# Matrixes Satisfying Siljak's Conjecture

Tian, Guoqiang

11 October 1981

第 2 卷 第 1 期　科 学 探 索　Vol.2 No.1
1982年 3 月　SCIENCE EXPLORATION　Mar. 1982

# Matrixes Satisfying Šiljak's Conjecture*

Tian Guoqiang

(Huazhong Institute of Technology)

**Abstract**

*Šiljak's conjecture on the existence of a symmetric positive definite matrix V having a specified structure and satisfying Liapunov's matrix equation* $A^*V+VA=-W$ *is shown to be true in cases when A is an orthogonal matrix; when A is a symmetric matrix; when A is a normal matrix or A is the linear combination of nonnegative coefficients of all these matrixes.*

## § 1. Introduction

It is known that a differential equation $\dot{X}=AX$ is said to be stable if and only if every eigenvalue of $A$ has no nonnegative real part. A square matrix with this property is called a stable matrix. In order to study the stability of $\dot{X}=AX$, a well-known theorem wes proposed by Liapunov in distinguishing whether an equation is stable[1]. An $n\times n$ order complex matrix $A$ is stable if and only if for a positive definite matrix $W$ ($W$ can be any positive definite matrix) a positive definite matrix V satisfying

$$A^*V+VA=-W$$

is determined uniquely.

It is important to consider the stability of a system. In some systems, however, even if we know they are stable, $V$ is required that not only to be positive definite, but to every element of $V$ is also required to be nonnegative. Thus in 1972, Šiljak[2] gave the following

**CONJECTURE**: for Liapunov's algebra matrix equation,

$$A'V+VA=-W \tag{1}$$

where $A$, $V$, $W\in R^{n\times n}$ and $V$ and $W$ are positive definite matrixes. If the two conditions

Received October 11, 1981.

* This work is done under the direction of Mr. Li Chulin. The author wishes to express his sincere thanks.

1. $A$ is a stable matrix;

2. in each column of $A$ there is at least one negative element

are satisfied, then there exists a positive definite matrix $W$ such that the solution $V$ to $A'V+VA=-W$ is a nonnegative positive definite matrix, i. e.

$$V=V' \quad (v_{ij}=v_{ji},\ v_{ii}>0,\ v_{ij}\geqslant 0,\ i\neq j \text{ for any } i,j\in N,\ N=\{1,2,\cdots,n\}). \quad (2)$$

If the above conjecture holds, it will be very useful in studying large-scale system, mathematical models (whose state variables are nonnegative) of biological system, economic system and chemical system, etc.

At first, Šiljak[3] wanted to show the existence of the solution $V$ for his conjecture only under the condition1. But a simple counter example[2] demonstrated that it did not hold. Taking

$$A=\begin{bmatrix}1 & -2\\ 3 & -4\end{bmatrix},\quad V=\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}$$

$$(v_{11}>0,\ v_{22}>0,\ v_{12}=v_{21}\geqslant 0),$$

we have

$$-(A'V+VA)=-\begin{bmatrix}2v_{11}+6v_{12}, & -2v_{11}-3v_{12}+3v_{22}\\ -2v_{11}-3v_{12}+3v_{22}, & -4v_{12}-8v_{22}\end{bmatrix}. \quad (3)$$

Then for any $v_{11}>0$, $v_{12}\geqslant 0$, there is always $-(2v_{11}+6v_{12})<0$, thus it is impossible for the matrix (3) to be positive definite. So Šiljak added the condition 2 to his conjecture. Generally, it is very difficult to prove this conjecture. It was not until 1975 that some other conditions were added by Montemayer and womack[4], i. e. 1) $|A|\neq 0$; 2) $|A-\mu I|=0\Rightarrow v_i v_j\neq 1$ for any $i$ and $j$; 3) $|B-\lambda I|=|(I-AA)^{-1}A-\lambda I|=0\Rightarrow \mathrm{Re}(\lambda)<0$, to show the existence of the conjecture. Two years later, Datta[5] demonstrated its possibility for companion matrix of polynomial and Schwarz matrix, etc. In 1980, two Japanese[6] showed that the conjecture hold in cases when $n=2$ (for $n>2$, it remained unsettled); or when off-diagonal elements are nonnegative or when $A$ is a tridiagonal matrix. By now, no one can give a definition for its existence. What the author tries to do is to prove its truth in cases when $A$ is an orthogonal matrix; when $A$ is a symmetric matrix; when $A$ is a normal matrix or when $A$ is the linear combination of nonnegative coefficients of all these matrixes.

## § 2. The solution satisfying Šiljak's conjecture with $A$ being an orthogonal

Shown in the following are the main theorems we are going to propose and prove.

**THEOREM 1:** It is supposed that when $A\in R^{n\times n}$, $A$ satisfies

1 $A$ is a stable matrix;

2 There is at Ieast one negative element in each column in $A$;

3 $A$ is a orthogonal matrix, i. e. $A'A=\mathrm{I}$, $A'=A^{-1}$.

Hence, there is a solution $V$ satisfying Šiljak's conjecture, and $V=I$, where $I$ is an $n\times n$ order identity matrix.

To prove the theorem, it is necessary to prove Lemma 1 first.

**LEMMA 1**: For a stable orthogonal matrix $A$, the eigenvalue of $A'+A$ can only be $\lambda_i+\frac{1}{\lambda_i}(i=1,2,\cdots,m)$, where $\lambda_i$ is the eigenvalue of $A$. And if $\lambda_i$ is the $n_i$ multiple eigenvalue of $A$, then $\lambda_i+\frac{1}{\lambda_i}(i=1,2,\cdots,m)$ is the $n_i$ multiple eigenvalue of $A'+A$, where $n_1+n_2+\cdots+n_n=n$.

**PROOF**. It is known that $A$ must be similar to Jordon matrix, that is, there must be a nonsingular matrix $P$, such that

$$P^{-1}AP=J \tag{4}$$

where

$$J=\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_n \end{pmatrix}. \tag{5}$$

By taking inverse on both sides of (4)

$$(P^{-1}AP)^{-1}=J^{-1},$$

that is

$$P^{-1}A^{-1}P=P^{-1}A'P=J^{-1}=\begin{pmatrix} J_1^{-1} & & & \\ & J_2^{-1} & & \\ & & \ddots & \\ & & & J_m^{-1} \end{pmatrix}. \tag{6}$$

Thus, from (4) and (6), we have

$$P^{-1}(A'+A)P=J^{-1}+J=\begin{pmatrix} J_1+J_1^{-1} & & & \\ & J_2+J_2^{-1} & & \\ & & \ddots & \\ & & & J_m+J_m^{-1} \end{pmatrix} \tag{7}$$

where

$$J_i=\begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}_{n_i\times n_i} \tag{8}$$

($\lambda_i$ are the $n_i$ multiple eigenvalue of $A$). Now it is easy to verify that $J_i^{-1}$ has the following form

$$J_i^{-1}=\begin{pmatrix}\frac{1}{\lambda_i} & -\frac{1}{\lambda_i^2} & \frac{1}{\lambda_i^3} & \cdots & (-1)^{n_i}\frac{1}{\lambda_i^{n_i-1}} & (-1)^{n_i+1}\frac{1}{\lambda_i^{n_i}} \\ 0 & \frac{1}{\lambda_i} & -\frac{1}{\lambda_i^2} & \cdots & (-1)^{n_i-1}\frac{1}{\lambda_i^{n_i-2}} & (-1)^{n_i}\frac{1}{\lambda_i^{n_i-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\lambda_i} & -\frac{1}{\lambda_i^2} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{\lambda_i}\end{pmatrix}. \tag{9}$$

Hence, $J_i+J_i^{-1}$ is still an upper triangular matrix, and its diagonal elements are $\lambda_i+\frac{1}{\lambda_i}(i=1,2,\cdots,m)$.

Let $J^*_i=J_i+J_i^{-1}$, (10)

$J^*=J+J^{-1}$, (11)

then
$$|\lambda I_n-J^*|=|\lambda I_{n_1}-J_1^*|\,|\lambda I_{n_2}-J_2^*|\cdots|\lambda I_{n_m}-J_m^*|$$
$$=\prod_{i=1}^{m}|\lambda I_{n_i}-J_i^*|=\prod_{i=1}^{m}\left(\lambda-\lambda_i-\frac{1}{\lambda_i}\right)^{n_i}. \tag{12}$$

Thus, if $\lambda_i$ is the $n_i$ multiple eigenvalue of $A$, then $\lambda_i+\frac{1}{\lambda_i}$ is the $n_i$ multiple eigenvalue of $J+J^{-1}$ $(i=1,2,\cdots,m)$. In addition, by $P^{-1}(A'+A)P=J+J^{-1}$, it is known $A'+A$ is similar to $J+J^{-1}$, Hence $A'+A$ and $J+J^{-1}$ have the same characteristic polynomial, i. e. the same eigenvalues. Hence $\lambda_i+\frac{1}{\lambda_i}$ is the $n_i$ multiple eigenvalue of $A'+A$, i. e. the eigenvalue of $A'+A$ must take the form of $\lambda_i+\frac{1}{\lambda_i}$. The proof is completed.

By the above lemma, we can prove Theorem 1.

**PROOF.** First we prove that $A'+A$ is a negative definite matrix. Suppose $\lambda$ is any eigenvalue of $A$. As $A$ is stable, we know that $\lambda$ has a negative real part, i. e.

$$\lambda=a+ib\quad(a<0), \tag{13}$$

and we have
$$\frac{1}{\lambda}=\frac{1}{a+ib}=\frac{a-ib}{a^2+b^2}. \tag{14}$$

Thus, from Lemma 1, we know

$$\lambda+\frac{1}{\lambda}=a+ib+\frac{a-ib}{a^2+b^2}=a\left(1+\frac{1}{a^2+b^2}\right)+ib\left(1-\frac{1}{a^2+b^2}\right)$$

is the eigenvalue of $A'+A$. As $A$ is a real matrix, $A'+A$ is a real symmetric matrix. It is known that the eigenvalue of the real symmetric matrix must be a real number[7], so $\lambda+\frac{1}{\lambda}$ is a real number, i. e.

$$\lambda+\frac{1}{\lambda}=a\left(1+\frac{1}{a^2+b^2}\right)<0. \tag{15}$$

Hence, any eigenvalue of $A'+A$ is less than zero, as $A'+A$ is a negative definite matrix if and only if each eigenvalue of $A'+A$ is less than zero[8]. So $A'+A$ is a negative definite matrix while $-(A'+A)$ is a positive definite matrix. Let $W=-(A'+A)$, $V=I$, we have $A'V+VA=A'+A=-[-(A'+A)]=-W$, i. e. $V=I$ is a solution satisfying Šiljak's conjecture. The proof is completed.

## §3. The solution satisfying Šiljak's conjecture with $A$ being a symmertic matrix

**THEOREM 2**: If $A$ satisfies, in addition to the conditions 1 and 2, the condition of $A$ being a symmetric matrix, then we have the solution $V$ satisfying Šiljak's conjecture, and $V=I$.

**PROOF.** Knowing that $A$ is a stable real symmetric matrix, it can be easily seen that each eigenvalue of $A$ is less than zero, and each eigenvalue of $A'+A=2A$ is also less than zero. Thus, $A'+A$ is a negative definite matrix; $-(A'+A)=-2A$ is a positive definite matrix. Taking $W=-2A$, $V=I$, we have $A'V+VA=A'+A=2A=-(-2A)=-W$, i. e. $V=I$ is the solution satisfying Šiljak's conjecture. This completes the proof.

## §4. The solution of Šiljak's conjecture with $A$ being a normal matrix

The discussion made above is concerned with the real matrix $A$. However, when $A$ is a complex matrix, the result is exactly the same. We will show in the following that when $A$ is a normal matrix, the conjecture holds.

It is said that $A$ is a normal matrix, which shows that $A^*A=AA^*$, where $A^*$ is the conjugate transpose of $A$.

**LEMMA 2**: For the normal matrix $A$, there exists a unitary matrix $U$ such that $UAU^*$ is a diagonal matrix. (See Theorem 4′ in[9] at page 273.)

Using the above Lemma, it is easy for us to prove

**THEOREM 3**: If $A$ satisfies, in addition to 1, 2, the condition that $A$ is a normal matrix, then we have the solution $V$ satisfying Šiljak's conjecture, and $V=I$.

**PROOF.** By Lemma 2, we have a unitary matrix $U$ for the normal matrix $A$ such that

$$UAU^*=\Lambda \tag{16}$$

where $\Lambda$ is a diagonal matrix. Taking the conjugate transpose on both sides, we have

$$(UAU^*)^*=UA^*U^*=\Lambda^*. \tag{17}$$

From (16) and (17), we get

$$U(A+A^*)U^*=\Lambda+\Lambda^* \tag{18}$$

which means if $\lambda$ is the eigenvalue of $A$, the eigenvalue of $A+A^*$ is $\lambda+\bar{\lambda}$. By knowing $A$ is a stable matrix, the real part "$a$" in $\lambda = a + ib$ is less than zero, hence, $\lambda + \bar{\lambda} = 2a < 0$, i. e. every eigenvalue of $A + A^*$ is less than zero. So we know that $A + A^*$ is a negative definite matrix; $-(A + A^*)$ is a positive definite matrix (see [10]). Letting $W = -(A + A^*)$, $V = I$, we have $A^*V + VA = A^* + A = -[-(A^* + A)] = -W$, i. e. $V = I$ satisfies the solution of Šiljak's conjecture. The proof is completed.

## § 5. The solution satisfying Šiljak's conjecture with $A$ being a linear combination of nonnegative coefficients of thses matrixes

We will discuss again the case that $A$ is a real matrix. Generally speaking, the sum of two orthogonal matrixes does not always mean an orthogonal matrix. The consideration of the following may serve as an example of it.

Taking
$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{19}$$

$$A_2 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2} \end{bmatrix}, \tag{20}$$

it is easily to verify that both $A_1$ and $A_2$ are orthogonal matrixes, but their sum

$$A = \begin{bmatrix} -\frac{2+\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}+2}{2} \end{bmatrix} \tag{21}$$

is not. In the following, we'll prove more kinds of matrix satisfying Šiljak's conjecture by using Theorem 1 and 2.

**THEOREM 4:** If $A$ satisfies, in addition to 1 and 2, $A = A_1 + A_2$ where $A_1$ and $A_2$ are stable orthogonal matrixes, then, there exists the solution $V$ satisfying Šiljak's conjecture, and $V = I$.

**PROOF.** As $A' + A = A_1' + A_2' + A_1 + A_2 = (A_1' + A_1) = (A_2' + A_2)$, by Theorem 1, we know $A_1' + A_1$ and $A_2' + A_2$ are both negative definite matrixes, because the sum of the two negative definite matrixes is also a negative definite matrix (this is because for any non-zero vector Y, if $Y'V_1Y < 0$, $Y'V_2Y < 0$, then $Y'(V_1 + V_2)Y = Y'V_1Y + Y'V_2Y < 0$, i. e. $V_1 + V_2$ is negative definite). Letting $W = -(A' + A)$, and $V = I$, we have $A'V + VA = A' + A = -[-(A' + A)] = -W$, i. e. $V = I$ is the solution satisfying Šiljak's conjecture. This Completes the proof.

Applying mathematical induction we can easily deduce from Theorem 4 the following result:

**COROLLARY 1**: As the matrix $A$ satisfyies the conditions 1 and 2 of Šiljak's conjecture, if $A$ can be divided into $A=\sum_{i=1}^{m}\alpha_iA_i$, where $\alpha_i\geqslant 0$, $\sum_{i=1}^{m}\alpha_i>0$, $A_i$ is a stable orthogonal matrix ($i=1, 2, \cdots, m$), then there is the solution $V$ satisfying Šiljak's conjecture, and $V=I$.

It has been proved that the nonnegative coefficients linear combination of the orthogonal matrixes satisfies the conjecture. In the following, we will prove the nonnegative coefficients linear combination of symmetric matrixes and orthogonal matrixes also satisfies the conjecture.

**THEOREM 5**: As the matrix $A$ satisfies conditions 1 and 2 of the conjecture, if $A$ can be divided into $A=B+C$, where $B$ is a stable orthogonal matrix, $C$ a negative definite matrix, we have a positive definite matrix $V$ satisfying th conjecture, and $V=I$.

**PROOF.** $A'+A=(B'+B)+(C'+C)$. By Theorems 1 and 2, we know both $(B'+B)$ and $(C'+C)$ are negative definite matrixes, thus, $A'+A$ is a negative definite matrix. Taking $W=-(A'+A)$, $V=I$, we have $V=I$ is the solution satisfying the conjecture. The proof is completed.

The negative definite matrix $C$ in Theorem 5 can also be weaken into a semi-negative definite matrix. Thus we have

**COROLLARY 2**: If $A$ satisfies, in addition to 1 and 2, $A=B+C$ where $B$ is a stable orthogonal matrix and $C$ is a semi-negative definite matrix, we have a solution $V$ satisfying the conjecture, and $V=I$.

Similar to Corollary 1, from Theorem 5 and Corollary 2, we have

**COROLLARY 3**: As the stable matrix $A$ satisfies conditions 1 and 2, if $A$ can be divided into $A=\sum_{i=1}^{m}\beta_iB_i+\sum_{j=1}^{k}\gamma_jC_j$ where $\beta_i\geqslant 0$ $\sum_{i=1}^{m}\beta_i>0$ $A_i$ is a stable orthogonal matrix ($i=1,2,\cdots,m$), $\gamma_j\geqslant 0$, $\sum_{j=1}^{k}\gamma_i>0$, $C_j$ is a negative definite (or negative semi-definite) matrix, ($j=1,2,\cdots,k$), then there exists the solution $V$ satisfying the conjecture, and $V=I$.

Generally, the sum of two normal matrixes is not always normal matrix. But, by applying the same method as is shown in proving Theorem 4, it is proved that their sum satisfies the conjecture.

**THEOREM 6**: If $A$ satisfies, in addition to conditions 1 and 2, $A=A_1+A_2$ where $A_1$ and $A_2$ are both stable normal matrixes, then we have a positive definite matrix $V$ satisfying the conjecture, and $V=I$.

Similarly, we have the following

**COROLLARY 4**: If $A$ not only satisfies the conditions 1 and 2, but can be divided into $A=\sum_{i=1}^{m}\alpha_iA_i$ where $\alpha_i\geqslant 0$, $\sum_{i=1}^{m}\alpha_i>0$, $A_i$ is a stable normal matrix ($i=1, 2,\cdots,m$), we have a positive matrix $V$ satisfying the conjecture, and $V=I$.

## §6. Notions

It is easily seen from the above that discussions are concerned with the classes of matrix satisfying the conditions 1 and 2. It is interesting that the condition 2 is naturally satisfied for the real matrixes discussed above. This is because when $A$ is an orthogonal matrix, $A'+A$ is negative positive, and its diagonal elements are all less than zero, i. e. $2a_{ii}<0$, or $a_{ii}<0$ $(i=1,2,\cdots,m)$. Hence, the condition that each column has at least one negative element is satisfied. For the negative poaitive matrix $A$, it is evident that its diagonal elements are all less than zero. This also satisfies the condition 2. For the linear combination of nonnegative coefficients of these matrixes, its diagonal elements are less than zero, too. Thus for the real matrix $A$ discussed above, the condition 2 is redundant and can be cancelled.

## References

〔1〕 N. J. Pullman, *Matrix Theory and Its Application*, (Selected Topics), New York, 1976 pp. 175-8.
〔2〕 L. T. Grujic and D. D. Šiljak, *Stability of Large-scale system with stable and Unstable subsystem*, Proc (1972) JACC, paper 17-3, pp. 550-3.
〔3〕 D.D. Šiljak, *Stability of Large-scale systems*, Proc. 5th, IFAC Congr., C-32 (1972), pp. 1-11.
〔4〕 J. J. Montemayer and et al. IEEE Trans. Automat Contr., Ac-20 (1975), pp. 572-3.
〔5〕 B. N. Datta, IEEE Trans, Automat, Contr, Ac-22 (1977), pp. 132-3.
〔6〕 桑武宏，桑原道义，システムヒ制御，24，4 (1980)，pp. 287-8.
〔7〕 北京大学数学力学系几何与代数教研室代数小组，高等代数，人民教育出版社，1978，p. 356.
〔8〕 ibid.p. 363.
〔9〕 Ф. p. 甘特马赫尔，矩阵论，柯召译，高等教育出版社，1955，p. 273.
〔10〕 Joel N. Franklin, *Matrix Theory*, Prentice-Hall Inc., Englewood Cliffs, (1978), p. 105.

# 满足Šiljak猜想的矩阵

田国强
(华中工学院数学系)

## 提　要

1972年Šiljak 提出了对大系统稳定性研究很重要的猜测：若$A$是稳定矩阵，每一列至少有一负元素，则存在正定矩阵$W$，使得$A^{*}V+VA=-W$有非负正定解$V$. 至今只对某类矩阵证得猜测成立(见〔4,5,6〕)，本文不同于已有工作，证明了对于正交矩阵、对称矩阵、正规矩阵以及它们的非负系数线性组合，Šiljak 猜测是成立的.

## 作者简介

田国强，生于1956年，1980年毕业于华中工学院数学系，现是华中工学院经济数学专业研究生，主要研究方向是计量经济学和数理经济中的数学问题.