An epsilon-based measure of efficiency in DEA - An alternative method for the affinity index

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ABSTRACT

Tone & Tsutsui (2010) introduced a hybrid model (epsilon-based measure, EBM), which combines both radial and non-radial measures in a unified framework [*Tone, K., & Tsutsui, M. (2010). An epsilon-based measure of efficiency in DEA - A third pole of technical efficiency. European Journal of Operational Research, 207, 1554-1563.*]. We find that their method to construct the affinity matrix may be questionable. Their method may not reflect the true degree of scattered distribution of inputs or outputs. Based on Tone & Tsutsui's idea, we introduced an alternative method of constructing the affinity matrix, which overcomes the drawback of their method.

KeyWords: Data Envelopment Analysis; Epsilon-based Measure (EBM); Affinity Index

1. Introduction

Data envelopment analysis (DEA) is a linear programming methodology to evaluate the technical efficiency for each member of a set of peer decision making units (DMUs) with multiple inputs and multiple outputs. There are mainly two types of approaches to measuring technical efficiency in DEA: radial and non-radial. The radial measure was first introduced by the CCR model developed by Charnes, Cooper, & Rhodes (1978). The non-radial model is represented by the widely used slacks-based measure (SBM model) developed by Tone (2001). On the basis of these two types of measures, Tone & Tsutsui (2010) introduced a hybrid model (epsilon-based measure, EBM), which combines both radial and non-radial measures in a unified framework [*Tone, K., & Tsutsui, M. (2010). An*

epsilon-based measure of efficiency in DEA - A third pole of technical efficiency. European Journal of Operational Research, 207, 1554-1563.]. In the EBM model, there are two parameters that need to be determined: a scalar named as epsilon indicating the relative importance of the non-radial measure over the radial measure, and a vector indicating the weights of inputs or outputs. To compute these two parameters, Tone & Tsutsui (2010) constructed an affinity matrix and then the largest eigenvalue of the defined affinity matrix was used to compute the epsilon, and its associated nonnegative eigenvector was used to compute the weights of inputs or outputs. However, we find their method to construct the affinity matrix may be questionable. Their method may not reflect the true degree of scattered distribution of inputs or outputs. Based on Tone & Tsutsui's idea, we introduced an alternative method of constructing the affinity matrix, which overcomes the drawback of their method.

2. An epsilon-based measure (EBM) of efficiency

The input-oriented EBM model under constant returns to scale technology is used for demonstration. It can be easily extended to out-orientation and variable returns to scale. The EBM model is defined as follows:

$$\gamma^* = \min \theta - \varepsilon_x \sum_{i=1}^{m} \frac{w_i^- s_i^-}{x_0}$$

$$\text{subject to } \theta x_0 - X\lambda - s^- = 0,$$

$$Y\lambda \geq y_0,$$

$$\lambda \geq 0,\ s^- \geq 0.$$

where w^i is the weight (relative importance) of input i and satisfies $\sum_{i=1}^{m} w_i = 1\ (w_i \geq 0\ \forall i)$, and ε_x is a key parameter which indicates the relative importance of the non-radial slacks over the radial θ. Parameters ε_x and w^i must be supplied prior to the efficiency measurement, and they should be units-invariant values.

To determine the two parameters, Tone & Tsutsui (2010) constructed an affinity index between two input vectors instead of the Pearson's correlation coefficient. The affinity index S(a,b) between the vector a and vector b has the following properties.

(P1) Identical: S(a,a) = 1,

(P2) Symmetric: S(a,b) = S(b,a),

(P3) Units-invariant : S(ta,b) = S(a,b) (t > 0), and

(P4) 1≥S(a,b)≥0.

The ‘‘affinity index” S(a,b) was defined as follows:

S(a,b)=1-2D(a,b)

D(a,b) is a diversity index, which indicates the degree of scattered distribution, and was defined as follows:

$$D(a,b)=\begin{cases}\dfrac{\sum_{j=1}^{n}|c_j-\bar{c}|}{n(c_{\max}-c_{\min})} & (if\ c_{\max}>c_{\min})\\ 0 & (if\ c_{\max}=c_{\min})\end{cases},$$

where $c_j=\ln\frac{b_j}{a_j}$, $\bar{c}=\frac{1}{n}\sum_{j=1}^{n}\ln\frac{b_j}{a_j}$, $c_{\max}=\max(c_j)$, and $c_{\min}=\min(c_j)$.

The largest value of eigenvalue ρ of the affinity matrix S was used to compute the epsilon, and its associated nonnegative eigenvector w_x was used to compute the weights, as follows.

$$\varepsilon_x=\frac{m-\max(\rho_x)}{m-1}, \text{ and}$$

$$w^{-}=\frac{w_x}{\sum_{i=1}^{m}w_{xi}}.$$

As discussed in the paper by Tone & Tsutsui (2010), in the narrow range case, the affinity index should approximate to 1; and in the widely scattered case, the affinity index should approximate to 0. In their paper, they give an widely scatted example (example 2 in their paper), the computed affinity index between x1 and x2 is 0. However their example is too simple to illustrate the rationale of the index, it has only 2 SBM-efficient DMUs. When we apply their method to a completely scattered case which has more than 2 efficient DMUS, we get questionable results.

The following example data have two inputs (x1 and x2) and one output (Table1). For the 5 efficient DMUs (A-E), a decrease in x1 is always accompanied by an equal amount of increase in x2, which shows the complete substitutability between the two inputs. The complete negative linear correlation exists between x1 and x2 in the frontier (Figure 1).

Table 1 Example data

DMU	x1	x2	y
A	1	5	1
B	2	4	1
C	3	3	1
D	4	2	1
E	5	1	1
F	3	4	1
G	4	3	1

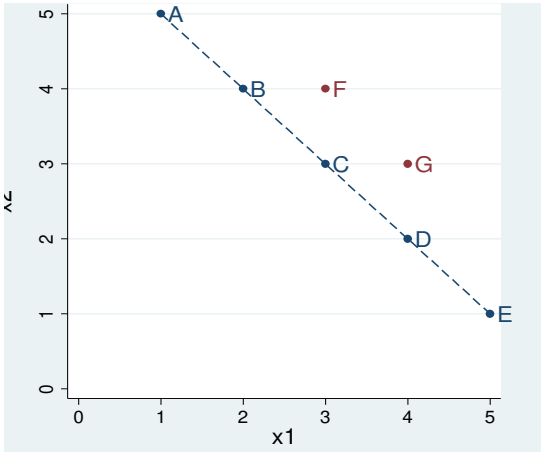

Figure 1 Example data (completely scattered)

According to the rationale of the affinity index, its value should be 0 in such an scattered case. However using the above method, the computed value of the affinity index between x1 and x2 is 0.428, and the epsilon is 0.572. This result contradicts the rationale of the affinity index. (Table 2 and table 3)

Table 2 Diversity matrix for the example data

DMU	x1	x2
x1	0	0.286
x2	0.286	0

Table 3 Affinity matrix for the example data

DMU	x1	x2
x1	1	0.428

x2	0.428	1

3. An alternative method of computing the affinity index

The questionable result of the affinity index using the method proposed by Tone & Tsutsui (2010) makes us turn back to the traditional Pearson's correlation coefficient. The Pearson's correlation coefficient conforms to the first three properties (P1-P3), but violates the last property (P4). Although the range of the Pearson's correlation coefficient is [-1, 1], it can be adjusted into [0, 1] according its relationship between the Pearson's correlation coefficient and the rationale of the affinity index(Figure 2). In the narrow range case, the two inputs are highly dependent on each other, the Pearson correlation coefficient is equal to or approximate to 1, and the affinity index should be equal to or approximate to 1; while in the widely scattered case, the two inputs are highly substitutable for each other, the Pearson correlation coefficient is equal to or approximate to -1, and the affinity index should be equal to or approximate to 0. According to the relationship, the new affinity index can be defined as

$S(a,b)=0.5+0.5r(a,b)$,

where r(a,b) is the Pearson correlation coefficient between a and b.

The new affinity index conforms to all the 4 properties, and overcomes the shortcoming of the method proposed by Tone & Tsutsui (2010) .

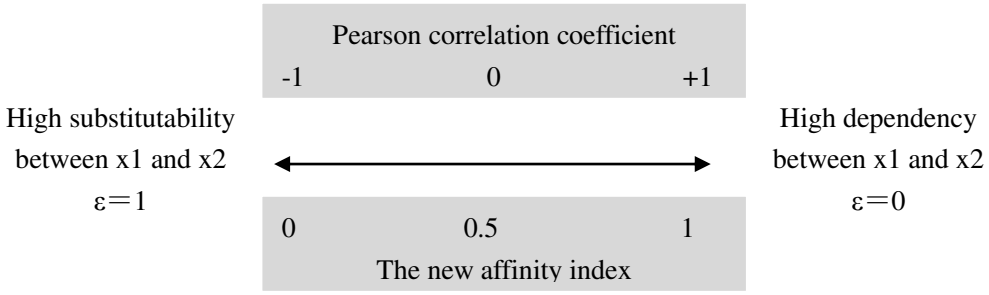

Figure 2 Relationship between the Pearson correlation coefficient and the new affinity index

The new affinity index between x1 and x2 of the example data in table 1 is

S(x1,x2)=0.5+0.5 r(x1, x2)=0.5+0.5(-1)=0.

The new affinity matrix is listed in table 4.

Table 4 New affinity matrix for the example data

DMU	x1	x2

x1	1	0
x2	0	1

The largest eigenvalue and eigenvector of the new affinity matrix are $\max(\rho_x) = 1$. So we have $\varepsilon_x = \frac{m - \max(\rho_x)}{m-1} = \frac{2-1}{2-1} = 1$. This result conforms to the rationale of the affinity index defined by Tone & Tsutsui (2010).

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