Condorcet's principle and the strong no-show paradoxes

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Abstract

We consider two no-show paradoxes, in which a voter obtains a preferable outcome by abstaining from a vote. One arises when the casting of a ballot that ranks a candidate in first causes that candidate to lose the election. The other arises when a ballot that ranks a candidate in last causes that candidate to win. We show that when there are at least four candidates and when voters may express indifference, every voting rule satisfying Condorcet's principle must generate both of these paradoxes.

1 Introduction

A Condorcet winner is a candidate for election who is preferred by a majority in all pairwise comparisons with the other candidates. Condorcet's principle says that a Condorcet winner must be elected whenever there is one (see [2]). This principle may have undesirable consequences.

Moulin's [6] no-show paradox arises when the addition of a ballot that ranks candidate x above candidate y may take victory away from x and give it to y. Moulin shows that this paradox is generated by every voting rule that satisfies Condorcet's principle when there are four or more candidates.[1]

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[1] The phrase "no-show paradox" was coined by Brams and Fishburn [1]. They note that paradoxes of this kind were described in 1910 by the Royal Commission Appointed to Enquire into Electoral Systems [9] and in 1913 by Meredith [5]. The Royal Commission observes that under the Single Transferable Vote system a party may win two seats instead of one if some of its supporters abstain from the vote. Meredith, also discussing that voting system, describes a paradoxical election:

> Suppose that D (Nationalist), M (Ind. Unionist) and Z (Unionist) are three [candidates] ... D has 410 votes, M 400, and Z 500. Then M is eliminated, and his votes may be supposed to be transferred to Z, who is elected. But if D were

In this paper we consider two special cases of the no-show paradox. One arises when the casting of a ballot that ranks a candidate in first causes that candidate to lose the election. The other arises when a ballot that ranks a candidate in last causes that candidate to win. We call these the strong no-show paradoxes after Pérez [8] and Nurmi [7]. One or both of these special cases of the paradox are also considered by Smith [12], Richelson [10], Brams and Fishburn [1], Saari [11] and Lepelley and Merlin [4].

Pérez [8] demonstrates that voting rules do exist that satisfy Condorcet's principle and that never generate either of the strong no-show paradoxes, no matter the number of candidates.

Moulin [6] and Pérez [8] both consider the aggregation of linear orderings. Crucially, in this paper we consider instead the aggregation of weak orderings. In other words, we allow voters to express indifference. We show that, in this case, both of the strong no-show paradoxes are generated by every rule that satisfies Condorcet's principle when there are four or more candidates.

2 Notation

Let A be a finite set of candidates, and let N_∞ be a finite or countably infinite set of potential voters. Every finite subset of N_∞ is called an electorate. Let $W(A)$ be the set of all weak orderings on A. By a weak ordering we mean a binary relation on A that is transitive, complete and reflexive.

A *profile* assigns a weak ordering to each voter in an electorate. For every electorate N there is a set of possible profiles $W(A)^N$. We write u_{-i} to denote the profile obtained by removing individual i from profile u.

A voting rule is a function S that assigns a candidate to every possible pair of electorate and profile. So, given an electorate N and a profile u in $W(A)^N$, $S(N, u)$ is the winning candidate.

Given electorate N and profile u in $W(A)^N$, let n_{ab} be the balance of the number of voters who prefer a to b to those who prefer b to a. Let m_a be the greatest value taken by n_{ba} over all b in $A \setminus \{a\}$. Candidate a is a Condorcet winner if and only if $m_a < 0$. If $m_a > 0$ then m_a is the margin of a's greatest pairwise defeat. And m_a is zero if a does not suffer any pairwise defeat but does tie with another candidate in a pairwise comparison.

3 Theorem

We first define *Condorcet Consistency.*

Condorcet Consistency. For all candidates a, all electorates N and all profiles u in $W(A)^N$, if $m_a < 0$ then $S(N, u) = a$.

eliminated before M, we may easily suppose that his votes would go to M, who would be elected. The injustice of the result appears even more striking when we reflect that, if D had had 11 votes less, his supporters would have succeeded in returning M instead of Z, as they desired to do.

Next we define *Positive Involvement* and *Negative Involvement*. A voting rule that is free from one of the two strong no-show paradoxes will have one of these two properties. And a voting rule free from both paradoxes will have both properties.

Positive Involvement. For all electorates N that contain at least two voters, all voters i in N and all profiles u in $W(A)^N$, if i weakly prefers $S(N \setminus \{i\}, u_{-i})$ to all other candidates then i weakly prefers $S(N, u)$ to all other candidates.

The Positive Involvement criterion requires that if a candidate ranked in first by individual i is elected when i does not participate, then a candidate ranked in first by i should also be elected when i does participate.

Negative Involvement. For all electorates N that contain at least two voters, all voters i in N and all profiles u in $W(A)^N$, if i weakly prefers every candidate to $S(N, u)$ then i weakly prefers every candidate to $S(N \setminus \{i\}, u_{-i})$.

The Negative Involvement criterion requires that if a candidate ranked in last by individual i is elected when i participates, then a candidate ranked in last by i should also be elected when i does not participate.

We conclude by stating and proving the theorem.

Theorem. *(i) If there are three candidates or fewer then there are Condorcet Consistent voting rules that satisfy both Positive Involvement and Negative Involvement.*

(ii) If there are at least four candidates and at least 37 (34) potential voters then there is no Condorcet Consistent voting rule that satisfies Positive (Negative) Involvement.

Proof. Following Moulin [6], we prove part (i) by giving the example of a voting rule that always elect a candidate from the Kramer set (see [3]).

Given (N, u), let K, the Kramer set, be the set of all candidates a that minimise m_a. That is, $K = \{a \in A \mid m_a \leq m_b \text{ for all } b \in A\}$. If K is a singleton set then voting rule S elects that candidate in K. And if K contains more than one candidate then S elects the candidate in K whose name comes first by lexicographic order.

If there is a Condorcet winner a then it follows that $m_a < 0$ and $m_b > 0$ for all other candidates b. Then $K = \{a\}$ and so a will be elected by S. So S is Condorcet Consistent. This is true no matter how many candidates there are.

We now establish that S satisfies Positive Involvement and Negative Involvement when there are three candidates. Let us label the candidates a, b and c. These labels mask the names of the candidates so that their lexicographic ordering by name is unknown to us. Suppose that S elects a at some profile. It follows that $m_a \leq m_b$ and $m_a \leq m_c$. Let us add a voter to this profile to form a new profile.

We consider three cases. The first case is that the new voter strictly prefers a to b and c. The second case is that the new voter weakly prefers b to a, and strictly prefers a to c. The final case is that the new voter weakly prefers both b and c to a.

In the first case m_a falls by one. Since we are adding just one new voter, all margins of pairwise defeat/victory can change by at most one. So m_b and m_c cannot fall by more than one. In other words, m_a remains equal to or falls below (or further below) each of m_b and m_c as a result of the additional voter. Therefore, the new Kramer set is a subset of the original Kramer set, and must contain a. Hence, S elects a at the new profile, consistent with both Positive Involvement and Negative Involvement.

In the second case m_c increases by one, and m_a cannot increase by more than one. In other words, m_c remains equal to or rises above (or further above) m_a. So c can only be in the new Kramer set if that set also contains a, and c was in the original Kramer set (implying that candidate a's name comes before c's by lexicographic order). Hence, S elects a or b at the new profile, as required by both Positive Involvement and Negative Involvement.

In the final case both Positive Involvement and Negative Involvement permit any outcome at the resulting profile, so no contradiction can arise.

When there are just two candidates then it is clear that simple majority rule with lexicographic tie-breaking will satisfy Positive Involvement and Negative Involvement.

We proceed to statement (ii) of the theorem.

Let us now assume that A contains at least four candidates and that S is Condorcet Consistent. We first prove that (iia) if N_∞ contains at least 37 potential voters then S does not satisfy Positive Involvement, and then we prove that (iib) if N_∞ contains at least 34 potential voters then S does not satisfy Negative Involvement. The proof of (iia) is based on the proof of statement (ii) in [6].

By way of contradiction, assume that N_∞ contains at least 37 potential voters and S satisfies Positive Involvement. We make the following claim. For all distinct candidates a and b, every electorate N and every profile u in $W(A)^N$,

$$m_a + 1 \leq 37 - |N| \text{ and } m_a + 2 \leq n_{ab} \text{ implies } S(N, u) \neq b. \tag{1}$$

To prove (1), take any electorate N, a profile u in $W(A)^N$ and candidates a and b such that $m_a + 1 \leq 37 - |N|$ and $m_a + 2 \leq n_{ab}$ and assume that $S(N, u) = b$. Since S is Condorcet Consistent, a cannot be a Condorcet winner. Therefore, $m_a \geq 0$. Let M and v be an electorate and profile obtained by adding $m_a + 1$ voters to (N, u), all of whom rank b alone in first and a alone in second.

At (N, u) the greatest margin of defeat suffered by a is m_a, and a defeats b by a margin of at least $m_a + 2$. So the addition of $m_a + 1$ voters who all rank b alone in first and a alone in second results in all of a's pairwise defeats being reversed, while a continues to pairwise defeat b. So candidate a is a Condorcet winner at (M, v). However, since b is elected at (N, u), Positive Involvement requires that b is elected at (M, v). This contradiction establishes (1).

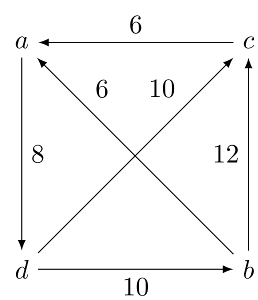
To complete the proof of (iia) we construct two profiles. Take any four candidates a, b, c and d. The first of the two profiles is described in Table 1.

6	3	8	7
a	a	d	b
d	d	b	c
c	b	c	a
b	c	a	d

Table 1: A profile.

Each number above the horizontal line indicates the number of voters who have submitted the ranking below that number. All other candidates (if there are any) are ranked below a, b, c and d by the voters.

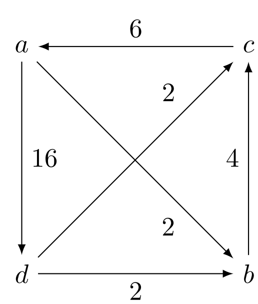
The following directed, weighted graph indicates the margins of pairwise victory and defeat amongst the top four candidates. An edge is directed from b to a and carries a weight of 6 to indicate that b pairwise defeats a by a margin of 6, and so on.



There are 24 participating voters, and we have $m_a = 6$ and $n_{ad} = 8$ so, by (1), S cannot elect d. We have $m_d = 8$ and $m_{db} = 10$ so S cannot elect b. For every candidate x in $A \setminus \{a, b, c, d\}$ we have $m_a = 6$ and $n_{ax} = 24$ so S cannot elect x. Hence the winner must be a or c.

Now let us add eight voters to that first profile to create a second profile. All eight of these new voters are indifferent between a and c, and rank those two candidates in joint first place. Their next most preferred candidate is b, followed by d, and they rank all other candidates (if there are any) below d.

Now the graph is as follows.



There are 32 participating voters, and we find that $m_c = 4$ and $n_{ca} = 6$, so, again by (1), S cannot elect a. We also have $m_b = 2$ and $n_{bc} = 4$, so S cannot elect c. However, Positive Involvement implies that S must elect a or c. This contradiction completes the proof of statement (iia).

Finally, we prove statement (iib).

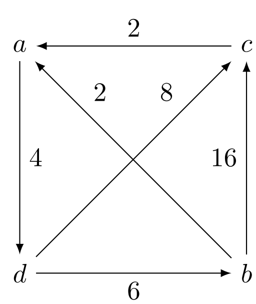
Let us now assume that N_∞ contains at least 34 potential voters and that S satisfies Negative Involvement (and not necessarily Positive Involvement). Take any four candidates a, b, c and d. A profile is described in Table 2. Let us call this profile u. In this table we use Greek letters to label the six weak

α	β	γ	δ	ϵ	ζ
4	5	1	6	4	6
b	a	d	a	d	b
c	c	a	d	b	d
a	d	b	b	c	c
dx	bx	cx	cx	ax	ax

Table 2: Another profile.

orderings that appear in the profile. We say that there are four α voters, five β voters and so on. The x in each column marks the position of all candidates x in $A \setminus \{a, b, c, d\}$ (if there are any), and it is written next to another letter to indicate indifference. For example, the four α voters are indifferent among the candidates in $A \setminus \{a, b, c\}$, and rank all of those candidates in last.

The following graph indicates the margins of pairwise victory and defeat amongst the top four candidates.



There are 26 participating voters. If three of the four α voters are deleted from profile u then a's pairwise defeats to b and c are reversed and a becomes a Condorcet winner. So, by Negative Involvement, a candidate in $\{a, b, c\}$ must be elected at profile u. If instead we delete the five β voters then d becomes a Condorcet winner. So b (ranked in last by those voters) cannot be elected at profile u. The winner must be a or c.

Now let us add eight voters to create a final profile v. All eight of these new voters rank c alone in first, a alone in second, and rank all other candidates in joint last place.

Now the graph is as follows (there is a pairwise tie between c and d).

$$\begin{tikzcd}
a \arrow[d, "12"] \arrow[dr, "6"'] & c \arrow[l, "10"'] \\
d \arrow[r, "6"'] & b \arrow[u, "8"]
\end{tikzcd}$$

There are 34 participating voters. If we delete the four ϵ voters and five of the six ζ voters then c becomes a Condorcet winner. So a cannot be elected at profile v. If instead we delete the single γ voter and the six δ voters then b becomes a Condorcet winner. So c cannot be elected at profile v. However, Negative Involvement implies that S must elect a or c.

This contradiction completes the proof of the theorem. □

References

[1] S. Brams and P. Fishburn, Paradoxes of preferential voting, Mathematics Magazine 56, No. 5 (1983), 207-214.

[2] A. N. Condorcet, Marquis de, Essai sur l'application de l'analyse à la probabilité des decisions rendues à la probabilité de voix, Paris, 1785.

[3] G. Kramer, A dynamical model of political equilibrium, Journal of Economic Theory 16 (1977), 310-334.

[4] D. Lepelley and V. Merlin, Scoring run-off paradoxes for variable electorates, Economic Theory 17 (2001), 53-80.

[5] J. C. Meredith, Proportional Representation in Ireland, University Press, Dublin, 1913.

[6] H. Moulin, Condorcet's principle implies the no show paradox, Journal of Economic Theory 45 (1988), 53-64.

[7] H. Nurmi, Voting Procedures under Uncertainty, Springer-Verlag, Berlin, 2002.

[8] J. Pérez, The strong no-show paradoxes are a common flaw in Condorcet voting correspondences, Social Choice and Welfare 33 (2001), 343-359.

[9] Report of the Royal Commission Appointed to Enquire into Electoral Systems, HMSO, London, 1910.

[10] J. T. Richelson, A comparative analysis of social choice functions III, Behavioral Science 23 (1978), 169-176.

[11] D. G. Saari, Basic Geometry of Voting, Springer-Verlag, New York, 1995.

[12] J. H. Smith, Aggregation of preferences with variable electorate, Econometrica 41, No. 6 (1973), 1027-1041.