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Abstract

We introduce new axioms for the class of all TU-games with a fixed but arbitrary player set, which require either invariance of an allocation rule or invariance of the payoff assigned by an allocation rule to a specified subset of players in two related TU-games. Comparisons with other axioms are provided. These new axioms are used to characterize the Shapley value, the equal division rule, the equal surplus division rule and the Banzhaf value. The classical axioms of efficiency, anonymity, symmetry and additivity are not used.

Keywords: uniform addition invariance, uniform transfer invariance, Shapley value, equal division rule, equal surplus division rule, Banzhaf value.
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1 Introduction

The principle of invariance takes on various forms in many fields of economic theory. The most known example is maybe the use of conservation laws in growth theory celebrated by Samuelson [17], who shows that the aggregate capital-output ratio is an invariant in a von Neumann economy. Sato [19] deepens the analogy with classical mechanics by applying the Noether theorem to growth theory. Conservation laws are also used by Kamiya and Talman [10] to study equilibrium properties in matching theory. Other examples of the principle of invariance are the invariance of tests to certain types of data transformations in econometric theory (Kemp [11]), the invariance of economic indexes to changes in units of measurement (Richter [16], Samuelson and Swamy [18]), the risk invariance of utility functions in decision theory (Willig [23]), and the invariant probability distributions characterizing the long run behavior in markovian economic models (Futia [6]).

In non-cooperative game theory, a principle of invariance is introduced by Kohlberg and Mertens [13], who require that a solution concept for normal-form games should be invariant to the deletion or addition of payoff-equivalent strategies. The axiom of independence of irrelevant alternatives, used by Arrow [1] in social choice theory and by Nash [14] in bargaining theory, relies on a similar principle of invariance. It states that the solution to a problem does not change as the set of alternatives is reduced, so long the solution still belongs to the set of feasible alternatives.

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In cooperative game theory, an informal definition of the principle of invariance is as follows. An (allocation) rule is called invariant between two TU-games if it assigns the same payoff vector in these TU-games. It is then crucial to delineate desirable transformations between two TU-games from which the invariance of a rule can be required. The first axiom of *uniform addition invariance* that we introduce is in this spirit. More specifically, it says that if the worths of all coalitions of a given size (except the grand coalition) fluctuate by the same amount, then the allocation rule should prescribe the same payoff vector. The rational behind this axiom is that any two players belong to the same number of coalitions of a given size, so that they should bear the same impact from a uniform variation of worth in all such coalitions. Since, in addition, the worth of the grand coalition is unchanged, it makes sense to require that the overall effect is null for every player.

A weaker principle of invariance for a rule requires the invariance of specific players' payoffs only. The second axiom of *uniform transfer invariance* that we propose is in this vein. It states that if the worth of two coalitions of the same size is affected by opposite amplitudes, then the payoffs of the players belonging to both coalitions should not vary. The intuition is that the per-capita impact within the two involved coalitions should be of opposite magnitudes. Thus it is reasonable to think that an exact compensation will occur for each player belonging to both coalitions.

These two axioms are satisfied by many natural rules. We show that any additive, efficient and symmetric rule satisfies uniform addition invariance, and that any additive and anonymous rule satisfies uniform transfer invariance (Proposition 1). Therefore, the equal division rule, the equal surplus division rule and the Shapley value satisfy these two axioms. The converse implications do not hold. Even if the axioms of invariance possess some flavor of efficiency and symmetry, it turns out that these axioms are not related. Nevertheless, the two axioms of invariance are useful to characterize many rules on the class of all TU-games with a fixed but arbitrary player set without relying on the classical axioms of efficiency, symmetry, anonymity and additivity.

Firstly, uniform addition invariance, uniform transfer invariance, the nullifying property and a restricted version of covariance characterize the equal division rule (Theorem 1). The nullifying player property requires that a player obtains a zero payoff if the worths of all coalitions to which he belongs is zero. Secondly, uniform addition invariance, uniform transfer invariance, covariance and the nullifying player property for zero-normalized TU-games characterize the equal surplus division rule (Theorem 2). Thirdly, uniform addition invariance, uniform transfer invariance and the dummy player property characterize the Shapley value (Theorem 3). The dummy player property requires that a player obtains his stand-alone worth if his (marginal) contribution to any coalition he belongs to is exactly his stand-alone worth. Fourthly, a stronger version of uniform transfer invariance in which the two involved coalitions can have different sizes is used together with the dummy player property to characterize the Banzhaf value (Theorem 4). To the best of our knowledge, Hamiache [7] and Driessen [5] are the only other articles in which the classical axioms of efficiency, anonymity, symmetry and additivity are not incorporated to the characterizing set of axioms.

Our characterizations of well-known rules are comparable since they have in common the two axioms of invariance. As such, our results are in the spirit of the comparable characterizations of the equal division rule and of the Shapley value studied in van den Brink [22] and Kamijo and Kongo [9]. Kamijo and Kongo [9] also use axioms of invariance. The main difference is that the invariance axioms compare the payoffs in a TU-game before and after the deletion of a player from the player set instead of keeping the same player set and altering the TU-game under consideration as in the present article. As such Kamijo and Kongo [9] have to consider a class of TU-games with a variable player set whereas we deal with a fixed but arbitrary player set. Nonetheless, both articles share the use of axioms such as the dummy player property and the nullifying player property in order to distinguish between egalitarian and marginalist rules.

Besides Kamijo and Kongo [9], different principles of invariance appear in the literature. The closest article is maybe Young [24], in which the axiom of marginality requires that a player's payoff is the same in two TU-games if all his contributions are identical in the two TU-games. Young [24] proves that marginality, anonymity and efficiency characterize the Shapley value. A similar article is Chun [4], in which the axiom of coalitional strategic equivalence is used in order to characterize the Shapley value. This axiom states that if a multiple of a unanimity TU-game on a coalition is added to a TU-game, then the payoff of each player outside of this coalition should be invariant in the constructed TU-game. It has been proved that marginality and coalitional strategic equivalence are equivalent. We show that each of our axioms of invariance does not imply nor is implied by marginality. It should also be noted that the characterizations of the Shapley value in [24] and [4] incorporate at least one of the classical axioms of efficiency, symmetry and additivity. The axioms of associated consistency in Hamiache [7] and $\mathcal{B}$-consistency in Driessen [5] require invariance of a rule if the worth of every coalition is adjusted in particularly specific ways, which allow much less freedom than our axiom of uniform addition invariance. Finally, our research can be connected to Kleinberg and Weiss [12] in which the set of TU-games for which the Shapley value recommends the null payoff vector is characterized. The link with our article is that requiring invariance for an additive rule between two TU-games is equivalent to impose that the rule specifies the null payoff vector in the TU-game obtained by taking the difference between the two starting TU-games.

Our techniques of proof are also new. They consist of an arbitrary selection of both a TU-game and a player. Then, the TU-game is transformed into other TU-games by a chain of modifications as described by the axioms of invariance. The chosen player plays a particular role in the ultimate TU-game of the chain in the sense that an axiom can be used to determine this player's payoff. For instance, he is a dummy player in the final TU-game that we construct in the proof of the characterization of the Shapley value, so that his payoff is given by the dummy player property. Then, the axioms of invariance are employed to conclude on the payoff of the chosen player in the original TU-game. Apart from the present article, the idea to wend through the space of TU-games by constructing chains of TU-games connected through some axioms is also used by Hamiache [7] and Driessen [5], and by Pintér [15] for an alternative proof of Young's [24] result. A drawback of the approach in Hamiache [7] and Driessen [5] is that the chain of TU-games asymptotically converges, which necessitates to use a technical axiom of continuity.

The rest of the article is organized as follows. Section 2 gives the game-theoretical definitions. The axioms used throughout the article are presented in section 3. Section 4 contains a study of the axioms of invariance. Section 5 provides the characterizations of the equal division rule, the equal surplus division rule, the Shapley value and the Banzhaf value. Section 6 concludes.

2 Preliminaries

Let $N = \{1, \ldots, n\}$ be a finite set of players, which is fixed for the rest of the article. A *cooperative game with transferable utility* or simply a *TU-game* on N is a *characteristic function* $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. For each $C \in 2^N$, $v(C)$ is the *worth* of *coalition* C and c its cardinality. Throughout this article, we consider the set of all TU-games on N, with typical elements denoted by v, w and u. For each $i \in N$, let $\mathcal{C}^i$ and $\mathcal{C}^{ic}$ be the sets of all coalitions containing i and of all coalitions of size $c \in \{1, \ldots, n\}$ containing i, respectively.

A TU-game v is *symmetric* if for each $C \in 2^N$ and each $C' \in 2^N$ such that $c' = c$ it holds that $v(C) = v(C')$. For each nonempty coalition $C \in 2^N$ and any real $a \in \mathbb{R}$, the *a-Dirac TU-game* on C is the TU-game 1_C^a such that, for each $C' \in 2^N$, $1_C^a(C') = a$ if $C = C'$, and $1_C^a(C') = 0$ otherwise. For any two TU-games v and w, the TU-game $(v + w)$ is defined, for each $C \in 2^N$, as $(v + w)(C) = v(C) + w(C)$. For any v, any real $a \in \mathbb{R}\backslash\{0\}$ and any $b \in \mathbb{R}^n$, the TU-game $(av + b)$ is such that, for each $C \in 2^N$, $(av + b)(C) = av(C) + \sum_{i\in C} b_i$. A permutation π on N assigns a position $\pi(i)$ to each player $i \in N$ and, for each $C \in 2^N$, $\pi(C) = \{\pi(i)\}_{i\in C}$. For a TU-game v and a permutation π, the TU-game πv is such that, for each $C \in 2^N$, $\pi v(\pi(C)) = v(C)$.

Two players $i, j \in N$ are *symmetric* in v if for each $S \subseteq N\backslash\{i, j\}$, $v(C \cup \{i\}) = v(C \cup \{j\})$. A player $i \in N$ is *dummy* in v if, for each $C \in \mathcal{C}^i$, $v(C) - v(C\backslash\{i\}) = v(\{i\})$. A player $i \in N$ is *nullifying* in v if, for each $C \in \mathcal{C}^i$, $v(C) = 0$.

A *payoff vector* $x \in \mathbb{R}^n$ on N is an n-dimensional vector giving a payoff $x_i \in \mathbb{R}$ to each player $i \in N$. An *allocation rule* or simply a *rule* is a function f that assigns to each TU-game v on N a payoff vector $f(v) \in \mathbb{R}^n$. The four following well-known rules will be used in the rest of the article.

The *equal division rule is the rule ED* such that:

$$\forall v,\ \forall i \in N, \qquad ED_i(v) = \frac{v(N)}{n}.$$

The *equal surplus division rule* is the rule ESD such that:

$$\forall v,\ \forall i \in N, \qquad ESD_i(v) = v(\{i\}) + \frac{v(N) - \sum_{j\in N} v(\{j\})}{n}.$$

The *Shapley value* (Shapley [20]) is the rule Sh such that:

$$\forall v,\ \forall i \in N, \qquad Sh_i(v) = \sum_{C\in\mathcal{C}^i} \frac{(n-c)!(c-1)!}{n!}\Big(v(C) - v(C\backslash\{i\})\Big).$$

The *Banzhaf value* (Banzhaf [2]) is the rule β such that:

$$\forall v,\ \forall i \in N, \qquad \beta_i(v) = \sum_{C\in\mathcal{C}^i} \frac{1}{2^{n-1}}\Big(v(C) - v(C\backslash\{i\})\Big).$$

3 Usual axioms and axioms of invariance

3.1 Usual axioms

We start by recalling a list of classical axioms used in the literature. Let f be any rule and v and w be any two TU-games.

Efficiency. $\sum_{i\in N} f_i(v) = v(N)$.

Anonymity. For each $i \in N$, $f_{\pi(i)}(\pi v) = f_i(v)$.

Symmetry. If $i \in N$ and $j \in N$ symmetric in v, then $f_i(v) = f_j(v)$.

Dummy player property. If $i \in N$ is a dummy player in v, then $f_i(v) = v(\{i\})$.

Nullifying player property. If $i \in N$ is a nullifying player in v, then $f_i(v) = 0$.

Additivity. $f(v + w) = f(v) + f(w)$.

Covariance. $f(av + b) = af(v) + b$.

Fairness. If $i \in N$ and $j \in N$ are symmetric in w, then $f_i(v + w) - f_i(v) = f_j(v + w) - f_j(v)$.

Marginality. If for each $C \in \mathcal{C}^i$, $v(C) - v(C\backslash\{i\}) = w(C) - w(C\backslash\{i\})$, then $f_i(v) = f_i(w)$.

The nullifying player property, fairness, marginality and are studied in van den Brink [22], van den Brink [21], Young [24] respectively. Casajus [3] proves that marginality and the axiom coalitional strategic equivalence introduced by Chun [4] are equivalent (see Proposition 3 and footnote 3 on page 169) and van den Brink [21] shows that fairness does not imply nor is implied by marginality. We also refer to Casajus [3] for a comparison with the closely related axiom of **differential marginality**, which is proved to be equivalent to fairness. Anonymity implies symmetry, while the converse implication is not true.

3.2 Axioms of invariance

In order to motivate two new axioms of invariance, let us start with a very intuitive property that many rules satisfy: a player's payoff should not decrease if the worth of one of his coalitions increases. This is exactly the idea of the axiom of **coalitional monotonicity** introduced by Young [24]. In a sense, one can see such a change in worth of a coalition as a modification of the total payoff that its members can claim as a whole. Let us go further in this direction on two aspects.

Firstly, we can think about many ways to reflect the change in worth of a coalition on its members. Nonetheless, since the worth of any other coalition remains the same, only the concerned coalition as a single entity can be judged responsible for its change in worth. As such, it is reasonable to think that any of its members is equally liable for this change. Therefore, a natural requirement would be that *each member bears the same payoff variation*.

Secondly, suppose that two coalitions of different sizes are affected by an identical change in worth. It makes sense to think that this change should not have a bigger impact on each member of the largest coalition. In other words, the size of the coalitions matters, just because each unit of surplus that a coalition can get has to be split among its members. Put differently, *if two coalitions have the same size, the per-capita effect of an identical change in worth on their members should be the same.*

We are going to incorporate these two natural principles into our axioms of invariance under the extra assumptions that the worth of the grand coalition if left unchanged and that changes in worth can occur simultaneously to multiple coalitions.

— In order to state our first axiom of invariance, suppose that the same variation of worth is applied to all coalitions of a given size. For any such coalition, the first above-mentioned principle states that each player should be affected similarly. Since any two players belong to the same number of coalitions of a given size, each of them endures the same effect as many times as any other player.

In other words, there is a uniform impact for each player. Since the worth of the grand coalition is assumed to be the same, the total payoff distributed should not vary, regardless of the fact that the rule under consideration satisfies efficiency or not. It is therefore intuitive to require that the overall effect on each player is null. This is precisely the condition incorporated to our first axiom of invariance. Consider any v, any size $s \in \{1, \ldots, n-1\}$, and any $t \in \mathbb{R}$. The TU-game w obtained from v by the (s,t)-*addition* is defined as:

$$\forall C \in 2^N, \quad w(C) = \begin{cases} v(C) + t & \text{if } c = s, \\ v(C) & \text{if } c \neq s. \end{cases}$$

Uniform addition invariance. If w is obtained from v by a (s,t)-addition, then $f(v) = f(w)$.

As an example, consider a symmetric weighted voting TU-game in which a coalition is winning if and only if it contains at least $k > n/2$ players, $k \in \mathbb{N}$. Any power index which satisfies symmetry will attribute the same voting power to any player. If in addition the power index satisfies efficiency, then each player's voting power would be $1/n$. After a change in the legislation, suppose that the larger quota $k+1$ is needed to reach a qualified majority. The new weighted voting TU-game remains symmetric, which means that any power index satisfying symmetry and efficiency should still attribute the same voting power $1/n$ to each player. The modification of the legislation and the absence of variation in the measure of voting power illustrate the axiom of uniform addition invariance since the new weighted voting TU-game is obtained from the original weighted voting TU-game by a $(k, -1)$-addition on all coalitions of size k.

— Now suppose that two coalitions C' and C'' with the same number of players are affected by opposite changes in worth. Which payoff variation can a player belonging to both coalitions expect? According to the two principles developed above, such a player should bear two payoff variations of opposite magnitudes. Therefore, it seems reasonable to think that these payoff variations should compensate for a player belonging to the two involved coalitions, leading to a null overall effect. This aspect is captured by the our second axiom of invariance. Consider any two TU-games v and w, any $i \in N$, any size $s \in \{2, \ldots, n-1\}$ and any $t \in \mathbb{R}$. The TU-game w is obtained from v by a (s,t)-*transfer* if there exist two distinct coalitions C' and C'' such that:

$$c' = c'' = s \quad \text{and} \quad \forall C \in 2^N, \quad w(C) = \begin{cases} v(C') + t & \text{if } C = C', \\ v(C'') - t & \text{if } C = C'', \\ v(C) & \text{if } C \in 2^N \backslash \{C', C''\}. \end{cases} \tag{1}$$

Uniform transfer invariance. If w is obtained from v by a (s,t)-transfer, then $f_i(v) = f_i(w)$ for each $i \in C' \cap C''$.

We write that a (s,t)-transfer involves a player $i \in N$ if $i \in C' \cap C''$ for the two coalitions C' and C'' concerned by this transfer.

Remark 1 The size of coalitions C' and C'' is neither 1 nor n since we only look for distinct coalitions. Observe also that N cannot be chosen as a coalition in the statements of uniform addition invariance and uniform transfer invariance. It should also be noted that it is necessary to have $n \geq 2$ to apply uniform addition invariance and $n \geq 3$ to apply uniform transfer invariance. □

Finally, we need the following weaker version of covariance.

Restricted covariance (RC). For each $a \in \mathbb{R}\backslash\{0\}$ and $b \in \mathbb{R}^n$ such that, for each $i, j \in N$, $b_i = b_j$, then $f(av + b) = af(v) + b$.

4 A study of the axioms of invariance

In this section, we aim at determining which rules satisfy the new axioms of invariance. This task is simplified by stating the proposition below. Then, we compare our axioms to the existing ones.

Proposition 1 *Let f be any rule.*

(i) If f satisfies efficiency, additivity and symmetry, then it satisfies uniform addition invariance.

(ii) If f satisfies additivity and anonymity, then it satisfies uniform transfer invariance.

(iii) If f satisfies uniform addition invariance and uniform transfer invariance, then it does not necessarily satisfy any of the following axioms: efficiency, additivity, anonymity and symmetry.

Proof. First of all, recall that any additive rule f is also an odd function, *i.e.* for each TU-game v, it holds that $-f(v) = f(-v)$.

(i) Consider any rule f that satisfies efficiency, additivity and symmetry. Then pick any two TU-games v and w such that w is obtained from v by some (s, t)-addition on all coalitions of size $s \in \{1, \ldots, n-1\}$. The TU-game $(v - w)$ is defined as:

$$\forall C \in 2^N, \quad (v-w)(C) = \begin{cases} 0 & \text{if } c \neq s, \\ -t & \text{if } c = s. \end{cases}$$

Thus, $(v - w)$ is a symmetric TU-game and $(v - w)(N) = 0$. By symmetry and efficiency, it follows that $f_i(v - w) = 0$ for each $i \in N$. Because f is an odd function, additivity yields $f(v - w) = f(v) + f(-w) = f(v) - f(w)$. Thus, $f_i(v) - f_i(w) = 0$ for each $i \in N$, which means that f satisfies uniform addition invariance.

(ii) Consider any rule f that satisfies additivity and anonymity, any player $i \in N$ and any two TU-games v and w such that w is obtained from v by some (s, t)-transfer. Denote by C' and C'' the two coalitions of size s involved in the transfer and assume that $C' \cap C'' \neq \emptyset$. We have to show that $f_j(v) = f_j(w)$ for each $j \in C' \cap C''$. Since f is additive, it is an odd function, so that this is equivalent to prove that $f_j(v - w) = 0$. By definition (1) of a (s, t)-transfer, the TU-game $(v - w)$ is such that:

$$(v-w)(C) = \begin{cases} -t & \text{if } C = C', \\ t & \text{if } C = C'', \\ 0 & \text{if } C \in 2^N\backslash\{C', C''\}. \end{cases}$$

In other words $(v - w) = 1^t_{C''} - 1^t_{C'}$, *i.e.* $(v - w)$ is the difference between the multiple of the two Dirac TU-games associated with coalitions C'' and C'. Since $c' = c''$, we can choose a permutation π on N such that $\pi(C') = C''$ and, for each $j \in C' \cap C''$, $\pi(j) = j$. To show: $\pi 1^t_{C'} = 1^t_{C''}$. Pick any nonempty coalition $C \in 2^N$. There are two cases.

– Suppose that $C = C'$. Then, $\pi(C) = \pi(C') = C''$, and we get:

$$\pi 1^t_{C'}(\pi(C)) = \pi 1^t_{C'}(\pi(C')) = 1^t_{C'}(C') = 1 = 1^t_{C''}(\pi(C)).$$

– Suppose that $C \neq C'$. Then $\pi(C) \neq \pi(C') = C''$, and we get:

$$\pi 1^t_{C'}(\pi(C)) = 1^t_{C'}(C') = 0 = 1^t_{C''}(\pi(C)).$$

Combining the two above cases, we conclude that $\pi 1^t_{C'} = 1^t_{C''}$. Anonymity of f implies that, for each $i \in N$, $f_{\pi(i)}(\pi 1^t_{C'}) = f_{\pi(i)}(1^t_{C''}) = f_i(1^t_{C'})$. Now pick any $j \in C' \cap C''$. From $\pi(j) = j$, we get $f_j(1^t_{C''}) = f_j(1^t_{C'})$. Since f is an odd function, additivity yields that $f_j(1^t_{C''} - 1^t_{C'}) = 0$, or equivalently $f_j(v - w) = 0$. Using again the additivity of f, we obtain $f_j(v) = f_j(w)$, which means that f satisfies uniform transfer invariance.

(iii) Choose any rule f satisfying both uniform addition invariance and uniform transfer invariance and any vector $b \in \mathbb{R}^n_+$ with distinct coordinates. Define the rule $g^{f,b}$ which assigns to each v and each $i \in N$, the payoff $g^{f,b}_i(v) = f_i(v) + b_i$. The rule $g^{f,b}$ satisfies uniform addition invariance and uniform transfer invariance but violates efficiency, additivity, anonymity and symmetry. ■

Remark 2 Observe that anonymity cannot be replaced in point *(ii)* of Proposition 1 by the weaker axiom of symmetry. As an example, pick any coalition C of size two. The rule g^C that assigns to each v and each $i \in N$ the payoff $g^C_i(v) = v(C)$, satisfies additivity and symmetry but not uniform transfer invariance, and thus not anonymity by point *(ii)* of Proposition 1. □

Remark 3 Efficiency cannot be dropped from point *(i)* of Proposition 1. Indeed, the combination of additivity and symmetry does not guarantee the satisfaction of uniform transfer invariance. The Banzhaf value β is a prominent example of a rule which satisfies additivity and symmetry but not efficiency, and which violates uniform addition invariance. To see this, consider the TU-game w that is built from an arbitrary TU-game v by a (s,t)-addition on all coalitions of a given size $s \in \{1, \ldots, n-1\}$. We have to prove that $\beta_i(w) - \beta_i(v) \neq 0$ for some player $i \in N$. Since β satisfies additivity, this is equivalent to show that $\beta_i(w - v) \neq 0$ for some $i \in N$, where $(w - v)$ is defined as:

$$\forall C \in 2^N, \quad (w-v)(C) = \begin{cases} t & \text{if } c = s, \\ 0 & \text{if } c \neq s. \end{cases}$$

For simplicity, assume that $n \geq 3$ and $s = 1$ in TU-game $(w - v)$. By definition of β, we obtain:

$$\forall i \in N, \quad \beta_i(w - v) = \frac{1}{2^{n-1}} \times t(2 - n) \neq 0.$$

This is due to the fact that $\mathcal{C}^{i2}$ contains $(n - 1)$ coalitions in which the contribution of i to each coalition is $-t$, whereas $\mathcal{C}^{i1}$ contains a unique coalition $\{i\}$ to which i's contribution is $+t$. □

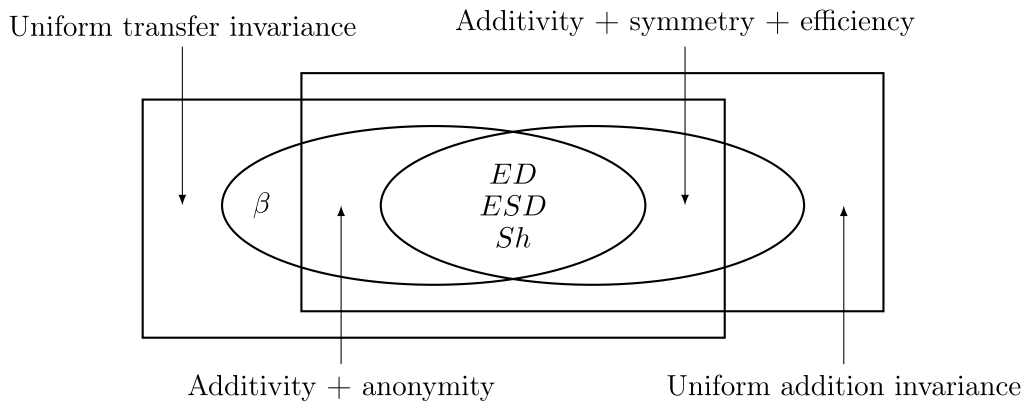
As a consequence of Proposition 1, any rule which satisfies additivity, efficiency and anonymity also satisfies the combination of uniform addition invariance and uniform transfer invariance. This is the case of many well-known rules such as the equal division rule, the equal surplus division rule and the Shapley value. This result is summarized in the following corollary.

Corollary 1 *The equal division rule ED, the equal surplus division rule ESD and the Shapley value Sh satisfy the combination of uniform addition invariance and uniform transfer invariance.*

Our new axioms do not imply nor are implied by the axioms of fairness introduced by van den Brink [21] or marginality proposed by Young [24]. The appendix proves these statements. The final remark of this section ensures that our two axioms of invariance are logically independent.

Remark 4 On the one hand, uniform addition invariance does not imply uniform transfer invariance. To see this, fix $n \geq 4$ and a size $s \in \{2, \ldots, n-1\}$. For each $i \in N$, choose two distinct coalitions C'_i and C''_i in $\mathcal{C}^{is}$. Define the rule g^{ED} that assigns to each TU-game v and each $i \in N$, the payoff $g^{ED}_i(v) = ED_i(v) + v(C'_i) - v(C''_i)$. This rule satisfies uniform addition invariance but not uniform transfer invariance. On the other hand, uniform transfer invariance does not imply uniform addition invariance since the Banzhaf value β satisfies uniform transfer invariance by point *(ii)* of Proposition 1 but violates uniform addition invariance by Remark 3. □

The following picture partially illustrates Proposition 1, Corollary 1 and Remarks 2 to 4.



5 Axiomatic characterizations

5.1 Characterization of the equal division rule

The first theorem below shows that uniform addition invariance, uniform transfer invariance, the nullifying player property and restricted covariance characterize the equal division rule. It is interesting to note that this result (and its proof) does not use the axioms of additivity, anonymity, symmetry and efficiency.

Theorem 1 *The equal division rule is the unique rule that satisfies uniform addition invariance, uniform transfer invariance, the nullifying player property and restricted covariance.*

Proof. It is easy to see that ED satisfies the nullifying player property. Corollary 1 establishes that ED satisfies uniform addition invariance and uniform transfer invariance. Regarding restricted covariance, pick $a \in \mathbb{R}\backslash\{0\}$ and $b \in \mathbb{R}^n$ such that, for each $i, j \in N$, $b_i = b_j$. Denote by $d \in \mathbb{R}$ the constant such that $b_i = d$ for each $i \in N$. To show: $ED_i(av + b) = av(N)/n + d$ for each v and each $i \in N$. For any v and any $i \in N$, we have:

$$ED_i(av + b) = \frac{(av + b)(N)}{n} = \frac{av(N) + nd}{n} = a\frac{v(N)}{n} + d = aED_i(v) + d.$$

It remains to prove the uniqueness part. So, consider any rule f that satisfies uniform addition invariance, uniform transfer invariance, the nullifying player property and restricted covariance. Pick any TU-game v and construct the TU-game v^0 as follows:

$$\forall C \in 2^N, \quad v^0(C) = v(C) - \frac{c}{n}v(N).$$

Note that both $v^0(N) = 0$ and $v^0 = av + b$ for $a = 1$ and $b_i = -v(N)/n$ for each $i \in N$. Next, fix a player, say $j \in N$, for the rest of the proof. The latter consists of using the modifications that are described by the axioms of uniform transfer invariance and uniform addition invariance in order to construct a TU-game in which j is a nullifying player. We proceed in three steps. In Step 1, we construct a TU-game $w^{(j)}$ that can be obtained from v^0 and j by some successive (s,t)-transfers involving player j as described by the axiom of uniform transfer invariance. The main characteristic of $w^{(j)}$ will be that two coalitions have the same worth if they both contain player j and have the same size. In Step 2, we construct a TU-game $u^{(j)}$ that can be obtained from $w^{(j)}$ and j by some successive (s,t)-additions as described by the axiom of uniform addition invariance. The main feature of TU-game $u^{(j)}$ will be that the worth of the coalitions containing j will be adjusted to zero. Finally, in Step 3, the four axioms are used to link the payoffs received by j in TU-game $u^{(j)}$ and in the original TU-game v. This last step can be summarized as follows:

$$v \xleftarrow{\text{restricted covariance}} v^0 \xleftarrow{\text{uniform transfer invariance}} w^{(j)} \xleftarrow{\text{uniform addition invariance}} u^{(j)} \xleftarrow{\text{nullifying player property}}$$

Step 1. For each $s \in \{1, \ldots, n\}$, define the rule d^s which assigns to each player, the average worth of the coalitions of size s to which he belongs:

$$\forall w, \forall i \in N, \quad d_i^s(w) = \binom{s-1}{n-1}^{-1} \sum_{C \in \mathcal{C}^{is}} w(C).$$

Clearly, $d_i^1(w) = w(\{i\})$ and $d_i^n(w) = w(N)$. Finally, define for each TU-game w the function θ_w on $\mathcal{C}^j$ as:

$$\forall C \in \mathcal{C}^j, \quad \theta_w(C) = d_j^c(w) - w(C),$$

and denote by $\Theta(w)$ the number of coalitions C in $\mathcal{C}^j$ such that $\theta_w(C) \neq 0$.

Claim. *For any TU-game w such that $\Theta(w) \neq 0$, there exist a size $s \in \{2, \ldots, n-1\}$ and a TU-game u obtained from w by a (s,t)-transfer involving j such that $\Theta(u) \leq \Theta(w) - 1$ and $d_j^{s'}(u) = d_j^{s'}(w)$ for each each size $s' \in \{1, \ldots, n\}$.*

The proof of this claim runs as follows. Consider any TU-game w such that $\Theta(w) \neq 0$. This means that there exists a size s and a coalition $C' \in \mathcal{C}^{js}$ such that $\theta_w(C') \neq 0$. By definition of the function θ_w, there exists another coalition $C'' \in \mathcal{C}^{js}$ such that $\theta_w(C'') \neq 0$, which in turn implies that $s \in \{2, \ldots n-1\}$. From w and the two coalitions C' and C'', construct the TU-game u by a $(s, \theta_w(C'))$-transfer:

$$\forall C \in 2^N, \quad u(C) = \begin{cases} w(C') + \theta_w(C') & \text{if } C = C', \\ w(C'') - \theta_w(C') & \text{if } C = C'', \\ w(C) & \text{if } C \in 2^N \backslash \{C', C''\}. \end{cases}$$

Observe that the rule d^s satisfies uniform transfer invariance, which implies $d_j^s(u) = d_j^s(w)$. For each other size $s' \neq s$, we also have $d_j^{s'}(u) = d_j^{s'}(w)$. Thus, for each $C \in 2^N \backslash \{C', C''\}$, $\theta_u(C) = \theta_w(C)$. Furthermore, it holds that $\theta_u(C') = 0$, which yields $\Theta(u) \leq \Theta(w) - 1$ as claimed.

To reach the announced TU-game $w^{(j)}$ through a sequence of (s,t)-transfers, it suffices to repeat the above argument by defining a finite sequence $(v^0, \ldots, v^q)$ of TU-games such that:

- The length q of the sequence is such that $q \leq \Theta(v)$;
- For any $k < q$, v^{k+1} is obtained from v^k by a $(s, \theta_{v^k}(C'))$-transfer involving j;
- For any $k \leq q$, it holds that $d_j^{s'}(v^k) = d_j^{s'}(v^0)$ for each $s' \in \{1, \dots, n\}$;
- The TU-game v^q satisfies $\Theta(v^q) = 0$.

Thus, in v^q, two coalitions of size $s' \in \{1, \dots, n\}$ and containing player j have the same worth $d_j^{s'}(v^q)$, as desired. In the remainder of the proof, $w^{(j)}$ stands for the TU-game v^q.

Step 2. From the TU-game $w^{(j)}$, construct the TU-game $u^{(j)}$ such that:

$$\forall C \in 2^N \quad , u^{(j)}(C) = \begin{cases} 0 & \text{if } C \in \mathcal{C}^j \backslash \{N\}, \\ w^{(j)}(C) - d_j^c(w^{(j)}) & \text{if } C \in 2^N \backslash \mathcal{C}^j, \\ w^{(j)}(N) & \text{if } C = N. \end{cases} \tag{2}$$

Claim. *The TU-game $u^{(j)}$ can be obtained from TU-game $w^{(j)}$ by the $n-1$ successive $(s, -d_j^s(w^{(j)}))$-additions for $s \in \{1, \dots, n-1\}$.*

To see this, construct the sequence of TU-games $(w^0, \dots, w^{n-1})$ such that $w^0 = w^{(j)}$ and, for each $s \in \{1, \dots, n-1\}$, the TU-game w^s is defined as:

$$\forall C \in 2^N, \quad w^s(C) = \begin{cases} w^{s-1}(C) - d_j^s(w^{(j)}) & \text{if } c = s, \\ w^{s-1}(C) & \text{if } c \neq s. \end{cases}$$

It is easy to check that $w^{n-1} = u^{(j)}$. Furthermore, observe that $u^{(j)}(N) = w^{(j)}(N) = v^0(N) = 0$.

Step 3. Using uniform addition invariance, uniform transfer invariance, the nullifying player property and restricted covariance, we complete the proof.

Claim. *The rule f coincides with the equal division rule ED.*

To prove this claim, note that by construction of $u^{(j)}$ in Step 2, the player j selected at the beginning of the proof is a nullifying player in $u^{(j)}$. By the nullifying player property, his payoff $f_i(u^{(j)}) = 0$ is uniquely determined. Because $u^{(j)}$ is built from $w^{(j)}$ by $n-1$ successive $(s, -d_j^s(w^{(j)}))$-additions for $s \in \{1, \dots, n-1\}$, we can apply uniform addition invariance $n-1$ times to get $f(w^{(j)}) = f(u^{(j)})$. In particular, we have $f_j(w^{(j)}) = f_j(u^{(j)}) = 0$. Because $w^{(j)}$ is constructed from v^0 by a sequence of (s,t)-transfers involving player j, we can apply uniform transfer invariance to any pair of consecutive TU-games in the sequence to obtain $f_j(v^0) = f_j(w^{(j)}) = 0$. Restricted covariance can be applied to v^0 and v since, by definition, $v^0 = av + b$ where $a = 1$ and $b_i = -v(N)/n$ for each $i \in N$. Therefore, we get $f_i(v^0) = f_i(v) - v(N)/n$ for each $i \in N$. For player j, this means that $f_j(v^0) = 0 = f_j(v) - v(N)/n$ or equivalently $f_j(v) = v(N)/n = ED_j(v)$. Since j and v were chosen arbitrarily, the proof is complete. ■

The following rules show the logical independence of the axioms.

- The Shapley value Sh satisfies uniform addition invariance, uniform transfer invariance and restricted covariance but not the nullifying player property.

– The rule $g^{(i)}$ that assigns to each v and each $i \in N$ the payoff $g_i^{(i)}(v) = v(\{i\})$ satisfies uniform transfer invariance, the nullifying player property and restricted covariance but not uniform addition invariance.
– For any real $a \in \mathbb{R}\backslash\{1\}$, the rule ED^a that assigns to each TU-game v and each $i \in N$ the payoff $ED_i^a(v) = a \times ED_i(v)$ satisfies uniform addition invariance, uniform transfer invariance and the nullifying player property but not restricted covariance.
– The rule g^{ED} defined in Remark 4 satisfies uniform addition invariance, restricted covariance and the nullifying player property but not uniform transfer invariance.

5.2 Characterization of the equal surplus division rule

Theorem 1 can be adapted to characterize the equal surplus division rule. It is sufficient to replace restricted covariance by covariance and to require the nullifying player property for zero-normalized TU-games only *i.e.* TU-games in which the stand-alone worth of each player is equal to zero.

Theorem 2 *The equal surplus division rule is the unique rule that satisfies uniform addition invariance, uniform transfer invariance, the nullifying player property for zero-normalized TU-games and covariance.*

Proof. It is well-known that ESD the nullifying player property for zero-normalized TU-games and covariance. Corollary 1 establishes that ESD satisfies uniform addition invariance and uniform transfer invariance. Next, let us prove the uniqueness part. Consider any rule f that satisfies uniform addition invariance, uniform transfer invariance, the nullifying player property for zero-normalized TU-games and covariance, and pick any TU-game v. The proof reuses most of the arguments of the proof of Theorem 1. The major difference is the following. The TU-game v^0 that appeared in the proof of Theorem 1 is now defined as:

$$\forall C \in 2^N, \quad v^0(C) = v(C) - \sum_{j \in C} ESD_j(v).$$

Remark that $v^0(N) = 0$ and $v^0 = av + b$ for $a = 1$ and $b_i = -ESD_i(v)$ for each $i \in N$. In particular, this implies that, for each $i \in N$, $v^0(\{i\})$ is a constant equal to:

$$-\frac{1}{n}\left(v(N) - \sum_{k \in N} v(\{k\})\right) \tag{3}$$

Next, fix a player, say $j \in N$ until the end of the proof. For each $s \in \{1, \ldots, n\}$, we consider the rule d^s, and, for each TU-game w, we consider the functions θ_w on $\mathcal{C}^j$ and $\Theta(w)$, as defined in Theorem 1. We shall only give a sketch of the rest of the proof since it shares many similarities with the proof of Theorem 1. From Step 1 in the proof of Theorem 1, we know that the TU-game v^0 can be transformed by a sequence of (s, t)-transfers involving player j into a TU-game $w^{(j)}$ such that $\Theta(w^{(j)}) = 0$. From Step 2 in the proof of Theorem 1, we can construct a TU-game $u^{(j)}$ that can be obtained from $w^{(j)}$ by $n - 1$ successive $(s, -d_j(w^{(j)})$-additions for $s \in \{1, \ldots, n-1\}$, as described by the axiom of uniform addition invariance (see equation (2) in the proof of Theorem 1). The main feature of the TU-game $u^{(j)}$ is therefore that $u^{(j)}(C) = 0$ for each $C \in \mathcal{C}^j$. The key difference with the proof of Theorem 1 is that we need to show that $u^{(j)}$ is zero-normalized in order to apply the nullifying player property to j. Firstly, note that $w^{(j)}(\{i\}) = v^0(\{i\})$ for each $i \in N\backslash\{j\}$ since any (s,t)-transfer involving player j does not alter the worth of any coalition in $2^N\backslash\mathcal{C}^j$. Secondly, both $v^0(\{i\})$, for $i \in N$, and $d^1_j(v^0)$ are equal to the constant (3), which yields:

$$u^{(j)}(\{i\}) = w^{(j)}(\{i\}) - d^1_j(v^0) = 0, \tag{4}$$

i.e. $u^{(j)}$ is a zero-normalized TU-game. As a consequence, the nullifying player property for zero-normalized TU-games yields that the nullifying player j obtains a payoff $f_j(u^{(j)}) = 0$. Since the TU-game $u^{(j)}$ is built from $w^{(j)}$ by $n-1$ successive $(s, -d^s_j(w^j))$-additions for $s \in \{1,\ldots,n-1\}$, we can apply uniform addition invariance $n-1$ times to get $f(w^{(j)}) = f(u^{(j)})$. In particular, we have $f_j(w^{(j)}) = f_j(u^{(j)}) = 0$. Since the TU-game $w^{(j)}$ can be constructed from v^0 by a sequence of (s,t)-transfers involving player j, we can apply uniform transfer invariance to any pair of consecutive TU-games in this sequence to obtain $f_j(v^0) = f_j(w^{(j)}) = 0$. By covariance and definition of v^0, we know that for each $i \in N$, $f_i(v^0) = f_i(v) - ESD_i(v)$. For player j, this means that $f_j(v^0) = 0 = f_j(v) - ESD_j(v)$ or equivalently $f_j(v) = ESD_j(v)$. Since both j and v were chosen arbitrarily, the proof is complete. ■

The following rules show the logical independence of the axioms.

- The equal division rule ED satisfies uniform addition invariance, uniform transfer invariance, the nullifying player property for zero-normalized TU-games but not covariance.
- The Shapley value Sh satisfies uniform addition invariance, uniform transfer invariance and covariance but not the nullifying player property for zero-normalized.
- The rule $g^{(i)}$ used after Theorem 1 satisfies uniform transfer invariance, the nullifying player property for zero-normalized and covariance but not uniform addition invariance.
- Fix $n \geq 4$. For each $i \in N$, choose two distinct coalitions $(C'_i, C''_i) \in \mathcal{C}^{i2} \times \mathcal{C}^{i2}$. Then define the rule g^{ESD} as follows:

$$\forall v, \forall i \in N, \quad g^{ESD}_i(v) = ESD_i(v) + \Big(v(C'_i) - v(C'_i\backslash\{i\}) - \big(v(C''_i) - v(C''_i\backslash\{i\})\big)\Big).$$

  This rule satisfies uniform addition invariance, the nullifying player property for zero-normalized and covariance but not uniform transfer invariance.

5.3 Characterization of the Shapley value

As for the equal division rule, the result below provides a characterization of the Shapley value that does not rely on any of the classical axioms efficiency, anonymity, symmetry and additivity. Although different and more sophisticated, the proof of this result relies on a mechanism which is similar to the one used in the proof of Theorem 1.

Theorem 3 *The Shapley value is the unique rule that satisfies uniform addition invariance, uniform transfer invariance and the dummy player property.*

Proof. It is well-known that the Sh satisfies the dummy player property. Corollary 1 establishes that Sh satisfies uniform addition invariance and uniform transfer invariance. It remains to prove the uniqueness part. So let f be any rule that satisfies the three axioms. Consider any TU-game v

and any player $j \in N$, which is fixed for the rest of proof. We consider steps described as follows. In Step 1, we construct a TU-game $w^{(j)}$ from v and j by some successive (s,t)-transfers involving player j as described by the axiom of uniform transfer invariance. The main characteristic of $w^{(j)}$ will be that player j has the same contribution to any two of his coalitions having the same size. In Step 2, we construct a TU-game $u^{(j)}$ that can be obtained from $w^{(j)}$ and j by some successive (s,t)-additions for $s \in \{1,\dots,n-1\}$, as described by the axiom of uniform addition invariance. The $n-1$ successive (s,t)-additions will be uniquely chosen so as to have player j being a dummy player in $u^{(j)}$. Finally, in Step 3, the three axioms are used to connect the payoffs received by j in TU-game $u^{(j)}$ and in the original TU-game v. This last step can be summarized as follows:

$$v \xleftarrow{\text{uniform transfer invariance}} w^{(j)} \xleftarrow{\text{uniform addition invariance}} u^{(j)} \xleftarrow{\text{dummy player property}}$$

Step 1. For each $s \in \{1,\dots,n\}$, define the rule b^s which assigns each TU-game and to each player, his average contribution to his coalitions of size $s-1$:

$$\forall w, \forall i \in N, \quad b_i^s(w) = \binom{s-1}{n-1}^{-1} \sum_{C \in \mathcal{C}^{is}} \Big(w(C) - w(C \backslash \{i\} \Big).$$

Obviously, $b_i^1(w) = w(\{i\})$ and $b_i^n(w) = w(N) - w(N\backslash\{i\})$. Define for each TU-game w the function δ_w on $\mathcal{C}^j$ as:

$$\forall C \in \mathcal{C}^j, \quad \delta_w(C) = b_j^c(w) - (w(C) - w(C\backslash\{j\}),$$

and denote by $\Delta(w)$ the number of coalitions C in $\mathcal{C}^j$ such that $\delta_w(C) \neq 0$.

Claim. *For any TU-game w such that $\Delta(w) \neq 0$, there exist a size $s \in \{2,\dots,n-1\}$ and a TU-game u obtained from w by a (s,t)-transfer involving j such that $\Delta(u) \leq \Delta(w) - 1$ and $b_j^{s'}(u) = b_j^{s'}(w)$ for each each size $s' \in \{1,\dots,n\}$.*

The proof of this claim is omitted since it is similar to the one of the corresponding claim in Theorem 1. It suffices to note that the rules b^s satisfy uniform transform invariance and to apply the same arguments. Next, the announced TU-game $w^{(j)}$ is easily constructed through a sequence of (s,t)-transfers by successive applications of the above claim. In fact, we get a finite sequence $(v^0,\dots,v^q)$ of TU-games such that:

- $v^0 = v$ and $q \leq \Delta(v)$;
- For any $k < q$, v^{k+1} is obtained from v^k by a $(s, \delta_{v^k}(C'))$-transfer involving j, where C' stands for one the two coalitions involved by the transfer from step k to step $k+1$;
- For any $k \leq q$, it holds that $b_j^{s'}(v^k) = b_j^{s'}(v^0)$ for each $s' \in \{1,\dots,n\}$;
- The TU-game v^q satisfies $\Delta(v^q) = 0$.

Thus, in v^q, the contribution of player j to two coalitions of size $s' \in \{1,\dots,n\}$ is equal to $b_j^{s'}(v^q)$, a desired. In the remainder of the proof, $w^{(j)}$ stands for v^q.

Step 2. From the TU-game $w^{(j)}$, construct the TU-game $u^{(j)}$ such that:

$$\forall C \in 2^N, \quad u^{(j)}(C) = w^{(j)}(C) + c \times Sh_j(v) - \sum_{k=1}^{c} b_j^k(v).$$

Claim. *The TU-game $u^{(j)}$ can be obtained from the TU-game $w^{(i)}$ by the $n-1$ successive $(s, s \times Sh_j(v) - \sum_{k=1}^{s} b_j^k(v))$-additions for $s \in \{1, \ldots, n-1\}$. In $u^{(j)}$, player j's contribution to any of his coalitions is equal to $Sh_j(v)$.*

To prove this claim, construct the sequence of TU-games $(w^0, \ldots, w^{n-1})$ such that $w^0 = w^{(j)}$ and, for each $s \in \{1, \ldots, n-1\}$, the TU-game w^s is defined as:

$$\forall C \in 2^N, \quad w^s(C) = \begin{cases} w^{s-1}(C) + s \times Sh_j(v) - \displaystyle\sum_{k=1}^{s} b_j^k(v) & \text{if } c = s, \\ w^{s-1}(C) & \text{if } c \neq s. \end{cases}$$

Note that $u^{(j)}(N) = w^{(j)}(N)$ since $n \times Sh_j(v) = \sum_{k=1}^{n} b_j^k(v)$. It is therefore straightforward to verify that $w^{n-1} = u^{(j)}$. Next, consider any $C \in \mathcal{C}^j$. Player'j contribution to C in $u^{(j)}$ is equal to:

$$\begin{aligned} u^{(j)}(C) - u^{(j)}(C\backslash\{j\}) &= w^{(j)}(C) + s \times Sh_j(v) - \sum_{k=1}^{s} b_j^k(v) \\ &\quad - \left(w^{(j)}(C\backslash\{j\}) + (s-1) \times Sh_j(v) - \sum_{k=1}^{s-1} b_j^k(v) \right) \\ &= w^{(j)}(C) - w^{(j)}(C\backslash\{i\}) + Sh_j(v) - b_j^s(v) \\ &= Sh_j(v) - \delta_{w^{(j)}}(C) \\ &= Sh_j(v). \end{aligned}$$

Step 3. Using uniform addition invariance, uniform transfer invariance and the dummy player property, we complete the proof.

Claim. *The rule f coincides with the Shapley value Sh.*

To prove this claim, observe that the construction of $u^{(j)}$ in Step 2 yields that the player j selected at the beginning of the proof is a dummy player in $u^{(j)}$. Therefore, his payoff $f_j(u^{(j)}) = u^{(j)}(\{j\}) = Sh_j(v)$ in $u^{(j)}$ is uniquely determined by the dummy player property. Since $u^{(j)}$ is built from $w^{(j)}$ by $n-1$ successive $(s, s \times Sh_j(v) - \sum_{k=1}^{s} b_s^k(v))$-additions for $s \in \{1, \ldots, n-1\}$, we can apply uniform addition invariance $n-1$ times to get $f(w^{(j)}) = f(u^{(j)})$. In particular, we have $f_j(w^{(j)}) = f_j(u^{(j)}) = u^{(j)}(\{j\})$. Since $w^{(j)}$ can be constructed from v by a sequence of (s,t)-transfers involving player j, we can apply uniform transfer invariance to any pair of consecutive TU-games in the sequence to obtain $f_j(v) = f_j(w^{(j)}) = Sh_j(v)$. Thus, the payoff of player j in TU-game v is uniquely determined. Because j and v were chosen arbitrarily, the proof is complete. ■

The following rules demonstrate the logical independence of the three axioms.

- The equal division rule ED satisfies uniform addition invariance and uniform transfer invariance but not the dummy player property.
- The Banzhaf value β satisfies uniform transfer invariance and the dummy player property but not uniform addition invariance.
- Suppose $n \geq 5$ and, for each player $i \in N$, consider any two distinct coalitions of size 2 containing player j. Denote by C_i' and C_i'' these two coalitions. Note that $n \geq 5$ ensures that

the size of coalitions $C_i'\backslash\{i\}$, $N\backslash C_i'$, $(N\backslash C_i')\cup\{i\}$ as well as $C_i''\backslash\{i\}$, $N\backslash C_i''$ and $(N\backslash C_i'')\cup\{i\}$ is different from two. Define the rule g^2 as follows:

$$\begin{aligned}\forall v\in 2^N, \forall i\in N,\quad g_i^2(v) = \;& Sh_i(v) + \Big(v(C_i') - v(C_i'\backslash\{i\}) + v((N\backslash C_i')\cup\{i\}) - v(N\backslash C_i')\Big)\\ & -\Big(v(C_i'') - v(C_i''\backslash\{i\}) + v((N\backslash C_i'')\cup\{i\}) - v(N\backslash C_i'')\Big).\end{aligned}$$

The rule g^2 satisfies uniform addition invariance and the dummy player property but not uniform transfer invariance.

The characterizations of the equal division rule and of the Shapley value in Theorems 1 and 3 are comparable. Both rules satisfy uniform addition invariance, uniform transfer invariance, and restricted covariance as shown in Corollary 1. Among the rules satisfying these three axioms, the equal division rule is the only one that satisfies the nullifying property (Theorem 1) while the Shapley value is the only one that satisfies the dummy player property (Theorem 3). This comparison between the equal division rule and the Shapley value is in the spirit of the comparative axiomatic characterizations studied by van den Brink [22] and Kamijo and Kongo [9]. In particular, as in these articles, it is enough to replace the nullifying player property by the dummy player property in the set of axioms characterizing the equal surplus division rule to obtain a set of axioms that characterizes the Shapley value.

More generally, the results in Theorems 1 to 3 can be summarized by the following table, in which a "+" means that the rule satisfies the axiom, a "−" has the converse meaning and the "⊕" symbols indicate the characterizing sets of axioms.

	ED (Th. 1)	ESD (Th. 2)	Sh (Th. 3)
Uniform addition invariance	⊕	⊕	⊕
Uniform transfer invariance	⊕	⊕	⊕
Covariance	−	⊕	+
Restricted covariance	⊕	+	+
Nullifying player property	⊕	−	−
Nullifying player property for zero-normalized TU-games	+	⊕	−
Dummy player property	−	−	⊕

As a final remark, recall from point *(iii)* in Proposition 1 that the combination of these axioms of invariance do not imply the classical axioms of efficiency, additivity, anonymity or symmetry. This is important to show that the uniqueness parts in the proofs of Theorems 1 to 3 cannot be shortened by invoking other characterizations in which the axioms of efficiency, additivity, anonymity or symmetry are used. In other words, these classical axioms that are usually employed to characterize the Shapley value, the equal division rule and the equal surplus division rule cannot be derived from our axioms of invariance without the addition of extra axioms such as the nullifying player property (Theorem 1) or the dummy player property (Theorem 3).

5.4 Characterization of the Banzhaf value

The combination of uniform addition invariance and uniform transfer invariance is crucial to prove Theorems 1 to 3. Since the Banzhaf value fails to satisfy uniform addition invariance by Remark 4, one may ask whether strengthening the other axiom of invariance would lead to an axiomatization of the Banzhaf value. This section shows that such a result is possible. More specifically, we consider the axiom of transfer invariance, which is the variation of the axiom of uniform transfer invariance obtained by relaxing the assumption that the two coalitions involved in the transfer should have the same size. Formally, the TU-game w is obtained from another TU-game v by a *t-transfer* if there exists a real $t \in \mathbb{R}$ and a pair of distinct coalitions C' and C'', such that:

$$\forall C \in 2^N, \quad w(C) = \begin{cases} v(C') + t & \text{if } C = C', \\ v(C'') - t & \text{if } C = C'', \\ v(C) & \text{if } C \in 2^N \backslash \{C', C''\}. \end{cases} \tag{5}$$

Transfer invariance. If w is obtained from v by a t-transfer, then $f_i(v) = f_i(w)$ for each $i \in C' \cap C''$.

Note that the grand coalition is allowed to be part of a t-transfer. The next result shows that the Banzhaf value is characterized by transfer invariance and the dummy player property.

Theorem 4 *The Banzhaf value is the unique rule that satisfies transfer invariance and the dummy player property.*

Proof. It is well known that the Banzhaf value satisfies the dummy player property. Further, note that for any TU-game v and any player $i \in N$, $\beta_i(v)$ only depends on the sum of the contributions to all i's coalitions. This sum is not not altered by a t-transfer involving i, which implies that β satisfies transfer invariance. For the uniqueness part, consider any rule f that satisfies transfer invariance and the dummy player property, any v, and any $j \in N$ which is fixed for the rest of proof. For each TU-game w, define the function γ_w on $\mathcal{C}^j$ as:

$$\forall C \in \mathcal{C}^j, \quad \gamma_w(C) = \beta_j(v) - (w(C) - w(C \backslash \{j\})).$$

Proceeding as in the proofs of Theorems 1, 2 and 3, it is possible to construct a sequence of TU-games $(v^0, \ldots, v^q)$ through successive $(\gamma_{v^k}(C'))$-transfers involving player j, $k \in \{1, \ldots, q\}$, and such that:

$$v^0 = v, \text{ and } \forall C \in \mathcal{C}^j, \quad v^q(C) - v^q(C \backslash \{j\}) = \beta_j(v).$$

Player j is clearly a dummy player in the final TU-game v^q. By the dummy player property, $f_j(v^q) = v^q(\{j\}) = \beta_j(v)$. Because v^q is constructed from v by a sequence of t-transfers involving j, we can apply transfer invariance to any pair of consecutive TU-games in the sequence to obtain $f_j(v) = f_j(v^q) = \beta_j(v)$. Since j and v were chosen arbitrarily, the proof is complete. ■

Logical independence of the two axioms is demonstrated as follows:

- The Shapley value Sh satisfies the dummy player property but not transfer invariance.
- The rule g^{sum} defined as:

$$\forall v, \forall i \in N, \quad g_i^{\text{sum}}(v) = i \times \left(\sum_{C \in \mathcal{C}^i} v(C) \right),$$

  satisfies transfer invariance but not the dummy player property.

6 Conclusion

We conclude this article by discussing two additional aspects of our study.

Firstly, the reader might wonder whether our characterizations still hold when $n \in \{1, 2\}$ since some of the axioms of invariance cannot be applied for such TU-games. It turns out that Theorems 1 to 3 are still valid for small player sets. If $n \in \{1, 2\}$, the nullifying player property and restricted covariance are sufficient in Theorem 1 to characterize the equal division rule. Similarly, if $n \in \{1, 2\}$, the nullifying player property for zero-normalized TU-games and covariance are sufficient in Theorem 2 to characterize the equal surplus division rule. Lastly, if $n = 2$, uniform addition invariance and the dummy player property are sufficient in Theorem 3 to characterize the Shapley value, whereas the dummy player property alone provides the characterization if $n = 1$.

Secondly, the technique of proof that is used in our Theorems can be replicated to obtain extra results. As an example, the proof of Theorem 3 can be mimic to characterize the consensus value introduced by Ju, Borm and Ruys [8]. Indeed it is enough to replace the dummy player property in Theorem 3 by the **neutral dummy property** used by Ju, Borm and Ruys [8] to obtain a characterization of the consensus value.

Appendix

Remark 5 Let us show that uniform addition invariance neither implies nor is implied by fairness or marginality. The Banzhaf value β satisfies fairness but not uniform addition invariance by Remark 3. Next, for any player $j \in N$, the rule E^j that assigns to each TU-game v the payoffs $E^j_j(v) = v(N)$ and $E^j_i(v) = 0$ for each player $i \in N\backslash\{j\}$ satisfies uniform addition invariance but not fairness. The Banzhaf value β also satisfies marginality but not uniform addition invariance by Remark 3, while the equal division rule ED satisfies uniform addition invariance but not marginality. $\square$

Remark 6 Let us prove that uniform transfer invariance neither implies nor is implied by fairness or marginality. The rule g^C defined after Proposition 1 satisfies fairness but violates uniform transfer invariance. Next, fix $n \geq 3$ and choose a nonempty coalition $C' \in 2^N$ such that $c' \in \{2, \ldots, n-1\}$. The rule $g^{-C'}$ defined as:

$$\forall v, \forall i \in N, \quad g_i^{-C'}(v) = \sum_{C \in \mathcal{C}^i \backslash \{C'\}} \big(v(C) - v(C\backslash\{i\})\big),$$

satisfies marginality but violates uniform transfer invariance. Finally, the rule g^{sum} defined after Theorem 4 satisfies uniform transfer invariance but violates fairness and marginality. $\square$

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