# Leverage, skewness and amplitude asymmetric cycles

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# Leverage, Skewness and Amplitude Asymmetric Cycles

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**Abstract**

The leverage parameter is shown to turn up as part of the third-order moment when a stochastic volatility process is linearly filtered. If the filter is of the autoregressive class and possesses complex-valued roots or is a Gegenbauer long-memory filter, the leverage effect plays a determinant role in producing Amplitude Asymmetric Cycles, in which the degree of asymmetry depends on the persistence of the process at both levels (conditional mean and variance), the variance of the shocks to the volatility and the value of their inter-temporal correlation with the shocks to the levels.

**Keywords:** Leverage, stochastic volatility, skewness, amplitude asymmetric cycles

**JEL Classifications:** C22 E37

## 1 Introduction

Business cycle data are characterized by a number of non-linearities among which the Amplitude Asymmetry, that is, differing lengths from the mean at

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peaks or troughs, is one of the least understood by macroeconometricians. When a series displays signs of negative (positive) deepness asymmetry, it is skewed negatively (positively), troughs are further (shorter) below mean than peaks are above and there are more (less) observations above it than below. Therefore, negative (positive) deepness asymmetry is the cyclical expression of negative (positive) skewness in the series. Following this approach, Sichel (1993) proposes a test for amplitude asymmetry based on the sample skewness and reports evidence of negative deepness asymmetry in the unemployment and industrial production of US.

This asymmetry takes place in a scenery of strong persistence as oscillatory deviations from the trend level of the economic activity can remain present for long periods of time. Collard (1998), Candelon and Gil-Alaña (2004), Arteche and Velasco (2005), Caporale and Gil-Alaña (2006, 2007) and Ferrara and Guegan (2008) have proposed cyclical and seasonal long memory models in order to capture this periodic dependence in many macroeconomic series.

On the other hand, heteroscedasticity in the cyclical behaviour has been a key concept of business cycle modelling since the seminal paper by Engle (1982), who proposes an autoregressive conditional heteroscedasticity (ARCH) model to capture the time-varying volatility of inflation rates in the United Kingdom. Bollerslev (1986) proposes a natural extension, the Generalized ARCH (GARCH) models, to allow for past conditional variances in the conditional variance equation that avoids the use of highly parametrized models. Hamori (2000) fits GARCH(1,1) processes to the volatilities of the GDP growth rates for US, UK, and Japan and finds evidence of lower volatility during expansions.

An alternative approach to the GARCH family, the Stochastic Volatility (SV) models, were introduced by Taylor (1986) with the purpose of capturing the statistical properties of daily asset returns. In SV models the volatility depends on an unobservable stochastic process with innovations different to those affecting the levels, which allows for a more flexible definition of the stationarity conditions including long memory (Harvey, 1998; Breidt et al., 1998), whereas Giraitis et al. (2000) show that stationarity and long memory of squares are incompatible in GARCH models so strong persistence in the squared observations necessarily implies explosive variance. SV long memory models have been fitted to the volatility of Spanish inflation (Arteche, 2011).

A second property of the volatility of the returns of many economic series is the asymmetric response to positive and negative shocks. This behaviour, firstly described for financial series by Black (1976), is known as leverage effect for its consequences in this area. Ho and Tsui (2003) find significant evidence of conditional volatility asymmetry in the growth rates of Canada and the United States. Ho et al. (2009) fit five VC-QGARCH(1,1) models to the five sectors of the US index of industrial production (consumer good, investment good, manufacturing, non-durables and raw materials). Their results show a negative leverage in all the five series implying that negative shocks induce greater future volatilities on IIP than positive shocks of the same magnitude.

This paper links the first one and the last one of the aforementioned observed stylized facts in business cycle data and shows that a non-zero intertemporal leverage effect in the stochastic volatility process of innovations produces a non-zero skewness in the series, which turns out as amplitude asymmetry in cyclical models. This effect can be accommodated for both short and long memory processes therefore providing the second order autoregressive amplitude asymmetric models and the Gegenbauer long memory amplitude asymmetric models.

The structure of the paper is as follows. Section 2 shows the general forms of the third order and some other moments of linearly filtered leverage stochastic volatility processes. Section 3 analyses some specific cases of Amplitude Asymmetric cycles, starting from the simplest AR(2)-LSV($\delta$) with no serial dependence in the volatility, and explores the behaviour of the Amplitude Asymmetry as a function of the dependence structures of both levels and volatility. Section 4 illustrates the results of previous sections by fitting an amplitude asymmetric model to a version of the US seasonally adjusted Index of Industrial Production and finally Section 5 concludes.

# 2 Linearly filtered Leverage Stochastic Volatility process

Let

$$\mathit{\Phi}_r\left(L\right)h_t=\mathit{\Theta}_s\left(L\right)\eta_t, \tag{1}$$

be a stationary and invertible process with $\eta_t$ innovations, where $\mathit{\Phi}_r\left(L\right)=1+\varphi_1L+\varphi_2L^2+\ldots+\varphi_rL^r$ and $\mathit{\Theta}_s\left(L\right)=1+\vartheta_1L+\vartheta_2L^2+\ldots+\vartheta_sL^s$, and by Wold's decomposition theorem let $h_t=K(L)\,\eta_t$ be its MA representation with $K(L)=1+\varkappa_1L+\varkappa_2L^2+\varkappa_3L^3+\ldots.$ and $w\in\{s,\infty\}$.

Then let

$$y_t = \sigma_* \exp\left(\frac{h_t}{2}\right)\varepsilon_t, \tag{2}$$

be a Stochastic Volatility process with $\sigma_*$ a scale factor and $\varepsilon_t$ innovations distributed conjointly with $\eta_t$ as

$$\begin{pmatrix} \eta_{t+1} \\ \varepsilon_t \end{pmatrix} \to \mathcal{N}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_\eta^2 & \delta\sigma_\eta \\ \delta\sigma_\eta & 1 \end{pmatrix}\right]. \tag{3}$$

where $\delta \equiv corr\left(\eta_{t+1}, \varepsilon_t\right)$ is the leverage parameter that represents a non-zero correlation between the shocks to the volatility and the one-lagged shocks to the levels.

In this paper the leverage effect is modelled inter-temporally between both processes of innovations. Jacquier et al. (2004) allow contemporaneous correlation as $\delta = corr\left(\varepsilon_t, \eta_t\right)$ and Yang (2008) explores the first four moments of the resulting process $y_t$ for the simplest case of a non-serially dependent $h_t$ in (2). He shows that $\delta$ plays a role in the mean, variance, skewness and kurtosis although the effect of the leverage becomes negligible in even moments for small values of this parameter.

However, as argued by Yu (2005), under contemporaneous correlation the process $y_t$ is not a martingale difference sequence. Alternatively, Harvey and Shephard (1996) propose to introduce an inter-temporal correlation as in (3). With this specification, the variance and kurtosis of $y_t$ are independent of $\delta$ and the process has zero odd moments.

Nevertheless, the correlation between $\varepsilon_t$ and $\eta_{t+1}$ induces a correlation between the squares of $y_t$ and its lagged values. Specifically, the third-order generalized autocovariances[1] of $y_t$ are defined as

$$\mathbb{E}\left[y_t^2 y_{t-k}\right] = \sigma_*^3 \delta \varkappa_{k-1} \sigma_\eta \exp\left\{\frac{\sigma_h^2\left[5 + 4\rho_h(k)\right]}{8}\right\}, \tag{4}$$

where $\varkappa_0 = 1$ and $\varkappa_k = 0$ for $k > w$ and $\rho_h(k)$ is the autocorrelation function of $h_t$. The finiteness of $\mathbb{E}\left[y_t^2 y_{t-k}\right]$ is guaranteed by the usual stationary restrictions on $h_t$ and the Gaussianity of $\varepsilon_t$ and $\eta_t$.

Finally let

$$x_t = \frac{\Theta_q(L)}{\Phi_p(L)} y_t = \Psi_v(L)\, y_t, \tag{5}$$

[1] In the terminology of Welsh and Jernigan (1983).

with $\Theta_q = 1 - \theta_1 L - \ldots - \theta_q L^q$ and $\Phi_p = 1 - \phi_1 L - \ldots - \phi_p L^p$ two polynomials with all roots outside the unit circle and $\Psi_v(L) = \Theta_q(L)\Phi_p(L)^{-1} = 1 + \psi_1 L + \psi_2 L^2 + \ldots$ with $v \in \{q, \infty\}$.

Then,

$$\begin{aligned}\mathbb{E}\left[x_t^3\right] = 3\sigma_*^3 \delta \sigma_\eta \exp\left(\frac{5\sigma_h^2}{8}\right) \\ \cdot \sum_{k=1}^{\min\{v,w+1\}} \left\{\varkappa_{k-1} \exp\left[\frac{\sigma_h^2 \rho_h(k)}{2}\right] \sum_{i=0}^{v-k} \psi_i^2 \psi_{i+k}\right\},\end{aligned} \tag{6}$$

with $\psi_0 = 1$.

Eq. (6) shows that the skewness of $x_t$, defined as $\gamma_3 = \mathbb{E}[x_t^3] / \mathbb{E}[x_t^2]^{\frac{3}{2}}$, is a weighted sum of third-order generalized autocovariances of the process $y_t$ and is determined by the variance of the process of innovations to the volatility, the dependence structure at both levels of the series and the leverage coefficient $\delta$. Note that $\delta = 0$ or $\Psi_v(L) = 1$ imply $\gamma_3 = 0$ whereas $\delta \neq 0$ and $\Psi_v(L) \neq 1$ do not necessarily imply $\gamma_3 \neq 0$ but on the contrary a certain resonance between the dependence structures of the volatility and the levels of the series is required.

Some other moments of $x_t$ that can become non-zero or are altered by the presence of a non-zero leverage effect are the third-order generalized autocovariances of $x_t$, defined as

$$\begin{aligned}\mathbb{E}\left[x_t^2 x_{t-k}\right] = \sum_{i=k+1}^{v+k} \sum_{j=0}^{v-i+k} \psi_j^2 \psi_{j+i-k} \mathbb{E}\left[y_t^2 y_{t-i}\right] \\ + \sum_{i=max\{1,k-v\}}^{k} \sum_{j=0}^{v-k+i} \psi_j \psi_{j+k-i}^2 \mathbb{E}\left[y_t^2 y_{t-i}\right] \\ + 2\ \mathbf{1}_{\mathbb{N}^*}(v-k) \sum_{i=1}^{v-k} \sum_{j=0}^{v-i-k} \psi_j \psi_{j+k} \psi_{j+k+i} \mathbb{E}\left[y_t^2 y_{t-i}\right]\end{aligned}$$

where $\mathbf{1}_{\mathbb{N}^*}(\nu - k)$ is an indicator function taking value 1 if $v - k > 0$ and 0 otherwise, and the fourth-order moment of $x_t$ which is

$$\begin{aligned}\mathbb{E}\left[x_t^4\right] = \sum_{i=0}^{v} \psi_i^2 \mathbb{E}\left[y_t^4\right] + 6 \sum_{i=1}^{v-1} \psi_{i-1}^2 \sum_{j=i}^{v} \psi_j^2\ \mathbb{E}\left[y_t^2 y_{t-(j-i+1)}^2\right] \\ + 12 \sum_{i=1}^{v-2} \psi_{i-1}^2 \sum_{j=i}^{v-1} \psi_j \sum_{k=j+1}^{v} \psi_k\ \mathbb{E}\left[y_t^2 y_{t-(j-i+1)} y_{t-(k-j+1)}\right]\end{aligned}$$

with the fourth-order moment and generalized autocovariances of $y_t$ defined as

$$\begin{aligned}\mathbb{E}\left[y_t^2 y_{t-j} y_{t-k}\right] = \sigma_*^4 \delta^2 \sigma_\eta^2 \varkappa_{j-1} \left(\varkappa_{k-1} + \frac{\varkappa_{k-j-1}}{2}\right) \\ \cdot \exp\left\{\frac{\sigma_h^2}{2}\left[\frac{3}{2} + \rho_h(j) + \rho_h(k) + \frac{\rho_h(k-j)}{2}\right]\right\},\end{aligned} \tag{7}$$

$$\mathbb{E}\left[y_t^2 y_{t-k}^2\right] = \sigma_*^4 \left(\varkappa_{k-1}^2 \delta^2 \sigma_\eta^2 + 1\right) \exp\left\{\sigma_h^2 \left[1 + \rho_h(k)\right]\right\}, \tag{8}$$

and

$$\mathbb{E}\left[y_t^4\right] = 3\sigma_*^4 \exp\left(2\sigma_h^2\right). \tag{9}$$

# 3 Amplitude Asymmetric Cycles

## 3.1 AR(2)-LSV($\delta$)

Second order autoregressive processes with complex-valued roots (which implies $-2\sqrt{-\phi_2} < \phi_1 < 2\sqrt{-\phi_2}$) display what has been denoted as pseudo-cyclical behaviour that takes place at the frequency $\omega = \cos^{-1}\left(\phi_1/2\sqrt{-\phi_2}\right)$ (Yule, 1927). If the innovations are generated from a SV process with inter-temporal leverage effect as defined in (2) and (3), a skewed stochastic cyclical model that shows amplitude asymmetry is obtained.

The simplest specification, the AR(2)-LSV($\delta$) process, is built with Eqs. 1 to 3 and 5 where $h_t = \eta_t$ and $(1 - \phi_1 L - \phi_2 L^2)\, x_t = y_t$. In this case,

$$\mathbb{E}\left[x_t^3\right] = \frac{3\phi_1(1-\phi_1\phi_2)\,\sigma_*^3 \delta \sigma_\eta \exp\left(\frac{5\sigma_\eta^2}{8}\right)}{(\phi_1+\phi_2-1)(\phi_2^3-\phi_1\phi_2-1)(\phi_1^2-\phi_1\phi_2+\phi_1+\phi_2^2+\phi_2+1)}. \tag{10}$$

Figure 1 shows the skewness of the AR(2)-LSV($\delta$) process $x_t$ for several combinations of the AR parameters below the curve $\phi_1^2 + 4\phi_2 = 0$. In this figure $\phi_2$ ranges from -0.05 up to -0.95, decrementing by -0.1, and $\phi_1 = -7\sqrt{-\phi_2}/4, \ldots, 7\sqrt{-\phi_2}/4$ incrementing by $\sqrt{-\phi_2}/4$[2]. When $\phi_1 = -2\sqrt{-\phi_2}$

[2]In this figure, as in Figure 3, $\delta = \sigma_* = \sigma_\eta^2 = 1$.

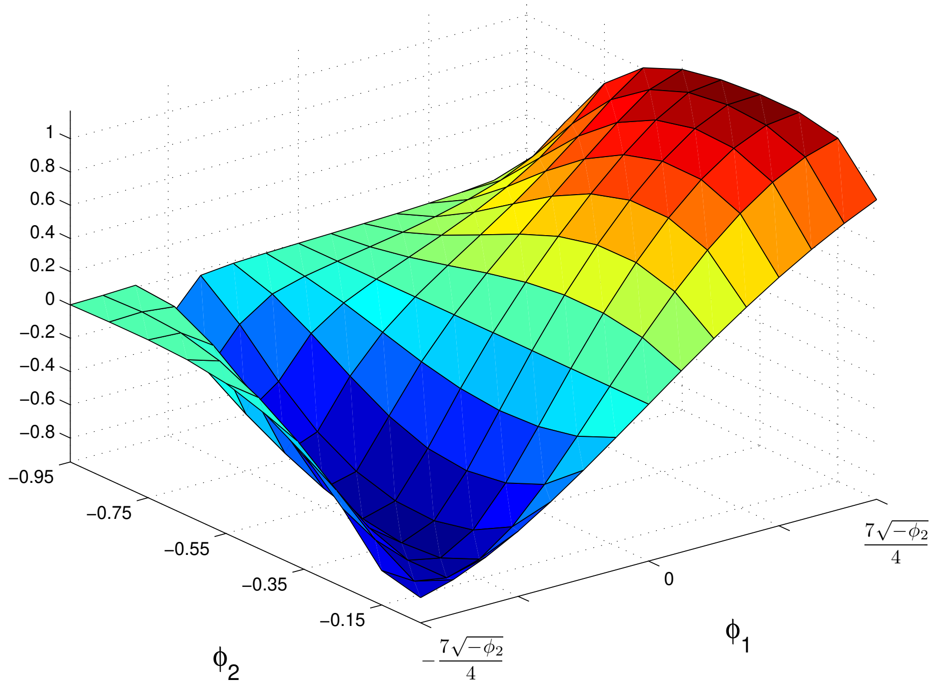
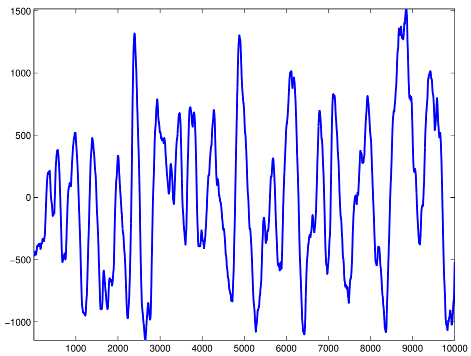


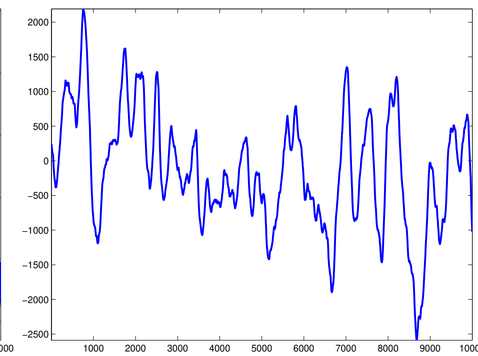

Figure 1: Skewness of an AR(2)-LSV($\delta$) process as a function of $\phi_1$ and $\phi_2$.

the process behaves pseudo-cyclically with frequency $\pi$. The frequency decreases as the value of $\phi_1$ increases and equals $\omega = \pi/2$ for $\phi_1 = 0$ and $\omega = 0$ for $\phi_1 = 2\sqrt{-\phi_2}$. The values of $\gamma_3$ for $\phi_1 = 7\sqrt{-\phi_2}/4$ in Figure 1 are representative, therefore, of the behaviour of this measure of asymmetry for business cycle frequencies. As can be seen, for these values of $\phi_1$ the amplitude asymmetry is relatively larger, reaches a maximum for intermediate values of $\phi_2$ and displays identical sign as the leverage parameter.

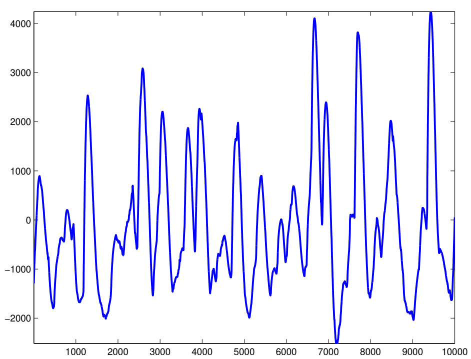
Figure 2 depicts six examples of cyclical AR(2) processes with parameters $\phi_2 = -0.9875$ and $\omega = 0.01$ and LSV innovations with $\delta = \pm 0.95$, no dependence structure in the cases of the first row, a first order autoregressive process with parameter $\varphi = 0.9$ in the second row and a long memory at-the-origin process in the last one. It can be observed a moderate deviation from amplitude symmetry when the volatility only possesses a leverage effect that is consistent with the sign of $\delta$. Figures 2(c) to 2(f) show that this deviation enlarges in a considerable manner when the process of volatility has a stronger persistence.



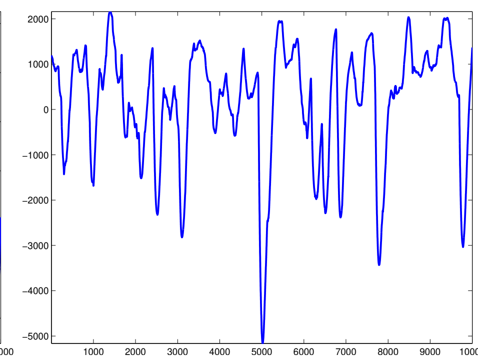

(a) AR(2)-LSV($\delta = 0.95$)



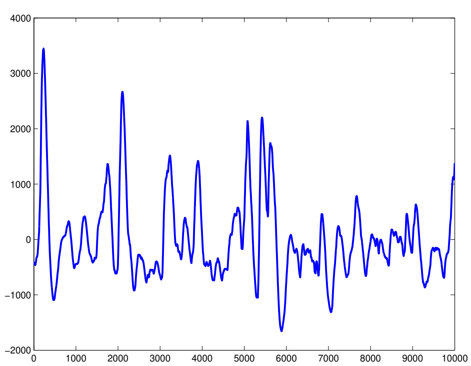

(b) AR(2)-LSV($\delta = -0.95$)



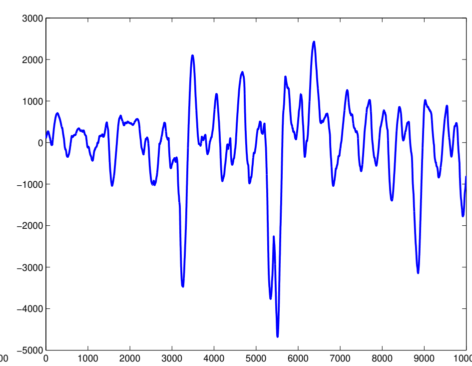

(c) AR(2)-ARLSV($\varphi = 0.9$,$\delta = 0.95$)




(d) AR(2)-ARLSV($\varphi = 0.9$,$\delta = -0.95$)




(e) AR(2)-LMLSV($d = 0.45$,$\delta = 0.95$)




(f) AR(2)-LMLSV($d = 0.45$,$\delta = -0.95$)

Figure 2: AR(2) examples with LSV innovations. $\phi_2 = -0.9875$ and $\omega = 0.01$.

## 3.2 GLM(d, $\omega$)-LSV($\delta$)

The interaction between a LSV($\delta$) process and a Gegenbauer long memory filter, defined as $(1 - 2L\cos\omega + L^2)^d$, produces a long memory ampli-

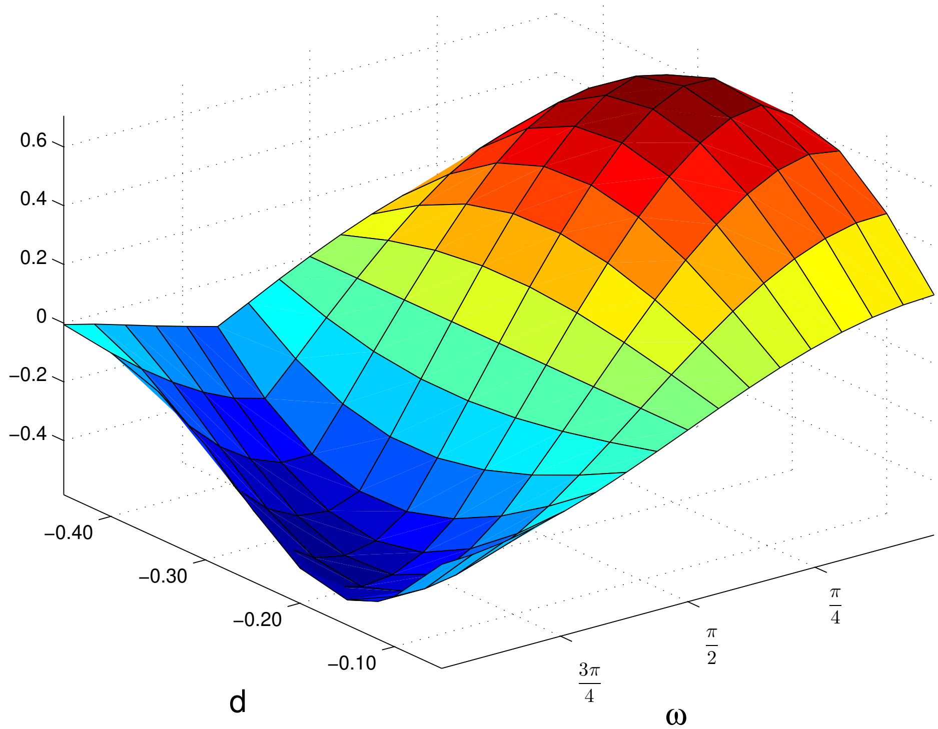



Figure 3: Skewness of a GLM($d$,$\omega$)-LSV($\delta$) process as a function of $d$ and $\omega$.

tude asymmetric cycle with period $\tau = \frac{2\pi}{\omega}$. Specifically if $h_t = \eta_t$ and $\left(1 - 2L\cos\omega + L^2\right)^d x_t = y_t$ is imposed in Eqs. 1 to 3 and 5, the third-order moment of $x_t$ is

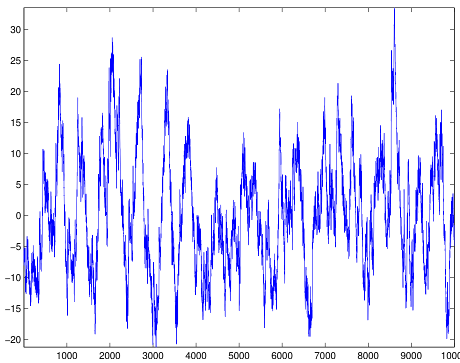
$$\mathbb{E}\left[x_t^3\right] = 3\sigma_*^3 \delta \sigma_\eta \exp\left(\frac{5\sigma_\eta^2}{8}\right) \sum_{i=0}^{\infty} \left[G_i^{(d)}\left(\cos\omega\right)\right]^2 G_{i+1}^{(d)}\left(\cos\omega\right), \qquad (11)$$

where $G_i^{(d)}\left(\cos\omega\right)$ are the inverse Gegenbauer coefficients which are defined as

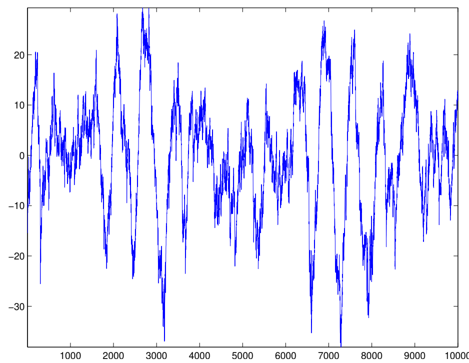
$$G_i^{(d)}\left(\xi\right) = \sum_{k=0}^{[i/2]} \frac{\left(-1\right)^k \Gamma\left(i - k + d\right)\left(2\xi\right)^{i-2k}}{\Gamma\left(k+1\right)\Gamma\left(i - 2k + 1\right)\Gamma\left(d\right)}.$$

Figure 3[3] shows the skewness of the GLM-LSV($\delta$) process $x_t$ with $\omega$ ranging from $\frac{59\pi}{60}$ up to $\frac{\pi}{60}$, decrementing by $\frac{3.625\pi}{60}$ and memory parameter

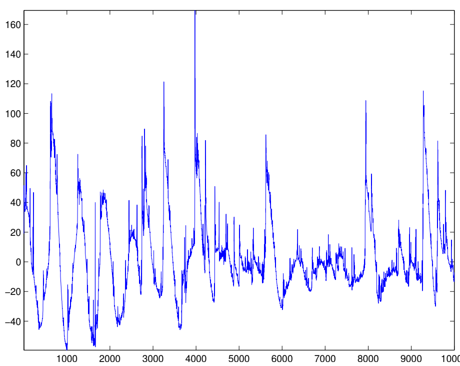
[3]The infinite sum in (11) is truncated at $i = 40000$ for this representation.



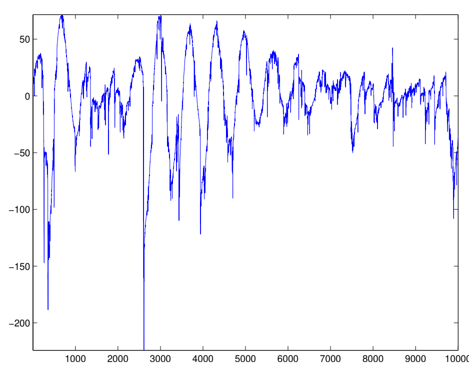
(a) GLM-LSV($\delta = 0.95$)



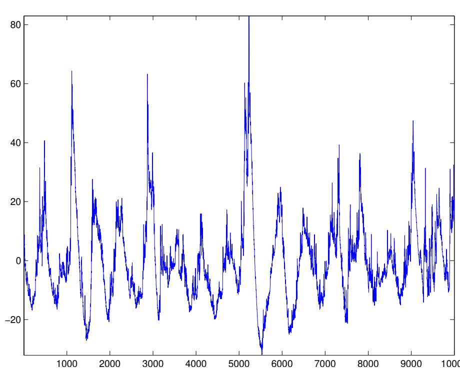
(b) GLM-LSV($\delta = -0.95$)



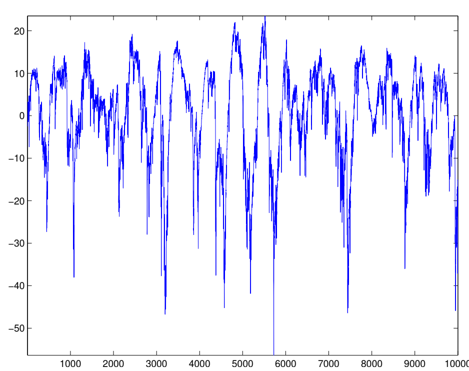
(c) GLM-ARLSV($\varphi = 0.9, \delta = 0.95$)



(d) GLM-ARLSV($\varphi = 0.9, \delta = -0.95$)



(e) GLM-GLMLSV($d = 0.25, \omega = 0.01,$
$\delta = 0.95$)



(f) GLM-GLMLSV($d = 0.25, \omega = 0.01,$
$\delta = -0.95$)

Figure 4: GLM($d = 0.45, \omega = 0.01$) examples with LSV innovations.

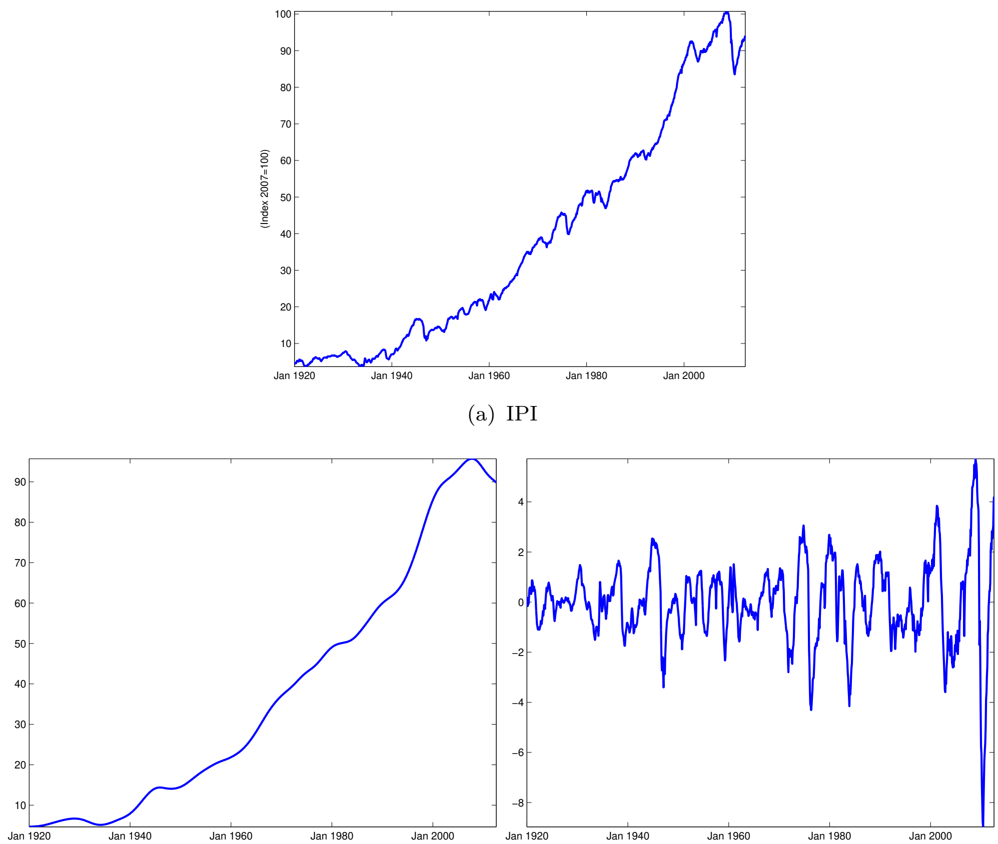
$d = 0.05, \ldots, 0.45$ incrementing by 0.05. It can be seen that the skewness behaves in accordance with the sign of the leverage parameter when $\omega < \frac{\pi}{2}$ (which implies business cycle frequencies) and opposite sign when $\omega > \frac{\pi}{2}$. It can be also appreciated that the skewness is larger in absolute value for low (business cycle) frequencies.

Figure 4 depicts six examples of GLM processes with parameters $d = 0.45$ and $\omega = 0.01$ and LSV innovations with $\delta = \pm 0.95$, no dependence structure in the two first cases, a first-order autoregressive process with parameter $\varphi = 0.9$ in the two second ones and another GLM process with parameters $d = 0.25$ and $\omega = 0.01$ in the two last ones. Again, only a slight deviation from amplitude symmetry is observed when the volatility process is characterized by no serial dependence but a leverage effect, that is consistent with the sign of $\delta$. Figures 4(c) and (d) show that this deviation increases considerably when the process of volatility obeys to a persistent AR(1) component and even more when the volatility displays long-memory (Figs. 4(e) and (f)).

# 4 Empirical illustration

The empirical illustration focuses on the analysis of a series of the Industrial Production Index (IPI). In this case we deal with a monthly version of the US seasonally adjusted IPI compiled by the Federal Reserve Bank of St. Louis. The sample spans from January 1919 to August 2011 with a length of $T = 1112$ months. The series, displayed in Figure 5(a), is characterized by a long-term increasing pattern altered by a series of fluctuations. The long-term component (Figure 5(b)) and the short-term (cyclical) component (Figure 5(c)) are separated using a Hodrick-Prescott filter with smoothing parameter $\lambda = 120000$ (see Maravall and del Rio, 2001). Sichel's test reports the presence of negative amplitude asymmetry (skewness $\hat{\gamma}_3 = -0.5738$) at the 10% level in the short-term component of this indicator.

The parametric maximum likelihood estimation of SV models in the time domain is very difficult to implement due to the non-linearity of the model and the presence of two different innovations, which in this case is aggravated by the inter-temporal correlation between them and the dependence structure in the levels. In the frequency domain it is substantially easier to implement the asymptotic approximation to maximum likelihood based on the so called Whittle function. Zaffaroni (2003) proves the validity of this approach for the estimation of the ARMA parameters in levels when volatility takes the




(a) IPI

(b) Long-Term Component

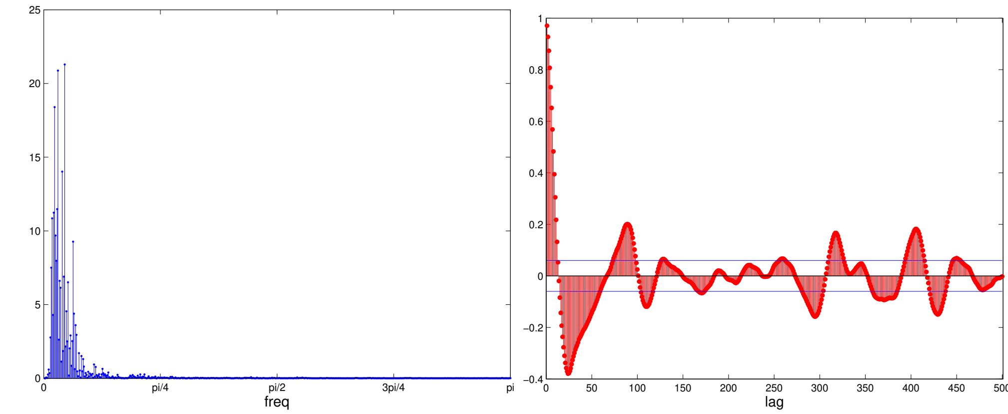
(c) Cyclical Component ($x_t$)

Figure 5: US SA Industrial Production Index

form of a non-linear moving average, closely related to SV models, that allows for a wide range of forms of persistence in volatility. This method is implemented here for the analysis of the levels of the series. Once the dependence structure for the levels has been extracted the parameters of the volatility process are estimated on the residuals by the Generalized Method of Moments (Hansen, 1982). Simulation analysis not included in this work[4], show that this sequential approach yields controlled variances and almost zero biases in finite samples.

Figure 6 depicts the correlogram and the periodogram of the extracted short-term component $x_t$ shown in Figure 5(c). The spectral peak that ap-

[4]But available upon request.




(a) Periodogram

(b) Correlogram

Figure 6: Periodogram and Correlogram of $x_t$.

pears in the periodogram at the low frequencies (Fig. 6(a)) can be considered as pertaining to a second-order autoregressive process. Once an AR(2) process is fitted to the series a MA(1) model is found appropriate for the remaining serial dependence. The joint Whittle estimation of all the parameters of the ARMA(2,1) model yields the results of $\hat{\phi}_1 = 1.9158$, $\hat{\phi}_2 = -0.9320$ and $\hat{\theta}_1 = 0.7929$. The estimated process is characterized by cyclicity, specifically at the frequency $\hat{\omega} = 0.1121$ ($\hat{\tau} = 56.0498$ months).

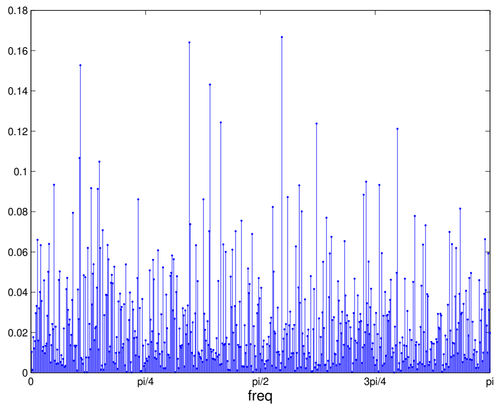
The periodogram and correlogram of the residuals $\hat{y}_t = \frac{1-1.9158L+0.9320L^2}{1-0.7929L} x_t$ and of the logarithm of the squares of the residuals $\log\left(\hat{y}_t^2\right)$ are displayed in Figure 7. In Figs. 7(a) and 7(b) it can be appreciated no remaining serial dependence in the levels of the series[5].

Figure 7(d) shows a persistent set of positive hyperbolically decreasing values in the correlogram of $\log\left(\hat{y}_t^2\right)$, which indicates that a long-memory process at the origin is appropriate for the volatility of the series. This statement is confirmed by the spectral peak in Figure 7(c).

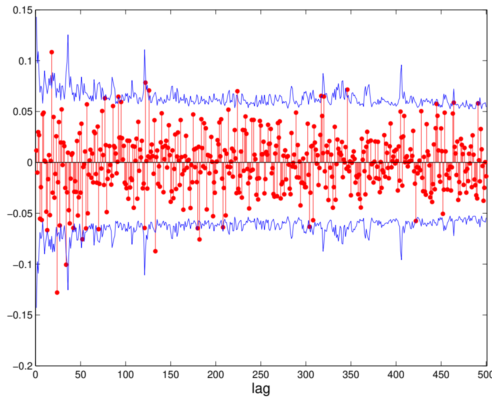
[5] Pérez (2000) shows that the asymptotic autocovariances of SV processes are defined as $\gamma_y(k) = \frac{\exp[\gamma_h(k)]}{T}$. However, as the process $h_t$ is unobservable, the asymptotic autocovariances of ARCH(q) processes as shown by Milhøj (1985) and Krämer and Runde (1994) and defined as

$$\gamma_y(k) = \frac{1}{T}\left[1 + \frac{\gamma_{y^2}(k)}{\sigma_y^2}\right]$$

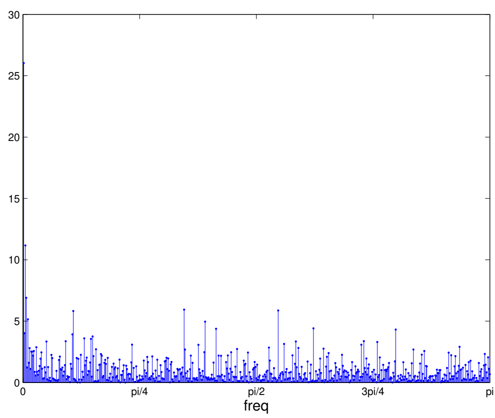
are used in the confidence intervals of the correlogram in Fig. 7(b) where $\gamma_{y^2}(k)$ and $\sigma_y^2$ are substituted by their sample counterparts calculated on the residuals $\hat{y}_t$.



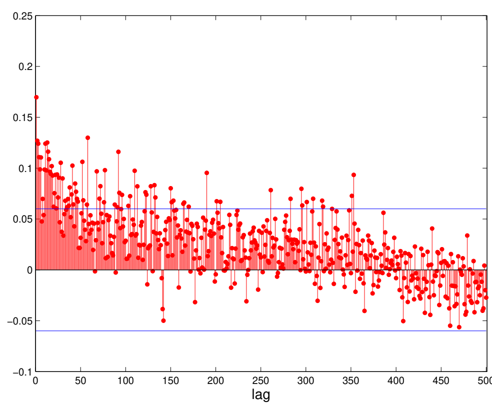

(a) Periodogram of $\hat{y}_t$




(b) Correlogram of $\hat{y}_t$




(c) Periodogram of $\log\left(\hat{y}_t^2\right)$




(d) Correlogram of $\log\left(\hat{y}_t^2\right)$

Figure 7: Dynamics of the residuals $\hat{y}_t$.

The estimation of all the parameters of the LMLSV model for the volatility (namely $\delta$, $d$, $\sigma_\eta$ and $\sigma_*$) is carried out jointly with the Generalized Method of Moments (see Hamilton, 1994, page 409 and following pages for a detailed description). The selected 16 targeting moments for the procedure are $\mathbb{E}\left[y_t^2\right] = \sigma_*^2 \exp\left(\sigma_h^2/2\right)$ and $\mathbb{E}\left[y_t^2 y_{t-k}\right]$ for $k = 1, \ldots, 4$, $\mathbb{E}\left[y_t^2 y_{t-j} y_{t-k}\right]$ for $j = 1, 2, 3$ and $k = j+1, \ldots, 4$, $\mathbb{E}\left[y_t^2 y_{t-k}^2\right]$ for $k = 1, \ldots, 4$ and $\mathbb{E}\left[y_t^4\right]$, calculated as in (4), (7), (8) and (9) respectively with $\sigma_h^2 = \sigma_\eta^2 \frac{\Gamma(1-2d)}{\Gamma(1-d)^2}$ and $\rho_h(k) = \frac{\Gamma(1-d)\Gamma(k+d)}{\Gamma(d)\Gamma(k+1-d)}$.

The GMM results of estimation of the parameters of the volatility are $\hat{d} = 0.4315$, $\hat{\sigma}_\eta^2 = 0.5731$, $\hat{\sigma}_* = 0.2256$ and $\hat{\delta} = -0.675$. The value of $\hat{\delta} = -0.675$ provides outstanding empirical evidence supporting the existence of a negative leverage effect in the volatility of the US IPI.

The volatility process $h_t$ can be estimated by means of the smoothing method proposed by Harvey (1998). This method requires the inversion of the Variance-Covariance matrix of $\log(y_t^2)$ which has T×T non-null entries due to the long memory components in $h_t$. However, in order to make the operation tractable and considering that the autocovariances at widely separated lags are negligible, the estimated Variance-Covariance Matrix is truncated here to a size of $151 \times 151$.

The standardised residuals $\hat{\varepsilon}_t$ of the complete ARMA(2,1)-LMLSV model are obtained from this smoothed estimation of the volatility as

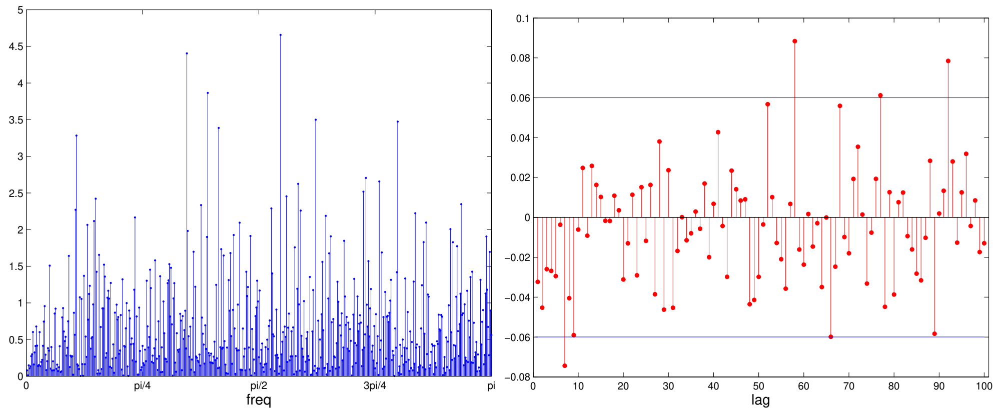
$$\hat{\varepsilon}_t = \frac{\hat{y}_t}{\hat{\sigma}_* \exp\left(\frac{\hat{h}_t}{2}\right)}.$$

Figure 8 shows the periodogram and correlogram of $\log\left(\hat{\varepsilon}_t^2\right)$, where no remaining dependence in the volatility can be appreciated. Therefore, the short-term cyclical component of the US IPI is coherent with an ARMA(2,1)-LMLSV process. The overall estimated model (with the SE, calculated via Monte Carlo in 10000 replications, in parenthesis) is

$$\left(1 - \underset{(0.0672)}{1.9158} L + \underset{(0.0656)}{0.9320} L^2\right) x_t = \left(1 - \underset{(0.0967)}{0.7929} L\right) y_t,$$

with

$$y_t = \underset{(0.0359)}{0.2256} \cdot e^{\frac{1}{2}(1-L)^{-\underset{(0.0707)}{0.4315}} \eta_t} \varepsilon_t,$$




(a) Periodogram

(b) Correlogram

Figure 8: Dynamics of $\log\left(\hat{\varepsilon}_t^2\right)$.

and

$$\widehat{Var}\left[\left(\begin{array}{c}\eta_{t+1}\\ \varepsilon_t\end{array}\right)\right]=\left[\begin{array}{cc}\underset{(0.3944)}{0.5731} & \underset{(0.1653)}{-0.6753}\sqrt{0.5731}\\ -0.6753\sqrt{0.5731} & 1\end{array}\right].$$

The estimated third-order moment $\widehat{\mathbb{E}\left[x_t^3\right]}=-2.1365$ closely matches the sampling $M_3=-2.4722$.

# 5 Conclusions

This paper provides a modelling strategy of the deepness or amplitude cyclical asymmetry, which is characterized by an oscillatory fluctuation that reaches different distances from the mean at peaks or troughs and constitutes a common phenomenon in macroeconomic data regarding the business cycle. Specifically, it is shown that the amplitude asymmetry is stochastically induced by the presence of a leverage effect or correlation between the shocks to the levels and the one-leaded shocks to the volatility when a Stochastic Volatility model is linearly filtered, which provides a flexible and global definition for both short and long memory amplitude asymmetric models. The degree and sign of asymmetry is governed by the degree and sign of the leverage correlation, the variance of the shocks to the volatility and the dependence structures at both levels and volatility.

Two cases are analysed in detail, the second-order autoregressive amplitude asymmetric model and the Gegenbauer long memory amplitude asymmetric model. It is shown that for these models the skewness (and consequently the amplitude asymmetry) is relatively larger at low (business cycle) frequencies for identical degrees of persistence and variance of the innovations.

The empirical application focuses on the US seasonally adjusted Index of Industrial Production. We find remarkable evidence of negative leverage in the volatilities of the cyclical component of this indicator, which accounts for the negative deepness asymmetry in the oscillations of the economic activity.

It is still unclear what causes the asymmetric volatility at the level of industrial production. Ho and Tsui (2003) point out the existence of heterogeneous beliefs and short-run supply-side constraints as one possible explanation. When a recessionary shock takes place, economic agents may tend to reduce investment, which leads to a further contraction. This effect may be aggravated if there exist heterogeneous beliefs about the future prognosis of the economy which produces a greater sense of uncertainty that induce a more prudent behaviour in (risk-averse) economic agents. On the contrary, if an expansionary shock is perceived, the subsequent desire to increase investment expenditure may be limited by the potential productive capacity of the economy in the short-run. The main policy implication for these asymmetric effects is constituted by a vindication of the government's role in stabilising the macroeconomic environment, above all in the onset of recessions, that may mitigate the adverse impact of negative shocks and in the end balance the fluctuation of the economic activity around the trend level on condition that the countercyclical measures are effective.

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