lambda(y)$ is a monotonically increasing function, and thus the monotonic equilibrium is unique. When $F(.)$ is strictly convex, we can also show the uniqueness of monotonic equilibrium and the existence and uniqueness of non-monotonic equilibrium for some types of convex distribution functions.

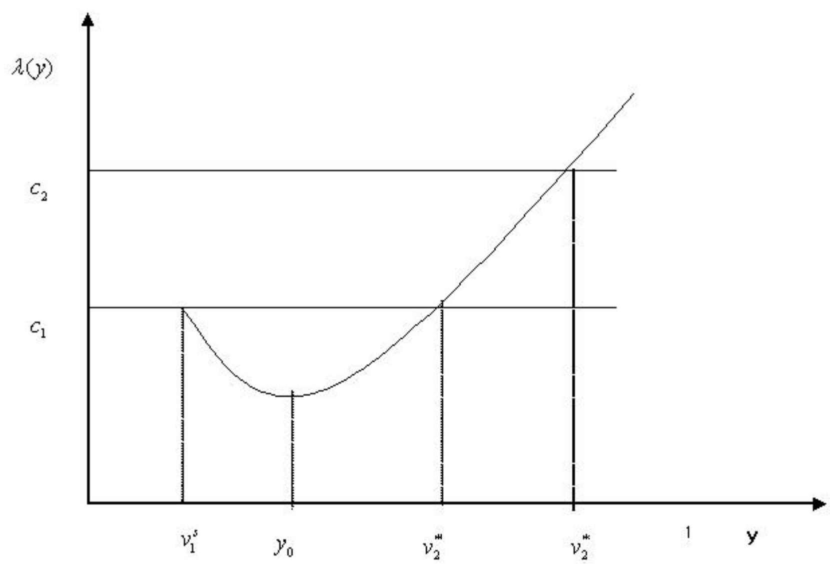

Figure 1: Uniqueness for Convex Case

Remark 4 It is worthwhile to point out some remarks on Proposition 1:

1. For any power functions $F(.)$ that are convex, the reverse hazard rate is a non-increasing function. Thus, the set of such strictly convex functions is not empty.

To understand why there is a unique monotonic equilibrium for this type of strictly convex distribution, see Figure 1. $\lambda(y)$ starts from v_1^s with negative slope. When $\lambda'(y)$ equals 0 at most once, $\lambda(y)$ intersects with c_2 at most once, indicating that the monotonic equilibrium is unique.

2. From the proof in the appendix, one can see $c_2 > c_m$. Then, as long as $c_2 - c_1$ is sufficiently small, we have $c_2 > c_1 > c_m$. Thus, we can conclude that when $c_2 - c_1$ is sufficiently small, there are two non-monotonic equilibria that are given by (x_1, y_1) and (x_2, y_2) with $y_1 = y(x_1)$, $y_2 = y(x_2)$, and $y_1 < y_2 < v_2^s < x_1 < x_m < x_2 < x_0$. Thus, when $F(.)$ is strictly convex, the existence of non-monotonic equilibrium depends on the difference of participation costs, $c_2 - c_1$. For instance, in Example 1, we have one monotonic equilibrium and two non-monotonic equilibria. However, when $c_1 = 0.3$ and $c_2 = 0.4$, there is only one equilibrium $(0.3003, 0.9994)$ and it is monotonic.

3. Figure 2 can help us understand the proof in the appendix and the points mentioned above. $\phi(x)$ starts from $y = v_2^s$ with negative slope. When $c_2 - c_1$ is small enough, it intersects with c_1; i.e., a non-monotonic equilibrium exists. When $c_2 - c_1$ is big enough so that $c_1 < c_m$, $\phi(x)$ and c_1 cannot intersect; i.e., no non-monotonic equilibrium exists. From the figure, when c_1 is close to c_2, there are at least two intersection points for $y = \phi(x)$ and $y = c_1$, which means there are at least two non-monotonic equilibria, say, (x_1, y_1) and (x_2, y_2).

4. Campbell (1998) and Tan and Yilankaya (2006) showed that there exist asymmetric equilibria when distribution functions are strictly convex. However, our result shows that the strict convexity of the distribution function alone is not a sufficient condition for the existence of a non-monotonic equilibrium, unless the difference $c_2 - c_1$ is small enough, which implies that one can refine equilibria and always eliminate non-equilibria by making participation costs for bidders sufficiently different when necessary.

5. In the proof of Proposition 1, the condition that $F(.)$ is concave can be weakened to $\frac{F(v)}{v}$ non-decreases for all $v \in [c_1, 1]$, and the condition that $F(.)$ is strictly convex can be weakened to $\frac{F(v)}{v}$ decreases with v for all $v \in [c_2, 1]$.

6. When multiple equilibria exist and $F(.)$ is non-atomic, there cannot exist mixed strategies in which a bidder uses different cutoffs with positive probability. Indeed, if one bidder behaves in this way, the expected payoff from participating of

his opponent can be uniquely determined, which is still a non-decreasing function of his valuation and thus there is only one cutoff.

7. A non-truth-telling equilibrium may exist when bidders do not use weakly dominant strategies. Suppose bidder 1 bids zero and bidder 2 bids 1 when they participate. Bidder 1 wins only when bidder 2 does not enter, hence at equilibrium $v_1^* F(v_2^*) = c_1$. Bidder 2 always wins once he enters and pays nothing. At equilibrium we have $v_2^* = c_2$. Thus $v_1^* = \frac{c_1}{F(c_2)}$. Therefore, if bidders do not use dominant bidding strategy, we have other cutoff equilibria.

8. Existence of a reserve price r does not affect part (1), (2) of Proposition 1. It can be easily shown that the condition of strict convexity of $F(.)$ in part (3) needs to be replaced by $F(v_2'^s) - (v_2'^s - r)f(v_2'^s) < 0$, where $v_2'^s$ is the revised symmetric cutoff equilibrium when both bidders have participation cost c_2, which is defined by $(v_2'^s - r)F(v_2'^s) = c_2$.

9. Letting $H(.) = \frac{c_2}{c_1}F(.)$, we can rewrite Equation (1) as $c_2 = v_1^* H(v_2^*)$. Then, the technique adopted in Tan and Yilankaya (2006) can be used to show the existence of monotonic equilibria[8]. However, when such a technique is used to show the existence of non-monotonic equilibria, as in Tan and Yilankaya (2006), one needs to impose a different condition that c_1 is sufficiently large. As such, in our opinion, it is a restrictive assumption.

The intuition for the existence of non-monotonic equilibria when $F(.)$ is strictly convex and $c_2 - c_1$ is sufficiently small is as follows. Rewrite equation (7) as

$$v_1^* F(v_1^*) - \int_{v_2^*}^{v_1^*} v f(v) dv - c_1 \leq 0, \tag{9}$$

where the equality holds whenever $v_1^* < 1$. The first term is the expected gross payoff for bidder 1 with value v_1^*, and the second term is the expected payment to the seller. Combine (6) and (9), to have a non-monotonic equilibrium, we need

$$(v_1^* - v_2^*)F(v_1^*) - \int_{v_2^*}^{v_1^*} v f(v) dv \leq c_1 - c_2 < 0. \tag{10}$$

When $F(.)$ is concave, $(v_1^* - v_2^*)F(v_1^*) - \int_{v_2^*}^{v_1^*} v f(v) dv$ is strictly positive. Bidder 1 need not pay much to the seller since relatively bidder 2 has a low valuation. Thus there is no non-monotonic equilibrium. However, when $F(.)$ is strictly convex, bidder 2 is more likely to have

[8]We would like to thank an anonymous referee for pointing out such a transformation.

a high valuation and thus bidder 1's expected payment to the seller is high and the above inequality may be satisfied. Indeed, consider the extreme case where $c_2 - c_1 = 0$, for any given v_2^*, we can always find $v_1^* > v_2^*$ such that equation (10) holds. To see this, let $\rho(v_1^*) = (v_1^* - v_2^*)F(v_1^*) - \int_{v_2^*}^{v_1^*} vf(v)dv$ be a function of v_1^*, holding v_2^* constant. It can be checked that $\rho(v_2^*) = 0$ and $\rho'(v_2^*) = F(v_2^*) - v_2^* f(v_2^*) < 0$ as $F(.)$ is strictly convex and thus for any given v_2^*, there exists a $v_1^* > v_2^*$, such that $\rho(v_1^*) < 0$. Note that $\rho(v_1^*)$ is bounded as a continuous function in the interval $[v_2^*, 1]$. When c_1 differs too much from c_2, (10) cannot hold and thus there is no non-monotonic equilibrium.

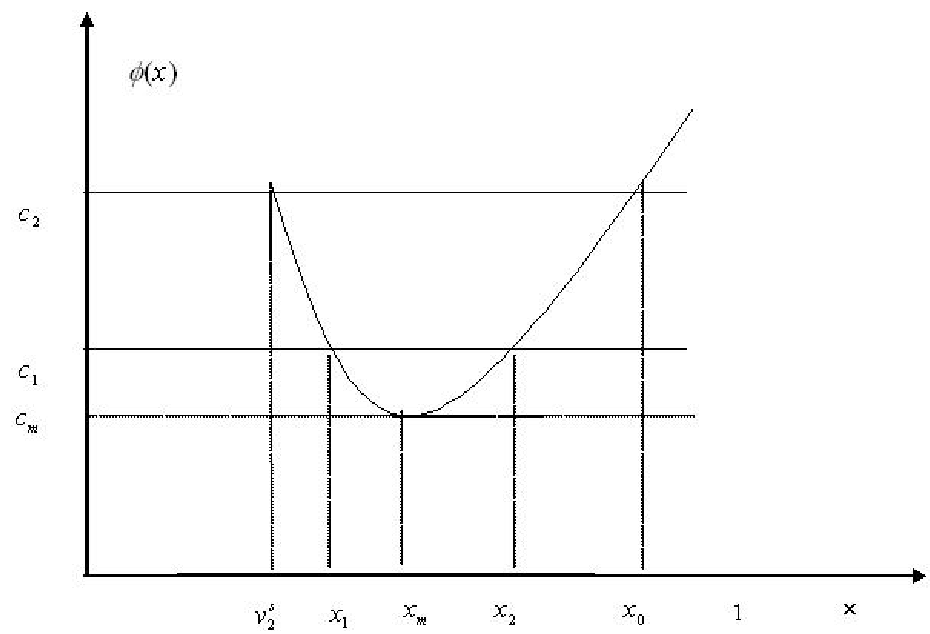

Figure 2: Existence of Non-Monotonic Equilibria For Convex Case

One may wonder what would happen at the limits of monotonic and non-monotonic equilibria as $c_2 - c_1 \to 0$. Should a monotonic equilibrium converge to a symmetric equilibrium or a non-monotonic equilibrium converge to an asymmetric equilibrium when $c_2 \to c_1$?

For instance, suppose c_1 is constant at 0.30, and let c_2 decrease from some point, say, 0.38, until $c_2 = c_1 = 0.3$. Will there be any convergence for monotonic and non-monotonic equilibria in this case? Do they converge to a symmetric equilibrium or an asymmetric equilibrium (if it exists) for a given distribution function? Some numerical experiments are given in Table 1.

Table 1 Sequences of Monotonic and Non-Monotonic Equilibria

c_1	c_2	$F(v)=\sqrt{v}$	$F(v)=v^2$		
0.3	0.3	(0.4481, 0.4481)	(0.6694, 0.6694)	(0.3425, 0.9358)	(0.9358, 0.3425)
0.3	0.31	(0.4387, 0.4675)	(0.6845, 0.6616)	(0.3327, 0.9426)	(0.9318, 0.3570)
0.3	0.32	(0.4303, 0.4861)	(0.7000, 0.6530)	(0.3237, 0.9627)	(0.9271, 0.3723)
0.3	0.33	(0.4226, 0.5038)	(0.7162, 0.6434)	(0.3155, 0.9751)	(0.9216, 0.3886)
0.3	0.34	(0.4156, 0.5210)	(0.7332, 0.6323)	(0.3079, 0.9870)	(0.9150, 0.4061)
0.3	0.35	(0.4091, 0.5376)	(0.7517, 0.6193)	(0.3009, 0.9985)	(0.9068, 0.4256)
0.3	0.36	(0.4032, 0.5537)	(0.7725, 0.6038)	(0.3000, 1.0000)	(0.8963, 0.4418)
0.3	0.37	(0.3976, 0.5694)	(0.7984, 0.5804)	(0.3000, 1.0000)	(0.8805, 0.4773)
0.3	0.38	(0.3923, 0.5847)	NA	(0.3000, 1.0000)	NA

From the table, when $F(v)=\sqrt{v}$, which is concave, we only have the monotonic equilibrium and it is unique. Tan and Yilanyaka (2006) proved that when $F(.)$ is concave, there is a unique symmetric equilibrium and no asymmetric equilibrium. Then we naturally conjecture that, when c_2 converges to c_1, the unique monotonic equilibrium converges to the unique symmetric equilibrium, as can be seen from Table 1.

However, when $F(v)=v^2$, which is strictly convex, we can see from the table that when c_2-c_1 is small enough, there exist one monotonic and two non-monotonic equilibria, but when c_2-c_1 is big enough, monotonic equilibrium is the only equilibrium. Somewhat surprisingly, we can see that unlike the monotonic equilibrium, one sequence of non-monotonic equilibria converges to the symmetric equilibrium, while the other sequence of monotonic equilibria converges to the asymmetric equilibrium. Thus, the notion of monotonic/non-monotonic equilibrium is not a trivial generalization of symmetric/asymmetric equilibria.

Actually, these limiting relationships among monotonic/non-monotonic equilibria and symmetric/asymmetric equilibria are true for general concave and strictly convex functions.

Proposition 2 (Limit Theorem) *For the independent private values economic environment with two bidders having participation costs $c_2>c_1$, we have the following conclusions:*

(1) Suppose $F(.)$ is concave. The unique monotonic equilibrium (no non-monotonic equilibrium) converges to the unique symmetric equilibrium as $c_2-c_1\to 0$.

(2) Suppose $F(.)$ is strictly convex with non-increasing reverse hazard rate. The unique monotonic equilibrium converges to an asymmetric equilibrium as $c_2 - c_1 \to 0$.

(3) Suppose $F(.)$ is strictly convex. When $c_2 - c_1 \to 0$, there are two non-monotonic equilibria, of which one converges to the unique symmetric equilibrium and the other converges to an asymmetric equilibrium.

The intuition can be given for the convergence results of the equilibria. By the continuity of the reaction function, as the participation costs c_1 and c_2 converge, the set of equilibria will converge to the set of equilibria when $c_1 = c_2$. In particular, if we focus on the equilibrium in which bidder 1 uses the smallest cutoff among all bidder 1's equilibrium cutoffs (which is necessarily a monotonic equilibrium), this will converge to the equilibrium for $c_1 = c_2$ in which bidder 1 uses the smallest cutoff among all of bidder 1's equilibrium cutoffs. Thus, if the equilibrium is unique when $c_1 = c_2$, and there is a unique monotonic equilibrium for all c_1 and c_2 in the sequence, that equilibrium sequence must converge to the symmetric equilibrium. However, if there are asymmetric equilibria when $c_1 = c_2$, then the equilibrium in which bidder 1 uses the smallest cutoff must converge to the asymmetric equilibrium in which bidder 1 uses the smaller cutoff. Hence, if the monotonic equilibrium is unique, then it will converge to an asymmetric euqilibrium, and the equilibrium that converges to the symmetric equilibrium must be non-monotonic.

From Figures 1 and 2, one can see that, as $c_2 - c_1 \to 0$, any monotonic/non-monotonic equilibrium converges along the bidders' reaction curves determined by $\lambda(y)$ and $\phi(x)$ to the nearest equilibrium, whether it is symmetric or asymmetric.

Before finishing this section, we examine the effects of changes in participation costs on equilibrium behavior.

Proposition 3 (Comparative Static Theorem) *For the independent private values economic environment with two bidders, suppose the values of bidders are drawn from a distribution function $F(.)$ and the participation costs c_1 and c_2 are common knowledge. Then for the monotonic equilibrium, an increase in participation cost c_i increases i's cutoff v_i^* but decreases the opponent's cutoff v_j^* for $j \neq i$.*

Specially, when $F(.)$ is concave, which gives us a unique and monotonic equilibrium, an increase in participation cost c_i increases i's cutoff v_i^* but decreases the opponent's cutoff v_j^* for $j \neq i$.

In fact, when $F(.)$ is uniform, we can derive and analyze the unique equilibrium explicitly. The condition for $v_2^* > 1$ implies $c_2 > \frac{1}{2} + \frac{1}{2}c_1^2$. In Figure 3, in the area above the parabola $c_2 = \frac{1}{2} + \frac{1}{2}c_1^2$ and inside the square (the shaded area), bidder 2 never participates ($v_2^* > 1$) and bidder 1 uses $v_1^* = c_1$ as his cutoff. In the area between $c_1 = c_2$ and the parabola, we have $c_1 < c_2 \leq \frac{1}{2} + \frac{1}{2}c_1^2$. In this case, there is a unique monotonic equilibrium with $v_1^* \leq 1$ and $v_2^* \leq 1$ that can be solved explicitly.

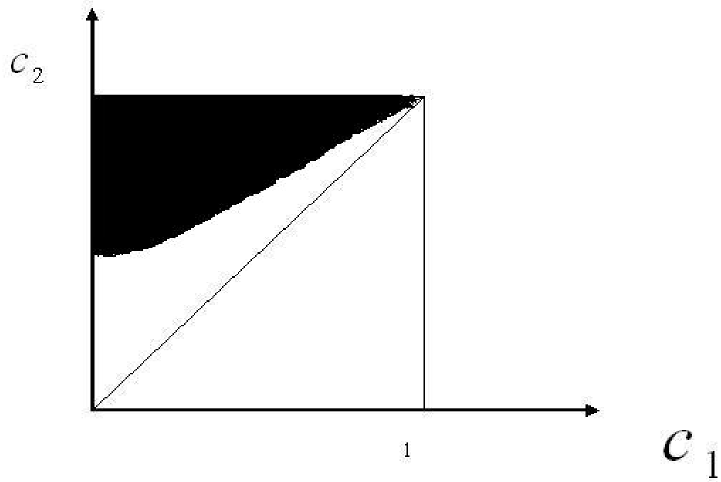

Figure 3: An Example of Uniform Case

Using (1) and (3) under the uniform distribution, we have $v_1^* = \frac{1}{2}(\sqrt{2(c_1+c_2)} - \sqrt{2(c_2-c_1)})$ and $v_2^* = \frac{1}{2}(\sqrt{2(c_1+c_2)} + \sqrt{2(c_2-c_1)})$. Since $\frac{\partial v_1^*}{\partial c_1} = (2\sqrt{c_1+c_2})^{-1} + (2\sqrt{c_2-c_1})^{-1} > 0$ and $\frac{\partial v_2^*}{\partial c_2} = (2\sqrt{c_1+c_2})^{-1} + (2\sqrt{c_2+c_1})^{-1} > 0$, the equilibrium cutoffs are increasing functions of their own participation costs; the higher a bidder's own participation cost is, the less likely he will participate in the auction and submit the bid. It can be checked that $\frac{\partial v_1^*}{\partial c_2} = (2\sqrt{c_1+c_2})^{-1} - (2\sqrt{c_2-c_1})^{-1} < 0$ and $\frac{\partial v_2^*}{\partial c_1} = (2\sqrt{c_1+c_2})^{-1} - (2\sqrt{c_2-c_1})^{-1} < 0$. The cutoff of each bidder is a decreasing function of the other's participation cost.

4 Extensions

The discussions in the previous section can be easily extended to the case in which we have two types of bidders and bidders in the same type use the same cutoffs. The results are similar to those of two bidders for the existence and uniqueness as well as the limiting properties of the equilibria. Beyond that we find when the number of both types of bidders increases, the cutoffs for both types of bidders increase. The intuition is that more bidders in the auction will increase the competition among the bidders, and this will reduce the expected payoff to each bidder. Thus, bidders will be less likely to participate in the auction, and their value cutoffs will increase. Detailed analysis and related proofs can be found in supplemental material and obtained from the authors upon request. In the following we make some other extensions.

4.1 Type Asymmetric Equilibria

In this subsection we give a brief discussion on allowing asymmetric cutoffs within a group. To allow such a possibility, we consider the simplest economy with three bidders in the two groups, assuming $c_1 = c_2 < c_3$ and $F_1(.) = F_2(.) = F_3(.) = F(.)$. Let v_1^* and v_2^* be the corresponding cutoffs for the two bidders in type 1 and v_3^* be the cutoff for type 2 bidder. Assume $v_1^* < v_2^*$. There are three cases to be considered.

Case 1: $v_1^* < v_2^* < v_3^*$. Then we have

$$
\begin{aligned}
c_1 &= v_1^* F(v_2^*) F(v_3^*), \\
c_2 &\geq v_2^* F(v_1^*) F(v_3^*) + F(v_3^*) \int_{v_1^*}^{v_2^*} (v_2^* - v) dF(v), \\
c_3 &\geq v_3^* F(v_1^*) F(v_2^*) + F(v_2^*) \int_{v_1^*}^{v_2^*} (v_3^* - v) dF(v) + \int_{v_2^*}^{v_3^*} (v_3^* - v) dF(v)^2.
\end{aligned}
$$

The above equations hold with equality whenever $v_i^* \leq 1$. On the right side of the third equation, the first term is the revenue bidder 3 receives when the other two bidders do not participate in the auction. The second term is the revenue he receives when the highest bid of the other two is less than v_2^*, which happens when bidder 2 does not participate in the auction. The third term is the revenue when the others' highest bid is greater than v_2^* and less than v_3^*.

When $F(.)$ is concave, we cannot have such an equilibrium. To see this, from the first two equations, we have $v_1^* F(v_2^*) F(v_3^*) > v_2^* F(v_1^*) F(v_3^*)$; i.e., we have $\frac{F(v_2^*)}{v_2^*} > \frac{F(v_1^*)}{v_1^*}$ with $v_2^* > v_1^*$, which cannot be true when $F(.)$ is concave.

When $F(.)$ is strictly convex, from the first two equations, we treat v_3^* as a constant. Then it seems as if bidder 1 and bidder 2 possess participation costs $\frac{c_1}{F(v_3^*)}$. We know there is an equilibrium in which $v_1^* < v_2^*$ and the equilibrium is a function of v_3^*. Inserting into the third equation, we can get v_3^*. In particular, when $v_3^* > 1$, bidder 3 never participates in the auction.

Case 2: $v_1^* < v_3^* < v_2^*$. Then we have

$$\begin{aligned} c_1 &= v_1^* F(v_2^*) F(v_3^*), \\ c_3 &\geq v_3^* F(v_1^*) F(v_2^*) + F(v_2^*) \int_{v_1^*}^{v_3^*} (v_3^* - v) dF(v), \\ c_2 &\geq v_2^* F(v_1^*) F(v_3^*) + F(v_3^*) \int_{v_1^*}^{v_3^*} (v_2^* - v) dF(v) + \int_{v_3^*}^{v_2^*} (v_2^* - v) dF(v)^2. \end{aligned}$$

When $F(.)$ is concave, from the first and third equations above, we have $v_1^* F(v_2^*) F(v_3^*) > v_2^* F(v_1^*) F(v_3^*)$, which again cannot be true for $v_1^* < v_2^*$. So $v_1^* = v_3^*$. The problem can be reduced to the type-symmetric equilibrium. When $F(.)$ is strictly convex, we can treat v_1^* in the second and third equations as a constant. From the discussion in Section 3, we know that when $c_3 - c_1$ is sufficiently small, there exists an equilibrium in which $v_2^* < v_3^*$. A limiting case is when $c_3 = c_1$. As Tan and Yilankaya (2006) pointed out, when $F(.)$ is strictly convex but not log-concave, there may exist equilibria with three or more cutoffs.

Case 3: $v_3^* < v_1^* < v_2^*$. The discussion for this is similar to that for Case 2.

Summarizing our discussion above and the results we obtain in Section 3, we have the following proposition:

Proposition 4 *For the independent private values economy with two groups and three bidders, when $F(.)$ is concave, we only have the unique type-symmetric monotonic equilibrium. When $F(.)$ is strictly convex, type-asymmetric equilibria exist.*

Remark 5 From the discussions above, it can be easily checked that when there are n bidders with potentially different costs, a sufficient condition for the uniqueness of equilibrium (necessarily monotone) is that $F(.)$ is concave.

4.2 Bidders with Different Valuation Distributions

We consider an economy where bidders have different valuation distributions $F_1(.)$ and $F_2(.)$. Here, we allow both valuation distribution functions and participation costs of bidders to be different. Again, we assume $c_1 < c_2$ and use x and y to denote the cutoffs used by bidders 1 and 2, respectively. We investigate the existence of equilibria and equilibrium behavior.

To find a monotonic equilibrium, consider the following two equations:

$$c_1 = x F_2(y)$$

$$c_2 \geq x F_1(x) + \int_x^y F_1(v) dv.$$

Again, the first equation implicitly defines x as a decreasing function of y, denoted by $x(y)$. We then have $\frac{dx}{dy} = -\frac{xf_2(y)}{F_2(y)}$. We know $x(y)$ has a fixed point $v_1^s \neq 0$ determined by $c_1 = v_1^s F_2(v_1^s)$. Since $x(y)$ is monotonically decreasing, we have $x < v_1^s$ and $y > v_1^s$.

Inserting $x(y)$ into the second equation and letting $\lambda(y) = xF_1(x) + \int_x^y F_1(v)dv$ with $x < y$, we have

$$\lambda'(y) = F_1(y) + xf_1(x)\frac{dx}{dy} = \frac{F_1(y)F_2(y) - x^2 f_1(x)f_2(y)}{F_2(y)}.$$

When $F_1(.)$ and $F_2(.)$ are both concave, we have

$$\lambda'(y) > \frac{F_1(y)F_2(y) - xyf_1(x)f_2(y)}{F_2(y)} > \frac{F_1(y)F_2(y) - F_1(x)F_2(y)}{F_2(y)} > 0,$$

which indicates that $\lambda(y)$ is a monotonically increasing function.

For the existence of a non-monotonic equilibrium, consider the following two equations:

$$c_2 = yF_1(x)$$

$$c_1 \geq yF_2(y) + \int_y^x F_2(v)dv.$$

From the first equation we have $y = \frac{c_2}{F_1(x)}$. Inserting it into the right side of the second equation and letting $\phi(x) = yF_2(y) + \int_y^x F_2(v)dv$ with $x \geq y$, by the same reasoning as before, we have $\phi'(x) > 0$ when both $F_1(.)$ and $F_2(.)$ are concave. $y = \frac{c_2}{F_1(x)}$ has a fixed point v_2^s determined by $c_2 = v_2^s F_1(v_2^s)$. Since $x(y)$ is monotonically decreasing, we have $y < v_2^s$ and $x > v_2^s$.

We then have the following proposition:

Proposition 5 (Existence and Uniqueness Theorem) For a two-bidder economy with different continuously differentiable distribution functions $F_1(v)$ and $F_2(v)$ and different costs $c_1 < c_2$, we have the following results:

(1) There always exists an equilibrium (v_1^*, v_2^*).

(2) Suppose $F_1(.)$ and $F_2(.)$ are both concave and $F_1(v) < F_2(v)$ for all $v \in (0, 1)$. Then there exists a unique equilibrium and it is monotonic.

(3) Suppose $F_1(.)$ and $F_2(.)$ are both concave and $F_1(v) > F_2(v)$ for all $v \in (0, 1)$. Let v_1^s and v_2^s satisfy $c_1 = v_1^s F_2(v_1^s)$ and $c_2 = v_2^s F_1(v_2^s)$, respectively. Then, we have

i) If $v_1^s < v_2^s$, there is a unique equilibrium and it is monotonic;

ii) If $v_1^s > v_2^s$, there is a unique equilibrium and it is non-monotonic, satisfying $v_1^* > v_2^*$;

iii) If $v_1^s = v_2^s = v^s$, there is a unique equilibrium and it is a special non-monotonic equilibrium, satisfying $v_1^* = v_2^* = v^s$.

Remark 6 Here we give some remarks on Proposition 5:

(1) Tan and Yilankaya (2006) showed that when the "weak" bidders' distribution is concave, there cannot exist an equilibrium in which the "weak" bidders choose a low cutoff. However our results show that depending on the magnitude of participation costs, the bidder who is "strong" in valuation distribution may choose a cutoff that is lower than, or higher than, or even equal to that of the bidder who is "weak" in valuation distribution, which is more general. Their conclusion is consistent with ours and can be treated as a special case of ours. To see this, note that when $c_1 = c_2$, the only possible case is $v_1^s > v_2^s$ since $F_1(v) > F_2(v)$ for all $v \in (0, 1)$, which fits (3ii) of the proposition above.

(2) $F_1(v) < F_2(v)$ for all $v \in [0, 1]$ means that bidder 1 is a bidder strong in valuation distribution in the sense that there is a high probability that his valuation is higher than bidder 2's valuation. A higher valuation together with a smaller participation cost makes bidder 1 choose a lower cutoff. However, when $F_1(v) > F_2(v)$ for all $v \in (0, 1)$, bidders with higher participation costs may have lower or identical cutoffs even though their participation costs are higher. Now bidder 2 has an advantage in the value and a disadvantage in the participation cost. When the advantage is dominant, bidder 2 has a lower cutoff, rather than a higher one, resulting in the nonexistence of a monotonic equilibrium. We can interpret this in another way. $F_1(v) > F_2(v)$ implies that $F_1(.)$ is more concave than $F_2(.)$ and bidder 1 is more risk-averse than bidder 2. This reduces his entrance probability by leading him to choose a higher cutoff.

(3) Here we see another manifestation of the result that the differences in valuations have the similar role to differences in bidding costs (Baye, Kovenock and Vries (1996)). The advantage of a bidder's valuation distribution can be weakened by his participation cost.

(4) Unlike the results obtained in Section 3, (3.iii) shows that when bidders' distribution functions are different, $v_1^* = v_2^*$ can be an equilibrium although bidders' participation costs are different. That is, we have a special non-monotonic equilibrium with $v_1^* = v_2^*$ even when $c_1 < c_2$. When bidders' distribution functions are the same, as in Section 3, this is impossible.

Thus, when bidders have different distributions of valuations, some of the previous results no longer hold true. The distributions of valuations have substantial effects on types of equilibria.

4.3 Positive Lower Bound of Supports

The support of valuations affects the existence of equilibria. When the lower bound of the support of the valuation is not zero, there may be an equilibrium in which one bidder always participates in the auction and the other never participates in the auction.

Suppose the support of the distribution function $F(.)$ is $[v_l, v_h]$. There are six cases for consideration.

Case 1. $v_l < v_h < c_1 < c_2$. It is clear that both bidders never participate in the auction.

Case 2. $v_l < c_1 < v_h < c_2$. Bidder 2 never participates in the auction. Bidder 1 participates in the auction if $v_1 \geq c_1$ and does not participate otherwise.

Case 3. $c_1 < v_l < v_h < c_2$. Bidder 2 never participates, and bidder 1 always participates.

Case 4. $v_l < c_1 < c_2 < v_h$. The analysis and results are the same as those in Section 3 that deal with the special case where $v_l = 0$ and $v_h = 1$.

Case 5. $c_1 < v_l < c_2 < v_h$. We may have an equilibrium in which bidder 1 always participates, and bidder 2 never participates. For this to be true, we need $v_h - v_l < c_2$; that is, the maximum revenue bidder 2 gets from participating in the auction must be smaller than his participation cost. When $c_2 \leq v_h - v_l$, bidder 2 will choose a cutoff $v_2^* \in [c_2, v_h]$. If there is an equilibrium in which bidder 1 never participates, then bidder 2 uses $v_2^* = c_2$. To have such an equilibrium, we need

$$v_h F(c_2) + \int_{c_2}^{v_h} (v_h - v) dF(v) = c_2 F(c_2) + \int_{c_2}^{v_h} F(v) dv < c_1.$$

A sufficient condition for this is $v_h + c_2 F(c_2) < c_1 + c_2$.

Case 6. $c_1 < c_2 < v_l < v_h$. It is possible to have an equilibrium in which bidder 1 always participates in the auction, and bidder 2 never participates. For this to be an equilibrium, we need $v_h - v_l < c_2$. Another possible equilibrium is bidder 2 always participates in the auction, and bidder 1 never participates. For this to be an equilibrium, we need $v_h - v_l < c_1$. When both bidders choose a cutoff inside the support of valuations, we can use the same analysis as in Section 3 to investigate the equilibrium and the corresponding properties.

5 Conclusion

This paper investigates equilibria of second price auctions when bidders have private valuations and different participation costs that are common knowledge. We identify two types of equilibria:

monotonic and non-monotonic equilibria. We show that there always exists an equilibrium that is monotonic, and further that, it is unique when $F(.)$ is concave or strictly convex with non-increasing reverse hazard rate.

For the non-monotonic equilibria, we show that when the distribution function of valuation is strictly convex and the difference of participation costs is sufficiently small, there exist non-monotonic equilibria. One implication is that in the land leasehold auctions, if the favored bidder gets sufficient help from the land bureau officials, he is more likely to participate in the auction for sure. We also show that when the difference in participation costs goes to zero, the monotonic equilibrium of concave valuation distribution converges to the symmetric equilibrium, while the monotonic equilibrium of convex valuation distributions converges to an asymmetric equilibrium.

We provide some comparative static analysis. We show that the cutoff is increasing in one's own participation cost but decreasing in the opponents' participation costs. As the number of bidders increases, the cutoffs for all bidders increase. This is consistent with the idea that more potential buyers will increase competition among bidders and thus reduce the expected payoff of each bidder, with the natural consequence of reduced bidder's participation.

We consider some extensions of our basic model. For example, when bidders are allowed to have different valuation distributions, some results for the basic model no longer hold. We also extend the basic model to the one with a positive lower bound of the support. In this case, we may have an equilibrium in which some bidders always participate and some never participate.

Appendix: Proofs

Proof of Lemma 1:

Suppose not. All bidders never participate in the auction (i.e., $v_i^* > 1$ for all bidders i). When bidder n knows the other $n-1$ bidders will not participate in the auction regardless of their valuations, bidder n participates in the auction when his value is greater than or equal to his participation cost. Then we have $v_n^* = c_n \leq 1$, a contradiction.

Proof of Lemma 2:

First note that $v_1^s < v_2^s$ by the monotonicity of $vF(v)$. When bidder 2 chooses never to participate, $v_1^* = c_1 < v_1^s$ and $v_2^* > 1$. Lemma 2 holds obviously.

Now suppose $v_2^* \leq 1$. We have

$$v_1^* F(v_1^*) + \int_{v_1^*}^{v_2^*} F(v)dv = c_2 = v_2^s F(v_2^s).$$

Since $v_1^* F(v_1^*) + \int_{v_1^*}^{v_2^*} F(v)dv = v_2^* F(v_2^*) - \int_{v_1^*}^{v_2^*} vf(v)dv$, we have

$$v_2^* F(v_2^*) - \int_{v_1^*}^{v_2^*} vf(v)dv = v_2^s F(v_2^s).$$

Then $v_2^s F(v_2^s) < v_2^* F(v_2^*)$. We must have $v_2^s < v_2^*$ by the monotonicity of $vf(v)$. Also, since we have $v_2^* > v_1^s$ and $c_1 = v_1^* F(v_2^*) = v_1^s F(v_1^s)$, for this equation to be true, we must have $v_1^* < v_1^s$. Otherwise we have $v_1^* F(v_2^*) > v_1^s F(v_1^s)$, a contradiction. So $v_1^* < v_1^s$. Thus, we prove $v_2^* > v_2^s > v_1^s > v_1^*$.

Proof of Proposition 1:

The proof of Proposition 1 is based on the following five lemmas (from Lemma 3 to Lemma 7).

Lemma 3 *For the economic environment with two bidders, there always exists an equilibrium that is monotonic; i.e., for $c_2 > c_1$, there exists a cutoff vector (v_1^*, v_2^*) such that $v_2^* > v_1^*$.*

Proof. When $c_1 F(c_1) + \int_{c_1}^1 F(v)dv < c_2$, bidder 2 will never participate in the auction and thus $v_1^* = c_1$ and $v_2^* > 1$ constitute a monotonic equilibrium. Now we consider the case of $c_1 F(c_1) + \int_{c_1}^1 F(v)dv \geq c_2$.

Given v_1^s determined by $c_1 = v_1^s F(v_1^s)$, we have $x < v_1^s$ and $y > v_1^s$ by noting that $y = y(x)$ is a decreasing function. Since $h(c_1) = c_1 F(c_1) + \int_{c_1}^1 F(v)dv - c_2 \geq 0$ and $h(v_1^s) = c_1 - c_2 < 0$, there exists a $v_1^* \in [c_1, v_1^s)$ such that $h(v_1^*) = 0$. Thus, $v_1^* < v_1^s$ and $v_2^* = y(v_1^*) > v_1^s$ constitute a monotonic equilibrium. ■

Lemma 4 *If $F(.)$ is concave, there is a unique monotonic equilibrium.*

Proof. Since $F(.)$ is concave, we have $F(v) \geq vF'(v) = vf(v)$ for any $v \in [0,1]$, and by noting $y > x$, we have

$$\lambda'(y) = F(y) - \frac{x^2}{F(y)}f(y)f(x) > F(y) - \frac{F(x)xf(y)}{F(y)} > F(y) - \frac{F(x)yf(y)}{F(y)} \geq F(y) - F(x) > 0,$$

which indicates that $\lambda(y)$ is monotonically increasing. First consider the case where $\lambda(1) = c_1F(c_1) + \int_{c_1}^{1} F(v)dv \geq c_2$. Since $\lambda(v_1^s) - c_2 = c_1 - c_2 < 0$, then, by the monotonicity and continuity of λ and $x(y)$, $y = v_2^* \in (v_1^s, 1]$ is uniquely determined by $\lambda(y) - c_2 = 0$, as is $x = v_1^* < v_1^s$. Thus, the monotonic equilibrium is unique. Now suppose $\lambda(1) < c_2$. Then bidder 2 will never participate in the auction; thus $x = v_1^* = c_1$ and $v_2^* > 1$ will again be the unique monotonic equilibrium. ■

Lemma 5 *If $F(.)$ is concave, there is no non-monotonic equilibrium, and thus the equilibrium is unique and monotonic.*

Proof. We first prove there is no non-monotonic equilibrium in which $v_1^* > 1$. To see this, notice that $v_1^* > 1$ requires $c_1 > c_2F(c_2) + \int_{c_2}^{1} F(v)dv$. However, when $F(.)$ is concave, we have

$$c_1 > c_2F(c_2) + \int_{c_2}^{1} F(v)dv \geq c_2F(c_2) + (1-c_2)F(c_2) = F(c_2) \geq c_2$$

by noting that $F(c_2) \geq c_2$ since $F(c) = F(c \times 1 + (1-c)0) \geq cF(1) + (1-c)F(0) = c$. This contradicts the fact that $c_1 < c_2$.

We now show that there does not exist any non-monotonic equilibrium with $v_1^* \leq 1$ either. Suppose not. Then we have

$$c_1 = v_2^*F(v_2^*) + \int_{v_2^*}^{v_1^*} F(v)dv \geq v_1^*F(v_2^*)$$

by noting that $F(.)$ is non-decreasing. $c_1 < c_2$ implies that $v_2^*F(v_1^*) > v_1^*F(v_2^*)$. Thus we have $v_2^* < v_1^*$ and $\frac{F(v_1^*)}{v_1^*} > \frac{F(v_2^*)}{v_2^*}$, which contradicts the fact that $\frac{F(v)}{v}$ is a non-increasing function when $F(.)$ is a concave function. Thus, there does not exist any non-monotonic equilibrium in either case. Consequently, by Lemma 4, the equilibrium is unique, which is monotonic. ■

Lemma 6 *Suppose $F(.)$ is strictly convex and the reverse hazard rate of $F(.)$ is non-increasing. Then, there is a unique monotonic equilibrium.*

Proof. Notice that $\lambda'(y)$ can be written as

$$\lambda'(y) = F(y) - \frac{x^2}{F(y)}f(y)f(x) = F(y)[1 - \frac{x^2f(x)f(y)}{F(y)^2}].$$

$\frac{x^2 f(x) f(y)}{F(y)^2}$ is a decreasing function in y. This is true since $f(.)$ is an increasing function by the strict convexity of $F(.)$ with $x = \frac{c_1}{F(y)}$ and the reverse hazard rate of $F(.)$ defined by $\frac{f(.)}{F(.)}$ is non-increasing. Then $1 - \frac{x^2 f(x) f(y)}{F(y)^2}$ is an increasing function in y, so is $\lambda'(y)$. Thus, there is at most one $y = y_0$, if any, satisfying $\lambda'(y_0) = 0$. Notice that, when $x = y = v_1^s$,

$$\lambda'(v_1^s) = F(v_1^s) - \frac{v_1^{s2}}{F(v_1^s)} f(v_1^s) f(v_1^s) < 0$$

by the strict convexity of $F(.)$. Then $\lambda(y)$ either decreases over the entire interval $[v_1^s, 1]$ (in this case $y_0 > 1$) or first decreases over $[v_1^s, y_0]$ and then increases over $[y_0, 1]$ if $y_0 \leq 1$. If $\lambda(y)$ decreases over the entire interval $[v_1^s, 1]$, then $\lambda(y) < c_2$ for all $y \in [v_1^s, 1]$, which means bidder 2 never participates in the auction. Thus we have a unique monotonic equilibrium with $v_1^* = c_1$ and $v_2^* > 1$. On the other hand, if $y_0 \leq 1$, $\lambda(y)$ first decreases over $[v_1^s, y_0]$ and then increases over $[y_0, 1]$. Thus $\lambda(y) = c_2 > c_1$ has at most one solution v_2^*. If the solution exists, we have a unique monotonic equilibrium with $v_1^* \leq 1$ and $v_2^* \leq 1$; otherwise the unique monotonic equilibrium is given by $v_1^* = c_1$ and $v_2^* > 1$. ■

Lemma 7 *Suppose $F(.)$ is strictly convex. There exists a non-monotonic equilibrium when $c_1 = c_m$ and at least two non-monotonic equilibria when $c_1 > c_m$. There is no non-monotonic equilibrium when $c_1 < c_m$.*

Proof. Since

$$\phi'(x) = F(x) + y(x) f(y(x)) y'(x)$$

and

$$y'(x) = -\frac{y f(x)}{F(x)},$$

we have

$$\phi'(v_2^s) = F(v_2^s) - v_2^s f(v_2^s) \frac{v_2^s f(v_2^s)}{F(v_2^s)} = \frac{F^2(v_2^s) - (v_2^s f(v_2^s))^2}{F(v_2^s)} < 0$$

by noting that $v_2^s f(v_2^s) > F(v_2^s)$ by $F(v) < v f(v)$ for all $v \in [c_2, 1]$ and $v_2^s \geq c_2$, which indicates that $\phi(x)$ is decreasing at $x = x_2^s$. Then $\phi(x)$ has a minimum value $c_m < c_2$ in the interval $[v_2^s, 1]$ since $\phi(v_2^s) = c_2$. Let $\phi(x_m) = c_m$.

When $c_1 < c_m$, we have $\phi(x) > c_1$ in the interval $[v_2^s, 1]$. Thus, there is no non-monotonic equilibrium with $v_1^* \leq 1$ since the set $\{x | \phi(x) = c_1, v_2^s \leq x \leq 1\}$ is empty. On the other hand, since $\phi(1) = c_2 F(c_2) + \int_{c_2}^{1} F(v) dv \geq c_m > c_1$, we do not have a non-monotonic equilibrium at which bidder 1 never participates so that $v_1^* > 1$ is not an equilibrium strategy for bidder 1.

When $c_1 = c_m$, since $\phi(x_m) = c_m$, then $x = x_m, y = c_2 / F(x_m)$ is the unique non-monotonic equilibrium. Note that when $c_1 = c_m$, we do not have an equilibrium at which bidder 1 never participates since $\phi(1) \geq c_m = c_1$.

When $c_m < c_1 < c_2$, we have at least two non-monotonic equilibria. To see this, first notice that there exists an $x_1 \in (v_2^s, x_m)$ such that $\phi(x_1) = c_1$ by the continuity of $\phi(x)$ and $\phi(x_m) = c_m < c_1$, $\phi(v_2^s) = c_2 > c_1$. If $\phi(1) < c_1$, we have a non-monotonic equilibrium at which bidder 1 never participates and bidder 2's equilibrium strategy is $v_2^* = c_2$. Otherwise if we have $\phi(1) \geq c_1$, we can find an $x_2 \in (x_m, 1]$ such that $\phi(x_2) = c_1$ by the continuity of $\phi(x)$ on $x_2 \in (x_m, 1]$, $\phi(1) > c_1$ and $\phi(x_m) = c_m < c_1$. Then $(x_1, c_2/F(x_1))$ and $(x_2, c_2/F(x_2))$ will be two non-monotonic equilibria. ■

Proof of Proposition 2:

2.(1) When $F(.)$ is concave, the monotonic equilibrium and symmetric equilibrium are both unique, so we have the result.

2.(2) From the proof of Lemma 6, we know that, when $F(.)$ is strictly convex with non-increasing reverse hazard rate, there is at most one y_0 such that $\lambda'(y_0) = 0$; $\lambda(y)$ either decreases over the entire interval $[v_1^s, 1]$ or first decreases over $[v_1^s, y_0]$ and then increases over $[y_0, 1]$ if $y_0 \leq 1$. Thus, there is a unique monotonic equilibrium, which is either given by $v_1^* = c_1$ and $v_2^* > 1$ when $\lambda(y)$ and c_2 have no intersection, or given by (v_1^*, v_2^*) with $v_1^* < v_1^s < y_0 < v_2^* \leq 1$ when $\lambda(y)$ and c_2 have an intersection. Here v_2^* is determined by $\lambda(v_2^*) = c_2$ and $v_1^* = c_1/F(v_2^*)$. Thus, from Figure 1, one can see that, when $c_2 \to c_1$, we have an equilibrium given by an asymmetric equilibrium $(v_1^{*\prime}, v_2^{*\prime})$ with $v_1^{*\prime} < v_1^s < y_0 < v_2^{*\prime} \leq 1$, where $v_2^{*\prime}$ is determined by $\lambda(v_2^{*\prime}) = c_1$ and $v_1^{*\prime} = c_1/F(v_2^{*\prime})$, so the unique monotonic equilibrium converges to an asymmetric equilibrium.

2.(3) When $F(.)$ is strictly convex and $c_2 - c_1$ is sufficiently small, there are two non-monotonic equilibria (x_1, y_1) and (x_2, y_2) with $y_1 = y(x_1)$, $y_2 = y(x_2)$, and $y_1 < y_2 < v_2^s < x_1 < x_m < x_2 < x_0$ as shown in Lemma 7. Thus, from Figure 2, as $c_1 \to c_2$, the non-monotonic equilibrium (x_1, y_1) converges to the symmetric equilibrium (v_2^s, v_2^s), and the other non-monotonic equilibrium (x_2, y_2) converges to the asymmetric equilibrium (x_0, y_0).

Proof of Proposition 3:

Suppose we have a monotonic equilibrium (v_1^*, v_2^*) for the costs (c_1, c_2). Now choose (c_1', c_2') satisfying $c_1' \geq c_1$ and $c_2' \leq c_2$. Note that bidder 1's best response when bidder 2's cutoff is in $[c_2, v_2^*]$ must lie in $[v_1^*, 1]$ since c_1 has weakly increased and bidder 2's best response when bidder 1's cutoff is in $[v_1^*, 1]$ must lie in $[c_2, v_2^*]$ since c_2 has weakly decreased. Thus for costs (c_1', c_2'), the sets $[v_1^*, 1]$ for bidder 1 and $[c_2, v_2^*]$ for bidder 2 are closed under best response. So there must be an equilibrium in which each bidder uses a cutoff from his specified set. That is, when one bidder's cost increases and the other's decreases, there is necessarily a new equilibrium in which the former uses a greater cutoff and the latter uses a smaller cutoff. Thus an increase in participation cost c_i increases i's cutoff v_i^* but decreases the opponent's cutoff v_j^* for $j \neq i$.

Proof of Proposition 5:

5.(1) Suppose by contradiction that there does not exist any type of equilibrium. We then have no monotonic equilibrium. Thus, $\lambda(y) > c_2 = v_2^s F_1(v_2^s)$ for all $y \in [v_1^s, 1]$, and particularly, $\lambda(v_1^s) = v_1^s F_1(v_1^s) > v_2^s F_1(v_2^s)$. Then we have $\frac{v_1^s}{v_2^s} > \frac{F_1(v_2^s)}{F_1(v_1^s)}$. Since there is no non-monotonic equilibrium either, we have $\phi(x) > c_1 = v_1^s F_2(v_1^s)$ for all $x \in [v_2^s, 1]$, and particularly, $\phi(v_2^s) = v_2^s F_2(v_2^s) > v_1^s F_2(v_2^s)$. Then $\frac{v_1^s}{v_2^s} < \frac{F_2(v_2^s)}{F_2(v_1^s)}$. Combining these two cases, we have

$$\frac{F_1(v_2^s)}{F_1(v_1^s)} < \frac{v_1^s}{v_2^s} < \frac{F_2(v_2^s)}{F_2(v_1^s)}.$$

Now we prove that these two inequalities cannot hold simultaneously. Indeed, if $v_1^s \leq v_2^s$, then $1 \leq \frac{F_1(v_2^s)}{F_1(v_1^s)} < \frac{v_1^s}{v_2^s} \leq 1$, which is impossible. On the other hand, if $v_1^s > v_2^s$, $1 < \frac{v_1^s}{v_2^s} < \frac{F_2(v_2^s)}{F_2(v_1^s)} < 1$, which is also impossible. Thus, there must exist an equilibrium for any $F_1(v)$ and $F_2(v)$ under consideration.

5.(2) First note that $\lambda(v_1^s) = v_1^s F_1(v_1^s) < v_1^s F_2(v_2^s) = c_1 < c_2$ by $F_1(v) < F_2(v)$ and $\lambda(y)$ is monotonically increasing by the concavity of $F_1(.)$ and $F_2(.)$. Thus, if $\lambda(y) < c_2$ for all $y \in (v_1^s, 1]$, bidder 2 will never participate (i.e., $v_2^* > 1$), and bidder 1 uses $v_1^* = c_1$ as the cutoff. Otherwise bidder 2 will use $v_2^* > v_1^s$ which is determined by $\lambda(y) = c_2$. Thus, in both cases, (v_1^*, v_2^*) constitute a monotonic equilibrium. Since $\lambda(y)$ is monotonically increasing, such a monotonic equilibrium must be unique.

Finally, we show there does not exist any non-monotonic equilibrium. To do so, we only need to focus on $\phi(x)$ with $x > v_2^s$. Since $\phi(v_2^s) = v_2^s F_2(v_2^s) > v_2^s F_1(v_1^s) = c_1$ and $\phi(x)$ is monotonically increasing, $\phi(x) > c_1$ for all $x \in (v_2^s, 1]$. Thus we do not have a non-monotonic equilibrium. Hence, there is a unique equilibrium and it is monotonic.

5.(3.i) Suppose $v_1^s < v_2^s$. We have $\phi(v_2^s) = v_2^s F_2(v_2^s) > v_1^s F_2(v_1^s) = c_1$. By $\phi'(x) > 0$ we have $\phi(x) > c_1$ for all $x \in (v_2^s, 1]$. Thus no non-monotonic equilibrium exists. We have $\lambda(v_1^s) = v_1^s F_1(v_1^s) < v_2^s F_1(v_2^s) = c_2$. Then by the monotonicity of $\lambda(y)$, there is a unique equilibrium and it is monotonic.

5.(3.ii) Suppose $v_1^s > v_2^s$. We have $\lambda(v_1^s) = v_1^s F_1(v_1^s) > v_1^s F_2(v_1^s) > v_2^s F_2(v_2^s) = c_2$. By $\lambda'(y) > 0$ we have $\lambda(y) > c_2$ for all $y \in (v_1^s, 1]$, so no monotonic equilibrium exists. On the other hand, we have $\phi(v_2^s) = v_2^s F_2(v_2^s) < v_1^s F_2(v_1^s) = c_1$. By $\phi'(x) > 0$, if for all $x \in (v_2^s, 1]$, we have $\phi(x) < c_1$, bidder 1 never participates in the auction (i.e., $v_1^* > 1$). Thus, $v_1^* > 1$ and $v_2^* = c_2$

will be the unique non-monotonic equilibrium. Otherwise $v_2^* > v_2^s$ is uniquely determined by $\phi(x) = c_1$. Then $v_1^* < v_2^s$ and $v_2^* > v_2^s$ is the unique non-monotonic equilibrium. Thus we have a unique equilibrium and it is non-monotonic.

5.(3.iii) Now suppose $v_1^s = v_2^s = v^s$. We then have $c_1 = v^s F_2(v^s)$ and $c_2 = v^s F_1(v^s)$. Then $\lambda(v^s) = v^s F_1(v^s) = c_2$ and $\phi(v^s) = v^s F_2(v^s) = c_1$. Thus $v_1^* = v_2^* = v^s$ is the equilibrium that is a special non-monotonic equilibrium. The uniqueness comes from the monotonicity of $\lambda(y)$ and $\phi(x)$.

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