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Breitmoser, Yves

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Yves Breitmoser*
EUV Frankfurt (Oder)

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Abstract

If Cournot oligopolists may sell their output prior to its production (forward trading), competition intensifies. Potentially, it may intensify so far as to imply convergence to the Bertrand equilibrium, as shown by Allaz and Vila (1993) for the case of linear demand and costs. The present paper analyzes the limiting outcome if demand or costs are non-linear, which still are open problems. Specifically, I consider a general family of convex demands and increasing marginal costs. In both cases, the limiting outcomes are strictly between Cournot and Bertrand. This shows that competitive futures markets improve welfare (upon Cournot) also for non-linear costs or demands, but they do generally not imply social efficiency.

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*I thank Friedel Bolle for helpful comments. Address: Europa Universität Viadrina; Postfach 1786; 15207 Frankfurt(Oder); Germany. email: breitmoser@euv-frankfurt-o.de; Telephone: +335 5534 2291; Fax: +335 5534 2390.

1 Introduction

In many industries, firms conclude forward contracts to sell their eventual output. Theoretically, forward trading is made for at least two reasons. On the one hand, it allows to hedge positions in markets with uncertain spot prices (Allaz, 1992; Hughes and Kao, 1997). On the other hand, strategic forward trading allows firms to increase profits unilaterally (Allaz and Vila, 1993; Bolle, 1993; Powell, 1993). For, forward trading increases a firm's marginal revenue in the production period, and as a result, marginal revenue equates with marginal costs at higher quantities. Thus, by forward trading, firm i creates an incentive for itself to increase its quantity in the production period, which is anticipated by its competitors. They reduce their quantities in response, and due to this reduction of its opponents' quantities, i benefits (Mahenc and Salanié, 2004, show that the logic inverts in case of strategic complements). In equilibrium, all firms trade forward, competition intensifies, and all firms are worse off. In their seminal analysis, Allaz and Vila (1993) show that the resulting equilibrium outcome approaches the Bertrand outcome (and thus social efficiency) as the number of forward trading periods tends to infinity, assuming costs and demand are linear.

This finding has sparked off much interest in forward trades and their ability to induce competitive pricing. In the process, various mitigating factors have been identified. Ferreira (2003) shows that the equilibrium may exhibit tacit collusion if there is no terminal trading period, and in repeated oligopoly, Liski and Montero (2006) show that forward trades simplify penal strategies. Thille (2003) shows that storage weakens the implications of forward trades, and Breitmoser (2012) shows that heterogeneity of goods weakens them. The experimental analyses of Le Coq and Orzen (2006), Brandts et al. (2008), and Ferreira et al. (2009) also suggest that forward trading is weaker empirically than theoretically.

These cases, where mitigating factors have been identified, have in common that Bertrand competition does not induce competitive pricing either. This raises the question as to whether the limiting Allaz-Vila outcome induces social efficiency in cases where factors that are generally known to be mitigating are not in effect. The arguably most relevant such cases are those with non-linear demand or costs, which surprisingly enough, have not yet been analyzed conclusively. Since non-linearity of demands and costs is widespread empirically, understanding the properties of Allaz-

Vila competition in these conditions even seems desirable. Bertrand competition induces competitive pricing also if say demands are non-linear, and intuitively Allaz-Vila competition appears to induce "undercutting" very similar to that of Bertrand competition as long as goods are homogenous. The limiting Allaz-Vila outcomes are difficult to analyze, however, as the closed-form solutions of T-round games with $T > 2$ are fairly cumbersome in non-linear cases. This is illustrated, for example, by Allaz (1987) and Bushnell (2007) who analyze non-linear two-round games.

I circumvent the known problems by transforming the induction on equilibrium prices into an induction on the underlying conjectural variations, generalizing the approach of Breitmoser (2012). This transformation enables me to derive the limiting equilibria of duopolies with inverse demands $P = \left(1 - a\sum_i x_i\right)^b$, which contains linear, quadratic, log-linear, and exponential demands as special or limiting cases, and to derive the limiting equilibria of duopolists with quadratic costs. I characterize the respective outcomes in closed form and find that competitive pricing does not obtain for either form of non-linearity. Forward trading still intensifies competition in relation to the basic Cournot case, but it does not converge to Bertrand as the number of forward trading periods approaches infinity. Thus, futures markets do not generally suffice to induce social efficiency even in "clean conditions" where Bertrand suffices, and Allaz-Vila competition in general is an intermediate form of competition, differing distinctly from both Bertrand and Cournot.

Section 2 introduces the required notation and terms. Sections 3 and 4 analyze non-linearity of demand and costs, respectively. Section 5 concludes.

2 Notation and definitions

The set of firms is $N = \{1,2\}$ with typical elements $i, j \in N$. The firms produce perfect substitutes for consumers with aggregate inverse demand $P(x)$ for quantity x. P is monotonically decreasing and weakly convex. Firm i's quantity is denoted as x_i, its aggregate costs as $C_i(x_i)$, which are continuous and satisfy $C_i(0) = 0$, and its average costs as $c_i(x_i) := C_i(x_i)/x_i \geq 0$. I will suppress the arguments of P and c_i when doing so may not cause confusion.

The interaction of the firms proceeds in rounds. Production takes place in $t = 0$, e.g. the year 2013. At time $-T$, the futures market opens, and in the T periods prior to $t = 0$, the firms may sell or buy futures contracts for delivery in $t = 0$. The firms' aggregate trade balances become common knowledge after each round. In round $t = 0$, the firms choose quantities and produce, and finally the market clears in Cournot fashion. Production cannot take place prior to 2013, either because the good is non-durable or because production in 2012 is needed to satisfy demand in 2012 (and demand in 2012 is left unmodeled). The futures markets are competitive, i.e. there are rational expectations about the eventual market price and hence forward selling or forward buying is not profitable in itself. This is a standard assumption that allows us to focus on the strategic implications of forward trading.

To keep notation simple, the firms are assumed to play Markov strategies. Thus, strategies may depend on the present state, which is characterized by the current balances of forward trades, but not on the specific history of actions that led to it. This assumption is actually made without loss of generality, as the extensive-form game is finite and the Markov perfect equilibrium (MPE) turns out to be unique. Hypothetically, without uniqueness of MPE, subgame perfect equilibria (SPEs) exist that are not Markov perfect and may exhibit tacit collusion, similarly to finitely repeated games when constituent games have multiple Nash equilibria (Benoit and Krishna, 1985).

The set of states is $\mathbb{R}^2$, and typical states will be denoted as $\mathbf{f} = (f_1, f_2) \in \mathbb{R}^2$. Here, f_i denotes i's aggregate amount of forward *sales*. If $f_i > 0$, then i has gone short (i.e. it has sold some of its future production); if $f_i < 0$, then i has gone long.

The game with $T \geq 0$ rounds of forward trading is denoted as Γ^T. The (sub-) game with $T \geq 0$ rounds of forward trading and initial state $\mathbf{f}$ is denoted as $\Gamma^T(\mathbf{f})$.

A strategy of $i \in N$ is denoted as (x_i, y_i) with $y_i = \left(y_{i,t}\right)_{t=-T}^{-1}$, where $x_i : \mathbb{R}^2 \to \mathbb{R}_+$ and $y_{i,t} : \mathbb{R}^2 \to \mathbb{R}$ for all $t \in \{-T, -T+1, \dots, -1\}$. Here, $x_i(\mathbf{f})$ is i's quantity conditional on the profile $\mathbf{f}$ of forward sales in $t = 0$, and $y_{i,t}(\mathbf{f})$ are the respective amounts of forward sales after period t conditional on $\mathbf{f}$. Note that i may forward sell even more than it will eventually produce, i.e. both $x_i(\mathbf{f}) \geq f_i$ and $x_i(\mathbf{f}) < f_i$ are admissible, and that i may switch without restrictions between forward selling and buying between periods, i.e. between $y_{i,t}(\mathbf{f}) \geq f_i$ and $y_{i,t}(\mathbf{f}) < f_i$, respectively.

Denoting strategy profiles as $(\mathbf{x}, \mathbf{y}) = \big((x_1, x_2), (y_1, y_2)\big)$, the equilibrium profits are defined as follows. Consider round $t \in \{-T, -T+1, \dots, 0\}$ is reached in state $\mathbf{f}_t$. The forward trades in the subsequent rounds $t' = t+1, t+2, \dots, 0$ can be resolved recursively as $f_{i,t'} = y_{i,t'-1}(\mathbf{f}_{t'-1})$ for all $i \in N$. The eventual balance of forward trades is $\mathbf{f}_0 = (f_{1,0}, f_{2,0})$, and overall the profits as anticipated in round t are

$$\Pi_i(\mathbf{x}, \mathbf{y} \,|\, t, \mathbf{f}_t) = \big[x_i(\mathbf{f}_0) - f_{i,t}\big] \cdot P\big[x_i(\mathbf{f}_0) + x_j(\mathbf{f}_0)\big] - C_i\big[x_i(\mathbf{f}_0)\big]. \tag{1}$$

Note that this expression ignores the sunk revenue $f_{i,t} \cdot P$ from forward trades concluded in previous stages. Finally, the definition of MPEs is as usual (Maskin and Tirole, 2001).

3 Non-linear demand

In this section, I analyze the implications of non-linear demand in relation to the standard linear case. That is, linearity of C_i is maintained, and $c_i = C_i/x_i \equiv C_i'$ is therefore constant. The analysis proceeds by backward induction, starting with $t = 0$, i.e. with the Cournot game $\Gamma^0(\mathbf{f})$ where the firms may have concluded forward trades. In $\Gamma^0(\mathbf{f})$, firm i enters the production period $T = 0$ with quantity $f_i \in \mathbb{R}$ being sold forward and its profit function is

$$\Pi_i = \big(x_i - f_i\big) \cdot P(x_i + x_j) - c_i x_i. \tag{2}$$

The first-order conditions in $\Gamma^0(\mathbf{f})$ therefore are

$$-(x_i - f_i)\, P' = P - c_i \qquad \forall\, i \in N. \tag{3}$$

Under typical assumptions on P and c_i, the larger f_i, the larger is the equilibrium quantity x_i and the lower is x_j.[1] This represents the strategic motive underlying forward trades discussed in the introduction. Condition (3) can be expressed as

$$-(x_i - f_i) \cdot \mu_{i,T} = P - c_i \qquad \forall\, i \in N \tag{4}$$

[1] In the case of constant average/marginal costs, a sufficient condition is $P' + P'' \cdot (x_i - f_i) < 0$ for all $i \in N$, see Eqs. (8) and (9) below using $\mu_i = \mu_j = P'$. For example, this condition holds for linear demand, $P'' = 0$, and non-linear demands such as those considered below (for all relevant x_i, f_i).

with $\mu_{i,0} = P'$ for all i. By induction, I show that the equilibrium conditions of $\Gamma^T(\mathbf{f})$ for all $T \geq 0$ can be expressed through (4), and Lemma 3.1 derives how $\mu_{i,T}$ is to be updated as the time horizon T increases. Note that this updating rule will be derived for general inverse demands P, which is notationally less cumbersome than using specific parametric forms such as $P = (1 - a * (x_1 + x_2))^b$. This particular, but general family of non-linear demand functions will be used subsequently, in turn, as an abstract treatment of non-linear demand functions seems to be intractable due to the terms $\frac{\partial \mu_{j,T}}{\partial x_i}$ and $\frac{\partial \mu_{j,T}}{\partial x_j}$ in the updating rule.

Lemma 3.1. *If the equilibrium quantities* (x_i) *in* $\Gamma^T(\mathbf{f})$, $T \geq 0$, *satisfy Eq.* (4), *then the equilibrium quantities in* $\Gamma^{T+1}(\mathbf{f}')$ *satisfy Eq.* (4) *with*

$$\mu_{i,T+1} = \frac{\frac{\partial \mu_{j,T}}{\partial x_j}\left(P - c_j\right)P' - \frac{\partial \mu_{j,T}}{\partial x_i}\left(P - c_j\right)P' - \mu_{j,T}^2 P'}{\frac{\partial \mu_{j,T}}{\partial x_j}\left(P - c_j\right) - \mu_{j,T}\left(P' + \mu_{j,T}\right)} \qquad \forall i \in N. \tag{5}$$

Proof. Define $\mu_i := \mu_{i,T}$ for all $i \in N$. Totally differentiating the induction assumption $P - c_i + \mu_i(x_i - f_i) = 0$ with respect to (x_i, x_j, f_i) yields

$$\left[P' + \frac{\partial \mu_i}{\partial x_i}(x_i - f_i) + \mu_i\right] \cdot dx_i + \left[P' + \frac{\partial \mu_i}{\partial x_j}(x_i - f_i)\right] \cdot dx_j - \mu_i\, df_i = 0 \tag{6}$$

and totally differentiating the corresponding assumption $P - c_j + \mu_j(x_j - f_j) = 0$ with respect to (x_i, x_j, f_i) yields

$$\left[P' + \frac{\partial \mu_j}{\partial x_i}(x_j - f_j)\right] \cdot dx_i + \left[P' + \frac{\partial \mu_j}{\partial x_j}(x_j - f_j) + \mu_j\right] \cdot dx_j = 0. \tag{7}$$

Solving Eqs. (6) and (7) with respect to dx_i/df_i and dx_j/df_i yields

$$\frac{dx_i}{df_i} = -\frac{\mu_i\left(P' - \frac{\partial \mu_j}{\partial x_j}\left(f_j - x_j\right) + \mu_j\right)}{\substack{\frac{\partial \mu_i}{\partial x_i}(f_i - x_i)\left(P' + \mu_j\right) + \frac{\partial \mu_j}{\partial x_j}\left(f_j - x_j\right)(P' + \mu_i) - \frac{\partial \mu_j}{\partial x_i}\left(f_j - x_j\right)P' - \frac{\partial \mu_i}{\partial x_j}(f_i - x_i)P' \\ -\left(\mu_j + \mu_i\right)P' + \left(\frac{\partial \mu_i}{\partial x_j}\frac{\partial \mu_j}{\partial x_i} - \frac{\partial \mu_i}{\partial x_i}\frac{\partial \mu_j}{\partial x_j}\right)(f_i - x_i)\left(f_j - x_j\right) - \mu_i \mu_j}} \tag{8}$$

$$\frac{dx_j}{df_i} = \frac{\mu_i\left(P' - \frac{\partial \mu_j}{\partial x_i}\left(f_j - x_j\right)\right)}{\substack{\frac{\partial \mu_i}{\partial x_i}(f_i - x_i)\left(P' + \mu_j\right) + \frac{\partial \mu_j}{\partial x_j}\left(f_j - x_j\right)(P' + \mu_i) - \frac{\partial \mu_j}{\partial x_i}\left(f_j - x_j\right)P' - \frac{\partial \mu_i}{\partial x_j}(f_i - x_i)P' \\ -\left(\mu_j + \mu_i\right)P' + \left(\frac{\partial \mu_i}{\partial x_j}\frac{\partial \mu_j}{\partial x_i} - \frac{\partial \mu_i}{\partial x_i}\frac{\partial \mu_j}{\partial x_j}\right)(f_i - x_i)\left(f_j - x_j\right) - \mu_i \mu_j}}. \tag{9}$$

Conditional on $\mathbf{f}'$, the anticipated profits in $\Gamma^{T+1}(\mathbf{f}')$ are $\Pi_i^{T+1} = (x_i - f_i')(P - c_i)$, and the respective first-order conditions for an equilibrium in (f_i, f_j) chosen in $T+1$ are

$$\frac{d\Pi_i^{T+1}}{df_i} = \left(x_i - f_i'\right)\left(\frac{dx_j}{df_i}P' + \frac{dx_i}{df_i}P'\right) + \frac{dx_i}{df_i}(P - c_i) = 0 \tag{10}$$

for all i. Substituting $\frac{dx_i}{df_i}$ and $\frac{dx_j}{df_i}$ by Eqs. (8) and (9), rearranging terms and focusing on the numerator, we obtain

$$\begin{aligned}-(P-c_i)P' - \frac{\partial\mu_j}{\partial x_j}\left(f_i' - x_i\right)\left(f_j - x_j\right)P' + \frac{\partial\mu_j}{\partial x_i}\left(f_i' - x_i\right)\left(f_j - x_j\right)P' \\ +\mu_j\left(f_i' - x_i\right)P' + \frac{\partial\mu_j}{\partial x_j}\left(f_j - x_j\right)(P - c_i) - \mu_j(P - c_i) = 0.\end{aligned} \tag{11}$$

Substituting $x_j - f_j$ by $-(P-c_j)/\mu_j$, as implied the induction assumption Eq. (4), and rearranging terms yields

$$\begin{aligned}\left(\frac{\partial\mu_j}{\partial x_i} - \frac{\partial\mu_j}{\partial x_j}\right)\left(f_i' - x_i\right)(P - c_j)P' - \mu_j(P - c_i)P' - \mu_j(P - c_i)\mu_j \\ +\mu_j\mu_j\left(f_i' - x_i\right)P' + \frac{\partial\mu_j}{\partial x_j}(P - c_i)(P - c_j) = 0.\end{aligned} \tag{12}$$

Finally, factorizing with respect to $(x_i - f_i')$ and $(P - c_i)$ yields an expression of the form $\alpha\cdot(x_i - f_i') = \beta\cdot(P - c_i)$ where

$$\alpha = \frac{\partial\mu_j}{\partial x_j}\left(P - c_j\right)P' - \frac{\partial\mu_j}{\partial x_i}\left(P - c_j\right)P' - \mu_j^2 P', \quad \beta = \frac{\partial\mu_j}{\partial x_j}\left(P - c_j\right) - \mu_j\left(P' + \mu_j\right).$$

□

Using Lemma 3.1, the following analysis transforms the induction on prices and quantities into an induction on first-order conditions characterized via (μ_i), which allows me to derive tractable characterizations of the outcomes in cases with non-linear demands or costs. A similar approach was applied previously to the case of linear demands and costs in Breitmoser (2012). Note how (μ_i) relate to conjectural derivatives. If firm i maximizes $x_i \cdot (P - c_i)$ assuming the conjectural derivative

$dx_j/dx_i = \kappa_i$, the first-order condition is

$$-x_i(1+\kappa_i)P' = P - c_i. \tag{13}$$

Thus, $\mu_i = (1+\kappa_i)P'$. The difference to conjectural derivatives is that μ_i is endogenous while κ_i is exogenous, and in this sense, the Allaz-Vila model rationalizes intermediate conjectural derivatives between Cournot and Bertrand.

In order to illustrate the induction, let us first apply Lemma 3.1 to the linear case $P = 1 - a*(x_1+x_2)$ and $c_i = b$ for all i. By Eq. (3), the induction starts with $\mu_{i,0} = P'$ for all i in Eq. (4), and by Lemma 3.1, for all $T > 0$ and all $i \neq j$,

$$\frac{\partial \mu_{j,T}}{\partial x_j} = \frac{\partial \mu_{j,T}}{\partial x_i} = 0 \qquad\qquad \mu_{i,T+1} = \frac{\mu_{j,T}P'}{P'+\mu_{j,T}} = \frac{\mu_{j,T}/P'}{1+\mu_{j,T}/P'} \cdot P'. \tag{14}$$

Unraveling the iteration yields $\mu_{i,T} = \frac{P'}{1+T}$, and as T tends to ∞, $\mu_{i,T}$ converges to 0. Thus, competitive pricing $P = c_i$ obtains in the limit. The resulting equilibrium quantities of Γ^T, $T \geq 0$, are $x_{i,T} = (1+T)(1-b)/(3+2*T)a$ for all i. Of course, these are nothing but the results of Allaz and Vila (1993). However, they show how the linear case is technically convenient. For all $T > 0$, $\mu_{i,T}$ is the product of a factor $\lambda_{i,T} = 1/(1+T)$ and P'. Critically, $\partial\lambda_i/\partial x_i = \partial\lambda_i/\partial x_j = 0$, as it implies $\partial\mu_{i,T}/\partial x_i = \partial\mu_{i,T}/\partial x_j = 0$ in the linear case.

As a result, the sequence $(\mu_{i,T})$ simplifies enormously, toward $\mu_{i,0} = P'$, $\mu_{i,-1} = P'/2$, $\mu_{i,-2} = P'/3$, and so on, which in turn implies that closed-form expressions for equilibrium quantities and prices are straightforward even for $T > 0$.

The derivatives of μ_i do not disappear if demand or costs are non-linear. In these cases, closed-form expressions for equilibrium prices and quantities may become intractable already after a few induction steps. In contrast, the induction on (μ_i) rather than prices/quantities continues to be possible, as I show for non-linear demands $P = (1 - a*(x_1+x_2))^b$ now. This family of inverse demands contains the linear one as a special case ($b = 1$), the relation to which will be used in illustrations. In addition, it contains many other forms used in empirical analysis as special cases, such as quadratic, log-linear, and exponential demands, and thus it constitutes the arguably most relevant generalization of linearity in the present context (for further discussion, let me refer to Genesove and Mullin, 1998).

Proposition 3.2. *If inverse demand is $P = (1 - a * (x_1 + x_2))^b$ with $a > 0$ and $b > 1$ and costs are $c_i = 0$ for all $i \in N$, then the equilibrium price of Γ^T converges to $\left(\frac{b-1}{b+1}\right)^b > 0$ as T approaches ∞.*

Proof. The first-order condition if the game starts in $T = 0$ is Eq. (3), and hence it satisfies Eq. (4) with $\mu_{i,0} = P'$ for all i. I claim that for all $T > 0$, the first-order condition is Eq. (4) with

$$\mu_{i,T} = \lambda_{i,T} P' \qquad \text{with} \quad \lambda_{i,T} = \frac{b^T}{\sum_{t=0}^{T} b^t} \qquad \forall i \in N. \tag{15}$$

The claim is satisfied for $T = 0$. Next I show that if it holds in Γ^T, $T \geq 0$, then also for Γ^{T+1}. By Lemma 3.1, in particular Eq. (5), $c_i = c_j = 0$, and $\mu_{j,T} = \lambda_{j,T} P'$,

$$\mu_{i,T+1} = \frac{\frac{\partial \mu_{j,T}}{\partial x_j} P P' - \frac{\partial \mu_{j,T}}{\partial x_i} P P' - \mu_{j,T}\, \mu_{j,T}\, P'}{\frac{\partial \mu_{j,T}}{\partial x_j} P - \mu_{j,T} \left(P' + \mu_{j,T}\right)} = \frac{-\lambda_{j,T}\, b P'}{(b-1) - (1 + \lambda_{j,T})\, b}. \tag{16}$$

Now, the claim follows from the induction assumption on $\lambda_{i,T}$. The sequences $(\mu_{i,T}) = (\mu_{j,T})$ are equal and unique, and they converge to

$$\lim_{T \to \infty} \frac{b^T}{\sum_{t=0}^{T} b^t} \cdot P' = \lim_{T \to \infty} \frac{b^{T+1} - b^T}{b^{T+1} - 1} \cdot P' = \frac{b-1}{b} \cdot P'. \tag{17}$$

By Eq. (4), the limiting quantities x_i, x_j are equal and solve

$$-\frac{b-1}{b} \cdot x_i P' = P \qquad \Rightarrow \qquad (b-1) \cdot a x_i = (1 - 2 a x_i). \tag{18}$$

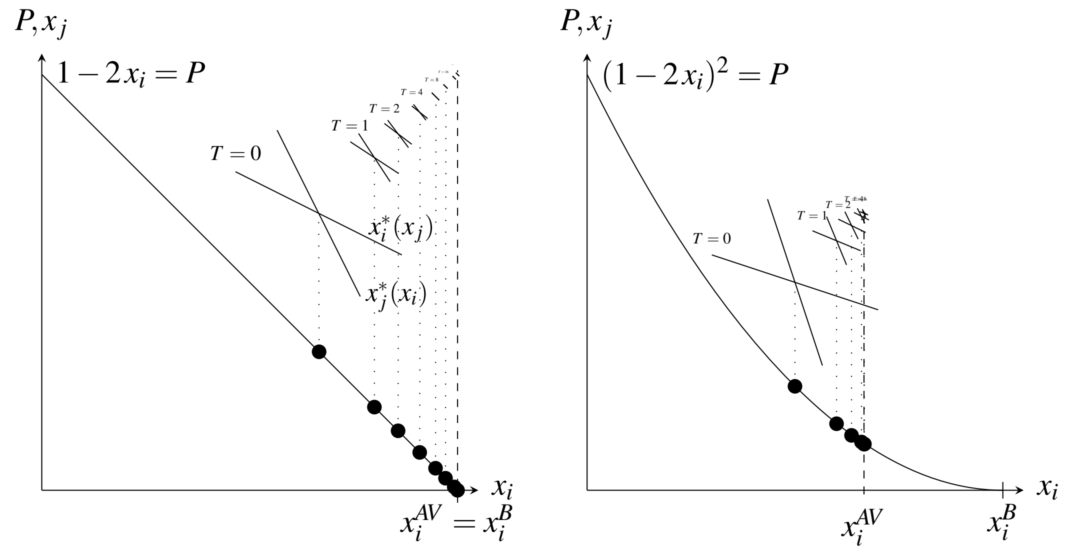
Hence, $x_i = 1/(ab + a)$ in the limit, and $P = \left(\frac{b-1}{b+1}\right)^b$. It remains to verify the sufficient conditions. On the one hand, let $F_i = P - c_i + x_i \cdot \mu_{i,T}$ for all $i \in N$ denote the conditions constituting the induction assumption. The determinant of the Jacobian at the equilibrium quantities for $T \geq 0$ is

$$\begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{vmatrix} = \left(\left(a^2 b^2 \lambda_{i,T} + a^2 b \right) \lambda_{i,T} + a^2 b \lambda_{i,T} \right) \mathrm{P}^{\frac{2b-2}{b}} \neq 0, \tag{19}$$

exploiting that $\mu_{i,T} = \mu_{j,T} = \lambda_{i,T} P'$, with $\lambda_{i,T} > 0$, and $a, b, P > 0$. Since the deriva-

Figure 1: Comparison linear and quadratic demands (with $c_i = 0$)

(a) Linear demand $P = 1 - x$ (b) Quadratic demand $P = (1-x)^2$



Note: These are plots of the equilibrium prices of Γ^T as $T \to \infty$ and the corresponding "best responses" Eq. (20). Correspondingly, the vertical axes have two scales, price P and opponents' quantity x_j. x_i^B denotes the Bertrand quantities (i.e. the $x_i = x_j$ such that $P = c_i = c_j$), and x_i^{AV} denotes i's limiting Allaz-Vila quantity (as $T \to \infty$).

tives are also continuous, the conditions of the implicit function theorem used above are therefore satisfied. On the other hand, the sufficient condition for the maximum derived from Eq. (10) can be expressed as, again exploiting $\mu_{i,T} = \mu_{j,T} = \lambda_{i,T} P'$ and using the optimal x_i and $P \geq \left(\frac{b-1}{b+1}\right)^b$,

$$\frac{d^2\Pi_i^{T+1}}{dy_i^2} < -\frac{a\,(b-1)^{b-1}\, b^2\, (b+1)^{1-b}\, \lambda_{i,T}^2 \lambda_{j,T}\, \left(ab^2\lambda_{j,T} f_i' + ab\lambda_{j,T} f_i' + b\lambda_{j,T} + 2\right)}{\left(b\lambda_{i,T}\lambda_{j,T} + \lambda_{j,T} + \lambda_{i,T}\right)^2} < 0,$$

since $a, \lambda_i, \lambda_j > 0$ and $b > 1$. For, $\lambda_{i,T}$ being decreasing as T increases implies that the quantities sold forward are monotonically increasing along the equilibrium path, i.e. $y_{i,t}(\mathbf{f}) \geq f_i$ for all $t \geq -T$. Hence, the cumulated forward trades are non-negative in all rounds and $f_i' \geq 0$ applies. $\square$

Figure 1 illustrates the difference between the linear case and the non-linear one, for the inverse demand $P = (1 - x_1 - x_2)^b$. By Eq. (17), the equilibrium conduct parameters are $\mu_{i,T} = \left(b^{T+1} - b^{T}\right)/\left(b^{T+1} - 1\right)$. In conjunction with the first-order condition Eq. (4), this allows us to express the optimal x_i as a function of x_j in Γ^T.

$$x_i = \frac{\left(b^{T+1} - 1\right)\left(1 - x_j\right)}{b^{T+2} - 1} \tag{20}$$

This "best response" converges to $x_i = \left(1 - x_j\right)(T+1)/\left(T+2\right)$ in the linear case ($b \to 1$), and to the Cournot response for $T = 0$. These best response functions are displayed in Figure 1 for the linear case $b = 1$ and the quadratic case $b = 2$.

Notably, $x_i(x_j)$ is linear even if the demands are non-linear. As b increases, however, the slopes of the response functions dx_i/dx_j increase. That is, the response functions become flatter, which moves the equilibrium outcome of Γ^T inward and thus $b > 1$ mitigates the implications of forward trading. Formally, as $\mu_{i,T}$ is updated according to Eq. (16), $b > 1$ decelerates the updating and ultimately yields convergence to $\mu^* = (b-1)/b$ rather than convergence to competitive conduct $\mu^c = 0$. Thus, for all demand functions in the general non-linear family analyzed above, forward trading still improves welfare in relation to the Cournot case, but it does not maximize welfare, which results for linear demands.

Corollary 3.3. *Under the conditions of Proposition 3.2, the equilibrium price of* Γ^T *is strictly between Cournot and Bertrand prices, both for all finite* $T > 0$ *and in the limit as* T *approaches* ∞.

Proof. By Eq. (17), the first-order condition of Γ^T satisfies Eq. (4) with $\mu_{i,T} = \lambda_{i,T}P'$ with $\lambda_{i,T} = b^T / \sum_{t=0}^{T} b^t$. The Cournot equilibrium corresponds to $T = 0$, which yields $\lambda_{i,0} = 1$, and the Bertrand equilibrium corresponds to $\lambda_i = 0$, i.e. $P = 0$. Since $0 < \lambda_{i,T} < 1$ for all $T > 0$, the claim follows for all finite T, and in the limit, as $T \to \infty$, we obtain $\lambda_{i,\infty} = (b-1)/b \in (0,1)$ for all $b > 1$. □

4 Non-linear costs

Non-linearity of costs implies that profits from previous forward sales are not fully sunk until the production quantity is finally set. The eventual quantity decision affects average costs and thus also costs of previous forward trades. Due to this effect, the recursion conditions needs to be adapted when costs are non-linear. Starting with the profits in the Cournot case $\Gamma^0(\mathbf{f})$, see Eq. (2), the first-order condition in case $c_i' \neq 0$ can be written as

$$-(x_i - f_i)\,P' = P - c_i - x_i\,c_i' \qquad \forall\, i \in N. \tag{21}$$

The difference to Eq. (3) is the last term $x_i\,c_i'$ which represents the aforementioned effect. Now if we set $\mu_{i,0} = P'$ for all i, then the first-order condition of $\Gamma^0(\mathbf{f})$ is

$$-(x_i - f_i)\,\mu_{i,T} = P - c_i - x_i\,c_i' \qquad \forall\, i \in N, \tag{22}$$

and as I show below, this condition can be iterated on to analyze $\Gamma^T(\mathbf{f})$ for all $T > 0$ with quadratic costs. The key implication of assuming quadratic costs is that the profits continue to be quadratic (as in the linear case), thanks to which the property $\partial\mu_{i,T}/\partial x_i = \partial\mu_{i,T}/\partial x_j = 0$ continues to hold (as in the linear case) if the induction is based on Condition (22).[2] This is established next. Despite these similarities to the linear case, I will then show that even quadratic costs imply that prices fail to converge to Bertrand prices as T approaches ∞.

Lemma 4.1. *Assume $P'' = c'' = 0$. If the equilibrium quantities (x_i, x_j) satisfy Eq.* (22) *in $\Gamma^T(\mathbf{f})$, $T \geq 0$, with $\partial\mu_{i,T}/\partial x_i = \partial\mu_{i,T}/\partial x_j = 0$ for all $i, j \in N$ such that $i \neq j$, then they satisfy Eq.* (22) *in $\Gamma^{T+1}(\mathbf{f}')$ with*

$$\mu_{i,T+1} = \frac{\left(\mu_{j,T} - 2\,c_j'\right)\,P'}{P' + \mu_{j,T} - 2\,c_j'} \tag{23}$$

and $\partial\mu_{i,T+1}/\partial x_i = \partial\mu_{i,T+1}/\partial x_j = 0$ for all $i, j \in N$ such that $i \neq j$.

Proof. Define $\mu_i = \mu_{i,T}$ for all i. Totally differentiating the induction assumption Eq. (22) with respect to (x_i, x_j, f_i) yields (using $P'' = c'' = 0$ and $\partial\mu_i/\partial x_i = \partial\mu_i/\partial x_j = 0$)

$$dx_i\,\left(P' + \mu_i - 2\,c_i'\right) + dx_j\,P' - \mu_i\,df_i = 0. \tag{24}$$

Totally differentiating the corresponding condition on $j \neq i$ with respect to (x_i, x_j, f_i)

[2] A joint analysis of non-linear demands and non-linear costs is intractable precisely because $\partial\mu_{i,T}/\partial x_i \neq 0$ would follow, due to which an iteration on a condition such as (22) with $c_i' \neq 0$ appears to be impossible for general T. For this reason, previous analyses of non-linear costs and demands, such as Allaz (1987) and Bushnell (2007), focus on two-round games.

yields

$$dx_j\left(P'+\mu_j-2c'_j\right)+dx_iP'=0. \tag{25}$$

Solving these conditions for dx_i/df_i and dx_j/df_i yields

$$\frac{dx_i}{df_i}=\frac{\mu_i\left(P'+\mu_j-2c'_j\right)}{-2c'_i\left(P'+\mu_j\right)-2c'_j\left(P'+\mu_i\right)+\mu_jP'+\mu_iP'+\mu_i\mu_j+4c'_ic'_j} \tag{26}$$

$$\frac{dx_j}{df_i}=-\frac{\mu_iP'}{-2c'_i\left(P'+\mu_j\right)-2c'_j\left(P'+\mu_i\right)+\mu_jP'+\mu_iP'+\mu_i\mu_j+4c'_ic'_j}. \tag{27}$$

Conditional on $\mathbf{f}'$, the anticipated profits in $\Gamma^{T+1}(\mathbf{f}')$ are $\Pi_i^{T+1}=(x_i-f'_i)P-x_ic_i$, and the respective first-order conditions for an equilibrium in (f_i,f_j) in $T+1$ are

$$\frac{d\Pi_i^{T+1}}{df_i}=\left(x_i-f'_i\right)\left(\frac{dx_j}{df_i}P'+\frac{dx_i}{df_i}P'\right)+\frac{dx_i}{df_i}P-c'_ix_i\left(\frac{dx_i}{df_i}\right)-c_i\left(\frac{dx_i}{df_i}\right)=0$$

for all i. Substituting $\frac{dx_i}{df_i}$ and $\frac{dx_j}{df_i}$ by Eqs. (26) and (27), rearranging terms and focusing on the numerator, we obtain

$$\begin{aligned}2c'_j\left(f'_iP'-x_iP'-P+c_i\right)-c'_ix_i\left(P'+\mu_j\right)+PP'&\\ -\mu_jf'_iP'+\mu_jx_iP'-c_iP'+\mu_jP+2c'_ic'_jx_i-c_i\mu_j&=0\end{aligned} \tag{28}$$

and thus $-\left(\mu_j-2c'_j\right)(x_i-f'_i)P'=(P-c'_ix_i-c_i)\left(P'+\mu_j-2c'_j\right)$. □

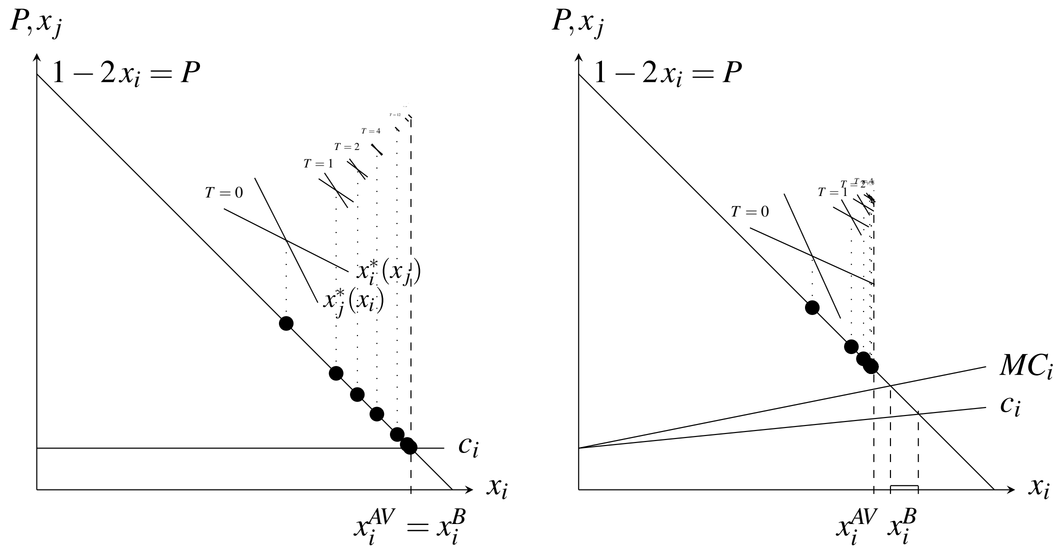
Having established that an induction based on Eq. (22) is possible, let us look at the implications. Substituting inverse demand $P=1-a*(x_1+x_2)$ and average costs $c_i=b_1+b_2x_i$ into Eq. (23), it follows that μ_i is updated as

$$\mu_{i,T+1}=-\frac{a\left(\mu_{j,T}-2b_2\right)}{\mu_{j,T}-2b_2-a}. \tag{29}$$

In relation to the linear case, which obtains for $b_2=0$, this shows again that non-linearity (now concavity of costs) mitigates the competition enhancing effect of forward trades. As in the case of non-linear demands, non-linearity of costs decelerates updating of μ_i, as $\partial\mu_{i,T+1}/\partial b_2<0$, and it also prevents price from equating with average costs in the limit. Figure 2 illustrates the deceleration effect, and the following

Figure 2: Comparison of linear and quadratic costs (with $P = 1 - x_1 - x_2$)

(a) Constant average/marginal costs $c_i = 1/10$ (b) Linear average costs $c_i = (1 + x_i)/10$



Note: As in Figure 1, the equilibrium prices and "best responses" Eq. (20) of Γ^T are plotted for various T. MC_i are i's marginal costs, x_i^{AV} are the limiting Allaz-Vila quantities, and x_i^B indicate the Bertrand quantities (linear case) and the range of Bertrand quantities (non-linear case).

result derives the relation of price and marginal costs in the limit. It shows that the price is above marginal costs, and since marginal costs are above average costs if the latter are increasing, the price is also above average costs. The relation to Cournot and Bertrand equilibria is discussed below.

Proposition 4.2. *Assume the inverse demand $P = 1 - a * (x_1 + x_2)$ and average costs $c_i = b_1 + b_2 x_i$ for all $i \in N$, with $a > 0$, $b_1 \in [0, 1)$ and $b_2 > 0$. Limiting equilibrium price and i's marginal costs MC_i at the equilibrium quantity in Γ^T, as T approaches ∞, are*

$$P = \frac{\sqrt{b_2\,(b_2 + 2a)} + b_2 + 2ab_1}{\sqrt{b_2\,(b_2 + 2a)} + b_2 + 2a} > \frac{b_1\sqrt{b_2\,(b_2 + 2a)} + (2 - b_1)\,b_2 + 2ab_1}{\sqrt{b_2\,(b_2 + 2a)} + b_2 + 2a} = MC_i. \tag{30}$$

Proof. The first-order condition in $\Gamma^0(\mathbf{f})$ is Eq. (3) and can thus be represented as Eq. (22) with $\mu_{i,0} = P'$ for all i. Lemma 4.1 implies, by induction, that the equilibrium conditions in any Γ^T, $T \geq 0$, satisfy Eq. (22) for some $\mu_{i,T} = \lambda_{i,T} P'$

where $\partial\lambda_{i,T}/\partial x_i = \partial\lambda_{i,T}/\partial x_j = 0$ for all i. By Eqs. (23) and (29), the sequences $(\mu_{i,T}) = (\mu_{j,T})$ are equal and have two possible fixed points: $b_2 - \sqrt{b_2^2 + 2ab_2}$ and $b_2 + \sqrt{b_2^2 + 2ab_2}$. The sequences converge to the former of these fixed points, which I will denote as $\mu^* = b_2 - \sqrt{b_2^2 + 2ab_2}$. For, $\mu_{i,T+1}$ is monotonically increasing in $\mu_{j,T}$,

$$\frac{d\mu_{i,T+1}}{d\mu_{j,T}} = \frac{a^2}{\left(\mu_{j,T} - 2b_2 - a\right)^2} > 0, \tag{31}$$

which implies that for all $T \geq 0$,

$$\mu_{i,T} < \mu^* \wedge \mu_{i,T} < \mu_{i,T+1} \quad \Rightarrow \quad \mu_{i,T+1} < \mu^*. \tag{32}$$

Since $\mu_{i,0} = -a < \mu^*$ and $\mu_{i,0} < \mu_{i,1}$ hold, convergence toward μ^* follows from Eq. (32) by induction. Since $\mu^* < 0$ and $-x_i\mu^* = P - c_i - x_i c_i'$ in the limiting equilibrium, $P > c_i$ follows. Solving the limiting condition for x_i yields

$$x_i^* = (1 - b_1)/\left(\sqrt{b_2^2 + 2ab_2} + b_2 + 2a\right) \tag{33}$$

and limiting equilibrium price and marginal costs $MC_i = b_1 + 2b_2x_i$ as claimed in Eq. (30). Their relation follows from $b_1, b_2, a > 0$, which implies

$$(1 - b_1)\sqrt{b_2\,(b_2 + 2a)} > (1 - b_1)\,b_2.$$

Finally, I verify the sufficient conditions again. Using $F_i = P - c_i - x_i c_i' + x_i \cdot \mu_{i,T}$ and $\mu_{i,T} = \lambda_{i,T}P'$ (which implies $\lambda_{i,T}$), the determinant of the Jacobian at the equilibrium of Γ^T, $T \geq 0$, is

$$\begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{vmatrix} = a^2\left(\lambda_{i,T}\lambda_{j,T} + \lambda_{j,T} + \lambda_{i,T}\right) + 2ab_2\left(\lambda_{j,T} + \lambda_{i,T} + 2\right) + 4b_2^2 \neq 0,$$

since $a, b_2, \lambda_i, \lambda_j > 0$. In addition to continuity, this establishes admissibility of the implicit function theorem. Similarly, the second-order condition for profit maximization is satisfied, as $d^2\Pi_i^{T+1}/df_i^2$ is (in the equilibrium for any $T \geq 0$)

$$-\frac{2a^2\lambda_{i,T}^2\left(a\left(\lambda_{j,T}+1\right)+2b_2\right)\left(ab_2\left(\lambda_{j,T}+3\right)+a^2\lambda_{j,T}+2b_2^2\right)}{\left(a^2\left(\lambda_{i,T}\lambda_{j,T}+\lambda_{j,T}+\lambda_{i,T}\right)+2ab_2\left(\lambda_{j,T}+\lambda_{i,T}+2\right)+4b_2^2\right)^2}$$

which is negative for all $a, b_2, \lambda_i, \lambda_j > 0$. □

Finally, again, I relate the limiting Allaz-Vila outcome to Bertrand and Cournot outcomes. The Cournot case is obvious, as the equilibrium price in Γ^T decreases monotonically as T increases. Since $T = 0$ represents Cournot competition, this shows that the Allaz-Vila prices ($T > 0$) are strictly below the Cournot prices for all T, which thus also holds in the limit. In turn, there are multiple Bertrand equilibria if average costs are non-linear (and increasing). A sufficient condition for price P being a Bertrand equilibrium price is that it is not greater than both firms' marginal costs at their respective quantities and not less than their respective average costs. It is easy to see that this condition is also necessary if the firms can make "offers while stocks lasts", i.e. if they can ration customers after undercutting their opponents.[3] Since the Allaz-Vila prices are monotonically decreasing in T and strictly above marginal costs even in the limit, as shown in Proposition 4.2, this shows that they are above all "rationing–proof" Bertrand prices. Therefore, I conclude as follows.

Corollary 4.3. *Under the conditions of Proposition 4.2, it also holds true that the equilibrium price of Γ^T is strictly between Cournot and Bertrand prices, for all finite $T > 0$ and in the limiting case as T approaches ∞.*

5 Conclusion

In this paper, I derived the limiting outcomes of Allaz-Vila competition if demand or costs are non-linear. I considered quadratic costs and inverse demands of the form $P = \left(1 - a\sum_i x_i\right)^b$, which contains linear demand and several other empirically relevant functional forms as special cases. In relation to the standard linear case, either form of non-linearity implies that the equilibrium price converges above marginal

[3]The possibility of rationing after undercutting opponents is realistic and standard practice, but even assuming rationing is impossible, the limiting Allaz-Vila price is above the highest Bertrand price for all $b_2 \leq 2a/3$ (i.e. if the curvature is not extremely strong).

costs when the number of forward trading periods approaches infinity. Solving for the limiting outcomes of Allaz-Vila competition under non-linearity was possible through transforming the induction on the equilibrium prices into an induction on "conduct parameters" relating to conjectural derivatives.

From a more general perspective, the results show that futures markets do not restore social efficiency in Cournot oligopolies if factors such as repeated interaction, storage, and product heterogeneity, which are known to be obstructive in general, are not at play. The convergence of the limiting Allaz-Vila outcome to the Bertrand outcome is specific to the assumption of linearity, while Allaz-Vila competition in general seems to be a distinct form of competition.

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