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Barnett, William A. and Serletis, Apostolos and Serletis, Demitre

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William A. Barnett
Department of Economics
University of Kansas
Lawrence, Kansas 66045-7585
U.S.A.

Apostolos Serletis*
Department of Economics
University of Calgary
Calgary, Alberta T2N 1N4
Canada
and

Demitre Serletis
Neurological Institute, Epilepsy Center
Cleveland Clinic
9500 Euclid Avenue
Cleveland, Ohio 44195
U.S.A.

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*Corresponding author. Phone: (403) 220-4092; Fax: (403) 282-5262; E-mail: Serletis@ucalgary.ca; Web: http://econ.ucalgary.ca/serletis.htm

Abstract:

This paper is an up-to-date survey of the state-of-the-art in dynamical systems theory relevant to high levels of dynamical complexity, characterizing chaos and near chaos, as commonly found in the physical sciences. The paper also surveys applications in economics and finance. This survey does not include bifurcation analyses at lower levels of dynamical complexity, such as Hopf and transcritical bifurcations, which arise closer to the stable region of the parameter space. We discuss the geometric approach (based on the theory of differential/difference equations) to dynamical systems and make the basic notions of complexity, chaos, and other related concepts precise, having in mind their (actual or potential) applications to economically motivated questions. We also introduce specific applications in microeconomics, macroeconomics, and finance, and discuss the policy relevancy of chaos.

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1 Introduction

According to an unsophisticated but perhaps still prevailing view, the output of deterministic dynamical systems can in principle be predicted exactly and, assuming that the model representing the real system is correct, errors in prediction will be of the same order of errors in observation and measurement of the variables. On the contrary, random processes describe systems of irreducible complexity owing to the presence of an indefinitely large number of degrees of freedom, whose behavior can only be predicted in probabilistic terms.

This simplifying view was completely upset by the discovery of chaos, i.e., deterministic systems with stochastic behavior. It is now well known that perfectly deterministic systems (i.e., systems with no stochastic components) of low dimensions (i.e., with a small number of state variables) and with simple nonlinearities (i.e., a single quadratic function) can have stochastic behavior. The discovery that such systems exist and are indeed ubiquitous has brought about a profound re-consideration of the issue of randomness.

Besides its obvious intellectual appeal, chaos is interesting in economics and finance, because of its ability to generate output that mimics the output of stochastic systems, thereby offering an alternative explanation for business cycles. Moreover, the possible existence of chaos could be exploitable and even invaluable. If, for example, chaos can be shown to exist in asset prices, the implication would be that profitable, nonlinearity-based trading rules exist (at least in the short run and provided the actual generating mechanism is known). Prediction, however, over long periods is all but impossible, due to the sensitive dependence on initial conditions property of chaos.

In the following sections, we will be discussing dynamical systems mainly from a geometric (or topological) point of view. This approach, being intuitively appealing and lending itself to suggestive graphical representations, has been tremendously successful in the study of low-dimensional systems, such as, for example, (discrete- and continuous-time) systems with one and perhaps two variables. For higher-dimensional systems, however, the ergodic approach, based on the axiomatic formulation of probability theory and aimed at the investigation of statistical properties of orbits, is more appealing. See Barnett *et al.* (1997) for a discussion of that approach.

While this paper does not survey bifurcation regions producing low levels of dynamical complexity, such as Hopf and transcritical bifurcation, those forms of bifurcation, close to the stable region, have been shown to be very common in even the most elementary dynamical macroeconomic models, including linear models. See Barnett and Duzhak (2007, 2010) and Barnett, Banerjee, Duzhak, and Gopalan (2011). With the growing importance of Euler equations models, having no analytical closed form solutions, intermediate levels of dynamical complexity, between the well-known low levels and the challenging chaotic level, are growing in importance. For example, Barnett and He (2010) have shown that singularity bifurcation arises in some Euler equations macroeconometric models.

In fact bifurcation can occur without leaving the stable region of the parameter space.

Monotonic stability and damped stability are within different bifurcation regions, and there is an infinite number of damped stable regions, beginning with single periodic, biperiodic, and an infinite number of multiperiodic damped regions — all stable.

2 Dynamical Systems

In general, in order to generate complex dynamics a deterministic model must have two essential properties: (i) there must be continuous- or discrete-time lags between variables and (ii) there must be some nonlinearity in the functional relationships of the model. In applied disciplines including economics, the first of these features is typically represented by means of systems of differential or difference equations and even though there exist other mathematical formulations of dynamics which are interesting and economically relevant, in this paper we shall concentrate our attention on them.

Typically, a system of ordinary differential equations will be written as

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{1}$$

where $f\colon U \to \mathbb{R}^n$ with U an open subset of $\mathbb{R}^n$ and $\dot{x} \equiv dx/dt$.[1] The vector x denotes the physical (economic) variables to be studied, or some appropriate transformations of them; $t \in \mathbb{R}$ indicates time. In this case, the space $\mathbb{R}^n$ of dependent variables is referred to as *phase space* (or *state space*), while $\mathbb{R}^n \times \mathbb{R}$ is called the *space of motions*.

Equation (1) is often referred to as a *vector field*, since a solution of (1) at each point x is a curve in $\mathbb{R}^n$, whose velocity vector is given by $f(x)$. A solution of equation (1) is a function

$$\psi : I \to \mathbb{R}^n$$

where I is an interval in $\mathbb{R}$ [in economic applications, typically $I = [0, +\infty)$], such that ψ is differentiable on I, $[\psi(t)] \in U$ for all $t \in I$, and

$$\dot{\psi}(t) = f[\psi(t)], \quad \forall\, t \in I.$$

The set $\{\psi(t)\,|t \in I\}$ is the *orbit* of ψ: it is contained in the phase space; the set $\{(t, \psi(t))\,|t \in I\}$ is the *trajectory* of ψ: it is contained in the space of motions. However, in applications, the terms ‘orbit’ and ‘trajectory’ are often used as synonyms. If we wish to indicate the

[1] Systems described by equation (1), in which f does not depend directly on the independent variable t are called *autonomous*. If f does depend on t directly, we shall write

$$\dot{x} = f(x, t), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}$$

and $f\colon U \to \mathbb{R}^n$ with U an open subset of $\mathbb{R}^n \times \mathbb{R}$. Equations of this type are called *non-autonomous*. In economics they are used, for example, to investigate technical progress.

dependence on initial conditions explicitly, then a solution of equation (1) passing through the point x_0 at time t_0 is denoted by

$$\psi(t, t_0, x_0)$$

(if t_0 is equal to zero it can be omitted). For a solution $\psi(t, x_0)$ to exist, continuity of f is sufficient. For such a solution to be unique, it is sufficient that f be continuous and differentiable in U.

We can also think of solutions of ordinary differential equations in a slightly different manner, which is now becoming prevalent in dynamical system theory and will be very helpful for understanding some of the concepts discussed in the following sections. Suppose we denote by $\psi_t(x)$ the point in $\mathbb{R}^n$ reached by the system at time t starting from the point x at time 0, under the action of the vector field f of equation (1). Then the totality of solutions of (1) can be represented by the *one-parameter family* of maps of the phase-space onto itself, $\psi_t : \mathbb{R}^n \to \mathbb{R}^n$, which is called *phase flow* (or, for short, *flow)* generated by the vector field f, by analogy with fluid flow where we think of the time evolution as a streamline.

If we now take t as a fixed parameter and considering that, for autonomous vector fields, time-translated solutions remain solutions [i.e., if $\psi(t)$ is a solution of equation (1), $\psi(t+\tau)$ is also a solution for any $\tau \in \mathbb{R}$], the problem may be formulated as

$$x_{t+1} = T(x_t), \quad x \in \mathbb{R}^n,\, t \in \mathbb{N} \tag{2}$$

where $T = \psi_\tau$ and τ is the fixed value of the parameter t, normalized so that $\tau = 1$.

Thus, a difference equation like (2) can be derived from a differential equation like (1). This need not be that case and many problems in economics as well as in other areas of research give rise directly to discrete-time dynamical systems. In fact, *non invertible* maps such as the celebrated logistic map extensively discussed later in this article could not be derived from a system of ordinary differential equations.

Equations like (2) are often referred to as *iterated maps*, since its orbit is obtained recursively given an initial condition x_t. For example, if we compose T with itself, then we get the second iterate

$$x_{t+2} = T \circ T(x_t) = T^2(x_t)$$

and by induction on n we get the nth iterate

$$x_{t+n} = T \circ T^{n-1}(x_t) = T^n(x_t).$$

Hence, by the notation $T^n(x)$, we mean T composed with itself $n-1$ times, not the nth derivative of T or the nth power of T.[2]

[2] As an example, if $T(x) = -x^3$, then $T^2(x) = T \circ T(x) = -(-x^3)^3 = x^9$ and $T^3(x) = T \circ T \circ T(x) = T \circ T^2(x) = -(x^9)^3 = -x^{27}$.

Notice the following difference between the orbits of continuous-time and those of discrete-time systems: the former are continuous *curves* in the state space, whereas the latter are *sequences of points* in space. Also, the fact that a map is a function implies that, starting from any given point in space, there exists only one forward orbit. If the function is non-invertible, however, backward orbits are not defined.[3]

It is also to be noted that dynamical systems (whether of a continuous or of a discrete type), can be classified into conservative and dissipative ones. See Medio (1992) for a detailed discussion. Conservative systems cannot have attracting regions in the phase space, i.e., there can never be asymptotically stable fixed points, or limit cycles, or strange attractors. Since strange attractors (to be defined later) are the main object of our investigation and conservative systems are relatively rare in economic applications, we shall not pursue their general study here. Unlike conservative ones, *dissipative* dynamical systems, on which most of this article concentrates, are characterized by contraction of phase space volumes with increasing time. Because of dissipation, the dynamics of a system whose phase space is n-dimensional, will eventually be confined to a subset of dimension smaller than n. In the case, for example, of an n-dimensional system of differential equations characterized by a unique, globally asymptotically stable equilibrium point, the flow will contract any n-dimensional set of initial conditions to a zero-dimensional final state, a point in $\mathbb{R}^n$.

The asymptotic, permanent regime of a dissipative system is the only observable behavior, in the sense that it is not ephemeral, can be repeated and therefore be 'seen' (i.e., on the screen of a computer), and is often easier to investigate than the overall orbit structure. Even though transients may sometimes last for a very long time and their behavior may be an interesting subject for investigation, for dissipative systems we shall concentrate instead on the long-run behavior of the system, ignoring the transient behavior associated with the start up of the system. That is, we shall consider only the attractor (or attractors, in general) to which trajectories from a range of initial conditions are attracted, to understand the asymptotic properties of a dynamical system. That is, we shall concentrate on the asymptotic properties of a dynamical system, devoting our attention mainly to the attractors of a system, i.e., to the sets of points to which trajectories starting from a range of initial conditions tend as time goes by. See Medio (1992, Chapter ?) for a more detailed discussion.

2.1 Strange Attractors

The simplest type of an attractor is a stable *fixed point* or (using a terminology more common in economics) a stable *equilibrium*. Ascertaining the existence of a fixed/equilibrium point mathematically amounts to finding the solutions of a system of algebraic equations. In the continuous-time case $\dot{x} = f(x)$, the set of equilibria is defined by $E = \{\bar{x} \,|\, f(\bar{x}) = 0\}$, i.e., the

[3] A map is invertible if and only if it is one-to-one. For example, the map $T\colon \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x^2$ is not one-to-one, since $T(1) = 1 = T(-1)$. However, the map $T\colon [0, \infty) \to \mathbb{R}$ defined by $T(x) = x^2$ is one-to-one (and therefore invertible).

set of values of x such that its rate of change in time is 0. Analogously, in the discrete-time case $x_{t+1} = T(x_t)$, we have $E = \{\bar{x}\,|\bar{x} - T(\bar{x}) = 0\}$, i.e., the values of x which are mapped to themselves by T. As an example, consider the logistic map

$$x_{t+1} = T_r(x_t) = rx_t(1 - x_t), \quad x \in [0, 1],\ r \in (0, 4]. \tag{3}$$

To find the fixed points of (3), we put $x_{t+1} = x_t = \bar{x}$ and solve for $\bar{x}$, finding $\bar{x}_1 = 0$ and $\bar{x}_2 = 1 - 1/r$, as can be seen in Figure 1.

To get some idea of the importance of fixed points, in Figure 2 we plot the phase diagram of the logistic map for different values of the *tuning* (or *control*) *parameter*, r. Notice that the height of the phase curve hill depends on the value r. For $r < 1$, the only fixed point in the interval $0 \le x \le 1$ is $\bar{x} = 0$, but for $r > 1$, there are two fixed points. Using graphical iteration (an algorithmic process of drawing vertical and horizontal segments first to the phase curve and then to the diagonal, $x_{t+1} = x_t$, which reflects it back to the curve), it is easy to show that all trajectories for starting values in the interval $0 \le x \le 1$ and for $r < 1$ approach the final value $\bar{x} = 0$. The point $\bar{x} = 0$ is the attractor for those orbits and the interval $0 \le x \le 1$ is the basin of attraction for that attractor.

In general, we can examine the dynamical information contained in the derivative of the map at the fixed point, $T'(\bar{x})$. If $|T'(\bar{x})| \neq 1$, $\bar{x}$ is called *hyperbolic fixed point*. In fact a fixed point $\bar{x}$ is *stable* (or *attracting*) if $|T'(\bar{x})| < 1$, *unstable* (or *repelling*) if $|T'(\bar{x})| > 1$, and *superstable* (or *superattractive*) if $|T'(\bar{x})| = 0$; superstable in the sense that convergence to the fixed point is very rapid. Fixed points whose derivatives are equal to one in absolute value are called *nonhyperbolic* (or *neutral*) *fixed points*.

Next in the scale of complexity of invariant sets, we consider *stable periodic solutions*, or *limit cycles*. For maps, a point $\bar{x}$ is a periodic point of T with period k, if $T^k(\bar{x}) = \bar{x}$ for $k > 1$ and $T^j(\bar{x}) \neq \bar{x}$ for $0 < j < k$. In other words, $\bar{x}$ is a periodic point of T with period k if it is a fixed point of T^k. In this case we say that $\bar{x}$ has period k under T, and the orbit is a sequence of k distinct points $\{\bar{x}, T(\bar{x}), \cdots, T^k(\bar{x})\}$ which, under the iterated action of T, are repeatedly visited by the system, always in the same order. Since all points between $\bar{x}$ and $T^k(\bar{x})$ are also period k points, the resulting sequence is known as a *period k cycle* or alternatively a *k-period cycle*. Notice that k is the *least* period; if $k = 1$, then $\bar{x}$ is a fixed point for T.[4]

The third basic type of attractor is called *quasiperiodic*. If we consider the motion of a dynamical system after all transients have died out, the simplest way of looking at a quasiperiodic attractor is to describe its dynamics as a mechanism consisting of two or more independent periodic motions. Quasiperiodic orbits can look quite complicated, since the motion never exactly repeats itself (hence, *quasi*), but the motion is not chaotic (as it was wrongly once conjectured). Notice that quasiperiodic dynamics have been found to occur in economically motivated dynamical models. See, for example, Medio (1992, Chapter 12).

[4]For example, the point 1 lies on a 2-cycle for $T(x) = -x^3$, since $T(1) = -1$ and $T(-1) = 1$. Similarly, the point 0 lies on a 3-cycle for $T(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1$, since $T(0) = 1$, $T(1) = 2$, and $T(2) = 0$.

Attractors with an orbit structure more complicated than that of periodic or quasiperiodic systems are called *chaotic* or *strange attractors*. The strangeness of an attractor mostly refers to its geometric characteristic of being a 'fractal' set, whereas chaotic is often referred to a dynamic property, known as 'sensitive dependence on initial conditions,' or equivalently, 'divergence of nearby orbits.' Notice that strangeness, as defined by fractal dimension, and chaoticity, as defined by sensitive dependence on initial conditions, are independent properties. Thus, we have chaotic attractors that are not fractal and strange attractors that are not chaotic.

As we shall see, separation of nearby orbits, or, equivalently, amplification of errors is the basic mechanism that makes accurate prediction of the future course of chaotic orbits impossible, except in the short run. On the other hand, as chaotic attractor are bounded objects, the expansion that characterizes their orbits must be accompanied by a 'folding' action that prevents them to escape to infinity. The coupling of 'stretching and folding' of orbits is the distinguishing feature of chaos and it is at the root of both the complexity of its dynamics and the 'strangeness' of its geometry.

In dissipative systems, a chaotic attractor typically arises when the overall contraction of volumes, which characterizes those systems, takes place by shrinking in some directions, accompanied by (less rapid) stretching in the others. However, one-dimensional, non-invertible maps that generate chaotic orbits characterized by sensitive dependence on initial conditions (such as, for example, the logistic map) pose a puzzling problem. Strictly speaking, they are not conservative or dissipative: they might indeed be called 'anti-dissipative.' These maps only have a stretching action and their output remains bounded due to the effect of the (nonmonotonic) nonlinearity. We could think of these maps as limit cases of (dissipative) two-dimensional, invertible maps with very strong contraction in one direction, so strong that, in the limit, only one dimension is left, along which nearby orbits separate.

In what follows, we shall discuss the 'fractal' property of chaotic attractors briefly, whereas the 'sensitive dependence on initial conditions' property of chaos will be given greater attention, since this property of chaos is, in our opinion, the most relevant to economics.

2.2 Fractal Dimension

The term 'fractal' was coined by Mandelbrot (1985) and it refers to geometrical objects characterized by non-integral dimensions and 'self-similarity.' The term *fractal* comes from the Latin *fractus* which means broken. Intuitively, a snowflake can be taken as a natural fractal. The problem of defining measurement criteria finer than the familiar Euclidean dimensions (length, area, volume) in order to quantify the geometric properties of 'broken' or 'porous' objects was tackled by mathematicians long before the name and properties of fractals became popular. There now exists a rather large number of criteria for measuring qualities that otherwise have no clear definition (such as, for example, the degree of roughness or brokenness of an object), but we shall limit ourselves here to discuss the simplest type concisely.

Let S be a set of points in a space of Euclidean dimension p (think, for example, of the points on the real line generated by the iterations of a one-dimensional map). We now consider certain boxes of side ϵ (or, equivalently, certain spheres of radius ϵ), and calculate the minimum number of such cells, $N(\epsilon)$, necessary to 'cover' S. Then, the *fractal dimension* D of the set S will be given by the following limit (assuming it exists)

$$D \equiv \lim_{\epsilon \to 0} \frac{\log(N(\epsilon))}{\log(1/\epsilon)}. \tag{4}$$

The quantity defined in Equation (4) is also called the (Kolmogorov) *capacity dimension*. It is easily seen that, for the most familiar geometrical objects, it provides perfectly intuitive results. For example, if S consists of just one point, $N(\epsilon) = 1$ and $D = 0$; if it is a segment of unit length, $N(\epsilon) = 1/\epsilon$, and $D = 1$; if it is a plane of unit area, $N(\epsilon) = 1/\epsilon^2$ and $D = 2$; finally, if S is a cube of unit area, $N(\epsilon) = 1/\epsilon^3$ and $D = 3$, etc. That is to say, for 'regular' geometric objects, dimension D does not differ from the usual Euclidean dimension, and, in particular, D is an integer.

The fractal dimension, however, is not always an integer. Let us consider the fractal called *Cantor set* (or *Cantor dust*), named after the German mathematician George Cantor (1845-1918). To make a Cantor set, start with a line segment of unit length. Remove the middle third and repeat this process without end, each time on twice as many line segments as before. The Cantor set is the set of points that remains, which are infinitely many but their total length is zero. What is the fractal dimension of the Cantor set? By making use of the notion of capacity dimension, we shall have $N(\epsilon) = 1$ for $\epsilon = 1$, $N(\epsilon) = 2$ for $\epsilon = 1/3$, and, generalizing, $N(\epsilon) = 2^n$ for $\epsilon = (1/3)^n$. Taking the limit for $n \to \infty$ (or, equivalently, taking the limit for $\epsilon \to 0$), yields

$$D = \lim_{\substack{n \to \infty \\ (\epsilon \to 0)}} \frac{\log 2^n}{\log 3^n} \simeq 0.63.$$

We have thus quantitatively characterized a geometric set that is more complex than the usual Euclidean objects. Indeed the dimension of the Cantor set is a non-integer. We might say that the Cantor dust is an object 'greater' than a point (dimension 0) but 'smaller' than a segment (dimension 1). It can also be verified that the Cantor set is characterized by self-similarity.

The concept of fractal dimension is useful in the geometric analysis of dynamical systems, because it can be conceived of as a measure of the way trajectories fill the phase space under the action of a flow or a map. A non-integer fractal dimension, for example, indicates that trajectories of a system fill up less than an integer subspace of the phase space. See Medio (1992, chapter 7) for a non-rigorous, but intuitive discussion. Also, the concept of fractal dimension is useful in the quantitative analysis of chaotic attractors. For example, the dimension of the attractor of a system [as measured by (4)] can be taken as an index of complexity, as indicated by the essential dimension of the system.

2.3 Lyapunov Exponents

To provide a rigorous characterization, as well as a way of measuring sensitive dependence on initial conditions, we shall now discuss a powerful conceptual tool known as Lyapunov exponents. They provide an extremely useful tool for characterizing the behavior of nonlinear dynamical systems. They measure the (infinitesimal) exponential rate at which nearby orbits are moving apart. A positive Lyapunov exponent is an operational definition of chaotic behavior. Notice, however, that it is possible to have sensitive dependence on initial conditions with orbit divergence less than exponential. In this case, no Lyapunov exponent will be positive.

Although Lyapunov exponents could be discussed in a rather general framework, we shall deal with the issue in the context of one-dimensional maps, since they are by far the most common type of dynamical system encountered in economic applications of chaos theory. Consider, therefore, the map given by equation (3), with $T : U \to \mathbb{R}$, U being a subset of $\mathbb{R}$. We want to describe the evolution in time of two orbits originating from two nearby points x_0 and $x_0 + \epsilon$ (where ϵ is the difference, assumed to be infinitesimally small, between x_0 and $x_0 + \epsilon$). If we apply the map function T, n times to each point, the difference between the results will be related to ϵ as follows

$$d_n = e^{n\lambda(x_0)}\epsilon$$

where d_n is the difference between the two points after they have been iterated by the map T, n times and $\lambda(x_0)$ is the rate of convergence or divergence.

Taking the logarithm of the above equation and solving for $\lambda(x_0)$ gives

$$\lambda(x_0) = \frac{1}{n}\log\left|\frac{d_n}{\epsilon}\right|.$$

Asymptotically, we shall have [since $d_n/\epsilon = T'(x_{n-1})\cdots T'(x_1)T'(x_0)$]

$$\begin{aligned}\lambda(x_0) &= \lim_{n\to\infty}\frac{1}{n}\log\left|\frac{d_n}{\epsilon}\right| \\ &= \lim_{n\to\infty}\frac{1}{n}\log\left|T'(x_{n-1})\cdots T'(x_1)T'(x_0)\right| \\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\left|T'(x_j)\right|. \qquad (5)\end{aligned}$$

The quantity $\lambda(x_0)$ is called Lyapunov exponent. Note that the right hand side of (5) is an average along an orbit (a time average) of the logarithm of the derivative. Notice that, in general, Lyapunov exponents depend on the selected initial conditions. From equation (5), the interpretation of $\lambda(x_0)$ is straightforward: it is the (local) asymptotic exponential rate of divergence of nearby orbits. It is local, since we evaluate the rate of separation in the limit, as $\epsilon \to 0$. It is asymptotic, since we evaluate it in the limit of an indefinitely large number of iterations, as $n \to \infty$, assuming that the limit exists.

As an example, let

$$T_\Lambda(x) = \begin{cases} 2x & \text{for} \quad 0 \leq x \leq 1/2 \\ 2(1-x) & \text{for} \quad 1/2 \leq x \leq 1 \end{cases} \tag{6}$$

be the symmetric 'tent' map. Clearly, $\lambda(x_0)$ is not defined if x_0 is such that $x_j = T_\Lambda^j(x_0) = 1/2$ for some j (because the derivative is not defined). For other points $x_0 \in [0,1]$, $|T'_\Lambda(x_j)| = 2$ for all j, so that $\lambda(x_0) = \log 2$.

As another example, consider the logistic map, $T_r(x)$, given by equation (3). Since $T'_r(x_j) = r(1-2x_j)$, the Lyapunov exponent is given by

$$\begin{aligned} \lambda(x_0) &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |r(1-2x_j)| \\ &= \log r + \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |1-2x_j| \,. \end{aligned}$$

Clearly, if $x_0 = 0$ or 1, then $\lambda(x_0) = \log r$. For points $x_0 \in (0,1)$ and for $r = 4$, $\lambda(x_0) = \log 2$.

The sign of Lyapunov exponents is especially important to classify different types of dynamical behavior. In particular, the presence of a positive Lyapunov exponent signals that nearby orbits diverge exponentially in the corresponding direction. In its turn, this indicates that observation errors will be amplified by the action of the map. We shall see in what follows that the presence of a positive Lyapunov exponent is intimately related to the lack of predictability of dynamical systems, and thus it is an essential feature of chaotic behavior. It is to be noted that the calculation of Lyapunov exponents in the general, multidimensional case is more complex and cannot be discussed here in any detail.

2.4 Transition to Chaos

In the previous sections, we have provided a classification of attractors and discussed the distinct properties of chaotic attractors. The relevance of these procedures would be greatly enhanced if, in addition, we could describe the qualitative *changes* in the orbit structure of the system which take place when the control parameters are varied. In this way, we would obtain not only a snapshot of chaotic dynamics, but also a description of its emergence. Moreover, if we could provide a rigorous and exhaustive classification of the ways in which complex behavior may appear, transition to chaos could be predicted theoretically, and potentially turbulent mechanisms could be detected in practical applications — and their undesirable effects could be avoided by acting on the relevant parameters.

Unfortunately, the present state of the art does not permit us to define the prerequisites of chaotic behavior with sufficient precision and generality. In order to forecast the appearance of chaos in a dynamical system, we are for the time being left with a limited number of theoretical predictive criteria and a list of certain typical (but by no means exclusive) 'routes to chaos'. Typically, transition to chaos takes place through *bifurcations*. A bifurcation is an essentially nonlinear phenomenon and describes a qualitative change in the orbit structure of a (discrete or continuous-time) dynamical system when one or more parameter is changed. Bifurcation theory is a vast and complex area and we shall consider it here only incidentally.

There exist various types of routes to chaos, generated by so-called codimension one bifurcations (that is, bifurcations depending on a single parameter). In what follows, we shall only (briefly) deal with *period-doubling*, probably the best known route to chaos at least in the economics literature — see, for example, Baumol and Benhabib (1989). For a discussion of other routes to chaos (such as intermittency and the quasiperiodic route), see Medio (1992, Chapter 9).

Period-doubling takes place in both discrete and continuous-time dynamical systems, and can be most simply described by considering the dynamics of the logistic map, $T_r(x)$ given by equation (3), for different values of r. If $r < 1$, the phase curve will lie entirely below the $x_{t+1} = x_t$ line in the positive quadrant [see Figure 3(a)] and $\bar{x} = 0$ is the only fixed point (in fact $\bar{x} = 0$ is an equilibrium for all r). Figures 3(a) and 3(b) give the phase and state space representations of $T_r(x)$ for $r = 0.6$ and $x_0 = 0.01$. Notice that the only fixed point is at $T_r(x) = \bar{x} = 0$.

As r increases beyond 1, $\bar{x} = 0$ loses stability, but a new (positive) fixed point, $\bar{x} = 1-1/r$, appears at the intersection of the $x_{t+1} = x_t$ line and the phase curve, as shown in Figure 4(a). In fact, for $r = 2$ the fixed point $\bar{x} = 1 - 1/r$ becomes superstable, since $T_2'(1/2) = 0$. Therefore, for $1 < r < 3$ there are two fixed points: $\bar{x} = 0$, which is unstable, and $\bar{x} = 1-1/r$, which is stable. From Figure 4(b) we see that the trajectory approaches some positive unique value (a so-called *single limit point*) between 0 and 1.

As r goes through $r = 3$, a bifurcation called 'flip' occurs and the situation changes. The fixed point $x = 1 - 1/r$ turns into a repeller, since $|T_r'(x)| > 1$, and a stable 2-*cycle* (or an *orbit of period* 2) is born: x, $T_r(x)$, $T_r^2(x) = x$. For example, for $r = 3.2360679775$, there is a superstable orbit of period 2: 0.5, 0.8090169943..., 0.5, as can be seen in the state diagram in Figure 5(b).

Let us briefly describe how this happens. For an orbit of period 2 we need to consider the function of $T_r \circ T_r(x)$, abbreviated $T_r^2(x)$, and the associated dynamic equation

$$\begin{aligned} T_r^2(x) &= T_r \circ T_r(x) \\ &= r^2 x(1-x)(1-rx(1-x)). \end{aligned} \tag{7}$$

This is again a nonlinear system and its dynamic behavior can be studied as r varies using the same analysis as before. In particular, the fixed points of $T_r^2(x)$ can be found by equating $T_r^2(x)$ with x and solving the resulting 4th order equation. Hence

$$\begin{aligned} x &= T_r^2(x) \\ &= r^2 x(1-x)(1-rx(1-x)) \\ &= -r^3x^4 + 2r^3x^3 - (r^2+r^3)x^2 + r^2 x \end{aligned}$$

whence we can derive the four fixed points, namely

$$\begin{aligned} \bar{x}_1 &= 0 \\ \bar{x}_2 &= 1 - 1/r \\ \bar{x}_3 &= \frac{1}{2r}\left(r + 1 + \sqrt{(r-3)(r+1)}\right) \\ \bar{x}_4 &= \frac{1}{2r}\left(r + 1 - \sqrt{(r-3)(r+1)}\right). \end{aligned}$$

Clearly, the four fixed points of $T_r^2(x)$ are the two fixed points of $T_r(x)$ and the two elements of the 2-cycle, which have no counterpart in $T_r(x)$; see the phase diagram in Figure 5(a).

The fixed points of the second-order system (7) are characterized by the derivative of $T_r^2(x)$, $(T_r^2)'(x)$. Since $(T_r^2)'(0) = r^2$ and $(T_r^2)'(1-1/r) = (2-r)^2$, for values of r between 3 and 3.45, each of the fixed points $\bar{x} = 0$ and $\bar{x} = 1 - 1/r$ (which are still present) are unstable. The other two fixed points, however, $\bar{x} = \frac{1}{2r}\left(r + 1 \pm \sqrt{(r-3)(r+1)}\right)$, are both stable, thus implying that each of them locally attracts the dynamics of the second-order system (7).

With respect to Figure 5(a), for r between 3 and 3.45, the trajectories of the first-order system (3) no longer converge to the fixed point $\bar{x} = 1 - 1/r$ (point B), but escape from it and diverge towards the pair of fixed points, $\bar{x} = \frac{1}{2r}\left(r + 1 \pm \sqrt{(r-3)(r+1)}\right)$ — points D and C, respectively. Any one of them is unstable under the first-order system (3), since $|T_r'(x)| > 1$ at both C and D, so that the trajectories once in any one of these points are initially repelled. Points C and D, however, are stable under the second-order system (7), since $(T_r^2)'(x)$ at both C and D is less than 1 in absolute value, so that after having moved away from each of C and D in the first step, trajectories come back to each of these points in the second step, thus making the dynamics of system (7) stable with respect to each of C and D. Summarizing, for r between 3 and 3.45, the trajectories of $T_r(x)$ oscillate in the set

{C, D}, giving rise to a stable 2-cycle for $T_r(x)$, as it is shown in Figure 5(b). In this case the system is said to undergo a *flip bifurcation*; see Guckenheimer and Holmes (1983).

If r is increased further, then the two stable fixed points of $T_r^2(x)$ become unstable. In particular, both fixed points of $T_r^2(x)$ will bifurcate at the same r value, leading to an orbit of period 4. In other words

$$\begin{aligned} T_r^2 \circ T_r^2(x) &= T_r^4(x) \\ &= T_r \circ T_r \circ T_r \circ T_r(x) \end{aligned}$$

will have eight fixed points, four of which will be stable. For example, for $r = 3.498561699$ there is a superstable orbit of period 4: 0.5, 0.874..., 0.383..., and 0.827...; see the phase and state space representations in Figures 6(a) and 6(b).

The same bifurcation scenario will repeat over and over again as r is increased, yielding orbits of period 16, 32, 64, and so on *ad infinitum*. However, the sequence $\{r_\kappa\}$ of values of r at which κ-cycles appear has a finite accumulation point $r_\infty \simeq 3.569946$, involving an infinite number of period doubling bifurcations. The values of r for which these transitions from one cycle to another cycle occur, are called *bifurcation points*, the transitions are called *bifurcations*, and the phenomenon is called *period-doubling*. The limit set corresponding to r_∞ is a geometric object with a non-integer fractal dimension $\simeq 0.538$ and a Lyapunov exponent equal to zero, and consequently the motion on it is not chaotic in the sense defined above. In fact, Feigenbaum (1978) discovered that convergence of r to r_∞ is controlled by the universal parameter $\delta \simeq 4.669202$, known as the *Feigenbaum attractor*. The computation of δ is based on the formula

$$\delta = \lim_{\kappa \to \infty} \left(\frac{r_\kappa - r_{\kappa-1}}{r_{\kappa+1} - r_\kappa} \right)$$

where $(r_\kappa - r_{\kappa-1})$ and $(r_{\kappa+1} - r_\kappa)$ are the distances on the real line between successive flip bifurcations.

Past r_∞, we enter what is usually called the 'chaotic zone'. For $r_\infty < r < 4$, the model will behave either periodically or aperiodically, in the latter case, the dynamics may be nonchaotic (zero Lyapunov exponent, no sensitive dependence on initial conditions) or chaotic (positive Lyapunov exponent, sensitive dependence on initial conditions). There is, for example, a tiny interval near $r = 3.83$ (a so-called *window of stability* or *periodicity*) where a stable 3-cycle occurs; see Figures 7(a) and 7(b). Just past $r = 3.83$, the period doubling occurs again, leading to orbits of period 6, 12, 24, and so on, also governed by the Feigenbaum constant. In fact, for r between r_∞ and 4 there is a denumerably infinite number of periodic windows and still an indenumerable number of values of r for which the model behaves aperiodically (chaotically or not). For $r = 4$, we have a completely chaotic orbit, as is illustrated in the state space diagram of Figure 8.

In fact, the different period lengths κ of stable periodic orbits appear in a universal order, with higher-period cycles being associated with higher values of r. In particular, if r_κ is the value of r at which a stable κ-cycle first appears as r is increased, then $r_\kappa > r_q$ if $\kappa \succ q$ (where $\kappa \succ q$ simply means that "κ is listed before q") in the following Sharkovski (1964) ordering (in which we first list the odd numbers except one, then 2 times the odds, 2^2 times the odds, etc., and at the end the powers of 2 in decreasing order, representing the period doubling)

$$\begin{aligned}3 \succ 5 \succ ... \succ 2 \cdot 3 \succ 2 \cdot 5 \succ ... \succ 2^2 \cdot 3 \succ 2^2 \cdot 5 \succ ...\\ \succ 2^3 \cdot 3 \succ 2^3 \cdot 5 \succ ... \succ 2^3 \succ 2^2 \succ 2 \succ 1\end{aligned}$$

This ordering seems strange, but it turns out to be the ordering which expresses which periods imply which other periods. For example, the minimum r value for an orbit of period $\kappa = 2 \cdot 3 = 6$ is larger than the minimum r value for an orbit of period $\kappa = 2^2 \cdot 3 = 12$, because $6 \succ 12$ in the Sharkovski ordering. One consequence of this ordering is that the existence of a stable κ $(= 3)$-cycle guarantees the existence of any other stable q-cycle for some $r_q < r_\kappa$; see, for example, Li and Yorke (1975).

3 Chaos in Dynamic Economic Models

Chaos represents a radical change of perspective on business cycles. Business cycles receive an endogenous explanation and are traced back to the strong nonlinear deterministic structure that can pervade the economic system. This is different from the (currently dominant) exogenous approach to economic fluctuations, based on the assumption that economic equilibria are determinate and intrinsically stable, so that in the absence of continuing exogenous shocks the economy tends towards a steady state, but because of stochastic shocks a stationary pattern of fluctuations is observed.

Goodwin (1951) was one of the first to understand the relevance of chaos theory for economics. Recently, however, there has been a revival of interest in dynamical systems theory, and there is a group of economists who look at economic fluctuations as deterministic phenomena, endogenously created by market forces, and aggregator (utility and production) functions. They agree with Goodwin that chaos theory has great implications for both theory and policy. For example, chaos could help unify different approaches to structural macroeconomics. As Grandmont (1985) has shown for different parameter values even the most classical of economic models can produce stable solutions (characterizing classical economics) or more complex solutions, such as cycles or even chaos (characterizing some Keynesian economics, including much post-Keynesian economics).

In what follows, we shall briefly review some representative theoretical microeconomic and macroeconomic models that predict cycles and chaos as outcomes of reasonable economic hypotheses. Our purpose is not to provide a complete survey of all existing dynamic economic models that predict chaos. The reader that is interested in a more exhaustive survey should also consult Brock (1988).

3.1 Rational Choice and Chaos

Benhabib and Day (1981), using a standard micro-framework, showed that rational choice can lead to erratic behavior when preferences depend on past experience. Following Benhabib and Day (1981), consider the (logarithmic representation of the) Cobb-Douglas utility function

$$u(x_1, x_2; \alpha) = \alpha \log x_1 + (1 - \alpha) \log x_2$$

with $0 < \alpha < 1$. Maximizing subject to (the usual budget constraint)

$$p_1 x_1 + p_2 x_2 = y \tag{8}$$

yields the Marshallian demand functions

$$x_1 = \alpha \frac{y}{p_1} \quad \text{and} \quad x_2 = (1 - \alpha) \frac{y}{p_2}. \tag{9}$$

Assuming, however, that preferences depend on past experience, as in Benhabib and Day (1981), according to a function

$$\alpha_t = r x_{1,t-1} x_{2,t-1} \tag{10}$$

where r is an 'experience dependence' parameter, then the demand for x_1 and x_2 is described by a first-order difference equation in x_1 and x_2, respectively. For example, by substituting (10) into (9) and exploiting the budget constraint (8), the demand for x_1 is obtained (under the assumption of constant prices) as

$$x_{1t} = \frac{ry}{p_1 p_2} x_{1,t-1}(y - p_1 x_{1,t-1}). \tag{11}$$

Clearly, equation (11) describes a one-humped curve like the logistic map (3). In fact, for $p_1 = p_2 = y = 1$, equation (11) reduces to equation (3). Therefore, the specification of experience dependent preferences generates chaotic behavior for appropriate values of the experience dependence parameter, r.

3.2 Descriptive Growth Theory and Chaos

Following Day (1982), we consider the descriptive one-sector model due to Solow (1956). Under the assumptions that aggregate saving equals gross investment and that the capital stock exists for exactly one period, this system can be written as a first-order system in discrete time as

$$(1 + \nu) k_{t+1} = s f(k_t) \tag{12}$$

where k is capital per worker, $f(k_t)$ a neoclassical production function, and the two parameters $\nu > -1$ and $s \in [0, 1]$ represent, respectively, the rates of population growth and saving.

Under the usual convexity assumptions, the phaseline of equation (12) is an increasing concave function through the origin, with two fixed points. The trivial steady state at 0 is asymptotically unstable while the other (positive) fixed point is globally stable, attracting orbits that start at any initial value $k_0 > 0$.

Day (1982) extended the above neoclassical one-sector model of capital accumulation by introducing a pollution effect that reduces productivity as in the following (Cobb-Douglas type) production function

$$f(k_t) = Bk_t^{\varphi}(\varsigma - k_t)^{\gamma} \tag{13}$$

where $k_t \leq \varsigma =$ constant (acting as a saturation level of capital per worker) and $(\varsigma - k_t)^{\gamma}$ reflects the effect of pollution on per capita output. In particular, when k increases, pollution also increases and less output can be produced with a given stock of capital than in the standard model. With (13), the neoclassical model (12) becomes

$$(1+\nu)k_{t+1} = sBk_t^{\varphi}(\varsigma - k_t)^{\gamma} \tag{14}$$

which for $B = \gamma = \varsigma = 1$ reduces to

$$k_{t+1} = rk_t(1 - k_t) \tag{15}$$

where $r = sB/(1+\nu)$. Equation (15) is formally identical with the logistic map (3). Hence, all properties of the logistic map apply here as well. Moreover, the general five-parameter map (14) is also chaotic for appropriate values of the parameters. See Day (1982) for details.

3.3 Optimal Monetary Growth Theory and Chaos

In this section we consider one version of the Sidrauski (1967) optimal growth model with money. It is assumed that the economy is composed of a large number of identical infinitely lived households, each maximizing (at time t) a lifetime utility function of the form

$$\sum_{t=0}^{\infty} \beta^t u(c_t, m_t)$$

where c and m are per capita consumption and real money balances. Ignoring capital accumulation, production, and interest-bearing public debt, the representative household's budget constraint for period t is assumed to be

$$P_t(m_t + c_t) = P_t y + H_t + P_{t-1}m_{t-1}$$

where y is a constant endowment and H_t is per capita lump-sum government transfers, assumed to be equal to μM_{t-1} (where $\mu > 0$ is the constant rate of money growth). Assuming additive instantaneous utility, the equilibrium fixed points for the system are obtained by solving the following first-order difference equation [see Azariadis (1993, section 26.3) for more details]

$$m_{t+1}u_c(y, m_{t+1}) = \frac{1+\mu}{\beta}\left[u_c(y, m_t) - u_m(y, m_t)\right] m_t. \tag{16}$$

If we drop the separability assumption and instead consider

$$u(c, m) = \frac{\left(c^{1/2}m^{1/2}\right)^{1-\sigma}}{1-\sigma}, \quad \sigma > 0, \sigma \neq 1$$

where σ is the reciprocal of the intertemporal elasticity of substitution between current and future values of the aggregate commodity $(cm)^{1/2}$, then equation (16) simplifies to

$$x_{t+1}^{\alpha} = \frac{1+\mu}{\beta} x_t^{\alpha}(1 - x_t) \tag{17}$$

where $x_t = y/m_t$ and $\alpha = (\sigma - 3)/2$, assumed to be positive. Equation (17) has a unique positive steady state

$$\bar{x} = 1 - \frac{\beta}{1+\mu}. \tag{18}$$

Substituting (18) into (17) to eliminate $(1+\mu)/\beta$, we obtain

$$x_{t+1} = x_t \left(\frac{1 - x_t}{1 - \bar{x}}\right)^{1/\alpha}$$

which for $\alpha = 1$ reduces to the logistic map (3). See Matsuyama (1991) or Azariadis (1993, section 26.4) for more details regarding the dynamic behavior of this system.

The implications for economics of the results just obtained are puzzling. For example, consider the case in which models of optimal growth give rise to dynamic, logistic-type equations with chaotic parameters. The sequences thus generated are optimal in the sense that they solve a problem of intertemporal maximization of rational agents, in an economy satisfying the requirements of competitive equilibrium at each point of time. In the absence of (exogenous) random disturbances, along optimal trajectories agents' expectations are supposed to be always fulfilled. While the latter assumption may be acceptable when the dynamics of the system are simple (i.e., convergence to a steady state or to a periodic orbit), it makes little sense if the dynamics are chaotic.

4 Efficient Markets and Chaos

The efficient market hypothesis and the notions connected with it have provided the basis for a great deal of research in financial economics. The hypothesis states that asset prices are rationally related to economic realities and always incorporate all the information available to the market. This implies the absence of exploitable excess profit opportunities. However, despite the widespread allegiance to the notion of market efficiency, a number of studies have suggested that certain asset prices are not rationally related to economic realities. It has been argued, for example, that market valuations differ substantially and persistently from rational valuations and that existing evidence (based on common techniques) does not establish that financial markets are efficient.

Motivated by these considerations, in this section we provide a review of the literature with respect to the efficient market hypothesis and consider the intersection between the efficient market theory and chaos theory.

4.1 The Martingale Model

Standard asset pricing models typically imply the *martingale model*, according to which tomorrow's price is expected to be the same as today's price. Symbolically, a stochastic process x_t follows a martingale if

$$E_t(x_{t+1}|\Omega_t) = x_t \tag{19}$$

where Ω_t is the time t information set, assumed to include x_t. Equation (19) says that if x_t follows a martingale the best forecast of x_{t+1} that could be constructed based on current information Ω_t would just equal x_t. Alternatively, the martingale model implies that $(x_{t+1} - x_t)$ is a *fair game* (a game which is neither in your favor nor your opponent's)

$$E_t[(x_{t+1} - x_t)|\Omega_t] = 0. \tag{20}$$

Clearly, x_t is a martingale if and only if $(x_{t+1} - x_t)$ is a fair game. It is for this reason that fair games are sometimes called *martingale differences*.

It is to be noted that the martingale process is a special case of the more general submartingale process. In particular, x_t is a *submartingale* if it has the property $E_t(x_{t+1}|\Omega_t) > x_t$. Note that the submartingale is also a fair game where x_{t+1} is expected to be greater than x_t. In terms of the $(x_{t+1} - x_t)$ process the submartingale model implies that $E_t[(x_{t+1} - x_t)|\Omega_t] > 0$. LeRoy (1989, pp. 1593-4) also offers an example in which $E_t[(x_{t+1} - x_t)|\Omega_t] < 0$, in which case x_t will be a *supermartingale*.

The martingale and fair game models are basically two names for the same characterization of equilibrium in financial markets. In fact, as LeRoy (1989, p. 1589) puts it, "rates of return are a fair game if and only if a series closely related to prices — that is, prices plus cumulated dividends, discounted back to the present — is a martingale." To see this, on the assumption that capital markets are perfect and investors are risk neutral, let R_t be the one-period rate of return and suppose that R_t, less a constant r, is a fair game

$$E_t(R_t\,|\Omega_t) - r = 0. \tag{21}$$

Using the definition of the rate of return as the sum of dividend yield plus capital gain, the fair game model (21) can be written as

$$\frac{E_t[(p_{t+1} + d_{t+1})\,|\Omega_t] - p_t}{p_t} - r = 0 \tag{22}$$

where d_t denotes dividends and p_t the stock price, both at time t. Equation (22) implies

$$p_t = \frac{E_t[(p_{t+1} + d_{t+1})\,|\Omega_t]}{(1+r)} \tag{23}$$

which says that the stock price today equals the sum of the expected future price and dividends, discounted back to the present at rate r.

It is to be noted that none of the variables defined so far is a martingale. In this regard LeRoy (1989) shows that the price itself, without dividends added in, is not generally a martingale and that the variable that is a martingale is the discounted value of a mutual fund which holds the stock [the price of which follows (23)] and reinvests dividend income in further share purchases. To see this, we follow LeRoy (1989) and let $v_t = p_t h_t/(1+r)^t$ be the discounted (to date zero) value of a mutual fund that holds h_t shares of stock at time t. The assumption that the mutual fund reinvests dividends (in further share purchases) implies that $p_{t+1}h_{t+1} = (p_{t+1} + d_{t+1})h_t$. Thus (neglecting Ω_t), we have

$$\begin{aligned} E_t(v_{t+1}) &= E_t\left[\frac{p_{t+1}h_{t+1}}{(1+r)^{t+1}}\right] = E_t\left[\frac{(p_{t+1}+d_{t+1})h_t}{(1+r)^{t+1}}\right] \\ &= \frac{1}{(1+r)^t}\frac{E_t(p_{t+1}+d_{t+1})}{(1+r)}h_t = \frac{1}{(1+r)^t}p_t h_t = v_t \end{aligned}$$

Hence, the discounted value of such a mutual fund is a martingale.

The fair game model (20) says that increments in value (changes in price adjusted for dividends) are unpredictable, conditional on the information set Ω_t. In this sense, information Ω_t is fully reflected in prices and hence useless in predicting rates of return. The hypothesis that prices fully reflect available information has come to be known as the *efficient market hypothesis*. In fact, Fama (1970) defined three types of (informational) capital market efficiency (not to be confused with allocational or Pareto-efficiency), each of which is based on a different notion of exactly what type of information is understood to be relevant. In particular, markets are weak-form, semistrong-form, and strong-form efficient if the information set includes past prices and returns alone, all public information, and any information public as well as private, respectively. Clearly, strong-form efficiency implies semistrong-form efficiency, which in turn implies weak-form efficiency, but the reverse implications do not follow, since a market easily could be weak-form efficient but not semistrong-form efficient or semistrong-form efficient but not strong-form efficient.

It should be noted that in perfect capital markets, risk neutral investors will always prefer to hold whichever asset generates the highest expected return (completely ignoring risk differences). In equilibrium (when all assets are held willingly), all assets earn the same expected rate of return, equal to the real interest rate which itself is constant over time. Therefore, returns follow the fair game model (20). However, relaxing any one of the (strong) assumptions required by a fair game process for asset returns — such as, risk neutrality or uniform and freely available information — is likely to negate the correspondence between the fair game hypothesis and informationally efficient markets. For example, allowing risk averse agents can result in nonzero expected excess returns in equilibrium, as time-varying compensation for bearing risk.

4.2 The Random Walk Model

The martingale model given by (19) can be written equivalently as

$$x_{t+1} = x_t + \varepsilon_t$$

where ε_t is the martingale difference. When written in this form the martingale looks identical to the *pure random walk* model (an AR(1) process with unit coefficient and zero drift), the forerunner of the theory of efficient capital markets.

The martingale, however, is less restrictive than the random walk. In particular, the martingale difference requires only independence of the conditional expectation of price changes from the available information, as risk neutrality implies, whereas the (more restrictive) random walk model requires this and also independence involving the higher conditional moments (i.e., variance, skewness, and kurtosis) of the probability distribution of price changes. By not requiring probabilistic independence between successive price changes, the martingale difference model is entirely consistent with the fact that price changes, although uncorrelated, tend not to be independent over time but to have clusters of volatility and tranquility (i.e., dependence in the higher conditional moments), a phenomenon originally noted for stock market prices by Mandelbrot (1963) and Fama (1965).

In this regard, as LeRoy (1989, p. 1592) puts it, "risk neutrality is consistent with nonzero serial correlation in conditional variances: The fact that future conditional variances are partly forecastable is irrelevant because risk neutrality implies that no one cares about these variances". In fact one promising approach to modelling the dynamic and distributional properties of stock price changes is to use Engle's (1982) autoregressive conditional heteroskedastic (ARCH) model and its extensions. These models have been successful in describing other financial market series. See, for example, Bollerslev (1986), Nelson (1991), and Engle *et al.* (1987).

4.3 The Present Value Model

Finally, the efficient market model (21) does not imply that prices are completely without structure. In fact (21) turns out to be exactly the same model as the expected present-value model [according to which stock prices (in the context of the stock market) equal the expected present value of future dividends] with which the theory of efficient capital markets is most often identified in the literature. This mathematical equivalence was first pointed out by Samuelson (1965).

In particular, replacing t by $t+1$ in (23) and using the resulting equation to eliminate p_{t+1} in (23), the stock price can be written [using the rule of iterated expectations that guarantees that $E_t(E_{t+1}(p_{t+2}\,|\Omega_{t+1}) = E_t(p_{t+2}\,|\Omega_t)$ and similarly for dividends] as

$$p_t = \frac{E_t(d_{t+1})}{(1+r)} + \frac{E_t(p_{t+2}+d_{t+2})}{(1+r)}.$$

Proceeding similarly n times and assuming that $E_t(p_{t+n})/(1+r)^n \to 0$ as $n \to \infty$, so as to rule out speculative bubbles, we obtain the expected present-value model

$$p_t = \sum_{j=1}^{\infty} \frac{1}{(1+r)^j} E_t(d_{t+j}) \tag{24}$$

which allows us to obtain the so-called fundamental value of the stock as the present value of expected future dividends. Moreover, the proof is completely reversible, implying that if (24) is satisfied, so is the fair game model (23), suggesting that it would be logically inconsistent to reject the expected present-value model while at the same time accepting the fair game model. What is striking here, however, is that even though dividend changes in (24) can be partly forecast, the efficient market model (21) implies that rates or return cannot be forecast.

Abandoning the convergence assumption, $E_t(p_{t+n})/(1+r)^n \to 0$ as $n \to \infty$, leads to an infinite number of solutions any one of which can be written as

$$p_t = \sum_{j=1}^{\infty} \frac{1}{(1+r)^j} E_t(d_{t+j}) + B_t \tag{25}$$

where the additional term, B_t, in (25) is called a 'rational bubble,' in the sense that it is entirely consistent with rational expectations and the time path of expected returns.

The implications for financial economics of the results just obtained are important. As Lucas (2009) put it , "one thing we are not going to have, now or ever, is a set of models that forecasts sudden falls in the value of financial assets, like the declines that followed the failure of Lehman Brothers in September [of 2008]." However, the possible existence of chaos implies that nonlinearity-based trading rules exist and that prediction over short periods is possible, provided that the deterministic part of the system is low-dimensional and its noisy part is of a small amplitude.

5 Policy Relevancy of Chaos

As it has been shown in Section 3, chaos can be produced, for some parameter settings, from even many of the most classical economic models, including models in which there is continuous market clearing, rational expectations, overlapping generations, perfect competition, no externalities, and no forms of market failure. The issue has been whether or not the parameter settings that can produce chaos are economically 'reasonable.' With large enough nonlinear, dynamic models to be viewed as possible approximations to reality, there are no currently available conclusions regarding the plausibility of the subset of the parameter set that can support chaos.

But there is also the question about whether or not we should care. In positive economics, there is good reason to care. Understanding the behavior of an economy that is chaotic is not possible with a model that is not chaotic, since chaotic solution paths have many properties that cannot be produced from nonchaotic solutions. But on the normative side, the usefulness of chaos is much less clear. Grandmont's (1985) model, for example, produces Pareto optimal chaotic solution paths. The fact that the solutions are chaotic does not alone provide any justification for government intervention, and indeed any such intervention could produce a stable, but Pareto inferior solution. In fact, Bullard and Butler (1993) have argued that the existing theoretical results on chaos have no policy relevance, since in chaotic models the justification for intervention always can be identified with a form of market failure entered into the structure of the model, and hence the chaos is an independent and policy-irrelevant feature of those models.

There is an exemption, however. Woodford (1989) has argued that chaos might produce increased Pareto-sensitivity to market failure. If that is the case, then there is an interaction between chaos and the policy implications of market failure, with small market failures producing increased Pareto loss, when the economy also is chaotic. This could be an important result and could result in high policy relevancy for chaos, but at present Woodford's speculation remains only a supposition, and has not been confirmed in theory or practice. Hence, at present, the policy relevance of chaos must remain in doubt.

6 Testability of Chaos within the Economy

From an empirical perspective it is difficult to distinguish between exogenous fluctuations produced by random shocks and endogenous fluctuations produced from the nonlinear structure of the economy. It is for this reason that there have been a great deal of studies in recent years investigating the basic features of chaotic phenomena in economic and financial time series, using a number of tests for nonlinearity, fractal attractors, and sensitivity to initial conditions. Some of the best known tests that have been used are the correlation dimension test for chaos [see Grassberger and Procaccia (1983)], the BDS test of the null hypothesis of whiteness [see Brock *et al.* (1996)], and a number of tests for chaos based on the calculation of Lyapunov exponents.

Barnett and Serletis (2000) devote a good deal of space to the empirical evidence on economic and financial data, look at the controversies that have arisen about the available results, address important questions regarding the power of some of the best known tests for nonlinearity or chaos against various alternatives, and raise the issue of whether dynamical systems theory is practical in economics. In this regard, as Barnett and Serletis (2000, pp. 721) put it,

> "in the field of economics, it is especially unwise to take a strong opinion (either pro or con) in that area of research. Contrary to popular opinion within the profession, there have been no published tests of chaos 'within the structure of the economic system,' and there is very little chance that any such tests will be available in this field for a very long time. Such tests are simply beyond the state of the art. Existing tests cannot tell whether the source of detected chaos comes from within the structure of the economy, or from chaotic external shocks, as from the weather. Thus, we do not have the slightest idea of whether or not asset prices exhibit chaotic nonlinear dynamics produced from the nonlinear structure of the economy (and hence we are not justified in excluding the possibility)."

7 Conclusion

We have reviewed a great deal of high quality research on nonlinear and complex dynamics. There are many reasons for this interest. Chaos, for example, represents a radical change of perspective on business cycles. Business cycles receive an endogenous explanation and are traced back to the strong nonlinear deterministic structure that can pervade the economic system. This is different from the (currently dominant) exogenous approach to economic fluctuations, based on the assumption that economic equilibria are determinate and intrinsically stable, so that in the absence of continuing exogenous shocks the economy tends towards a steady state, but because of stochastic shocks a stationary pattern of fluctuations is observed.

Chaos could also help unify different approaches to structural macroeconomics. As Grandmont (1985) has shown, for different parameter values even the most classical of economic models can produce stable solutions (characterizing classical economics) or more complex solutions, such as cycles or even chaos (characterizing much of Keynesian economics). Finally, if forecasting is a goal, the possible existence of chaos could be exploitable and even invaluable. If, for example, chaos can be shown to exist in asset prices, the implication would be that profitable, nonlinearity-based trading rules exist (at least in the short run and provided the actual generating mechanism is known). Prediction, however, over long periods is all but impossible, due to the 'sensitive dependence on initial conditions' property of chaos.

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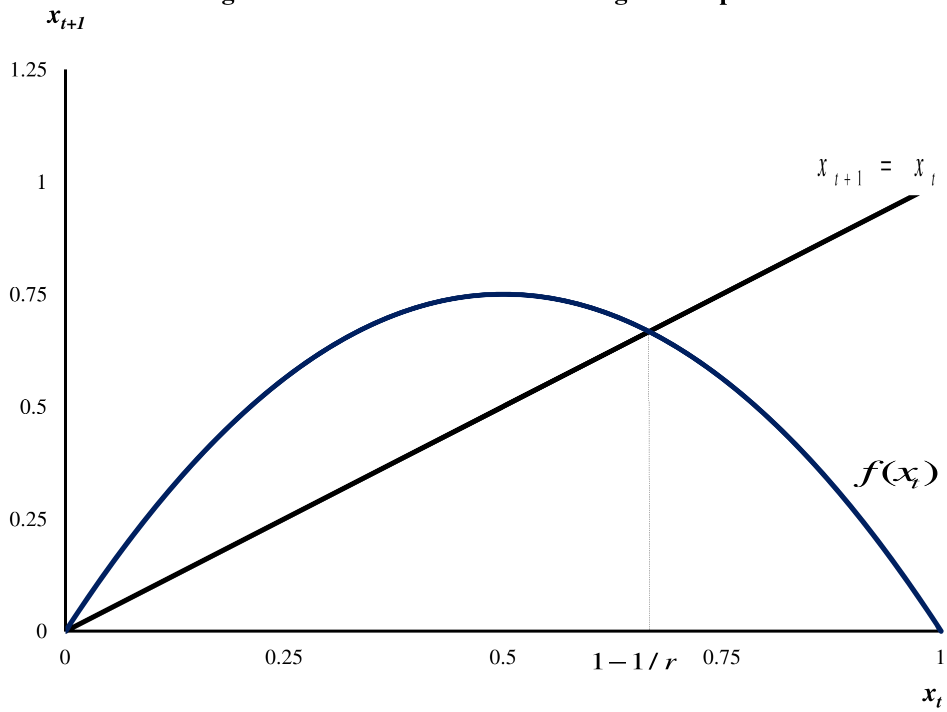
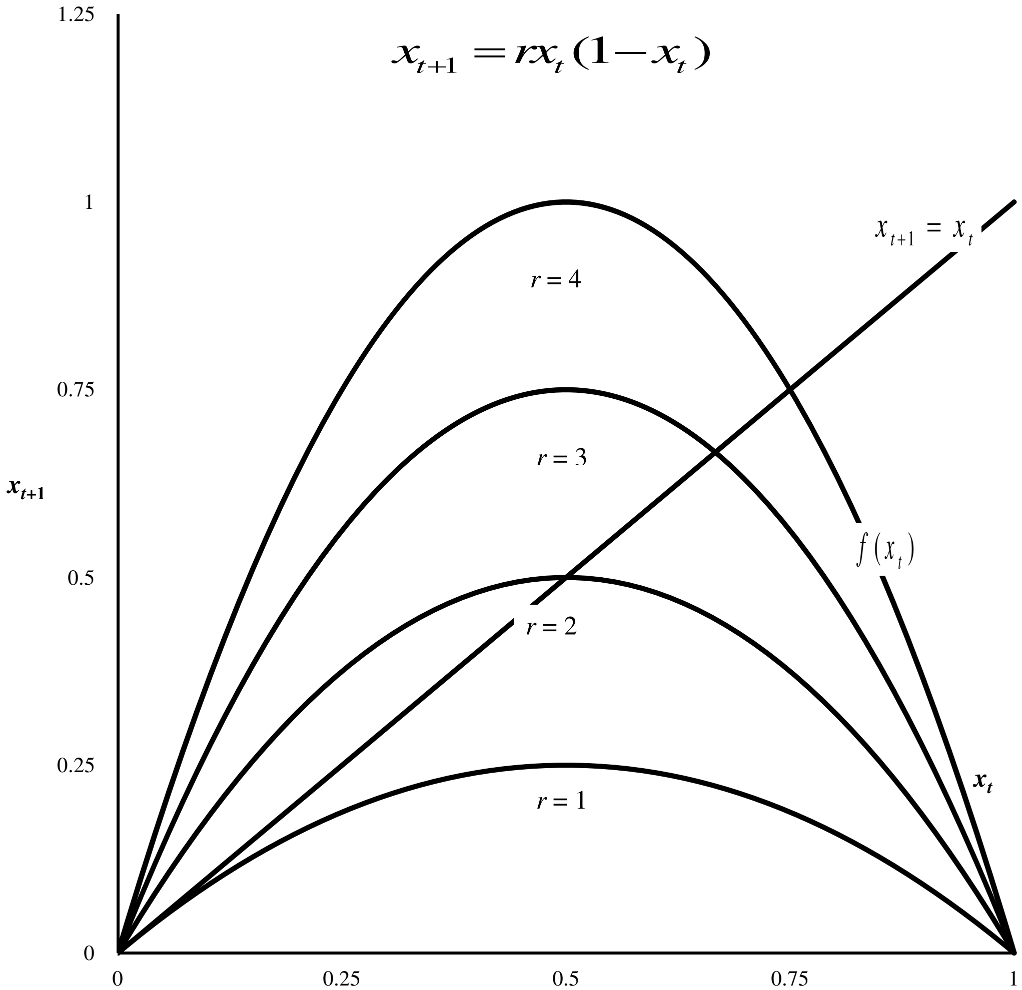
Figure 1. The Fixed Points of the Logistic Map



Note: Note the two fixed points: $x = 0$ and $1 - 1/r$.

Figure 2. Phase Diagram for the Logistic Map for Different Values of *r*



Note: The height of the phase curve depends on the value of the parameter, *r*.

Figure 3(a). Phase Diagram for the Logistic Equation

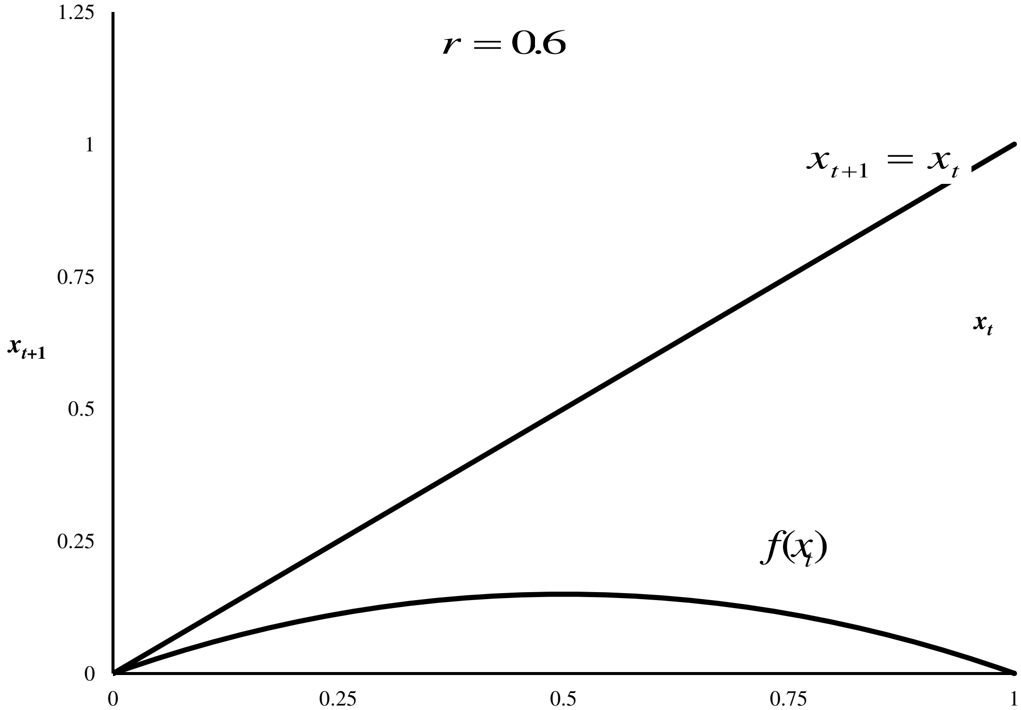
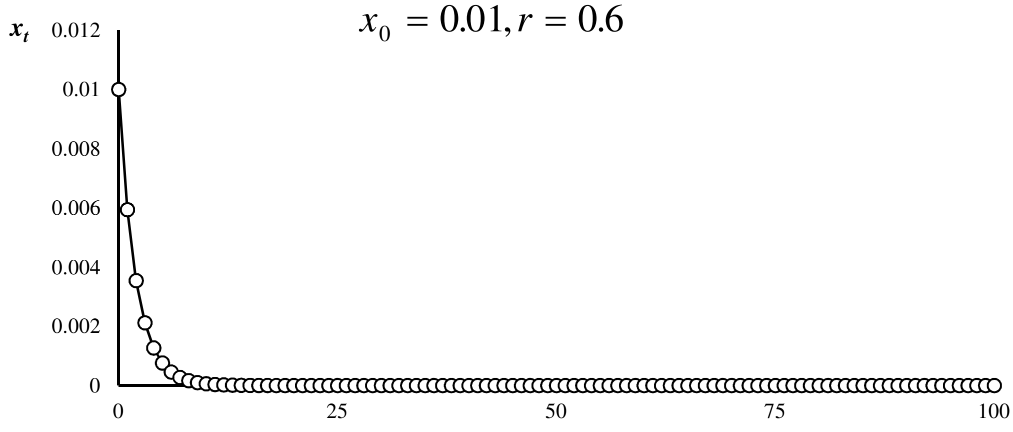


Figure 3(b). State Diagram for the Logistic Equation



Note: This graph has one possible final value, which is 0.

Figure 4(a). Phase Diagram for the Logistic Equation

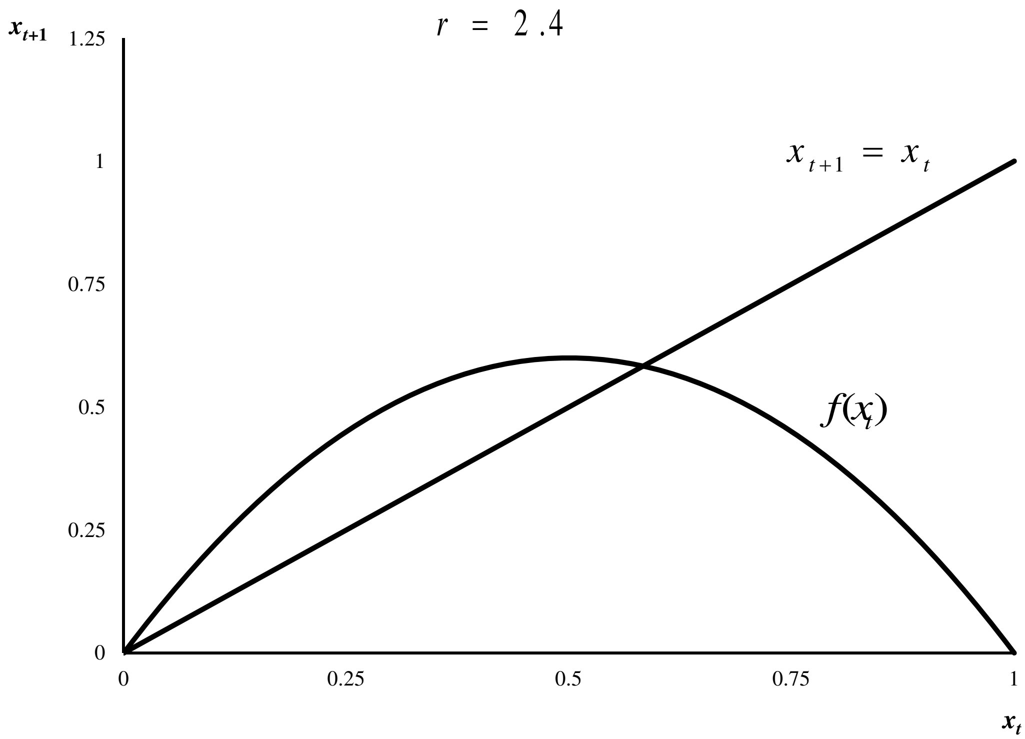
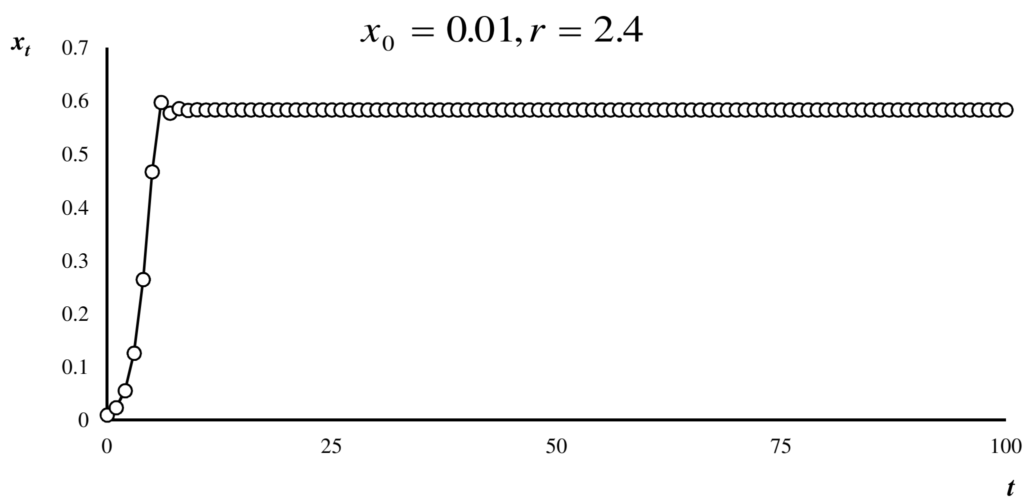


Figure 4(b). State Diagram for the Logistic Equation



Note: This graph has one possible final value, which is 0.583333333.

Figure 5(a). Phase Diagram for the Logistic Equation, Period 2 Cycle

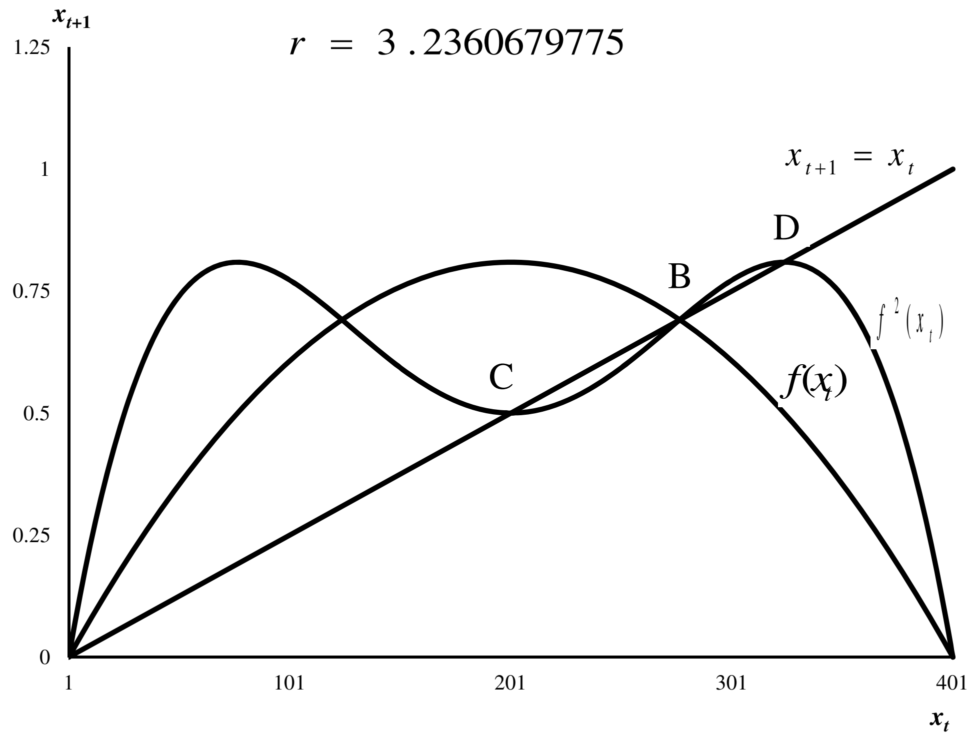
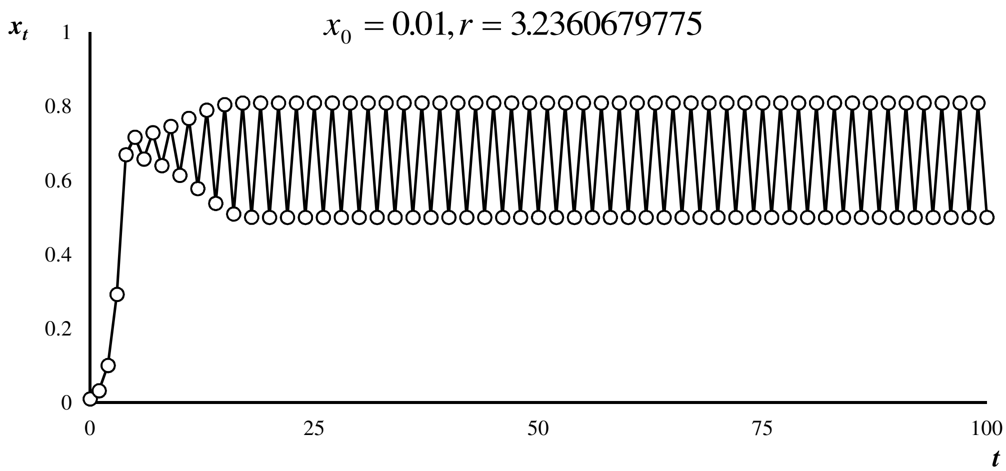


Figure 5(b). State Diagram for the Logistic Equation, Period 2 Cycle



Note: This graph has period 2 behavior, or two possible final values: 0.5 and 0.809016994.

Figure 6(a). Phase Diagram for the Logistic Equation, Period 4 Cycle

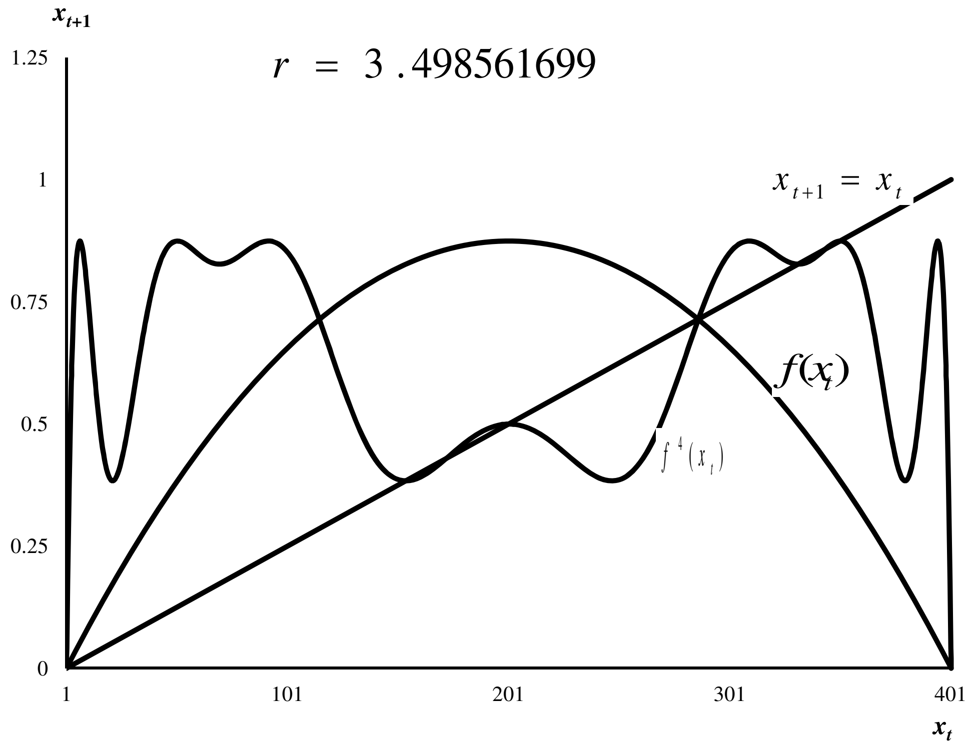
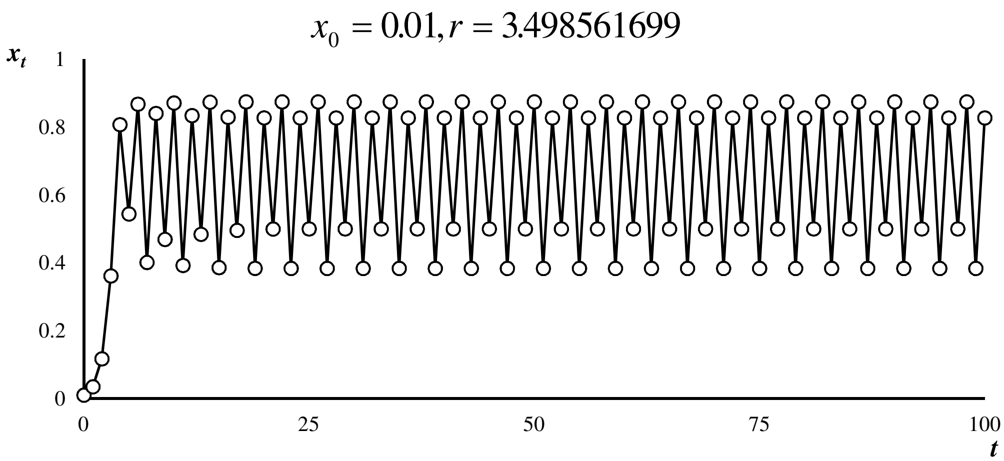


Figure 6(b). State Diagram for the Logistic Equation, Period 4 Cycle



Note: This graph has period 4 behavior, or four possible final values: 0.5, 0.874640425, 0.383598231, and 0.827237111.

Figure 7(a). Phase Diagram for the Logistic Equation, Period 3 Cycle

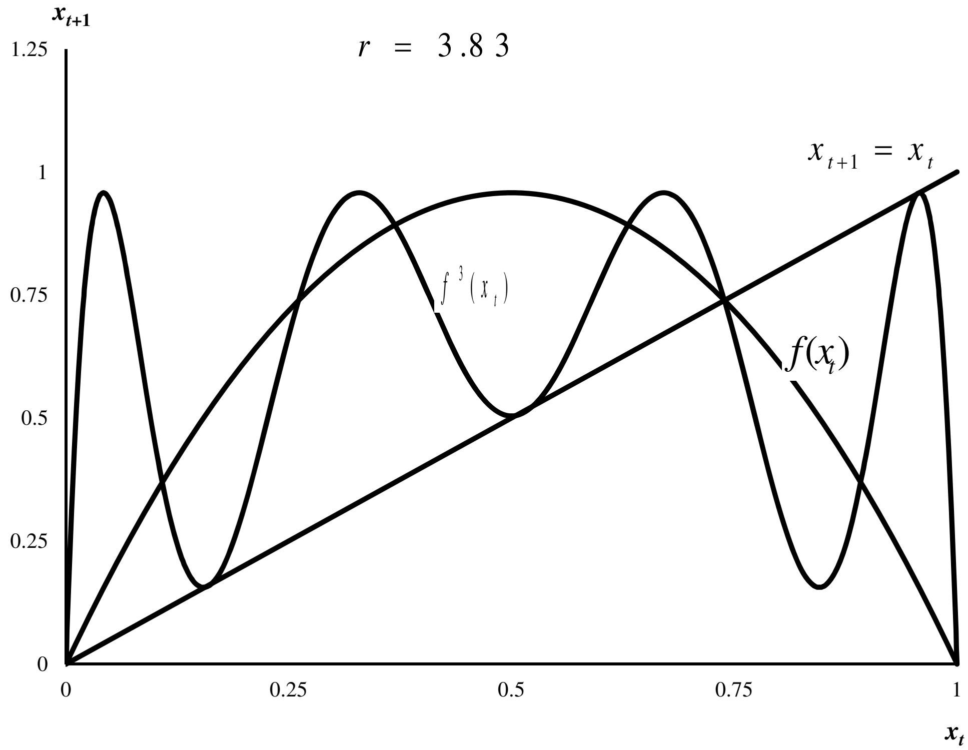
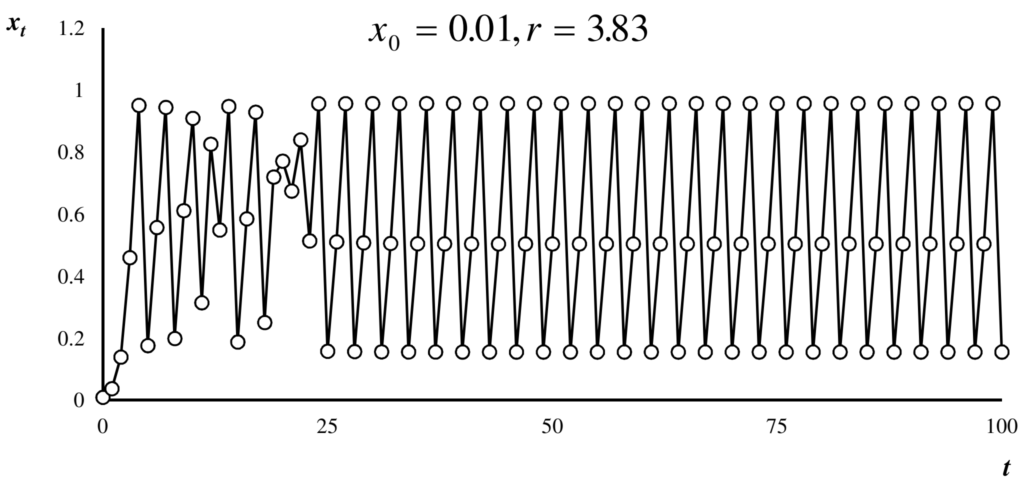
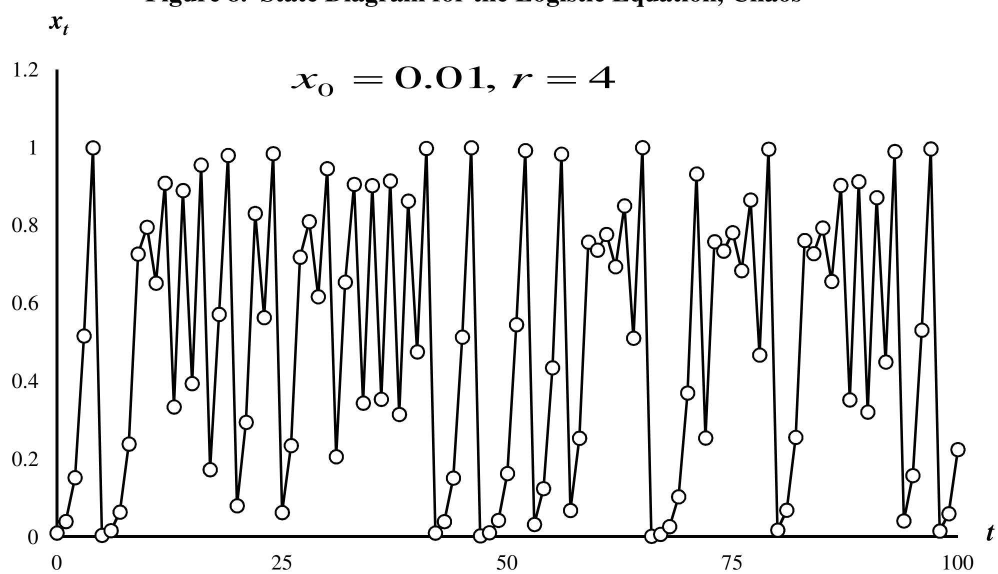


Figure 7(b). State Diagram for the Logistic Equation, Period 3 Cycle



Note: This graph has period 3 behavior, or three possible final values: 0.504666487, 0.957416598, and 0.156149316.

Figure 8. State Diagram for the Logistic Equation, Chaos



Note: This graph illustrates chaotic behavior, or an infinite number of possible final values.