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Abstract

In this paper we deal with the global distribution of capital and output across time. We supply empirical support to model it as a partial differential equation, if the support of the distribution is related to an initial ranking of the economies. If we consider a distributional extension of the AK model we prove that it displays both global endogenous growth and transitional convergence in a distributional sense. This property can also be shared by a distributional extension of the Ramsey model. We conduct a qualitative analysis of the distributional dynamics and prove that If the technology displays mild decreasing marginal returns we can have long run growth if a diffusion induced bifurcation is crossed. This means that global growth can exist even in the case in which the local production functions are homogeneous and display decreasing returns to scale.

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1 Introduction

Several areas in economics deal with the dynamics of distributions. Distributions of income within a country[1], distribution of GDP among nations, of distribution of activity among geographical locations share a common structure.

In the case of the distribution economic activity across geographical units, regions or countries, one would expect there should be a common ground between growth and geographical economics [2]. However, these two disciplines, when we consider one-sector economies that are similar in terms of preferences and technology, tend to reach contradictory results as regards the dynamics of the distribution of output: while growth economics predicts that different countries will grow across time and per capita capital will converge to a homogeneous distribution across countries (absolute convergence) [3], spatial economics emphasizes the fact that activity tends to agglomerate in some particular locations, that is, it converges to a heterogenous distribution across locations [4]. While growth economics does not deal with the joint dynamics of the distribution of capital accumulation between time and locations, beyond the two-country case, spatial economics models are generally partial equilibrium. Also, most of the spatial economics models do not fully integrate agglomeration with intertemporal optimization and do not address economic growth.

A recent strand in the literature deals jointly with the dynamics of growth and space, by representing the capital accumulation in time and space by a parabolic partial differential equation and by deriving the optimal distribution from intertemporal optimization: see Brito (2004) [5] for a first attempt, and Brock and Xepapadeas (2008), Boucekkine et al. (2009) and Boucekkine et al. (2010) for several alternatives and extensions. A survey and an assessement of the relevance of this model in the context of spatial economics can be found inDesmet

[1] See Stiglitz (1969), Chatterjee (1994), Caselli and Ventura (2000), and a survey in Bertola (2000).

[2] International economics deals with multi-sector economies which are differentiated by their relative factor endowments.

[3] See Barro and Sala-i-Martin (1992) and (Acemoglu, 2009, ch. 1), for an earlier contribution and a recent assessment of the convergence hypothesis in the empirical growth literature.

[4] See Quah (2002) Fujita and Thisse (2002).

[5] Isard and Liossatos (1979) can be seen as a precursor and a similar idea can be found in Chatterjee (1994).

and Rossi-Hansberg (2010) and a useful classification of the relevant papers is in Boucekkine et al. (2010).

In this paper we clarify in which sense can the spatial-time dynamics can be represented by a diffusion process, and we specify a model in which we may have both endogenous long run growth and stable distributional dynamics.

Our setup is based upon three main assumptions. The first assumption is related to the support for the distributions. We assume that there is a one-to-one correspondence between heterogeneous (or asymmetric) agents and their ranking by a particular order, or location in a particular point in space. Those supports (order or space) have metrics related to economic distance and not with hierarchical or geographical distance. The support is one-dimensional and serves as an indexing device as in Hotelling (1929). This means that if the capital endowments varies across locations, there will be a distributional dynamics for capital independently of the geographical distance. We assume further that the support is unbounded, which avoids spurious dynamics associated to the introduction of bounded domains.

The second assumption is related to the specification of the hierarchical- or spatial-temporal constraints. Although the equilibrium condition between savings and investment still holds, the time and distributional dynamics allocations of capital should be mutually consistent. Capital flows along the distributional support should be related to a particular gradient. With some mild assumptions we prove that the time-distributional dynamics is modeled by a forward parabolic partial differential equation, which represents the instantaneous budget constraint.

The third assumption is related to the specification of the distributional-temporal arbitrage conditions. We assume a centralized economy in which the planner decides on the spatial-temporal allocation of consumption that maximizes a social welfare Bergson-Samuelson utility functional [6]. In order to deal with the problems arising from the unbounded support for space, we assume a Millian average utility function in which the average utility

[6] In order to avoid the collective choice impossibility results (see Yaari (1981)), we assume that agents do not have preferences on distributions of consumption but rather on their own level of consumption.

is maximized and the central planner has a uniform spatial weighting distribution. A generalized Euler condition is represented by a backward parabolic partial differential equation.

At last, we assume that all the parameters are constant along the distributional support, because we want to concentrate on the endogenous dynamics of the distributed variables, the capital stock and consumption. The other features of the economy are benchmark: every location has an intertemporaly additive and instantaneously concave utility function and neoclassical production function. Thus, the model is a distributional generalization of Ramsey (1928), Cass (1965) and Koopmans (1965)[7]

We start by studying a simple AK model with iso-elastic preferences and are able to find a closed form solution. The optimal solution admits a separation between trend and transitional dynamics. It displays global endogenous long run growth and stability in a distributional sense. This means that the detrended distribution converges asymptotically to a homogeneous distribution. Boucekkine et al. (2010) obtained a similar result. The basic driving forces towards convergence is related to diffusion and to the concavity of preferences. As the marginal and average productivity is homogeneous in all the domain, differences in the rate of return of capital will not generate capital flows among locations.

If we assume instead a concave production function, the rate of return of capital will be a decreasing function of the stock of capital, for every location, and heterogeneity in the distribution of capital will create heterogeneity in the rates of return across space. As in this case the centralized problem does not have a closed form solution we study qualitatively both the asymptotic distribution and the local stability properties of the optimal solution. As in Brito (2004) and Brock and Xepapadeas (2008) a diffusion induced bifurcation may occur. This bifurcation separates values of the parameters such that the solution converges asymptotically to a homogeneous steady state distribution, from values of the parameters such that the solution is unbounded in time. Again, we prove that the solution can be separated, in a distributional sense, between a long run exponential and spatially homogeneous component and a detrended spatially-heterogeneous component that converges asymptotically to a ho-

[7]Our model is a particular problem of optimal control of parabolic partial differential equations, for which we derive heuristically a maximum principle.

mogeneous steady state distribution. As the transversality condition holds the unbounded distribution is indeed a solution of the centralized problem, which implies that this solution displays both endogenous growth and a time-varying but asymptotically homogeneous and stationary solution. We call the long run component the global balanced growth path and global rate of growth to the associated endogenous growth rate.

This is an important result of this strand of literature that has not been previously derived. As the diffusion-induced instability tends to occur when the technology is close to linear, this means that there is a trade-off between the dynamics of the rate of return to capital which works for stability and an instability mechanism which is introduced by diffusion. If the first effect is too strong the dynamics will be similar related a-spatial Ramsey-Cass-Koopmans model replicated across space. If the reaction of the rate of return to spatial heterogeneity is not too strong the instability introduced by diffusion will generate a global endogenous growth mechanism.

The type of instability is analogous to the Türing (1952) type of instability. (Fujita et al., 1999, ch. 6) also consider diffusion in a continuous space framework and report the emergence of spatial heterogeneity, as a result of the existence of Turing instability. They conclude that spatial interaction among symmetric regions may eventually lead to the emergence of agglomeration, through a mechanism of pattern formation.

In our model, when the optimal solution displays global endogenous growth a spatial pattern formation also emerges. However, differently from the previous case, it characterizes the conditional (distributional) stable optimal consumption and capital accumulation trajectories. This means that the detrended distribution will feature changes in the hierarchy between regions in the convergence to a global balanced growth path.

Next, in section 2 we discuss in which sense the distribution of GDP across US states can be seen as a diffusion process, in section 3 we present the optimal distribution problem for a central planer, in section 4 we solve the problem with a AK technology, in section 4 the concave technology case is studied and section **??** concludes.

2 Empirical GDP distribution: can it follow a diffusion process ?

In our model the distributional dynamics of capital, and therefore of per capita product, is represented by a parabolic partial differential equation (or a diffusion equation) [8]. The mechanism that drives the dynamics of capital distribution is related to the existence of initial differences in capital endowments, to asymmetric shocks to the distribution, and to the position in a particular ranking, and not to a particular location in geographical space.

In order to check the likelihood of this representation we present personal income data for the US states for the period 1955-2002 (FIPS database). In order to do this, we start by assigning a number in the sequence $\mathbb{X} = \{1, \ldots, n\}$ to every state, from their per capita 1955 level of income. That is, we build the support by making a mapping between the position of states in the 1955 income per capita distribution and the natural numbers. Then, for the years 1956 and afterwards we keep the same assignment for every state.

This ranking is only related to the initial distribution of income and is independent of their geographical location and the physical distance between different states. Let us denote $Y(x, t)$ the income per capita in state x at time t. Figure 1 shows $Y(x, t)$ from FIPS data.

`Figure 1 around here`

We readily observe that there is a distinctive trend. Figure 2 shows this by depicting the yearly average of income per capita, $\overline{Y}(t)$. It follows an exponential $\overline{Y}(t) \propto \exp(\gamma t)$ where $\gamma \approx 0.033$.

`Figure 2 around here`

If we perform the typical a trend-transition decomposition which is performed in standard growth accounting

$$Y(x, t) = y(x, t)e^{\gamma t}$$

[8] A diffusion equation is also been as a good representation of the long run link between economies Lucas (2009).

we obtain a detrended distribution of income per capita $y(x,t)$ as in figure 3. It seems that the distribution evolves roughly in a similar way as a diffusion process. If we run, for every year, the regression $y(x,t) \propto \exp(-\beta(t)x)$ we observe that the coefficient $\beta(t)$ will decrease in time (see figure 4). It starts from around 0.014 in 1955, falls sharply to around 0.009, in the beginning of the 1970's, and continues to fall, at a much smaller rate, afterwards. This means that the distribution of the US GDP per capital flattens out along time. This type of behavior is clearly compatible with a diffusion process with a low convergence rate towards a uniform distribution.

`Figure 3 around here`

`Figure 4 around here`

As the support variable x is independent of geographical distance the dynamics that drives the distribution is only related to the initial imbalance between GDP per capita among states. In other words, independently of geographical distance, an initial heterogeneity in the GDP per capita tends to be eliminated along time, although at a very slow rate.

This process is not incompatible with the existence of agglomerative forces, that tend to create an externality between regions which are geographically close. The increasing jaggedness of the detrended distribution, may be explained by those spatial agglomerative forces. However, as we will see in the rest of the paper, this type of behavior may be associated to the existence of pattern formation which is associated to the existence of diffusion induced instability, that is, to the existence of a potential for instability in the order-preservation of the dynamics of the distribution.

This section derives two features for the global growth dynamics: unboundedness of the distribution, related to the existence of positive growth rates, and convergence of the transient part towards a homogeneous distribution.

3 The optimal intertemporal distributed growth problem

There are three methodological dimensions to the problem: the choice of the positional support, the dynamic equation for capital flows connecting the positional and time dynamics, the planners objective function, and other technical assumptions regarding limits of the positional support.

We assume there is a homogeneous product which is produced, saved and consumed in different quantities in different locations x, for every point in time. The production in every location is a function of the location's stock of capital and the technology is homogeneous across space. The stock of capital in location x is financed by own and external savings. External savings are equal to capital flows into location x, which are assumed to be proportional to the marginal shift in the global distribution of capital to location x. Global capital markets are perfect, except for the existence of transaction costs which are not idiosyncratic. This implies that, for every location x, the product market equilibrium equation is a semi-linear autonomous parabolic partial differential equation (PDE).

Observe that the rate of return for every location x is equal to the local marginal productivity of capital.

All locations evaluate the paths of consumption by means of a intertemporal utility function. There is a centralized planner who determines a path for the global distribution of consumption in order to maximize social welfare function, and potentially reallocates consumption between locations. The welfare function assumes that the central planner aggregates the intertemporal utility functions for all locations with a uniform distribution function, i.e., it does not discriminate different locations. A global solvability condition holds.

3.1 Capital accumulation as a diffusion equation

The independent variables in our model are (x, t) where $x \in \mathbb{X} = (-\infty, \infty)$, $t \in \mathbb{T} = [0, \infty)$. The stock of capital at time t for region x is denoted by $K(t, x)$, the path the stock of capital in location x along time is denoted by $[K(t,x)]_{t\in\mathbb{T}}$. Then $(K(t,x))_{x\in\mathbb{X}}$ is the global distribution of the stock of capital at time $t \in \mathbb{T}$ and $[(K(t,x))_{x\in\mathbb{X}}]_{t\in\mathbb{T}}$ is path of the global distribution of the stock of capital along time. All the other variables are denoted analogously.

The technology at location x is represented by the production function $Y(x,t) = F(K(x,t))$. We consider two cases: First, the AK case in which the marginal productivity is independent of the capital stock, and therefore, the rate of return to capital is homogeneous across space. This is the benchmark case in one-sector endogenous growth models. Second, the concave case, in which $F^{''}(K(x,t)) < 0$: in this case heterogeneity in capital distribution entails heterogeneity in rates of return of capital across space. This is the benchmark case in one-sector exogenous growth models, if we add a exogenous source of productivity growth.

At location x, instantaneous net savings is $S(x,t) = Y(x,t) - C(x,t) - \delta K(x,t)$ and the equilibrium for the good market is $dK(x,t) = S(x,t)dt$, for a small time interval dt. Then, instantaneously,

$$\frac{dK(x,t)}{dt} = \frac{\partial K(x,t)}{\partial t} + \frac{\partial K(x,t)}{\partial x}\frac{dx}{dt} = S(x,t),\ x \in \mathbb{X}$$

savings finances the accumulation in the stock of capital which is used for production in location x and capital flows to other locations.

The capital inflow to x introduces a marginal shift in the distribution of capital among all the locations. We assume the net capital inflow to x is, instantaneously, proportional to the derivative of the distribution of capital across the global economy at x,

$$\frac{\partial K(x,t)}{\partial x}\frac{dx}{dt} = -\tau^2 \frac{\delta}{\delta x}\left(\int_{-\infty}^{\infty} \frac{\partial K(\xi,t)}{\partial \xi} d\xi\right) \tag{1}$$

where $\delta/\delta x$ is a functional derivative operator and τ^2 is an adjustment cost. We determine the functional derivative as is common in variational calculus: consider the distribution $K(x,t)$ and introduce a "spike" shift of height ϵ at location $y \in \mathbb{X}$. Then the distribution $[K(x,t)]_{x\in\mathbb{X}}$ changes to $[\tilde{K}(x,t)]_{x\in\mathbb{X}}$ where $\tilde{K}(x,t) = K(x,t) + \epsilon\delta(x-y)$ where $\delta(.)$ is Dirac's delta function. Then $\tilde{K}(x,t) = K(x,t)$ for $x \neq y$ and $\int \tilde{K}(x,t)dx - \int K(x,t)dx = \epsilon$ (see

(Gelfand and Fomin, 1963, p. 9)). Then the functional derivative is

$$
\begin{aligned}
\frac{\delta}{\delta x}\left(\int_{-\infty}^{\infty}\frac{\partial K(\xi,t)}{\partial \xi}d\xi\right) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{-\infty}^{\infty}\frac{\partial(K(\xi,t)+\epsilon\delta(x-\xi))}{\partial\xi}-\frac{\partial K(\xi,t)}{\partial\xi}d\xi \\
&= \int_{-\infty}^{\infty}\frac{\partial^2 K(\xi,t)}{\partial\xi^2}\delta(x-\xi)d\xi \\
&= \frac{\partial^2 K(x,t)}{\partial x^2}.
\end{aligned}
$$

Then

$$
\frac{\partial K(x,t)}{\partial x}\frac{dx}{dt} = -\tau^2\frac{\partial^2 K(x,t)}{\partial x^2}.
$$

This is equivalent to

$$
Kdx = \tau^2\sigma(K)dt
$$

where $\sigma(K) \equiv -K(\partial^2 K/\partial x^2)/(\partial K/\partial x)$ is the relative curvature of the distribution of K across space. This means that we are assuming implicitly that the variation of capital across locations, per unit of time, is proportional to the local relative curvature.

All things considered, the dynamics of the density $K(x,t)$ is governed by the quasi-linear diffusion equation

$$
\frac{\partial K(x,t)}{\partial t} = \tau^2\frac{\partial^2 K(x,t)}{\partial x^2} + F(K(x,t)) - C(x,t) - \delta K(x,t),\ (x,t)\in\mathbb{R}\times\mathbb{R}_+. \tag{2}
$$

3.2 Efficient consumption distribution

When goods and capital may be freely reallocated among heterogenous agents, we should have a single central planner that chooses not only the optimal intertemporal allocation of consumption, but also the optimal intratemporal distribution across different locations. The optimization criterium should consider not only aggregation of preferences across time but also across space.

The discounted and additively separable intertemporal utility function (**??**) presents a benchmark aggregation for utilities through time for the representative agent located in each point in space. Even when we consider intertemporally dependent preferences, the exponential time discounting would still present a natural weighting scheme.

Though we do not intend to dwell into the deep issues related to the definition of a collective preference relationship, we should observe that the choice of an aggregate utility function, when there is spatial asymmetry, is not as settled as for the case in which there is asymmetry. In order to stay close to a Pareto criterium, based upon the maximization of individual welfare, we will assume a Bergson-Samuelson social welfare function [9] and extend it to an intertemporal context.

Accepting an aggregate criterium, based upon a weighted sum of independent individual intertemporal utility functions, is only a first step. Next we have to address the problems of choosing a spatial weighting scheme and of dealing with the unboundedness of the spatial support.

Benthamian, Millian, von-Neumann-Morgenstern, egalitarian or Rawlsian utility functions [10] are based upon different weighting criteria, and verify reasonable ethical postulates. A Benthamian utility function would be defined, in our setup, as a simple, unweighted, sum of the individual intertemporal utility functions, for the representative households located in every point in space,

$$\int_{-\infty}^{+\infty} \int_{0}^{\infty} u(c(x,t))e^{-\delta t} dt dx.$$

This utility functional solves the aggregation problem but not the unboundedness problem: the intertemporal aggregate utility will be unbounded, even in the case in which all the admissible distributions would tend to a spatially homogeneous bounded steady state [11]

All the other collective utility functions introduce some type of spatial weighting. The simpler weighting schemes are based upon spatial discounting or averaging.

Spatial discounting introduces a symmetry between time and space, by penalizing dates and locations far away from $(x,t) = (0,0)$ [12]. For instance, space could be discounted in an

[9] Bergson (1938) and (Samuelson, 1947, p.219-229) presented an additive social welfare function as a sum of cardinal utility functions. Harsanyi (1955) showed that an aggregate social preference based upon ordinal utility functions and obeying some postulates (v.g, symmetry, independence, transitivity etc) would be represented as a weighted sum of individual cardinal (or Bernoullian) utility functions. These postulates also verify the two main Rawlsian criteria, impartiality and unanimity (see Mueller (2003)).

[10] See Atkinson and Stiglitz (1980) for the related static counterpart.

[11] This is the spatial counterpart of the unboundedness problem arising in the undiscounted Ramsey intertemporal utility function, $\int_0^\infty u(c(t))dt$.

[12] Boucekkine et al. (2009) consider this case.

exponential way, leading to the utility functional

$$\int_{-\infty}^{+\infty}\int_0^{\infty} u(c(x,t))e^{-(\delta t+\delta_x x^2)}dtdx,$$

where $\delta_w > 0$. However, spatial discounting has two unwelcome features: it introduces a preference relation over locations in space, which violates Harsanyis's symmetry postulate, and tends to force rejection of an homogeneous spatial distribution as an optimal distribution in the steady state (even in the case in which the other parameters of the model are spatially homogeneous).

Spatial averaging, weights all the locations in space by the inverse of their relative distance to $x = 0$. As weights are spatially homogeneous, then there is not an implicit preference of the central planner for any particular location in space. The following Millian intertemporal utility function may be seen as a collective utility function based upon an averaging criteria [13]

$$V := \lim_{x\to\infty} \frac{1}{2x}\int_{-x}^{x}\int_0^{\infty} u(c(y,t))e^{-\delta t}dtdy.$$

This utility functional will be bounded for steady state spatially symmetric distributions of consumption, i.e., $V = \frac{u(\overline{c})}{\rho}$, allowing for the comparison of alternative optimal distributional strategies. We will assume a Millian central planner from now on.

3.3 Boundary conditions

The solutions of partial differential equations depend on the specification of the boundary conditions. Three types of alternative boundary conditions can be found in the applied mathematics literature and be adapted to our model: free boundaries if $\lim_{x\to\pm\infty} k(x,t)$ and $\lim_{x\to\pm\infty} \frac{\partial k(x,t)}{\partial x}$ are not specified, or Cauchy or Dirichlet boundaries if $\lim_{x\to+\infty} k(x,t) = \overline{k}(t)$ and $\lim_{x\to-\infty} k(x,t) = \underline{k}(t)$ or if $\lim_{x\to+\infty} \frac{\partial k(x,t)}{\partial x} = \frac{\partial \overline{k}}{\partial x}(t)$ and $\lim_{x\to-\infty} \frac{\partial k(x,t)}{\partial x} = \frac{\partial \underline{k}}{\partial x}(t)$ were given, respectively. Neumann (or no-flux) boundaries $\lim_{x\to+\infty} \frac{\partial k(x,t)}{\partial x} = \lim_{x\to-\infty} \frac{\partial k(x,t)}{\partial x} = 0$ are a popular special case.

[13] It could also be seen as a von-Neumann-Morgenstern utility function for the case in which the particular location of a consumer is stochastic and equally probable.

The choice of the particular boundary condition depends on the type of heterogeneity, that we are considering. As, in our case, x is close to spatial location, the choice of the central place is arbitrary. A boundary condition that ties down the stock of capital of distant regions, would introduce incentives for the dispersion of capital and would imply that the choice of the optimal distribution of consumption would be determined exogenously. As our focus is related with the general properties of the dynamics of the capital distribution, it is irrelevant which geographical regions lay at a particular point in the distribution along time. Then Cauchy boundaries would be inappropriate.

Neumann boundaries would be a candidate for a state-space as $\mathbb{X} = [0, +\infty)$, where $x = 0$ could stand for the richest region and $x = +\infty$ for the poorest. The existence of a smooth distribution function would be possible if we would assume, tautologically, that capital could not move outside those boundaries. Neumann boundaries would allow for more flexibility than the Cauchy boundaries, because they would not eliminate the limit situation of homogeneity. However, from the application of Pontriyagin's maximum principle, the central planner would have a very strong incentive to allocate consumption to the extremes of the distribution. The dual boundary conditions would be $\lim_{k\to\pm\infty} \frac{\partial c(x,t)}{\partial x} = +\infty$.

We will assume, instead, the following boundary conditions,

$$\lim_{x\to\pm\infty} \frac{k(x,t)}{x} = 0, \quad \forall t \in \mathbb{T}.,$$

which is both weaker than the Neumann boundaries and is more realistic. It has the following property: the "tails" of the capital stock distribution are bounded functions of time and may be approximated by constant functions of space.

3.4 The optimal intertemporal distributions

The central planner's problem is built by assembling the elements of the model presented in the last section. It consists in determining an optimal distributive strategies for consumption and capital, $[C^*] \equiv [C^*(x,t)]_{(x,t)\in\mathbb{X}\times\mathbb{T}}$ and $[K^*] \equiv [K^*(x,t)]_{(x,t)\in\mathbb{X}\times\mathbb{T}}$, respectively, which solve the problem:

$$V([C]) \equiv \max_{[C]} \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_{0}^{\infty} u(C(\xi,t)) e^{-\rho t} dt d\xi. \tag{3}$$

subject to the accumulation equation

$$\frac{\partial K(x,t)}{\partial t} = \tau^2 \frac{\partial^2 K(x,t)}{\partial x^2} + F(K(x,t)) - C(x,t) - \delta K(x,t) \quad \forall (x,t) \in (\mathbb{X}, \mathbb{T}) \tag{4}$$

and the terminal, the boundary and the initial conditions

$$\begin{aligned} \lim_{t\to\infty} e^{-\int_0^t r(x,s)ds} K(x,t) &\geq 0, \quad \forall x \in \mathbb{X}, & (5)\\ \lim_{x\to\mp\infty} \frac{K(x,t)}{x} &= 0, \quad \forall t \in \mathbb{T} & (6)\\ K(x,0) &= \phi(x), \ \ \forall x \in \mathbb{X} \ \text{ given.} & (7) \end{aligned}$$

Proposition 1. *The necessary first order conditions for optimality are:*

$$u^{'}(C(x,t)) = Q(x,t), \ \forall (x,t) \in (\mathbb{X}, \mathbb{T}) \tag{8}$$

where $Q(x,t)$ is the generalized (in a distributional sense) co-state variable, the generalized Euler equation is a backward parsbolic PDE

$$\frac{\partial Q(x,t)}{\partial t} = -\tau^2 \frac{\partial^2 Q(x,t)}{\partial x^2} + Q(x,t)(\rho + \delta - F^{'}(K(x,t)) \tag{9}$$

give the following non-linear PDE forward-backward system in $C(x,t)$ and $K(x,t)$ for admissible paths $\{C\}$ $\{K\}$ verifying equations (4) *and* (5)*-*(7)

$$\begin{aligned} \frac{\partial C(x,t)}{\partial t} = -\tau^2 \left[\frac{\partial^2 C(x,t)}{\partial x^2} + \frac{u^{'''}(C(x,t))}{u^{''}(C(x,t))} \left(\frac{\partial C(x,t)}{\partial x} \right)^2 \right] + \\ + \frac{u^{'}(C(x,t))}{u^{''}(C(x,t))} \left(\rho + \delta - F^{'}(K(x,t)) \right), \quad x \in \mathbb{R}, \ t > 0 \end{aligned} \tag{10}$$

together with the boundary conditions

$$\lim_{x\to\pm\infty} e^{-\rho t} \frac{u^{'}(C(x,t))}{x} = 0, \ t > 0 \tag{11}$$

and the transversality condition

$$\lim_{t\to\infty} e^{-\rho t} \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} u^{'}(C(\xi,t)) K(\xi,t) d\xi = 0. \tag{12}$$

Next we will characterize the dynamics of the optimal distributional by considering two alternative technologies: a homogeneous linear technology and a homogeneous Cobb-Douglas technology.

4 A global AK model

In this section we assume that the production function is linear and homogeneous across the spatial support, $Y(x,t) = AK(x,t)$, meaning that both the average and marginal product are equal and homogeneous across locations. We assume further that the instantaneous utility function is homogeneous across locations and is iso-elastic: $u(C) = (1-\theta)^{-1}C^{1-\theta}$ for $\theta > 0$.

Then differences in the marginal productivity of capital cannot drive capital movements between locations. That is, if two different locations have different levels of capital stock, in the same point in time, then differences in marginal productivity of capital cannot be a source of distributional dynamics. However, the concavity of utility function may imply that differences in the levels of consumption generate optimal changes in the global consumption distribution which may be locationally inhomogeneous.

We would expect, from standard endogenous growth models, that an optimal solution would display unbounded growth in a global sense, that is involving all the distribution. However, two questions arise: does unbounded growth is admissible, in the sense of verifying the transversality condition ? Is there stable transitional dynamics, around a global balanced growth path ?

The first-order conditions are given by the forward-backward system of non-linear parabolic partial differential equations, (10) and (4), which become

$$\frac{\partial C}{\partial t} = -\tau^2 \left[\frac{\partial^2 C}{\partial x^2} - \frac{1+\theta}{C} \left(\frac{\partial C}{\partial x} \right)^2 \right] + \gamma C, \quad x \in \mathbb{R}, \ t > 0 \tag{13}$$

$$\frac{\partial K}{\partial t} = \tau^2 \frac{\partial^2 K}{\partial x^2} + rK - C, \quad x \in \mathbb{R}, \ t > 0 \tag{14}$$

$$\tag{15}$$

where

$$r \equiv A - \delta, \ \gamma \equiv \frac{r-\rho}{\theta}, \tag{16}$$

the transversality condition

$$\lim_{t\to\infty} \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} e^{-\rho t} K(\xi,t) C(\xi,t)^{-\theta} d\xi = 0, \tag{17}$$

and the dual boundary conditions

$$\lim_{x\to\pm\infty}\left(e^{\rho t}C(x,t)^{\theta}x\right)^{-1}=0,\ t\geq 0. \tag{18}$$

In system (13)-(14), r is the net total factor productivity and γ is equal to the endogenous growth rate in the benchmark homogeneous AK model. The initial condition $K(x,0)=\phi(x)$ and the boundary condition (6) should also hold.

Lemma 1. *Let r and γ be as in equation (16). Then the coupled system (13)-(14) has the closed form solution*

$$K(x,t)=e^{\gamma t}k(x,t),\ C(x,t)=e^{\gamma t}c(x,t),\ t\geq 0,\ x\in\mathbb{R} \tag{19}$$

where

$$k(x,t)=\frac{1}{2\tau\sqrt{\pi\theta t}}\int_{-\infty}^{\infty}\phi(\xi)e^{-\left(\frac{x-\xi}{2\tau}\right)^2\frac{1}{\theta t}}d\xi,\ t>0,\ x\in\mathbb{R}$$

and

$$c(x,t)=\frac{1}{2\tau\sqrt{\pi\theta t}}\int_{-\infty}^{\infty}\phi(\xi)\left[r-\gamma+(\theta-1)\left(\frac{1}{2\theta t}-\left(\frac{x-\xi}{2\tau\theta t}\right)^2\right)\right]e^{-\left(\frac{x-\xi}{2\tau}\right)^2\frac{1}{\theta t}}d\xi,\ t>0,\ x\in\mathbb{R}.$$

Lemma 2. *A necessary condition for a solution of the centralized problem is that $r>\gamma$.*

If we want to get an explicit solution of the centralized problem we need to check the boundary primal and the dual conditions. This poses some restriction on function $\phi(x)$. A function which is consistent with the empirical distribution in section **??** is

$$\phi(x)=k_0+e^{-\beta|x|},\ k_0>0,\ \beta>0. \tag{20}$$

Proposition 2. *Solution for the detrended variables of the global AK-model*

$$k(x,t)=k_0+\frac{1}{2}\left\{1-\mathrm{erf}\left(\frac{2\beta\theta\tau^2t-x}{2\tau\sqrt{\theta t}}\right)+e^{2\beta x}\left(1-\mathrm{erf}\left(\frac{2\beta\theta\tau^2t+x}{2\tau\sqrt{\theta t}}\right)\right)\right\}e^{\theta\tau^2\beta^2t-\beta x}$$

where $\mathrm{erf}(x)=(2/\sqrt{\pi})\int_0^x e^{-z^2}dz$, *and*

$$c(x,t)=(r-\gamma)k(x,t)+c_r(x,t)$$

where

$$c_r(x,t)=\frac{\tau\beta}{\sqrt{4\theta\pi t}}e^{-\beta x}\left\{\beta\tau\sqrt{\pi\theta t}e^{\theta t\tau^2\beta^2}\left[\left(\mathrm{erf}\left(\frac{2\beta\theta t\tau^2-x}{2\tau\sqrt{\theta t}}\right)-1\right)+\right.\right.$$
$$\left.\left.+e^{2\beta x}\left(\mathrm{erf}\left(\frac{2\beta\theta t\tau^2+x}{2\tau\sqrt{\theta t}}\right)-1\right)\right]+2e^{\frac{x(4\beta\tau^2\theta t-x)}{\theta t4\tau^2}}\right\} \tag{21}$$

`Figure 6 around here`

Therefore the model displays endogenous growth with a balanced growth path growing in all regions at the rate γ but the distributions among locations x is stationary across time and is equal to the initial distribution.

That is, in order to have both long run growth and convergence the rate of return of capital has to depend on the level of the stock of capital.

5 Global decreasing returns

The existence of unbounded growth generated by a homogeneous linear technology generating is a natural result. In this section we assume there is homogeneous decreasing returns in every location. A priori one would expect one of two results: First, decreasing returns would lead to an homogeneous Ramsey steady state for every point in space. In this case, the homogeneous stationary state would be stable in a distributional sense. Second, as shown by Brito (2004) and Brock and Xepapadeas (2008), there may be a diffusion-induced instability mechanism generated by spatial contact among locations which can work against the previous stability mechanism and generate unbounded growth. However, trajectories displaying unbounded growth may admissible if the transversality conditions and the boundary conditions are verified. As in the global linear case we have to check under which conditions an unbounded distribution is a solution for the centralized problem.

There is a crucial difference homogeneous linear and concave global production functions. If there is an initial in-homogeneous capital stock distribution, the marginal rates of return for capital are homogeneous throughout the distribution and they different across regions in the second. This introduces a new source of distributional dynamics which will be channeled through consumption.

In this section we assume there is a homogeneous Cobb-Douglas gloval production function,

$$Y(x,t) = F\left(K(x,t)\right) = AK(x,t)^{\alpha}, \quad (x,t) \in (\mathbb{R}, \mathbb{R}_{+})$$

where $0 \leq \alpha \leq 1$, and, again, we assume an isoelastic utility function.

It is easier to deal with coupled PDE system for (Q, K), which is derived from equations (4)-(9), and (8)

$$
\begin{aligned}
\frac{\partial Q(x,t)}{\partial t} &= -\tau^2 \frac{\partial^2 Q(x,t)}{\partial x^2} + Q(x,t)\left(\rho + \delta - F^{'}(K(x,t))\right), \\
\frac{\partial K(x,t)}{\partial t} &= \tau^2 \frac{\partial^2 K(x,t)}{\partial x^2} + F(K(x,t)) - (u^{'})^{-1}(Q(x,t)) - \delta K(x,t).
\end{aligned}
\tag{22}
$$

Next we will study the dynamics in the neighborhood of the homogeneous stationary solution, $[(Q^*, K^*)]_{x\in\mathbb{R}}$ such that $(Q(x,t), K(x,t)) = (Q^*, K^*)$ for all pairs (x,t).

The homogeneous stationary solution is

$$
F^{'}(K^*) = \rho + \delta,\ Q^* = u^{'}(C^*),\ C^* = \frac{\rho + (1-\alpha)\delta}{\alpha} K^*. \tag{23}
$$

5.1 Linearized system dynamics

Differently from the version in the previous section, the quasi-linear parabolic PDE system (22) in (Q, K) does not seem to have a closed form solution. Next, we will characterize the local distributional dynamics behavior by studying the approximated system in the neighborhood of the homogeneous stationary distribution $[(Q^*, K^*)_{x\in\mathbb{R}}]$ [14].

Let us introduce a small perturbation in the neighborhood of the stationary distribution by writing $Q(x,t) = Q^* + \tilde{Q}(x,t)$ and $K(x,t) = K^* + \tilde{K}(x,t)$. If we linearize the system (22) in the neighborhood of the homogeneous distribution we get a linear PDE forward-backward system in $\tilde{Q}(x,t)$ and $\tilde{K}(x,t)$

$$
\begin{aligned}
\frac{\partial \tilde{Q}(x,t)}{\partial t} &= -\tau^2 \frac{\partial^2 \tilde{Q}(x,t)}{\partial x^2} - Q^* F^{''}(K^*)\tilde{K}(x,t) \ \ x \in \mathbb{R},\ t > 0 \\
\frac{\partial \tilde{K}(x,t)}{\partial t} &= \tau^2 \frac{\partial^2 \tilde{K}(x,t)}{\partial x^2} - (u^{''}(C^*))^{-1}\tilde{Q}(x,t) + \rho \tilde{K}(x,t), x \in \mathbb{R},\ t > 0
\end{aligned}
\tag{24}
$$

together with the approximated boundary conditions

$$
\lim_{x\to\pm\infty} \frac{\tilde{K}(x,t)}{x} = 0,\ \lim_{x\to\pm\infty} e^{-\rho t}\frac{\tilde{Q}(x,t)}{x} = 0
$$

[14] See Henry (1981)

and the approximated transversality condition

$$\lim_{t\to\infty} e^{-\rho t} \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} Q^* \tilde{K}(y,t) + K^* \tilde{Q}(y,t) dy = 0. \tag{25}$$

We consider, again, initial perturbations $\tilde{K}(x,0) = \phi(x) = e^{-\beta|x|}$ where $\beta > 0$.

Let us introduce Fourier modes as trial solutions

$$\tilde{Q}(x,t) = \mathcal{Q}(\omega,t)\, e^{2\pi i\omega x},\; \tilde{K}(x,t) = \mathcal{K}(\omega,t)\, e^{2\pi i\omega x}$$

where $\omega \in \mathbb{R}$ are distributional frequencies.

Then, the PDE system (24) is transformed into the linear ODE

$$\begin{pmatrix} \frac{\partial \mathcal{Q}(\omega,t)}{\partial t} \\ \frac{\partial \mathcal{K}(\omega,t)}{\partial t} \end{pmatrix} = \begin{pmatrix} (2\tau\pi\omega)^2 & -Q^* F''(K^*) \\ (u''(C^*))^{-1} & \rho - (2\tau\pi\omega)^2 \end{pmatrix} \begin{pmatrix} \mathcal{Q}(\omega,t) \\ \mathcal{K}(\omega,t) \end{pmatrix},\ \omega \in \mathbb{R},\ t > 0. \tag{26}$$

We call $J(\omega)$ the Jacobian matrix which is parameterized by the frequencies ω. We call wave number to

$$\xi = \xi(\omega) \equiv (2\tau\pi\omega)^2 \geq 0,$$

and define

$$\mu \equiv -\frac{F''(K^*)Q^*}{u''(C^*)} = \frac{(\rho + \delta(1-\alpha))(\rho+\delta)(1-\alpha)}{A\alpha\theta} \geq 0, \tag{27}$$

which is independent from the distributional frequencies, is positive if the production function displays decreasing marginal returns and $\mu = 0$ if it features constant returns.

The jacobian matrix has a trace $\mathrm{tr}(J(\omega)) = \rho > 0$ and determinant $\det(J(\omega)) = (\rho - \xi)\xi + \mu$. Observe that the first is independent from the wave number and is positive, while the second varies with ω and has an ambiguous sign. However, the discriminant, though depending on the wave number, and therefore on the distributional frequencies, is always positive,

$$\Delta(\omega) \equiv \left(\frac{\rho}{2} - \xi(\omega)\right)^2 + \mu > 0,$$

because $\mu \geq 0$.

Then, the eigenvalues of $J(\omega)$ are termed eigenfunctions because they are functions of the frequencies ω

$$\lambda^{\mp}(\omega) = \frac{\rho}{2} \mp \Delta(\omega)^{1/2}. \tag{28}$$

We readily conclude that $\lambda^{+}(\omega) > 0$ for any $\omega \in \mathbb{R}$, but the sign of $\lambda^{-}(\omega)$ is, a priori, ambiguous.

Candidate solutions for the transformed system (26) are obtained from the generalized eigenspace associated to the eigenfunction $\lambda^{-}(\omega)$:

Lemma 3. *Candidate solutions for the transformed system* (26)

$$\begin{aligned} K(\omega,t) &= \mathcal{K}(\omega,0)e^{\lambda^{-}(\omega)t}, \;\; t>0 &(29)\\ Q(\omega,t) &= \mathcal{K}(\omega,0)\left(\frac{\rho}{2}-\xi(\omega)+\left[\left(\frac{\rho}{2}-\xi(\omega)\right)^2+\mu\right]^{1/2}\right)e^{\lambda^{-}(\omega)t}, \;\; t>0. &(30)\end{aligned}$$

We can obtain the solution in the original (x,t) space through the inverse transforms, because the space x is unbounded.

Lemma 4. *Candidate solutions for system* (24)

$$\tilde{K}(x,t) = \int_{-\infty}^{\infty} \phi(y) g_k(x-y,t)dy, \; t>0, \tag{31}$$

$$\tilde{Q}(x,t) = \int_{-\infty}^{\infty} \phi(y) g_c(x-y,t)dy, \; t>0, \tag{32}$$

where

$$\begin{aligned} g_k(x,t) &= \int_{-\infty}^{\infty} e^{\lambda^{-}(\omega)t+2\pi i\omega x}d\omega, &(33)\\ g_c(x,t) &= \int_{-\infty}^{\infty}\left(\frac{\rho}{2}-\xi(\omega)+\left[\left(\frac{\rho}{2}-\xi(\omega)\right)^2+\mu\right]^{1/2}\right)e^{\lambda^{-}(\omega)t+2\pi i\omega x}d\omega. &(34)\end{aligned}$$

Equations (31)-(32) would be approximate solutions of the centralized problem if the transversality condition (25) and the boundary conditions hold. This depends on the Green functions (33)-(34). However, as they do not have an explicit representation we will try to characterize them qualitatively by studying their implicit stability properties.

In order to determine the stability properties, we have to characterize the time behavior of equations (29)-(30). If the eigenfunction $\lambda^{-}(\omega)$ is negative for all $\omega \in \mathbb{R}$ then

$$\lim_{t\to\infty} \mathcal{K}(\omega,t) = \lim_{t\to\infty} \mathcal{Q}(\omega,t) = 0, \; \forall \omega \in \mathbb{R}.$$

Initial deviations from the steady state, $\mathcal{K}(\omega, 0)$, will die out asymptotically. The solutions of $\tilde{K}(x,t)$ and $\tilde{Q}(x,t)$ will converge asymptotically to a homogeneous stationary state equal to zero for any initial deviation. In this case the both the transversality condition and boundary conditions hold.

If there is a non-empty set of frequencies such that Ω^u such that $\lambda^-(\omega) > 0$ if $\omega \in \Omega^u$ then the associated frequencies for the state and co-state variables will be unbounded

$$\lim_{t\to\infty} \mathcal{K}(\omega, t) = \lim_{t\to\infty} \mathcal{Q}(\omega, t) = \infty, \text{ for } \omega \in \Omega^u.$$

Then, initial deviations from the steady state, $\mathcal{K}(\omega, 0)$, will also become unbounded. The solutions for $\tilde{K}(x,t)$ and $\tilde{Q}(x,t)$ may not verify both the transversality and the boundary conditions for any initial deviation. If there is an unbounded solution that verifies both the transversality condition and the boundary conditions, then the optimal distributional problem will have an unbounded solution, meaning that our model displays a new mechanism for endogenous growth. If not, the candidate functions are not a solution for the optimal distributional problem.

In order to study this we will perform a qualitative dynamics exercise in the transformed system and try to derive the implications for the initial perturbed variables.

5.2 Generalized stability and diffusion induced bifurcations

Consider again the wave number $\xi = (2\tau\pi\omega)^2$. A particular case of our model is the distributionally homogeneous Ramsey case where $\xi = 0$, in which the homogeneous stationary equilibrium (Q^*, K^*) is saddle point stable because

$$\lambda^-(0) = \frac{\rho}{2} - \left[\left(\frac{\rho}{2}\right)^2 + \frac{\rho + (1-\alpha)\delta}{\alpha}\right]^{1/2} < 0.$$

If $\xi > 0$ there is distribution heterogeneity, and differently from homogeneous case with concave technology and preferences, the homogeneous steady state may not always be saddle-point stable in a distributional sense:

Lemma 5. *(Stable eigenfunctions)*
Let

$$\lambda^c \equiv \frac{\rho}{2} - \mu^{1/2}, \tag{35}$$

and consider the stable eigenfunction $\lambda^-(\xi)$ *in equation* (28)*. Then:*

1. *If* $\rho/2 < \mu^{1/2}$ *then* $\lambda^-(\xi) < \lambda^c < 0$ *for every* $\xi \in \mathbb{R}_+$*;*
2. *If* $\rho/2 = \mu^{1/2}$ *then there is a critical wave number* $\xi^c = \rho/2 > 0$ *such that* $\lambda^- = \lambda^c = 0$ *if* $\xi = \xi^c$ *and* $\lambda^-(\xi) < \lambda^c = 0$ *if* $\xi \neq \xi^c$*;*
3. *If* $\rho/2 > \mu^{1/2}$ *then there are two wave numbers*

$$\xi^{\mp} = \frac{\rho}{2} \mp \left[\left(\frac{\rho}{2}\right)^2 - \mu \right]^{1/2}$$

verifying $\xi^+ > \xi^- > 0$*, such that,* $0 < \lambda^-(\xi) < \lambda^c$ *if* $\xi^- < \xi < \xi^+$*,* $\lambda^-(\xi) = 0$ *if* $\xi = \{\xi^-, \xi^+\}$*, and* $\lambda^-(\xi) < 0$ *if* $\xi \notin [\xi^-, \xi^+] \subset \mathbb{R}_+$*.*

Diffusion may be a source for instability if $\rho/2 > \mu^{1/2}$, for positive wave numbers $\xi > 0$. This means that this only occurs if there is heterogeneity in the distribution.

There are several instances of this phenomena in systems of parabolic PDE's. In the pattern formation literature, featuring systems of forward PDE's when the homogeneous case has two eigenvalues with negative real parts, this type of instability is termed Türing instability (Türing (1952)). In systems of forward-backward equations arising from the first order conditions of optimal control problems this has been showed by Brito (2004) and Brock and Xepapadeas (2008). We can call generically the change in the stability properties related to the existence of diffusion as *diffusion induced instability* (see Brock and Xepapadeas (2008)).

We say there is a *diffusion induced bifurcation* if for a critical parameter value and for a critical positive wave number (or associated frequencies) the dimension of the stable manifold changes dimension.

Proposition 3. *Let*

$$\theta_c \equiv \frac{4C(K^*)F''(K^*)}{\rho^2} \tag{36}$$

Then there is diffusion induced bifurcation for the critical parameter value and wave number (θ^c, ξ^c). *The homogeneous stationary point* (Q^*, K^*) *is saddle-point stable in a distributional sense if* $\theta < \theta^c$ *and it is unstable in a distributional sense if* $\theta > \theta^c$.

We call the critical frequencies to

$$\omega^c_{\mp} = \mp \frac{\sqrt{2\rho}}{4\pi\tau}.$$

As $\omega^c_{\pm} \neq 0$ and $\mathrm{Im}(\lambda^*(\xi^c, \theta^c)) = 0$, the pattern formation literature (see Cross and Hohenberg (1993)) calls the type of instability that may exist type I_s instability, which is associated to the formation of spatial patterns which are stationary in time.

The critical value for the bifurcating parameter

$$\theta^c = \frac{4(\rho + \delta(1-\alpha))(\rho+\delta)(1-\alpha)}{\rho^2 \alpha A}$$

is a function of the technological parameters and of the rate of time preference. Figure 6 plots the line $\theta = \theta_c(\alpha)$ in the graph (α, θ) for given values of the other parameters. We observe that below that line we have the case in which there is saddle-point stability $(\theta < \theta^c)$ and above that line there is distributional instability $(\theta > \theta^c)$, in the distributional sense.

`Figure 6 around here`

In this parametric case, the stable eigenfunction is a quadratic function of ξ,

$$\lambda^-(\xi) = \frac{\rho}{2} - \left[\left(\frac{\rho}{2} - \xi \right)^2 + \frac{(\rho + \delta(1-\alpha))(\rho+\delta)(1-\alpha)}{A\alpha\theta} \right]^{1/2},$$

and figure 7 depicts two examples of the two cases concerning distributional stability.

On those figures, several other observations can be made: first, there are "reasonable" values for the pair of parameters, featuring both concave utility and production functions such that there is distributional instability; however, second, distributional instability occurs for higher values of both parameters, that is if the production function is close to constant returns to scale instability and the utility function is more inelastic.

Then, diffusion induces instability even in the case in which there is decreasing marginal returns to capital along the support for x but more so if the production function is less concave.

The critical value for the stable eigenfunction

$$\lambda^c = \rho/2 - \left(\frac{(\rho + \delta(1-\alpha))(\rho+\delta)(1-\alpha)}{\alpha\theta A}\right)^{1/2}$$

clearly increases with the productivity parameter A.

`Figure 7 around here`

5.3 Approximating the stable manifold

As we already observed, the solution of the linearized system (31)-(32) does not admit an explicit representation. In particular, we cannot calculate the Green functions $g_k(x,t)$ and $g_c(x,t)$, for any values of the parameters. In order to get an analytical characterization of its behavior, and, in particular, to check if the transversality condition holds in the case in which there is distributional instability, we perform an approximation in the neighborhood of a the critical frequency, $\xi = \xi^c = \rho/2$. This is a frequency-independent value of the eigenvalue and is associated to the critical value of the eigenvalue, λ^c. This is the maximum value of the stable eigenvalue and defines an upper boundary for the solution.

The eigenvalue $\lambda^-(\xi)$ approximated in the neighborhood of ξ^c becomes a linear function of ξ (and therefore a quadratic function of ω

$$\lambda^-(\xi) \approx \frac{\rho}{2} - \mu^{1/2} - \mu^{-1/2}(\xi - \xi^c)^2 = \lambda^c - \mu^{-1/2}\left((2\tau\pi\omega)^2 - \rho/2\right)^2$$

Lemma 6. *(Approximated kernels) Let*

$$\gamma \equiv \frac{\rho}{2} - \sqrt{\mu}.$$

The kernels associated to the linearized dynamics are

$$g_k(x,t) = e^{\gamma t} h_k(x,t), \;\; g_q(x,t) = e^{\gamma t} h_q(x,t) \tag{37}$$

where

$$h_k(x,t) = \frac{\mu^{1/4}}{4\tau\sqrt{\rho\pi t}} \Big\{ \left[1 - \mathrm{erf}\left(\beta_1(x,t)\right)\right] e^{-\sqrt{\frac{\rho}{2}}\frac{ix}{\tau}} + \\ + \left[1 + \mathrm{erf}\left(\beta_2(x,t)\right)\right] e^{\sqrt{\frac{\rho}{2}}\frac{ix}{\tau}} \Big\} e^{-\frac{\sqrt{\mu}}{\rho t}\left(\frac{x}{2\tau}\right)^2} \tag{38}$$

and

$$h_q(x,t) = u^{'}(C^*)\left(h_{q,1}(x,t) + h_{q,2}(x,t)\right) \tag{39}$$

where

$$h_{q,1}(x,t) \equiv -\left\{\frac{1}{\sqrt{2\rho}}\left(\frac{\rho+\sqrt{\mu}}{4\pi\tau t}\right) + \left(\frac{\rho-\sqrt{\mu}}{2\pi\tau t}\right)\frac{\sqrt{\mu}}{(2\rho)^2\tau t}ix\right\} e^{-\frac{\rho^2 t}{2\sqrt{\mu}}} + \\ + \frac{\mu^{1/4}}{4\tau\sqrt{\pi\rho t}}\left[\sqrt{\mu} + \frac{\rho-\sqrt{\mu}}{2\rho t} + \sqrt{\frac{\mu}{2\rho}}\frac{i}{\tau t}x - \frac{\sqrt{\mu}(\rho-\sqrt{\mu})}{(2\rho\tau t)^2}x^2\right]\left[1-\operatorname{erf}\left(\beta_1(x,t)\right)\right] e^{-\frac{x}{2\tau}\left(\frac{\sqrt{\mu}}{2\rho\tau t}x+\sqrt{2\rho}i\right)} \tag{40}$$

and

$$h_{q,2}(x,t) \equiv -\left\{\frac{1}{\sqrt{2\rho}}\left(\frac{\rho+\sqrt{\mu}}{4\pi\tau t}\right) - \left(\frac{\rho-\sqrt{\mu}}{2\pi\tau t}\right)\frac{\sqrt{\mu}}{(2\rho)^2\tau t}ix\right\} e^{-\frac{\rho^2 t}{2\sqrt{\mu}}} + \\ + \frac{\mu^{1/4}}{4\tau\sqrt{\pi\rho t}}\left[\sqrt{\mu} + \frac{\rho-\sqrt{\mu}}{2\rho t} - \sqrt{\frac{\mu}{2\rho}}\frac{i}{\tau t}x - \frac{\sqrt{\mu}(\rho-\sqrt{\mu})}{(2\rho\tau t)^2}x^2\right]\left[1-\operatorname{erf}\left(\beta_2(x,t)\right)\right] e^{-\frac{x}{2\tau}\left(\frac{\sqrt{\mu}}{2\rho\tau t}x-\sqrt{2\rho}i\right)} \tag{41}$$

where

$$\beta_1(x,t) \equiv -\frac{\sqrt{2\rho^3}\tau t - \sqrt{\mu}xi}{2\tau\mu^{1/4}\sqrt{\rho t}} \tag{42}$$

$$\beta_2(x,t) \equiv \frac{\sqrt{2\rho^3}\tau t + \sqrt{\mu}xi}{2\tau\mu^{1/4}\sqrt{\rho t}} \tag{43}$$

Figure 8 depicts the kernels for k and q for the case in which there is no endogenous growth and there is distributional stability and figure 9 is for the case in which there is diffusion induced instability.

`Figure 8 around here`

`Figure 9 around here`

The approximated solution for the deviations from a homogeneous stationary state:

$$\tilde{K}(x,t) = e^{\lambda^c t}\int_{-\infty}^{\infty}\phi(\xi)h_k(x-\xi,t)d\xi,\ t>0$$

and

$$\tilde{Q}(x,t) = e^{\lambda^c t}\int_{-\infty}^{\infty}\phi(\xi)h_q(x-\xi,t)d\xi,\ t>0.$$

We consider, again the initial distribution

$$\phi(x) = e^{-\beta|x|},\ x \in \mathbb{R}$$

where $\beta > 0$.

Proposition 4. *The diffusion induced instability is a source of endogenous growth and the common long run rate of growth is equal to* λ^c

6 Conclusion

Diffusion induced instability can be a source of endogenous growth even in the case in which the production function displays decreasing marginal returns to capital. However, in this case, the production technology should be close to linear and/or the elasticity of intertemporal substitution should be very low. In benchmark homogeneous growth models there can be no endogenous growth with decreasing marginal returns to capital. In our model, the common long run rate of growth is equal to λ^c is different from the AK case but is still a positive function of the productivity parameter A.

Some models in math biology, chemistry, and physics also study the interaction between two diffusion processes, however they use a system of two forward PDE. in those models a diffusion induced instability, termed Türing instability, may also occur, and is associated to pattern formation. Our system is different, because it features a forward-backward system, but it also displays a diffusion induced instability, which has been already found by Brito (2004) and Brock and Xepapadeas (2008). This instability result leading to endogenous growth seems to be new.

References

Acemoglu, D. (2009). *Introduction to Modern Economic Growth*. Princeton University Press.

Atkinson, A. B. and Stiglitz, J. E. (1980). *Lectures on Public Economics*. McGraw-Hill.

Barro, R. and Sala-i-Martin, X. (1992). Convergence. *Journal of Political Economy*, 100(2):223–51.

Bergson, A. (1938). A Reformulation of Certain Aspects of Welfare Economics. *Quarterly Journal of Economics*, 52(2):310–334.

Bertola, G. (2000). Macroeconomics of distribution and growth. In Atkinson, A. B. and Bourguignon, F., editors, *Handbook of Income Distribution*, volume 1 of *Handbooks in Economics*, pages 479–541. Elsevier.

Boucekkine, R., Camacho, C., and Fabbri, G. (2010). Spatial dynamics and convergence: the spatial ak models. Working Papers 2010/06, Department of Economics, University of Glasgow.

Boucekkine, R., Camacho, C., and Zou, B. (2009). Bridging the gap between growth theory and the new economic geography: the spatial Ramsey model. *Macroeconomic Dynamics*, 13:20–45.

Brito, P. (2004). The dynamics of growth and distribution in a spatially heterogeneous world. Working Papers of the Department of Economics, ISEG-UTL http://ideas.repec.org/p/ise/isegwp/wp142004.html.

Brock, W. and Xepapadeas, A. (2008). Diffusion-induced instability and pattern formation in infinite horizon recursive optimal control. *Journal of Economic Dynamics and Control*, 32:2745–2787.

Butkovskiy, G. (1969). *Distributed Control Systems*. American Elsevier, New-York.

Carlson, D. A., Haurie, A. B., and Leizarowitz, A. (1996). *Infinite Horizon Optimal Control*. Springer-Verlag, 2nd edition edition.

Caselli, F. and Ventura, J. (2000). A representative consumer theory of distribution. *American Economic Review*, 90(4):909–26.

Cass, D. (1965). Optimum growth in an aggregative model of capital accumulation. *Review of Economic Studies*, 32:233–40.

Chatterjee, S. (1994). Transitional dynamics and the distribution of wealth in a Neoclassical model. *Journal of Public Economics*, 54(1):97–119.

Cross, M. and Greenside, H. (2009). *Pattern Formation and Dynamics in Nonequilibrium Systems*. Cambridge, Cambridge, UK.

Cross, M. C. and Hohenberg, P. C. (1993). Pattern formation outside of equilibrium. *Rev. Mod. Phys.*, 65:851.

Derzko, N. A., Sethi, S. P., and Thompson, G. L. (1984). Necessary and sufficient conditions for optimal control of quasilinear partial differential equations systems. *Journal of Optimization Theory and Applications*, 43(1):89–101.

Desmet, K. and Rossi-Hansberg, E. (2010). On spatial dynamics. *Journal of Regional Science*, 50(1):43–63.

Fujita, M., Krugman, P., and Venables, A. (1999). *The Spatial Economy. Cities, Regions and International Trade*. MIT Press.

Fujita, M. and Thisse, J.-F. (2002). *Economics of Agglomeration*. Cambridge.

Gelfand, I. M. and Fomin, S. V. (1963). *Calculus of Variations*. Dover.

Harsanyi, J. C. (1955). Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility. *Journal of Political Economy*, 63(4):309–321.

Henry, D. (1981). *Geometric Theory of Semilinear Parabolic Equations*. Number 840 in Lecture Notes in Mathematics. Springer, Berlin, Heidelberg, New York.

Hotelling, H. (1929). Stability in competition. *Economic Journal*, 39:41–57.

Isard, W. and Liossatos, P. (1979). *Spatial Dynamics and Optimal Space-Time Development*. North-Holland.

Kammler, D. W. (2000). *A First Course in Fourier Analysis*. Prentice-Hall.

Koopmans, T. (1965). On the concept of optimal economic growth. In *The Econometric Approach to Development Planning*. Pontificiae Acad. Sci., North-Holland.

Lions, J. L. (1971). *Optimal Control of Systems Governed by Partial Differential Equations*. Springer-verlag, New-York.

Lucas, R. E. (2009). Trade and the diffusion of the Industrial Revolution. *American Economic Journal: Macroeconomics*, 1(1):1–25.

Mueller, D. C. (2003). *Public Choice III*. Cambridge.

Neittaanmaki, P. and Tiba, D. (1994). *Optimal Control of Nonlinear Parabolic Systems*. Marcel Dekker.

Quah, D. (2002). Spatial Agglomeration Dynamics. *American Economic Review Papers and Proceedings*, 92(2):247–52.

Ramsey, F. P. (1928). A mathematical theory of saving. *Economic Journal*, 38(Dec):543–59.

Samuelson, P. (1947). *Foundations of Economic Analysis*, volume 80 of *Harvard Economic Studies*. Harvard University Press. 1965, Atheneum.

Stiglitz, J. E. (1969). Distribution of Income and Wealth Among Individuals. *Econometrica*, 37(3):382–97.

Türing, A. M. (1952). The chemical basis of morphogenesis. *Philosophical Transactions of the Royal Society of London B*, 237:37–72.

Yaari, M. (1981). Rawls, Edgeworth, Shapley, Nash: theories of distributive justice reexamined. *Journal of Economic Theory*, 24:1–39.

A Proofs

Proof of Proposition 1.

Proof. We have an optimal control problem of partial differential equations or an optimal distributed control problem [15]. Let us assume that there is a solution (C^*, K^*), for the problem, and define the value function as

$$V(C^*, K^*) = \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_{0}^{\infty} u(c^*(y,t)) e^{-\rho t} dt dy.$$

Consider a small continuous perturbation $(C(\epsilon), K(\epsilon)) = \{(c(x,t), k(x,t)) : \ (x,t) \in \mathbb{X} \times \mathbb{T}\}$, where ϵ is any positive constant, such that $c(x,t) = c^*(x,t) + \epsilon h_c(x,t)$ and $k(x,t) = k^*(x,t) + \epsilon h_k(x,t)$, for $t > 0$, and $h_c(x,0) = h_k(x,0) = 0$, for every $x \in \mathbb{X}$. The value of this strategy is

$$V(\epsilon) = \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_{0}^{\infty} u(c(y,t)) e^{-\rho t} dt dy.$$

But,

$$\begin{aligned} V(\epsilon) := & \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_{0}^{\infty} u(c(y,t)) e^{-\rho t} dt dy - \\ & - \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_{0}^{\infty} \lambda(y,t) \left[\frac{\partial k(y,t)}{\partial t} - \frac{\partial^2 k(y,t)}{\partial y^2} - Af(k(y,t)) + c(y,t) \right] dt dy + \\ & + \lim_{t\to\infty} \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} e^{-r(y,t)} \mu(y,t) k(y,t) dy \end{aligned} \quad (44)$$

where $\lambda(.)$ is the co-state variable and $\mu(.)$ is a Lagrange multiplier associated with the solvability condition. In the optimum, the Kuhn-Tucker condition should hold

$$\lim_{t\to\infty} \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} e^{-r(y,t)} \mu(y,t) k(y,t) dy = 0.$$

By using integration by parts we find that

$$\int_{0}^{\infty} \lambda(x,t) \frac{\partial k(x,t)}{\partial t} dt = \lambda(x,t) k(x,t)|_{t=0}^{\infty} - \int_{0}^{\infty} \frac{\partial \lambda(x,t)}{\partial t} k(x,t) dt$$

[15] Butkovskiy (1969), Lions (1971), Derzko et al. (1984) or Neittaanmaki and Tiba (1994) present optimality results with varying generality. We draw mainly upon the last two references. See also, for applications in economics Carlson et al. (1996, chap.9). Boucekkine et al. (2009) study the existence of solutions in a related problem with a linear utility function.

and that

$$\int_{-x}^{x}\int_{0}^{\infty}\lambda(y,t)\frac{\partial^2 k(y,t)}{\partial y^2}dtdy =$$

$$= \int_{0}^{\infty}\lambda(y,t)\frac{\partial k(y,t)}{\partial y}\bigg|_{y=-x}^{x} - k(y,t)\frac{\partial \lambda(y,t)}{\partial y}\bigg|_{y=-x}^{x} dt + \int_{-x}^{x}\int_{0}^{\infty}\frac{\partial^2 \lambda(y,t)}{\partial y^2}k(y,t)dtdy, \quad (45)$$

where the second term is canceled by the boundary conditions (6). Then

$$\begin{aligned} V(\epsilon) &= \lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x}\int_{0}^{\infty}\big(u(c(y,t))e^{-\rho t}+ \\ &+\frac{\partial \lambda(y,t)}{\partial t}k(y,t)+\frac{\partial^2\lambda(y,t)}{\partial y^2}k(y,t)+\lambda(y,t)[Af(k(y,t))-c(y,t)]\Big)\,dtdy - \\ &-\lim_{x\to\infty}\frac{1}{2x}\left(\int_{-x}^{x}\lambda(y,t)k(y,t)|_{t=0}^{\infty}\,dy+\int_{0}^{\infty}\lambda(y,t)\frac{\partial k(y,t)}{\partial y}\bigg|_{y=-x}^{x}dt\right) \end{aligned}$$

If an optimal solution exists, then we may characterize it by applying the variational principle,

$$\frac{\partial V(C^*,K^*)}{\partial \epsilon} = \lim_{\epsilon\to 0}\frac{V(C(\epsilon),K(\epsilon))-V(C^*,K^*)}{\epsilon} = 0.$$

But

$$\begin{aligned} \frac{\partial V}{\partial \epsilon} &= \lim_{x\to\infty}\frac{1}{2x}\Bigg\{\int_{-x}^{x}\int_{0}^{\infty}\Big[\Big(u^{'}(c^*(y,t))e^{-\rho t}-\lambda(y,t)\Big)h_c(y,t)+ \\ &+\left(\frac{\partial\lambda(y,t)}{\partial t}+\frac{\partial^2\lambda(y,t)}{\partial y^2}+\lambda(y,t)Af^{'}(k^*(y,t))\right)h_k(y,t)\Big] \\ &-\int_{-x}^{x}\lambda(y,t)h_k(y,t)\ |_{t=0}^{\infty}\,dy+\int_{0}^{\infty}\lambda(y,t)\frac{\partial h_k(y,t)}{\partial y}\ |_{y=-x}^{x}\,dt - \\ &-\lim_{t\to\infty}\int_{-x}^{x}\mu(y,t)e^{-r(y,t)}h_k(y,t)dy\Bigg\}. \end{aligned}$$

The last and the third to last expressions are canceled if $\lim_{t\to\infty}[\mu(x,t)e^{-r(x,t)}-\lambda(t,x)]=0$, and by the fact that $h_k(x,0)=0$, for any x. Then, substituting in the Kuhn-Tucker condition we get a generalized transversality condition. We get the first order conditions by equating to zero all the remaining components of $\frac{\partial V}{\partial\epsilon}$. Equations (**??**)-(**??**) are obtained by simply making $q(x,t)=e^{\rho t}\lambda(x,t)$. □

Proof of Lemma 1

Proof. The first order conditions (4) and (13) define a system of non-linear parabolic partial differential equations (PDE). In order to get a general solution we follow a method which is used in the pattern formation literature (see Cross and Greenside (2009)).

Let us introduce trial solutions of the system of parabolic pde's given by

$$C(x,t) = \mathcal{C}(\omega,t)e^{2\pi i\omega x},\ K(x,t) = \mathcal{K}(\omega,t)e^{2\pi i\omega x},$$

where ω is called the wave number with domain in $\mathbb{R}$, because the domain of the independent variable x is unbounded, $i = \sqrt{-1}$, and functions $\mathcal{C}(\omega,t)$ and $\mathcal{K}(\omega,t)$ will be determined by substitution upon system (4)-(13).

As

$$\frac{\partial C(x,t)}{\partial t} = \frac{\partial \mathcal{C}(\omega,t)}{\partial t}e^{2\pi i\omega x},\ \frac{\partial C(x,t)}{\partial x} = 2\pi i\omega\mathcal{C}(\omega,t)e^{2\pi i\omega x},\ \frac{\partial^2 C(x,t)}{\partial x^2} = -(2\pi\omega)^2\mathcal{C}(\omega,t)e^{2\pi i\omega x}$$

and analogously for $K(x,t)$, the non-linear parabolic PDE over $(C(x,t), K(x,t))$ is equivalent to the parameterized linear ODE (ordinary differential equation) over $(\mathcal{C}(\omega,t), \mathcal{K}(\omega,t))$

$$\frac{\partial \mathcal{C}(\omega,t)}{\partial t} = \left(\gamma - \theta(2\tau\pi\omega)^2\right)\mathcal{C}(\omega,t) \tag{46}$$

$$\frac{\partial \mathcal{K}(\omega,t)}{\partial t} = \left(r - (2\tau\pi\omega)^2\right)\mathcal{K}(\omega,t) - \mathcal{C}(\omega,t). \tag{47}$$

In order to solve this system, we introduce the conjecture that transformed consumption is a linear function of the transformed stock of capital: $\mathcal{C}(\omega,t) = \beta(\omega)\mathcal{K}(\omega,t)$, where $\beta(\omega)$ is arbitrary. Then $\partial\mathcal{C}(\omega,t)/\partial t = \beta(\omega)\partial\mathcal{K}(\omega,t)/\partial t$. If we substitute equations (46) and (47) we find that our conjecture is right if and only if

$$\beta(\omega) = r - \gamma + (\theta - 1)(2\tau\pi\omega)^2.$$

If we substitute consumption in equation (47) we get one uncoupled differential equation

$$\frac{\partial \mathcal{K}(\omega,t)}{\partial t} = \Gamma(\omega)\mathcal{K}(\omega,t),\ \text{where } \Gamma(\omega) \equiv \left(\gamma - \theta(2\tau\pi\omega)^2\right)$$

which has the solution

$$\mathcal{K}(\omega,t) = \mathcal{K}(\omega,0)G_k(\omega,t),\ \text{where } G_k(\omega,t) \equiv e^{\Gamma(\omega)t}.$$

Then the solution for consumption is

$$\mathcal{C}(\omega,t) = \mathcal{K}(\omega,0)G_c(\omega,t) \text{ where } G_c(\omega,t) = \left(r - \gamma + (\theta-1)(2\tau\pi\omega)^2\right)e^{\Gamma(w)t}.$$

These are particular solutions for the initial PDE system. As it has an infinite support, the general solution, for any $t > 0$ are

$$K(x,t) = \int_{-\infty}^{\infty}\mathcal{K}(\omega,t)e^{2\pi i\omega x}d\omega = \int_{-\infty}^{\infty}\mathcal{K}(\omega,0)G_k(\omega,t)e^{2\pi i\omega x}d\omega$$

and

$$C(x,t) = \int_{-\infty}^{\infty}\mathcal{K}(\omega,0)G_c(\omega,t)e^{2\pi i\omega x}d\omega.$$

Using the definitions in Kammler (2000), the Fourier transforms of $C(x,t)$ and $K(x,t)$ are, respectively,

$$\begin{aligned}\mathcal{F}[C(x,t)] &= \int_{-\infty}^{\infty} C(x,t)e^{-2\pi i\omega x}dx,\\ \mathcal{F}[K(x,t)] &= \int_{-\infty}^{\infty} K(x,t)e^{-2\pi i\omega x}dx\end{aligned}$$

and the related inverse Fourier transforms are

$$\begin{aligned}C(x,t) &= \mathcal{F}^{-1}[\mathcal{C}(\omega,t)] = \int_{-\infty}^{\infty} \mathcal{C}(\omega,t)e^{2\pi i\omega x}d\omega,\\ K(x,t) &= \mathcal{F}^{-1}[\mathcal{K}(\omega,t)] = \int_{-\infty}^{\infty} \mathcal{K}(\omega,t)e^{2\pi i\omega x}d\omega.\end{aligned}$$

Therefore, we can use the properties of the Fourier transforms to get explicit solutions for $C(x,t)$ and $K(x,t)$. First, observe that

$$K(x,0) = \int_{-\infty}^{\infty} \mathcal{K}(\omega,0)e^{2\pi i\omega x}d\omega = \phi(x)$$

which is the initial distribution for the capital stock which is given. Then we can write

$$\begin{aligned}K(x,t) &= \mathcal{F}^{-1}[\mathcal{K}(\omega,0)G_k(\omega,t)] =\\ &= K(x,0) * g_k(x,t)\end{aligned}$$

where $*$ is the convolution operator. Then, taking the initial condition we have

$$K(x,t) = \begin{cases}\phi(x), & \text{if } t=0\\ \int_{-\infty}^{\infty}\phi(\xi)g_k(x-\xi,t)d\xi, & \text{if } t>0\end{cases}$$

where $g_k(x,t) = \mathcal{F}^{-1}[G_k(\omega,t)]$ that is

$$g_k(x,t) = \int_{-\infty}^{\infty} e^{\Gamma(\omega)t}e^{2\pi ix\omega}d\omega$$

then

$$g_k(x,t) = \begin{cases}\delta(x), & \text{if } t=0\\ e^{\gamma t}\frac{1}{2\tau\sqrt{\pi\theta t}}e^{-\left(\frac{x}{2\tau}\right)^2\frac{1}{\theta t}}, & \text{if } t>0\end{cases} \tag{48}$$

where $\delta(.)$ is Dirac's delta function.

The solution for consumption is

$$\mathcal{C}(\omega,t) = \mathcal{K}(\omega,0)G_c(\omega,t),$$

where

$$G_c(\omega,t) = \left(r-\gamma+(\theta-1)(2\tau\pi\omega)^2\right)e^{\Gamma(w)t}.$$

Then,

$$C(x,t) = \phi * g_c(x,t) = \int_{-\infty}^{\infty}\phi(\xi)g_c(x-\xi,t)d\xi$$

where $g_c(x,t) = \mathcal{F}^{-1}[(r-\gamma+(\theta-1)(2\tau\pi\omega)^2)\,e^{\Gamma(w)t}]$, that is,

$$g_c(x,t) = e^{\gamma t}\frac{1}{2\tau\sqrt{\pi\theta t}}\left[r-\gamma+(\theta-1)\left(\frac{1}{2\theta t}-\left(\frac{x}{2\tau\theta t}\right)^2\right)\right]e^{-\left(\frac{x}{2\tau}\right)^2\frac{1}{\theta t}}.$$

□

Proof of Lemma 2

Proof. The first order conditions are

$$\frac{\partial c}{\partial t} = -\tau^2\left[\frac{\partial^2 c}{\partial x^2}-\frac{2}{c}\left(\frac{\partial c}{\partial x}\right)^2\right]+\gamma c \tag{49}$$

$$\frac{\partial k}{\partial t} = \tau^2\frac{\partial^2 k}{\partial x^2}+(\gamma+\rho)k-c \tag{50}$$

subject to $k(x,0)=\phi(x)$

$$\lim_{x\to\pm\infty}\left(e^{\rho t}c(x,t)\right)^{-1}=0$$

and the transversality condition

$$\lim_{t\to\infty}\lim_{x\to\pm\infty}\frac{1}{2x}\int_{-x}^{x}e^{-\rho t}\frac{k(x,t)}{c(x,t)}dx=0.$$

The co-state variable is $q(x,t)=1/c(x,t)$. Equation (49) becomes

$$\frac{\partial q}{\partial t} = -\tau^2\frac{\partial^2 q}{\partial x^2}-\gamma q.$$

The Fourier transforms for k and q are, $\mathcal{K}(\omega,t)=F[k(x,t)]=\int_{-\infty}^{\infty}k(x,t)e^{-2\pi i\omega x}d\omega$ and $\mathcal{C}(\omega,t)=F[c(x,t)]=\int_{-\infty}^{\infty}c(x,t)e^{-2\pi i\omega x}d\omega$, respectively. Then, then the PDE system, (49)-(50) is transformed into the ODE system

$$\begin{aligned}\frac{\partial \mathcal{Q}(\omega,t)}{\partial t} &= \left((2\tau\pi\omega)^2-\gamma\right)\mathcal{Q}(\omega,t)\\ \frac{\partial \mathcal{K}(\omega,t)}{\partial t} &= \left(\gamma+\rho-(2\tau\pi\omega)^2\right)\mathcal{K}(\omega,t)-\mathcal{Q}(\omega,t)^{-1}.\end{aligned}$$

Let us conjecture that the following relationship holds: $\mathcal{Q}(\omega,t)\mathcal{K}(\omega,t) = B(\omega)$. Taking time derivatives

$$\frac{\partial Q(\omega,t)}{\partial t}\mathcal{K}(\omega,t) + \frac{\partial K(\omega,t)}{\partial t}\mathcal{Q}(\omega,t) = 0$$

and substituting from equations (51)-eq:akfocw2 we get $B(\omega) = 1/\rho$. Then, equation (51) becomes a linear differential equation

$$\frac{\partial \mathcal{K}(\omega,t)}{\partial t} = \left(\gamma - (2\tau\pi\omega)^2\right)\mathcal{K}(\omega,t) = \Gamma(\omega)\mathcal{K}(\omega,t),$$

which has the solution

$$\mathcal{K}(\omega,t) = \mathcal{K}(\omega,0)G(\omega,t),\ G(\omega,t) = \exp\left(\Gamma(\omega)t\right).$$

As $G(\omega,t)$ is bounded in ω, we can take inverse Fourier transforms $k(x,t) = F^{-1}[\mathcal{K}(\omega,t)]$ to get

$$k(x,t) = k(x,0) * g(x,t) = \int_{-\infty}^{\infty} k(\xi,0)g(x-\xi,t)d\xi$$

where the inverse Fourier transform of the Green's function $G(\omega,t)$ is

$$\begin{aligned} g(y,t) &= e^{\gamma t}\int_{-\infty}^{\infty} e^{2\pi iy\omega-(2\pi\tau\omega)^2 t}d\omega \\ &= e^{\gamma t}\begin{cases}\delta(y), & \text{if } t=0\\ \frac{e^{-\left(\frac{y}{2\tau}\right)^2\frac{1}{t}}}{(4\tau^2\pi t)^{1/2}}, & \text{if } t>0.\end{cases}\end{aligned}$$

Using the initial given distribution for the capital stock we have

$$k(x,t) = \begin{cases}\phi(x), & \text{if } t=0\\ e^{\gamma t}\int_{-\infty}^{\infty}\phi(\xi)\frac{e^{-\left(\frac{x-\xi}{2\tau}\right)^2\frac{1}{t}}}{(4\tau^2\pi t)^{1/2}}d\xi, & \text{if } t>0.\end{cases}$$

Applying the inverse Fourier transform we get $c(x,t) = F^{-1}[\mathcal{C}(\omega,t)] = F^{-1}[\rho\mathcal{K}(\omega,t)] = \rho k(x,t)$. The transversality condition holds because

$$\lim_{t\to\infty}\lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x} e^{-\rho t}\frac{k(\xi,t)}{c(\xi,t)}d\xi = \lim_{t\to\infty}\lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x}\frac{e^{-\rho t}}{\rho}d\xi = \lim_{t\to\infty}\frac{e^{-\rho t}}{\rho} = 0.$$

The boundary condition becomes

$$\lim_{x\to\pm\infty}\left(e^{\rho t}\rho k(x,t)\right)^{-1} = 0$$

which depends on the initial distribution $\phi(x)$. □

Proof of Lemma 3

Proof. The Jordan canonical for associated to the the Jacobian $J(\omega)$ is $\Lambda(\omega)(\omega) = P^{-1}(\omega)J(\omega) = P(\omega)$ where $\Lambda(\omega)$ is a diagonal matrix with $(\lambda^+(\omega), \lambda^-(\omega))$ in the principal diagonal and $P(\omega)$ is the eigenvector functions matrix. Then, the solution of system (26) is

$$\begin{pmatrix} \mathcal{Q}(\omega,t) \\ \mathcal{K}(\omega,t) \end{pmatrix} = z^+(\omega)P^+(\omega)e^{\lambda^+(\omega)t} + z^-(\omega)P^-(\omega)e^{\lambda^-(\omega)t},$$

where $z^{\pm}(\omega)$ are two scalar functions of ω. As $\lambda^+(\omega) > \rho$, a necessary condition for the transversality condition to hold is that $z^+(\omega) = 0$ identically for all $\omega \in \mathbb{R}$. The eigenvector associated to the eigenvalue λ^- is $P^-(\omega) = (\rho - \xi - \lambda^-(\omega), 1)^\top$. We determine at time $t = 0$ by solving $z^-(\omega) = \mathcal{K}(\omega, 0)$ where

$$\mathcal{K}(\omega,0) = F[K(x,0)] = \int_{-\infty}^{\infty} \phi(x)e^{-2\pi i\omega x}dx$$

is the Fourier transform of the, given, initial distribution of the capital stock. Then $\mathcal{Q}(\omega, 0) = (\rho - \xi - \lambda^-(\omega))\mathcal{K}(\omega, 0)$ $\square$

Proof of Lemma 5

Proof. The discriminant $\Delta(\xi)$ is a convex function of ξ with a minimum at $\xi = \xi^* = \rho/2$, verifying $\Delta(\xi^c) = \mu$ if $\xi = \xi^c = \rho/2$ and $\Delta(\xi) > \mu$ if $\xi \neq \xi^c$. This implies that the eigenvalue $\lambda^-(\xi)$ is a concave function of ξ with maximum at $\xi = \xi^c$ verifying: $\lambda^- = \lambda^c \equiv \frac{\rho}{2} - \mu^{1/2}$ if $\xi = \xi^c$ and $\lambda^-(\xi) < \lambda^c$ if $\xi \neq \xi^c$.

However, the maximum for λ^-, λ^c, can be reached for negative or positive values: $\lambda^c < 0$ if $\rho < 2\mu^{1/2}$, $\lambda^c = 0$ if $\rho = 2\mu^{1/2}$ or $\lambda^c > 0$ if $\rho > 2\mu^{1/2}$. In this last case, as $\lambda^-(0) < 0$ and $\lambda^-(\infty) < 0$ then there is one interval of values of ξ such the eigenvalue is positive is closed. We can determine the limits of the interval $\xi^+ > \xi^- > 0$ by solving the quadratic equation over ξ, $\lambda^-(\xi) = 0$. Then $0 < \lambda^-(\xi) \leq \lambda^c$ if $\xi_1 < \xi < \xi_2$. $\square$

Proof of Lemma 5

Proof. As we already noted, the eigenfunction $\lambda^-(\omega)$ has two equal local maxima for $\omega = \omega_c = \pm\sqrt{2\rho}/4\tau\pi$. We perform quadratic approximation of eigenfunction (28)

$$\lambda^-(\omega) \approx \begin{cases} \lambda_1^-(\omega) = \frac{\rho}{2} - \mu^{1/2} - \rho(2\tau\pi)^2\mu^{-1/2}\left(\omega + \frac{\sqrt{2}\rho}{4\tau\pi}\right)^2, \ \omega \in (-\infty, 0) \\ \lambda_2^-(\omega) = \frac{\rho}{2} - \mu^{1/2} - (2\tau\pi)^2\rho\mu^{-1/2}\left(\omega - \frac{\sqrt{2}\rho}{4\tau\pi}\right)^2, \ \omega \in (0, \infty) \end{cases}$$

Then we approximate the Green functions by

$$g_k(x,t) = \int_{-\infty}^{0} e^{\lambda_1^-(\omega)t}e^{2\pi i\omega x}d\omega + \int_{0}^{\infty} e^{\lambda_2^-(\omega)t}e^{2\pi i\omega x}d\omega$$

$$g_c(x,t) = u''(C^*)\Big\{ \big(\rho - (2\tau\pi\omega)^2 - \lambda_1^-(\omega)\big) \int_{-\infty}^{0} e^{\lambda_1^-(\omega)t} e^{2\pi i\omega x} d\omega + \\ + \big(\rho - (2\tau\pi\omega)^2 - \lambda_2^-(\omega)\big) \int_{0}^{\infty} e^{\lambda_2^-(\omega)t} e^{2\pi i\omega x} d\omega \Big\}. \quad (51)$$

The Green functions g_k and g_c have explicit representations.

□

Proof of Proposition 4.

Proof. Properties of the kernels: if $\rho/2 < \mu^{1/2}$ then $\lim_{t\to\infty} g_k(x,t) = 0$ If $\rho/2 > \mu^{1/2}$ then $\lim_{t\to\infty} g_k(x,t) = \infty$ but $\lim_{t\to\infty} h_k(x,t) = 0$.

The transversality condition is an exponential function with exponent $\rho/2 - \mu^{1/2} - \rho = -(\rho/2 + \mu^{1/2}) < 0$ then the transversality condition holds if $\lim_{x\to\pm\infty} h_k(x,t)$ is bounded.

The verification of the boundary conditions depends on the initial distribution for capital $\phi(x)$. It is easy to see that both the transversality conditon and the boundary conditions hold, for any value of λ^c because $\lambda^c < \rho$.

□

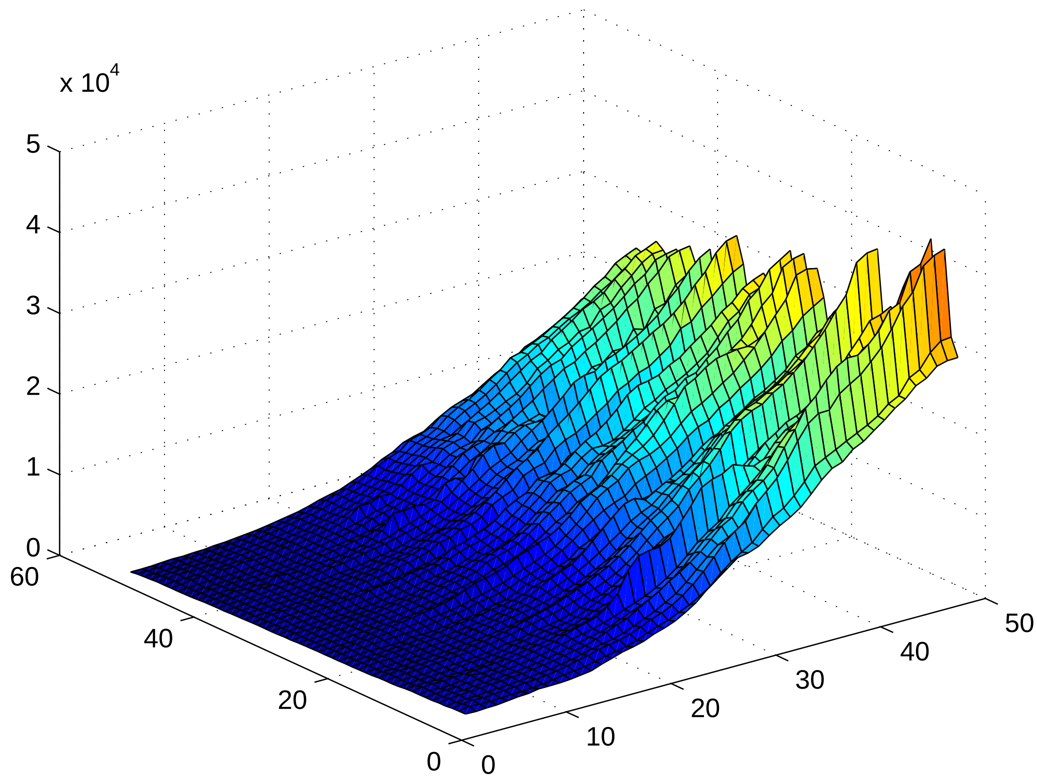

Figure 1: US states FIPS data 1955-2002. States are ranked in 1955 and we keep the same order afterwards.

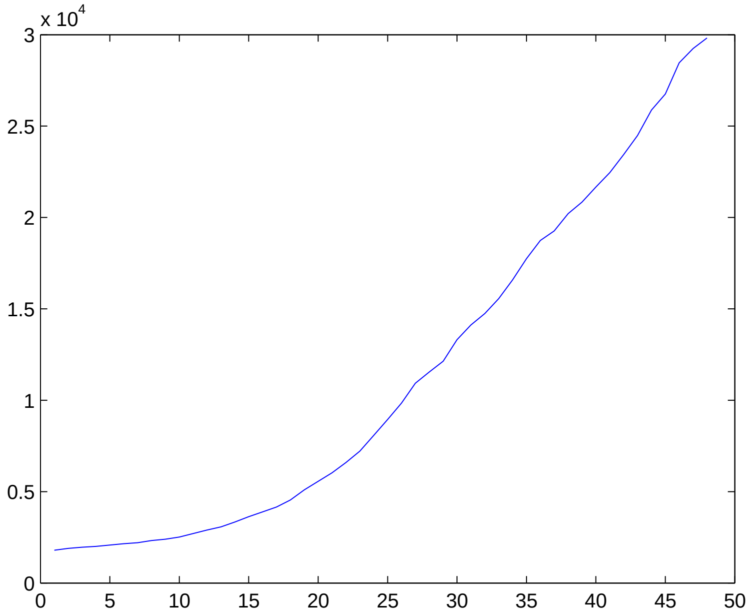

Figure 2: US states FIPS data 1955-2002.Trend of the yearly average income per capita

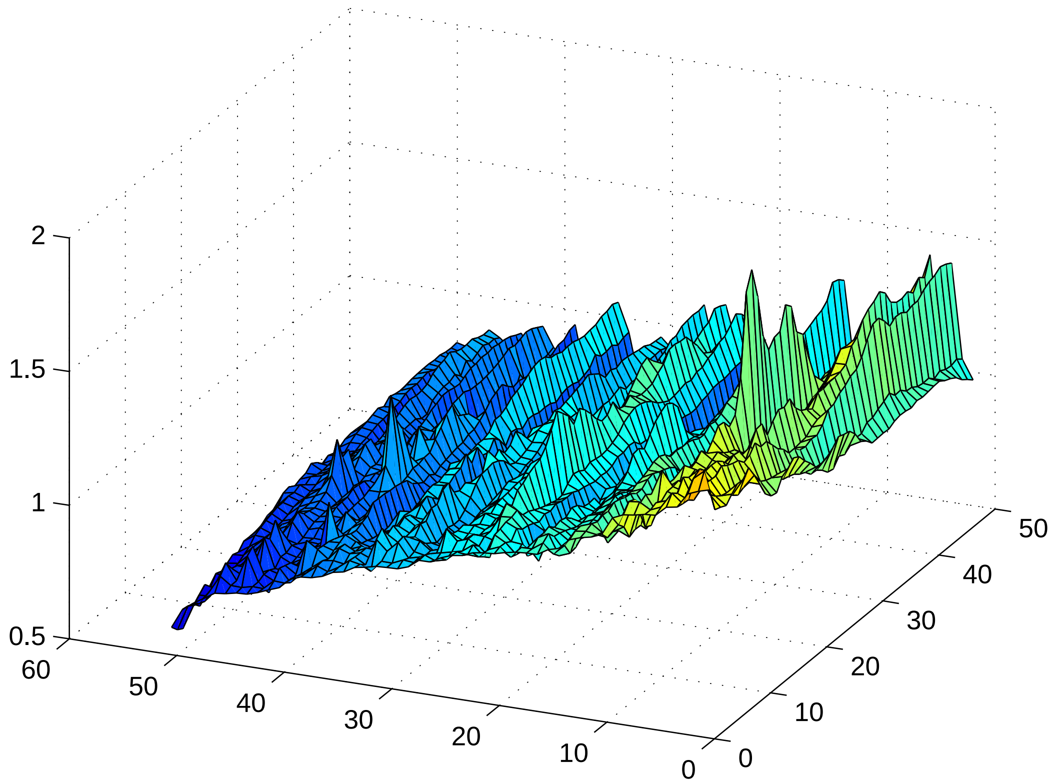

Figure 3: US states FIPS data 1955-2002. States are ranked in 1955 and we keep the same order afterwards. Detrended data

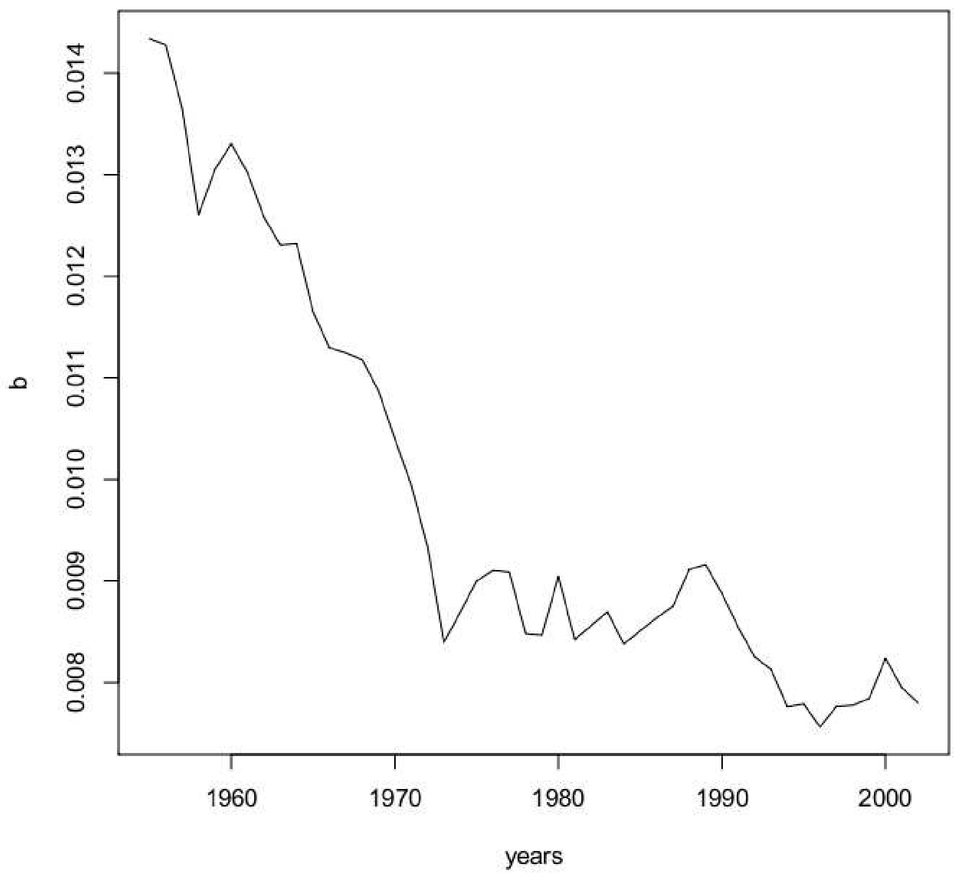

Figure 4: US states FIPS data 1955-2002: coefficient β of the regression $ln(y) = \beta_0 - \beta x$

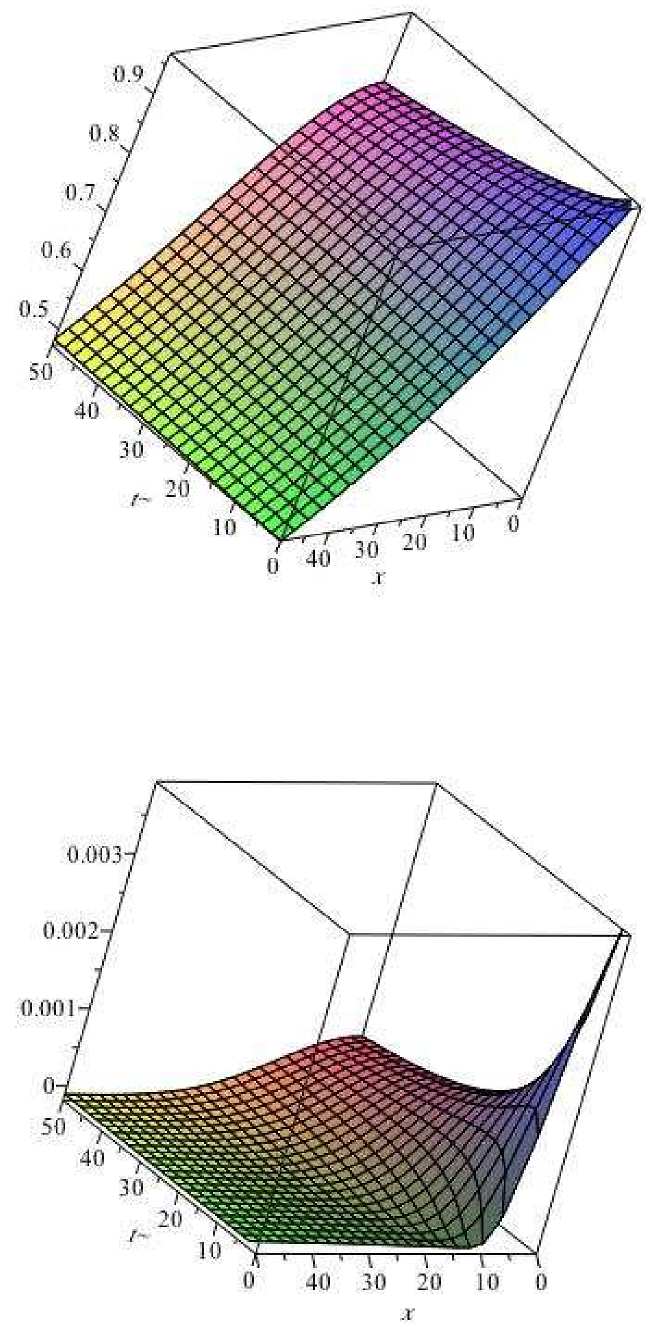

Figure 5: Convergence to a homogeneous stationary state: local dynamics for $k(x,t)$ and $c(x,t)$ for the case AK.

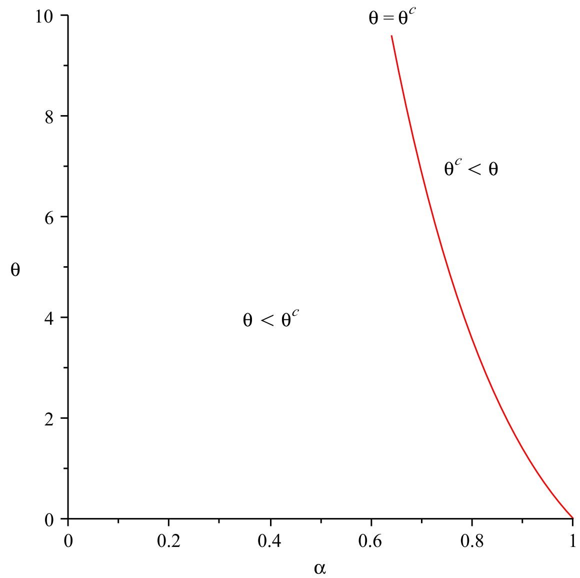

Figure 6: Bifurcation diagram in the space (α, θ) with the location of the line line $\theta = \theta^c$ for the parameter values $A = 1$, $\rho = 0.03$, and $\delta = 0.05$.

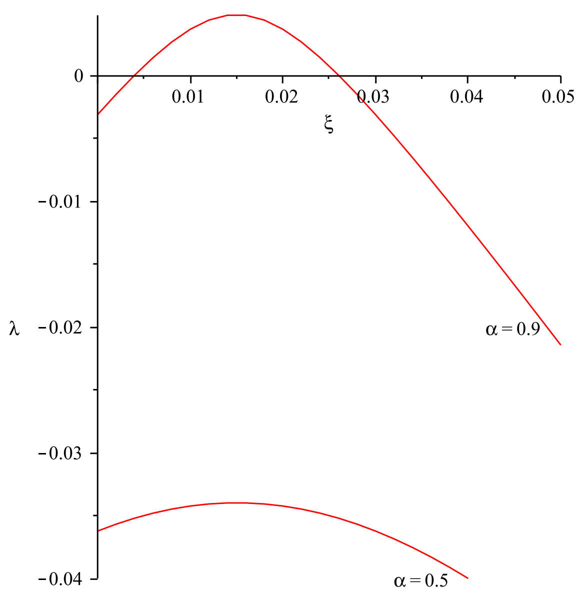

Figure 7: Eigenvalue λ^{-} as a function of ξ for the parameter values $A = 1$, $\rho = 0.03$, $\delta = 0.05$ and $\theta = 3$. The upper (lower) schedule is for case $\theta > \theta^c$ $(\theta < \theta^c)$.

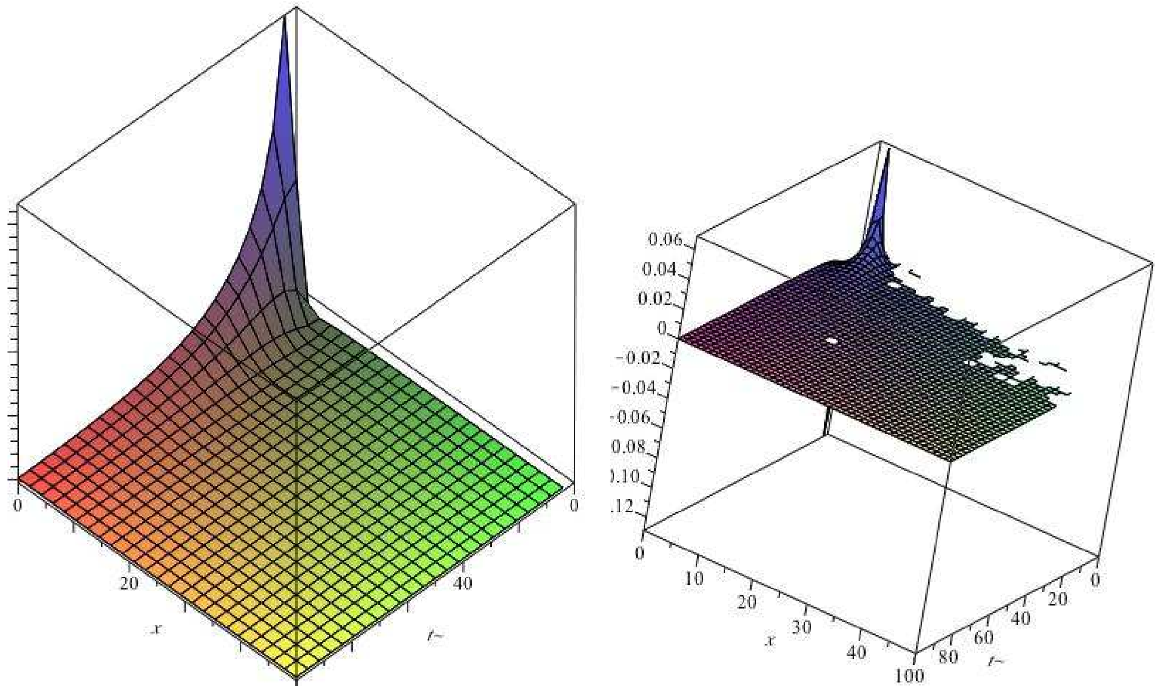

Figure 8: Convergence to a homogeneous stationary state: kernels $g_k(x,t)$ and $g_q(x,t)$ for the case $\theta < \theta^c$.

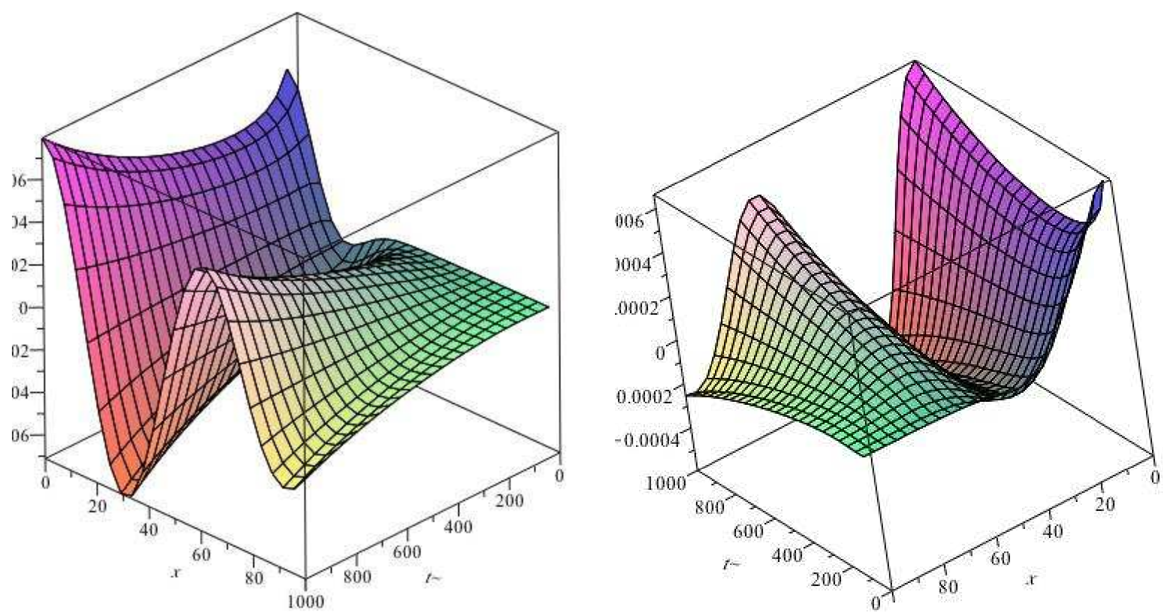

Figure 9: Convergence to endogenous growth induced by diffusion: detrended kernel $g_k(x,t)$ for the case $\theta > \theta^c$.