Mechanization, task assignment, and inequality

Yuki, Kazuhiro

Faculty of Economics, Kyoto University

7 October 2012

Mechanization, Task Assignment, and Inequality

Kazuhiro Yuki*

October 7, 2012

Abstract

Mechanization− the replacement by machines of humans engaged in production tasks− is a continuing process since the Industrial Revolution. As a result, humans have shifted to tasks machines cannot perform efficiently. The general trend until about the 1960s is the shift from manual tasks to analytical (cognitive) tasks, while, since the 1970s, because of the advancement of IT technologies, humans have shifted away from routine analytical tasks (such as simple information processing tasks) as well as routine manual tasks toward non-routine analytical tasks and non-routine manual tasks in services. Mechanization also has affected relative demands for workers of different skill levels and thus earnings levels and earnings inequality. The rising inequality has been the norm in economies with lightly regulated labor markets, although the inequality fell in periods when the relative supply of skilled workers grew rapidly.

This paper develops a Ricardian model of task assignment and examines how improvements of productivities of machines and an increase in the relative supply of skilled workers affect task assignment (which factors perform which tasks), earnings, earnings inequality, and aggregate output in order to understand these trends.

JEL Classification Numbers: J24, J31, N30, O14, O33

Keywords: mechanization, task assignment, earnings inequality, technical change

---

[*] Faculty of Economics, Kyoto University, Yoshida-hommachi, Sakyo-ku, Kyoto, 606-8501, Japan; Phone +81-75-753-3532; E-mail yuki@econ.kyoto-u.ac.jp. Comments and suggestions from seminar participants at Kobe University and Tohoku University are gratefully appreciated. Remaining errors are the author's own.

1 Introduction

Mechanization− the replacement by machines of humans (and animals) engaged in production tasks− is a continuing process since the Industrial Revolution. During the Industrial Revolution between the second half of the 18th century and the first half of the 19th century, mechanization progressed in tasks intensive in manual labor: in manufacturing, particularly in textile and metal working, machines and factory workers replaced artisans and farmers engaged in side jobs; in transportation, railroads and steamboats supplanted wagons and sailboats; and in agriculture, threshing machines and reapers reduced labor input greatly.[1] During the Second Industrial Revolution between the second half of the 19th century and World War I, with the utilization of electric power and internal combustion engines, mechanization proceeded further in manual tasks: in manufacturing, broader industries and production processes were mechanized with the introduction of mass production system; wider tasks were mechanized with tractors in agriculture and with automobiles and trucks in transportation. Further, some analytical (cognitive) tasks too were mechanized during the era: tabulating machines substituted workers engaged in data processing tasks and teleprinters replaced Morse code operators. In the post World War II era, especially since the 1970s, analytical tasks in much wider areas have been mechanized because of the rapid growth of IT technologies: computers replaced clerical workers engaged in information processing tasks; sensors mechanized inspection processes in manufacturing and services (particularly in commerce and distribution); and simple troubleshooting tasks in many sectors were mechanized with the construction of databases of known troubles.[2]

Consequently, humans have shifted to tasks machines cannot perform efficiently. The general trend until about the 1960s is the shift from manual tasks to analytical tasks: initially, they shifted from manual tasks at farms and cottages to manual tasks at factories and analytical tasks at offices and factories mainly associated with clerical, management, and technical jobs; after mechanization deepened in manufacturing, they shifted away from manual tasks at factories as well as at farms to the analytical tasks. Since the 1970s, as a result of the advancement of IT technologies, humans have shifted away from routine analytical tasks (such as simple information processing tasks) as well as manual tasks toward non-routine analytical tasks mainly associated with professional and technical jobs and non-routine manual tasks in services such as personal care, protective service, and cleaning.[3,4]

Further, as can be inferred from the shifts in tasks, mechanization has affected relative demands for workers of different skill levels and thus earnings levels and earnings inequality. In the early stage of industrialization, earnings of unskilled workers grew very moderately

---

[1] Works on the Industrial Revolution and the Second Industrial Revolution by economic historians include Landes (2003) and Mokyr (1985, 1999).

[2] Case studies of effects of IT technologies on the workplace include Autor, Levy, and Murnane (2002) on a commercial bank and Bartel, Ichniowski, and Shaw (2007) on a bulb manufacturing factory.

[3] Similarly to Autor, Levy, and Murnane (2003), routine tasks refer to tasks whose procedures are organized so that they can be performed by machines after relevant technologies are developed.

[4] Autor, Levy, and Murnane (2003) examine changes in the composition of tasks performed by humans in the U.S. economy between 1960 and 1998 and find that the advancement of IT technologies is important in explaining the changes after the 1970s. Relatedly, Acemoglu and Autor (2011) explore changes in occupational composition for the longer period, 1959−2007.

and earnings inequality between skilled and unskilled workers increased.[5] In later periods, unskilled workers have benefited more from mechanization, while, as before, the rising inequality has been the norm in economies with lightly regulated labor markets, except in periods of rapid growth of the relative supply of skilled workers and in the 1940s, when the inequality fell.[6] Since the 1990s, owing to the large shift away from routine analytical tasks, wage growth of middle-wage jobs has been weak relative to both high-wage and low-wage jobs and thus wage polarization (and job polarization) has been observed in economies including the U.S.[7,8]

This paper develops a Ricardian model of task assignment and examines how improvements of productivities of machines and an increase in the relative supply of skilled workers affect task assignment (which factors perform which tasks), earnings, earnings inequality, and aggregate output in order to understand the aforementioned long-run trends.

The model economy is a static small-open competitive economy where three kinds of factors of production− skilled workers, unskilled workers, and machines− are available. Each factor is characterized by *analytical ability* and *manual ability.* Skilled workers have a higher level of analytical ability than unskilled workers, while both types of workers have the same level of manual ability, reflecting the fact that there is no strong correlation between the two abilities, except in poorest countries.

The final good is produced from inputs of a continuum of *tasks* that are different in *the importance of analytical ability* and *the ease of codification* (*routinization*) using a Leontief technology.[9] The three factors are perfectly substitutable at each task, and a unit of each factor supplies a unit of time inelastically. Both types of abilities contribute to production at each task (except the most manual tasks and the most analytical tasks), but the relative contribution of analytical ability is higher in tasks of the greater importance of the ability. For given the ability's importance, machines are more productive in tasks with the

---

[5] Feinstein (1998) finds that real wages and the standard of living of British manual workers improved very moderately between the 1770s and the 1850s (they more or less stagnated until the 1830s). The finding suggests that earnings inequality between them and skilled workers such as white-collar employees, merchants, and professionals rose greatly during the period.

[6] Goldin and Katz (1998), based on data from 1909 to 1940, show econometrically that the introduction of particular mass production methods, continuous process and batch methods, raised the relative demand for skilled workers in U.S. manufacturing. Goldin and Katz (1999) document that returns to high school education in the U.S. fell considerably sometime between 1914 and 1939, when high school enrollment rates rose dramatically (from about 20% to over 70%), while thereafter the returns continued to rise except in the 1940s when they fell sharply. As for returns to college education in the U.S., after plummeting in the 1940s, they kept rising except in the 1970s when the relative supply of college educated workers grew rapidly due to the entry of baby boom cohorts into the labor market.

[7] Autor, Katz, and Kearney (2006) find the evidence of wage polarization for the U.S. economy between 1988 and 2004. OECD (2008) documents that, after the 1990s, wage inequality between middle-wage and high-wage workers enlarged in most developed economies studied, while the disparity between middle-wage and low-wage workers shrunk or was stable in the majority of the economies.

[8] Job polarization is the phenomenon where job growth is strong at high-wage and low-wage jobs and is weak at middle-wage jobs. It is identified first for the U.K. economy by Goos and Manning (2003). Later studies such as Autor, Katz, and Kearney (2006) and Goos, Manning, and Salomons (2010) find that it is observed in most developed economies.

[9] The term codify/routinize means "organize procedures of tasks systematically so that tasks can be performed by machines after relevant technologies are developed" in this paper.

greater ease of codification, while, for simplicity, workers' productivities are assumed to be independent of the ease of codification.

A competitive equilibrium determines task assignment, factor prices, task prices, and output etc. Comparative advantages of factors determine task assignment: unskilled (skilled) workers are assigned to relatively manual (analytical) tasks and machines are assigned to tasks that are easier to codify. Among tasks a given factor is employed, it is employed heavily in tasks in which its productivities are low.

Based on the model, the paper examines how task assignment, earnings, earnings inequality (relative earnings of skilled workers to unskilled workers), and output change over time, when analytical and manual abilities of machines improve exogenously over time. Section 4 analyzes a simpler case in which the two abilities grow proportionately and machines have comparative advantages in relatively manual tasks. The analysis shows that tasks and workers strongly affected by mechanization change over time. Mechanization starts from tasks that are highly manual and easy to routinize and gradually spreads to tasks that are more analytical and more difficult to routinize. Eventually, mechanization proceeds in highly analytical tasks, those previously performed by skilled workers, as well. Accordingly, workers shift to tasks that are more difficult to codify and, except at the final stage, more analytical. Skilled workers always benefit from mechanization, whereas the effect on earnings of unskilled workers is ambiguous while mechanization mainly affects them and the effect turns positive afterwards. Earnings inequality rises except at the final stage, where it does not change. And the output of the final good always increases. By contrast, an increase in the relative supply of skilled workers raises (lowers) earnings of unskilled (skilled) workers and thus lowers the inequality (it also raises output).

The results are consistent with long-run trends of task shifts, earnings, and earnings inequality described earlier, except job polariztion after the 1990s and the development of the latter two variables after around 1980 and in the wartime 1940s. However, the assumption that the two abilities grow proportionately, which made the analysis simple, is rather restrictive, considering that the growth of manual ability was faster in most periods of time, while analytical ability seems to have grown faster than manual ability recently. Hence, Section 5 analyzes the general case in which the two abilities may grow at different rates. Under realistic productivity growth, the model can explain long-run trends of the variables, except the development in the 1940s and the recent job and wage polarization, which is beyond the scope of the model with two types of workers, although the falling inequality predicted by the model may capture a part of the development, the falling inequality between low-skill and middle-skill workers. Finally, the model is used to examine possible future trends of the variables when the rapid growth of IT technologies continues.

The paper belongs to the literature on task (job) assignment model, which has been developed to analyze the distribution of earnings in labor economics (see Sattinger, 1993, for a review), and recently is used to examine broad issues, such as effects of technology on the labor market (Acemoglu and Autor, 2011), on cross-country productivity differences (Acemoglu and Zilibotti, 2001), and on organizational structure and wages (Garicano and Rossi-Hansberg, 2006), effects of international trade and offshoring on the labor market (Grossman and Rossi-Hansberg, 2008, and Costinot and Vogel, 2010), and inter-industry wage differentials and the effect of trade on wages (Sampson, 2011).

The most closely related is Acemoglu and Autor (2011), who argue that the conventional model fails to explain a large part of trends of task shifts, earnings, and earnings inequality after the 1980s, particularly job and wage polarization after the 1990s,[10] and develop a task assignment model with three types of workers (high skill, middle skill, low skill), which is a generalization of the Acemoglu and Zilibotti (2001) model with two types of workers. The final good is produced from inputs of a continuum of tasks that are different in the degree of 'complexity' using a Cobb-Douglas technology. High (middle) skill workers have comparative advantages in more complex tasks against middle (low) skilled workers. After examining task assignment, earnings, and relative earnings in an economy without capital, they analyze the situation where a part of tasks initially performed by middle skill workers come to be mechanized exogenously and find that a fraction of these workers shift to tasks previously performed by the other types of workers and relative earnings of high skill workers to middle skill workers rise and those of middle skill workers to low skill workers fall, reproducing job and wage polarization.[11]

The present paper builds on their work, particularly in the modeling, but there are several important differences. First, the paper is interested in long-run trends of task shifts, earnings, and earnings inequality since the Industrial Revolution, while they focus on the development after the 1980s, especially job and wage polarization after the 1990s. Second, the paper examines how tasks and workers strongly affected by mechanization change over time with improvements of machine abilities, whereas, because of their focus on job and wage polarization, they *assume* that mechanization occurs at tasks previously performed by middle skill workers. Third, in order to examine the dynamics of mechanization, the present model supposes that tasks are different in two dimensions, the importance of analytical ability and the ease of codification (routinization), while, in their model, tasks are different in one dimension, the degree of 'complexity'.

The paper is also related to the literature that theoretically examines the interaction between mechanization and economic growth, such as Givon (2006), Zeira (1998, 2006), and Peretto and Seater (2008). The literature is mainly interested in whether persistent growth is possible in models where economies grow through mechanization and whether the dynamics are consistent with stylized facts on growth. While the standard model assumes labor-augmenting technical change, which is labor-saving but *not* capital-using (and thus does not capture mechanization), Givon (2006) and Peretto and Seater (2008) consider technical change that is labor-saving *and* capital-using. By contrast, given technologies, Zeira (2006) examines interactions among capital accumulation, changes in factor prices, and

---

[10] Limitations of the conventional model, in which workers with different skill levels are imperfect substitutes in a macro production function, pointed out by Acemoglu and Autor include: the model cannot explain stagnant or negative earnings growth of particular groups in a growing economy; typically, workers are two type and thus it cannot examine phenomena such as 'wage polarization'; systematic changes in job (task) composition such as 'job polarization' cannot be analyzed; since all workers with a given skill level have the same 'job', shifts in jobs and tasks performed by particular groups cannot be examined; technical change is factor-augmenting, thus it does not model mechanization through technical change, which is also pointed out in the literature on growth models with mechanization reviewed below.

[11] They also examine the situation where a part of tasks initially performed by middle skill workers come to be offshored exogenously. Further, they analyze the effect of changes in factor supplies on technical change using a version of the model with endogenous factor-augmenting technical change.

mechanization. The Zeira (2006)'s model can be interpreted as a dynamic task assignment model after a slight modification of the production technology. However, the model assumes homogenous labor and constant productivity of machines and thus cannot examine the issue this paper focuses on.

The paper is organized as follows. Section 2 presents the model and Section 3 derives equilibrium allocations, given machine abilities. Section 4 examines effects of improvement of machine abilities on task assignment, earnings, earnings inequality, and aggregate output, when the two abilities improve proportionately. Section 5 examines the general case in which the abilities may improve at different rates, and Section 6 concludes. Appendix contains proofs of lemmas and propositions, except Propositions 4−7 whose proofs are very lengthy and thus are posted on the author's web site.[12]

2 Model

Consider a small open economy where three kinds of factors of production− skilled workers, unskilled workers, and machines− are available. All markets are perfectly competitive.

Abilities and productivities of factors: Each factor is characterized by *analytical ability* and *manual ability*. Denote analytical abilities of a skilled worker, an unskilled worker, and a machine by h, l_a, and k_a, respectively, where $h > l_a$, and their manual abilities by l_m, l_m, and k_m, respectively. Two types of workers have the same level of manual ability, reflecting the fact that there is no strong correlation between the two abilities, except in poorest countries. The final good is produced from inputs of a continuum of *tasks* that are different in *the importance of analytical ability*, $a \in [0, 1]$, and *the ease of codification* (*routinization*), $c \in [0, 1]$. Tasks are uniformly distributed over the (a, c) space and productivities of skilled workers, unskilled worker, and machines in task (a, c) are given by:

$$A_h(a) = ah + (1 - a)l_m, \tag{1}$$

$$A_l(a) = al_a + (1 - a)l_m, \tag{2}$$

$$cA_k(a) = c[ak_a + (1 - a)k_m]. \tag{3}$$

Except the most manual tasks ($a = 0$) and the most analytical tasks ($a = 1$), both abilities contribute to production in each task, but the relative contribution of analytical ability is greater in tasks with higher a.[13] For given a, machines are more productive in tasks with higher c, while workers are assumed to be equally productive for any c. Since $h > l_a$, skilled workers have comparative advantages in more analytical tasks relative to unskilled workers.

Production: At each task, the three factors are perfectly substitutable and thus the production function of task (a, c) is expressed as:

$$y(a, c) = A_h(a)n_h(a, c) + A_l(a)n_l(a, c) + cA_k(a)n_k(a, c), \tag{4}$$

---

[12] The address is http://www.econ.kyoto-u.ac.jp/˜yuki/english.html.

[13] One interpretation of the specification is that a task with certain a is composed of the proportion a of analytical subtasks, where only analytical ability is useful, and the proportion $1 - a$ of manual ones, and the two types of subtasks requiring different abilities are perfectly substitutable in the production of the task. (Due to indivisibility of subtasks and economies of scope, one needs to perform both types of subtasks.)

where $n_i(a,c)$ $(i = h, l, k)$ is the measure of factor i engaged in the task. The output of the task, $y(a,c)$, may be interpreted as either an intermediate good or a direct input in final good production, which is produced by either final good producers or separate entities.

The final good production function is Leontief with equal weights on all tasks, that is, all tasks are equally essential in the production:

$$Y = \min_{a,c}\{y(a,c)\}. \tag{5}$$

The Leontief specification is assumed for simplicity. Similar results would be obtained as long as different tasks are complementary in the production, although more general specifications seem to be analytically intractable.[14]

Factor markets: A unit of each factor supplies a unit of time inelastically. Let the final good be a numeraire and let the relative price of (the output of) task (a,c) be $p(a,c)$. Then, from profit maximization problems of intermediate producers,

$$p(a,c)A_h(a) = (\leq)\, w_h \text{ for any } (a,c) \text{ with } n_h(a,c) > (=)\, 0, \tag{6}$$

$$p(a',c')A_l(a') = (\leq)\, w_l \text{ for any } (a',c') \text{ with } n_l(a',c') > (=)\, 0, \tag{7}$$

$$p(a'',c'')c''A_k(a'') = (\leq)\, r \text{ for any } (a'',c'') \text{ with } n_k(a'',c'') > (=)\, 0, \tag{8}$$

where w_h (w_l) is earnings of a skilled (unskilled) worker, and r is exogenous interest rate.

From these equations, the basic pattern of *task assignment* can be derived (details are explained later). Because $\frac{A_h(a)}{A_l(a)} \leq (\geq) \frac{w_h}{w_l}$ for any (a,c) satisfying $n_h(a,c) = (>) 0$ and $n_l(a,c) > (=) 0$ and $\frac{A_h(a)}{A_l(a)}$ increases with a, there exists unique $a^* \in (0,1)$ satisfying $\frac{A_h(a^*)}{A_l(a^*)} = \frac{w_h}{w_l}$ and unskilled (skilled) workers are chosen over skilled (unskilled) workers for $a < (>) a^*$. That is, unskilled (skilled) workers are assigned to relatively manual (analytical) tasks, and, as $\frac{w_h}{w_l}$ increases, the range of tasks (in terms of a) performed by unskilled (skilled) workers expands (shrinks). Of course, which factor is employed in a given task depends on the relative profitability of workers to machines as well. For $a < a^*$, unskilled workers (machines) are assigned to tasks (a,c) with $\frac{A_l(a)}{cA_k(a)} > (<) \frac{w_l}{r}$, and for $a > a^*$, skilled workers (machines) are assigned to tasks (a,c) with $\frac{A_h(a)}{cA_k(a)} > (<) \frac{w_h}{r}$. Comparative advantages of factors and relative factor prices determine task assignment.

Task (intermediate) markets: Because each task (intermediate good) is equally essential in final good production, $y(a,c) = Y$ must hold for any (a,c). Thus, the following is true for any (a,c) with $n_h(a,c) > 0$, any (a',c') with $n_l(a',c') > 0$, and any (a'',c'') with $n_k(a'',c'') > 0$, except for the set of measure 0 tasks in which multiple factors are employed:

$$A_h(a)n_h(a,c) = A_l(a')n_l(a',c') = c''A_k(a'')n_k(a'',c'') = Y. \tag{9}$$

Given the task assignment, factors are employed heavily in low productivity tasks.

Denote the measure of total supply of factor i $(i = h, l, k)$ by N_i (N_k is endogenous). Then, by substituting (9) into $\iint_{n_i(a,c)>0} n_i(a,c)\,da\,dc = N_i$,

---

[14] The model with a Cobb-Douglas production function seems to be quite difficult to analyze. An advantage of the Leontief specification over the Cobb-Douglas one is that, as shown below, the former yields a realistic result that, among tasks a certain factor is employed, it is employed heavily in tasks in which their productivities are low.

$$\frac{N_h}{\iint_{n_h(a,c)>0}\frac{1}{A_h(a)}dadc}=\frac{N_l}{\iint_{n_l(a,c)>0}\frac{1}{A_l(a)}dadc}=\frac{N_k}{\iint_{n_k(a,c)>0}\frac{1}{cA_k(a)}dadc}=Y. \tag{10}$$

The first equality of the equation is one of the two key equations, which states that task assignment must be determined so that demands for two types of workers satisfy the equality.

Since the final good is a numeraire and a unit of the final good is produced from inputs of a unit of every task,

$$\iint p(a,c)dadc=1 \tag{11}$$

$$\Leftrightarrow w_l\iint_{n_l(a,c)>0}\frac{1}{A_l(a)}dadc+w_h\iint_{n_h(a,c)>0}\frac{1}{A_h(a)}dadc+r\iint_{n_k(a,c)>0}\frac{1}{cA_k(a)}dadc=1, \tag{12}$$

where the second equation is derived using (6)$-$(8) with the equal sign. (12) is the second key equation, which states that task assignment must be determined so that the unit production cost of the final good equals 1.

Equilibrium: An equilibrium is defined by (6)$-$(8), (9), (10), (12), and the task assignment conditions ($\frac{A_h(a^*)}{A_l(a^*)}=\frac{w_h}{w_l}$, $\frac{A_l(a)}{cA_k(a)}=\frac{w_l}{r}$, and $\frac{A_h(a)}{cA_k(a)}=\frac{w_h}{r}$). By using the task assignment conditions, the first equality of (10) and (12) are expressed as simultaneous equations of w_h and w_l. Once the factor prices and thus task assignment are determined, N_k and Y ($=y(a,c)$) are determined from the second and third equalities of (10), respectively; $n_i(a,c)$ ($i=h,l,k$) is determined from (9); and $p(a,c)$ is determined from (6) $-$ (8).

3 Analysis

This section derives task assignment and earnings explicitly, given machine abilities k_a and k_m. So far, no assumptions are imposed on comparative advantages of machines. Until Section 5, it is assumed that $\frac{k_a}{k_m}<\frac{l_a}{l_m}(<\frac{h}{l_m})$, that is, machines have comparative advantages in relatively manual tasks. Then, $\frac{A_l(a)}{A_k(a)}$ and $\frac{A_h(a)}{A_k(a)}$ increase with a. With this assumption, the task assignment conditions can be stated more explicitly.

3.1 Task assignment conditions

Remember that, for $a<a^*$, unskilled workers (machines) perform tasks (a,c) with $\frac{A_l(a)}{cA_k(a)}>(<)\frac{w_l}{r}$, and for $a>a^*$, skilled workers (machines) perform tasks (a,c) with $\frac{A_h(a)}{cA_k(a)}>(<)\frac{w_h}{r}$, where a^* is defined by $\frac{A_h(a^*)}{A_l(a^*)}=\frac{w_h}{w_l}$. Further, since $\frac{k_a}{k_m}<\frac{l_a}{l_m}(<\frac{h}{l_m})$, humans (machines) perform tasks with relatively high (low) a and low (high) c, and, for given c, machines perform tasks with $a>a^*$ only if they perform all tasks with $a\leq a^*$. Based on this basic pattern of assignment, critical variables and functions determining task assignment, $c_m, c^*, c_a, c_l(a)$, and $c_h(a)$, are defined next.

Unskilled workers vs. machines: From the above discussion, whenever $n_k(a,c)>0$ for some (a,c), $n_k(0,1)>0$, i.e. whenever machines are used in production, they perform the most manual and easiest-to-codify task. Define c_m as $\frac{A_l(0)}{c_mA_k(0)}=\frac{l_m}{c_mk_m}=\frac{w_l}{r}$, that is, c_m is the value of c such that hiring a machine and hiring an unskilled worker are equally profitable

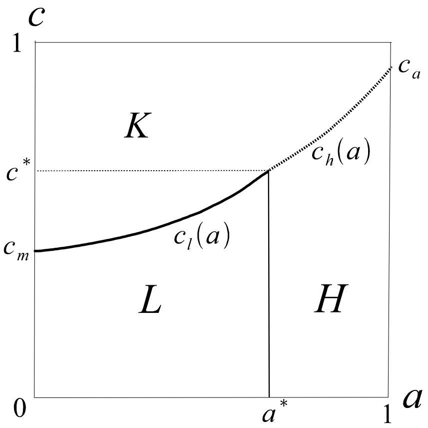


Figure 1: An example of task assignment when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ and $c_m < c^* < c_a < 1$

at task $(0, c_m)$. (Under the assumption $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, c_m is the lowest c satisfying $n_k(a, c) > 0$.) Then, other (a, c)s satisfying $\frac{A_l(a)}{cA_k(a)} = \frac{w_l}{r}$ is given by $\frac{A_l(a)}{cA_k(a)} = \frac{l_m}{c_m k_m}$. Let $c_l(a) \equiv \frac{k_m}{l_m}\frac{A_l(a)}{A_k(a)}c_m$. For given a, a machine and an unskilled worker are equally profitable at $c = c_l(a)$ and the former (latter) is chosen over the latter (former) for $c > (<) c_l(a)$. If there exists $c < 1$ such that they are equally profitable at $a = a^*$, i.e. $c_l(a^*) = \frac{k_m}{l_m}\frac{A_l(a^*)}{A_k(a^*)}c_m < 1$, machines perform some tasks with $a > a^*$. If $\frac{k_m}{l_m}\frac{A_l(a^*)}{A_k(a^*)}c_m \geq 1$, machines do not perform any tasks with $a > a^*$. Let $c^* \equiv \min\left\{\frac{k_m}{l_m}\frac{A_l(a^*)}{A_k(a^*)}c_m, 1\right\}$.

Skilled workers vs. machines: When $c^* < 1$, skilled workers and machines must be compared. Since $\frac{A_h(a^*)}{A_l(a^*)} = \frac{w_h}{w_l}$, (a, c)s satisfying $\frac{A_h(a)}{cA_k(a)} = \frac{w_h}{r}$ is given by $\frac{A_h(a)}{cA_k(a)} = \frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{1}{c_m}$ and let $c_h(a) \equiv \frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{A_h(a)}{A_k(a)}c_m$. For given a, hiring a skilled worker and hiring a machine are equally rewarding at $c = c_h(a)$. If there exists $c < 1$ such that both are equally profitable at $a = 1$, i.e. $c_h(1) = \frac{h}{k_a}\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m < 1$, machines perform some tasks with $a = 1$. Let $c_a \equiv \min\left\{\frac{h}{k_a}\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m, 1\right\}$.

Figure 1 illustrates $c_m, c^*, c_a, c_l(a)$, and $c_h(a)$ and thus task assignment on the (a, c) space, assuming that $c_m < c^* < c_a < 1$ holds. For given a, machines perform tasks with higher c. From the assumption that machines have comparative advantages at relatively manual tasks, for given c, they perform tasks with lower a and the proportion of tasks performed by machines decreases with a, i.e. $c_l(a)$ and $c_h(a)$ are upward sloping. (These properties are satisfied when $c_m < c^* < c_a < 1$ do not hold too.)

3.2 Key equations determining equilibrium

From their definitions, $c_l(a)$, $c_h(a)$, c^*, and c_a are functions of c_m and a^*:

$$c_l(a) = \frac{k_m}{l_m}\frac{A_l(a)}{A_k(a)}c_m, \quad c_h(a) = \frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{A_h(a)}{A_k(a)}c_m, \tag{13}$$

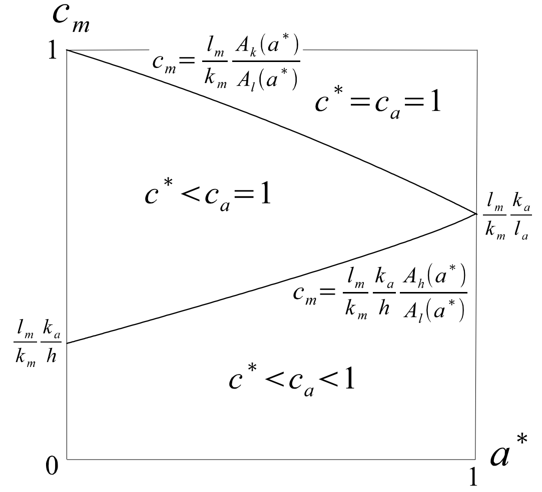


Figure 2: Values of c^* and c_a on the (a^*, c_m) space when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$

$$c^* = \min\left\{\frac{k_m}{l_m}\frac{A_l(a^*)}{A_k(a^*)}c_m, 1\right\}, \quad c_a = \min\left\{\frac{h}{k_a}\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m, 1\right\}. \tag{14}$$

From the equations defining a^* and c_m, earnings too are functions of c_m and a^*:

$$w_l = \frac{l_m}{k_m}\frac{r}{c_m}, \tag{15}$$

$$w_h = \frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{r}{c_m}. \tag{16}$$

Hence, the two key equations determining equilibrium, the first equality of (10) and (12), can be expressed as simultaneous equations of c_m and a^* (see Figure 1 for the derivation):

$$\frac{N_h}{N_l}\int_0^{a^*}\int_0^{\min\{c_l(a),1\}}\frac{1}{A_l(a)}dcda = \int_{a^*}^1\int_0^{\min\{c_h(a),1\}}\frac{1}{A_h(a)}dcda, \tag{HL}$$

$$\begin{aligned}\frac{l_m}{k_m}\frac{r}{c_m}\int_0^{a^*}\int_0^{\min\{c_l(a),1\}}\frac{dcda}{A_l(a)}+\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{r}{c_m}\int_{a^*}^1\int_0^{\min\{c_h(a),1\}}\frac{dcda}{A_h(a)}\\ +r\left[\int_0^{a^*}\int_{\min\{c_l(a),1\}}^1\frac{dcda}{cA_k(a)}+\int_{a^*}^1\int_{\min\{c_h(a),1\}}^1\frac{dadc}{cA_k(a)}\right]=1,\end{aligned} \tag{P}$$

Once $a^*$and c_m are determined from (HL) and (P), $c^*, c_a, c_l(a), c_h(a)$ and thus task assignment are determined. Then, earnings are determined from (15) and (16), and the remaining variables are determined as stated in the definition of equilibrium of the previous section.

The determination of equilibrium $a^*$and c_m can be illustrated graphically using a figure depicting graphs of the key equations on the (a^*, c_m) space. Since, as shown below, the shape of (HL) differs depending on whether c^* and c_a equal 1 or not, using (14), the (a^*, c_m) space is divided into three regions based on values of c^* and c_a (Figure 2).

When $c_m \geq \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} \Leftrightarrow \frac{A_l(a^*)}{1\times A_k(a^*)} \geq \frac{l_m}{c_m k_m} = \frac{w_l}{r}$, that is, when an unskilled worker is weakly chosen over a machine at task $(a, c) = (a^*, 1)$, machines are not used in any tasks with $a > a^*$ and thus $c^* = c_a = 1$ holds. When $c_m \geq \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow \frac{h}{1\times k_a} \geq \frac{l_m}{c_m k_m}\frac{A_h(a^*)}{A_l(a^*)} = \frac{w_h}{r}$ and $c_m < \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$, that is, when a skilled worker is weakly chosen over a machine at task $(a, c) = (1, 1)$ and a machine is strictly chosen over an unskilled worker at task $(a, c) = (a^*, 1)$, machines are employed in some tasks with $a > a^*$ but not in tasks with $a = 1$ and $c < 1$, thus $c^* < c_a = 1$ holds. Finally, when $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$, machines are employed in some tasks with $a = 1$ and $c < 1$ and thus $c^* < c_a < 1$ holds.

3.3 Shape of (HL) and its relations with exogenous variables

Now the shape of (HL) and its relations with exogenous variables are examined. Note that the results do *not* depend on the assumption $\frac{k_a}{k_m} < \frac{l_a}{l_m}$. Lemma 1 presents the result when $c^*, c_a < 1$ ($c^* < (>) c_a$ when $\frac{k_a}{k_m} < (>) \frac{h}{l_m}$), the area below $c_m = \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ of Figure 2. *No* assumptions are imposed regarding magnitude relations of analytical abilities to manual abilities, although presentations in the lemmas appear to suppose $h > l_m$, $l_m > l_a$, and $k_m > k_a$.

Lemma 1 *When* $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow c^*, c_a < 1$ ($c^* < (>) c_a$ *when* $\frac{k_a}{k_m} < (>) \frac{h}{l_m}$), *(HL) is expressed as*

$$\frac{N_h}{N_l}\ln\left(\frac{k_m}{A_k(a^*)}\right) = \frac{A_l(a^*)}{A_h(a^*)}\ln\left(\frac{A_k(a^*)}{k_a}\right), \quad \textit{when } \frac{k_a}{k_m} \neq 1, \tag{17}$$

$$\frac{N_h}{N_l}a^* = \frac{A_l(a^*)}{A_h(a^*)}(1-a^*), \quad \textit{when } \frac{k_a}{k_m} = 1. \tag{18}$$

a^* *satisfying the equation decreases with* $\frac{N_h}{N_l}$ *and* $\frac{k_a}{k_m}$.

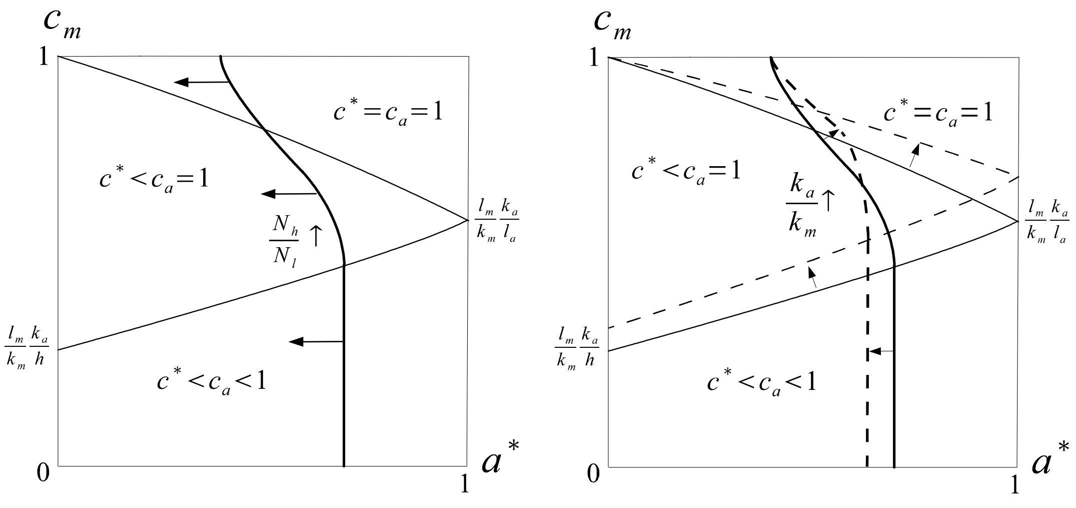
Unlike the other cases below, (HL) is independent of c_m. a^* satisfying the equation decreases with $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$. As will be seen, the relation with $\frac{N_h}{N_l}$ is negative in all the cases, while the one with $\frac{k_a}{k_m}$ differs in each case. The next lemma presents the result when $c^* < c_a = 1$, the area below $c_m = \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$ and on or above $c_m = \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ of Figure 2. This case arises only when $\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} > \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow \frac{k_a}{k_m} < \frac{h}{l_m}$.

Lemma 2 *When* $c_m \in \left[\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}, \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}\right) \Leftrightarrow c^* < c_a = 1$, *which arises only when* $\frac{k_a}{k_m} < \frac{h}{l_m}$, *(HL) is expressed as*

$$\textit{when } \frac{k_a}{k_m} \neq 1, \quad \frac{N_h}{N_l}\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\ln\left(\frac{k_m}{A_k(a^*)}\right)$$
$$= \frac{1}{h-l_m}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)} + (h-l_m)c_m}{\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}(hk_m - l_m k_a)}h\right] + \frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m-k_a}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)} + (h-l_m)c_m}{\frac{(hk_m-l_m k_a)c_m}{A_k(a^*)}}\right], \tag{19}$$

$$\textit{when } \frac{k_a}{k_m} = 1, \quad \frac{N_h}{N_l}\frac{c_m a^*}{l_m} = \frac{1}{h-l_m}\left\{\ln\left[\frac{h}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\right] - \frac{A_l(a^*)}{l_m}c_m + 1\right\}. \tag{20}$$

a^* *satisfying the equation decreases with* c_m *and* $\frac{N_h}{N_l}$ ($\frac{\partial a^*}{\partial c_m} = 0$ *at* $c_m = \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$), *and decreases (increases) with* $\frac{k_a}{k_m}$ *for small (large)* c_m.



(a) Relation of (HL) with $\frac{N_h}{N_l}$ (b) Relation of (HL) with $\frac{k_a}{k_m}$

Figure 3: Shape of (HL) and its relations with $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$

Unlike the previous case, a^* satisfying (HL) decreases with c_m (except at $c_m = \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$, where $\frac{\partial a^*}{\partial c_m} = 0$), and it increases with $\frac{k_a}{k_m}$ when c_m is large. Finally, the next lemma presents the result when $c^* = c_a = 1$, the area on or above $c_m = \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$ of Figure 2. This case arises only when $\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} < 1 \Leftrightarrow \frac{k_a}{k_m} < \frac{l_a}{l_m}$.

Lemma 3 *When $c_m \geq \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} \Leftrightarrow c^* = c_a = 1$, which arises only when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, (HL) is expressed as*

$$\frac{N_h}{N_l}\left\{\frac{1}{l_m-l_a}\ln\left[\frac{l_ak_m-l_mk_a}{(k_m-k_a)l_m-(l_m-l_a)k_mc_m}\frac{l_m}{A_l(a^*)}\right]+\frac{k_mc_m}{(k_m-k_a)l_m}\ln\left[\frac{(k_m-k_a)l_m-(l_m-l_a)k_mc_m}{(l_ak_m-l_mk_a)c_m}\right]\right\}$$
$$=\frac{1}{h-l_m}\ln\left(\frac{h}{A_h(a^*)}\right), \ \textit{when}\ \frac{k_a}{k_m}\neq 1, \tag{21}$$

$$\frac{N_h}{N_l}\frac{1}{l_a-l_m}\left\{\ln\left[\frac{c_mA_l(a^*)}{l_m}\right]+1-c_m\right\} = \frac{1}{h-l_m}\ln\left(\frac{h}{A_h(a^*)}\right), \ \textit{when}\ \frac{k_a}{k_m}=1, \tag{22}$$

where $a^* \in (0,1)$ holds for any c_m. a^* satisfying the equation decreases with c_m and $\frac{N_h}{N_l}$, and it increases with $\frac{k_a}{k_m}$ ($\lim_{c_m\to 1}\frac{\partial a^*}{\partial c_m} = \lim_{c_m\to 1}\frac{\partial a^*}{\partial \frac{k_a}{k_m}} = 0$).

a^* satisfying (HL) decreases with c_m as in the previous case, while it increases with $\frac{k_a}{k_m}$ ($\lim_{c_m\to 1}\frac{\partial a^*}{\partial c_m} = \lim_{c_m\to 1}\frac{\partial a^*}{\partial \frac{k_a}{k_m}} = 0$, though).

Figure 3 illustrates (HL) on the (a^*, c_m) space and shows its relations with $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$. The shape of (HL), i.e. negatively sloped when $c_a = 1$ and vertical when $c_a < 1$, can be explained intuitively for the case $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ as follows. A decrease in c_m lowers $c_l(a)$ and $c_h(a)$ from (13) and thus raises the proportion of tasks performed by machines (see Figure 1). When $c_a = 1$, that is, machines do not perform any tasks with $a = 1$ and $c < 1$, the mechanization mainly affects unskilled workers engaged in relatively manual tasks and thus they shift to more analytical tasks, i.e. a^* increases. By contrast, when $c_a < 1$, both types of workers are equally affected and thus a^* remains unchanged. Obviously, an increase in $\frac{N_h}{N_l}$ implies that a higher portion of tasks must be engaged by skilled workers and thus (HL) shifts to the left. Less straightforward is the effect of an increase in $\frac{k_a}{k_m}$, which shifts the locus to the right (left) when c_m is high (low), definitely so when $c^* = 1$ (when $c_a < 1$). An increase in $\frac{k_a}{k_m}$ weakens comparative advantages of humans in analytical tasks and thus lowers $c_l(a)$ and $c_h(a)$ (from equation 13) and the portion of tasks performed by humans (see Figure 1). When c_m (thus c^* and c_a too) is high, such mechanization mainly affects unskilled workers engaged in relatively manual tasks and thus a^* must increase, while the opposite is true when c_m is low.

3.4 Shape of (P) and its relations with exogenous variables

The next lemma presents the shape of (P) and its relations with k_m, k_a, and r.

Lemma 4 *c_m satisfying (P), which is positive, increases with a^* and r, and decreases with k_m and k_a.*

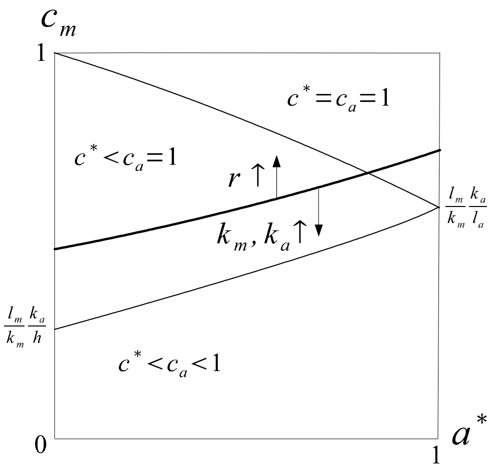


Figure 4: Shape of (P) and its relations with k_m, k_a, and r

Figure 4 illustrates the shape of (P) and its relations with the exogenous variables. Remember that, for (P) to hold, task assignment must be determined so that the unit production cost of the final good equals 1. When c_m increases, a^* must increase, that is, (P) is upward-sloping on the (a^*, c_m) plane, because, otherwise, both $w_l = \frac{l_m}{k_m}\frac{r}{c_m}$ and $w_h = \frac{A_h(a^*)}{A_l(a^*)} w_l$ fall and thus the unit production cost would decrease. An increase in r raises the cost of hiring machines and thus a higher portion of tasks are assigned to humans, i.e. the locus shifts upward, while the opposite holds when abilities of machines, k_m and k_a, increase. The locus never intersects with $c_m = 0$, because machines are completely useless and thus hiring machines are prohibitively expensive at the hardest-to-codify tasks.

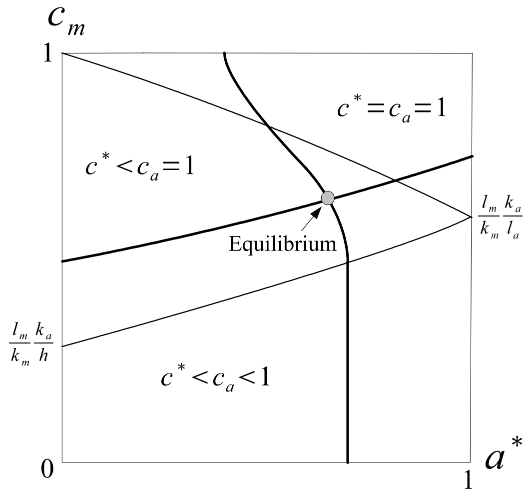


Figure 5: Determination of equilibrium a^* and c_m

As Figure 5 illustrates, equilibrium (a^*, c_m) is determined at the intersection of the two loci. Of course, the position of the intersection depends on exogenous variables such as k_m and k_a. The next two sections examine how increases in k_m, k_a, and $\frac{N_h}{N_l}$ affect the equilibrium, particularly, task assignment, earnings, earnings inequality, and aggregate output.

4 Mechanization with constant $\frac{k_a}{k_m}$

Suppose that abilities of machines, k_m and k_a, improve exogenously over time. This section examines effects of such productivity growth and an increase in $\frac{N_h}{N_l}$ on task assignment, earnings, earnings inequality, and output, when k_m and k_a satisfying $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ grow proportionately. As shown in Lemmas 1−3, (HL) does not shift under constant $\frac{k_a}{k_m}$ and thus the analysis is much simpler than the general case analyzed in the next section.

The next proposition presents the dynamics of the critical variables and functions determining task assignment of an economy undergoing the productivity growth.

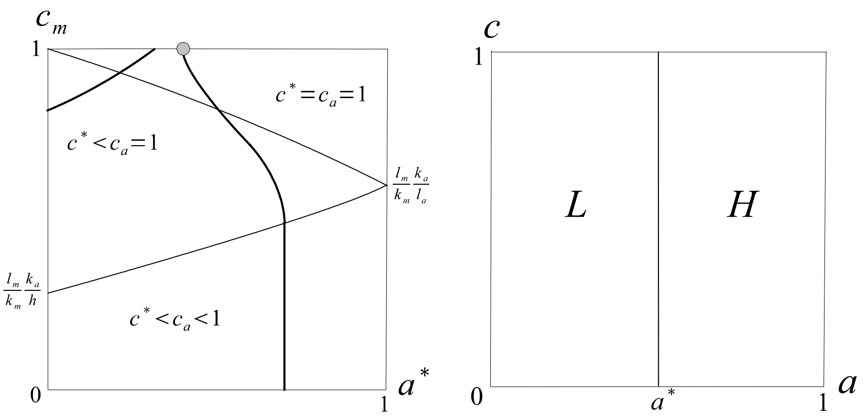
Proposition 1 *Suppose that k_m and k_a satisfying $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ grow over time with $\frac{k_a}{k_m}$ constant.*

(i) When k_m is very low initially, $c_m = c^* = c_a = 1$ is satisfied at first; at some point, $c_m < c^* = c_a = 1$ holds and thereafter c_m falls over time; then, $c_m < c^* < c_a = 1$ and c^* too falls; finally, $c_m < c^* < c_a < 1$ and c_a falls as well.

(ii) a^* increases over time when $c_m < c_a = 1$, while a^* is time-invariant when $c_a < 1$ (and when $c_m = 1$).

(iii) $c_l(a)$ and $c_h(a)$ (when $c^* < 1$) decrease over time when $c_m < 1$.

The results of this proposition can be understood graphically using figures similar to those in the previous section. When the level of k_m is very low, there are no (a^*, c_m) satisfying (P), or (P) is located at the left side of (HL) on the (a^*, c_m) plane (see Figure 6 (a)). Hence, the two loci do not intersect and an equilibrium with $c_m < 1$ does not exist. Because the manual



(a) Equilibrium (b) Task assignment

Figure 6: Equilibrium and task assignment when $c_m = c^* = c_a = 1$

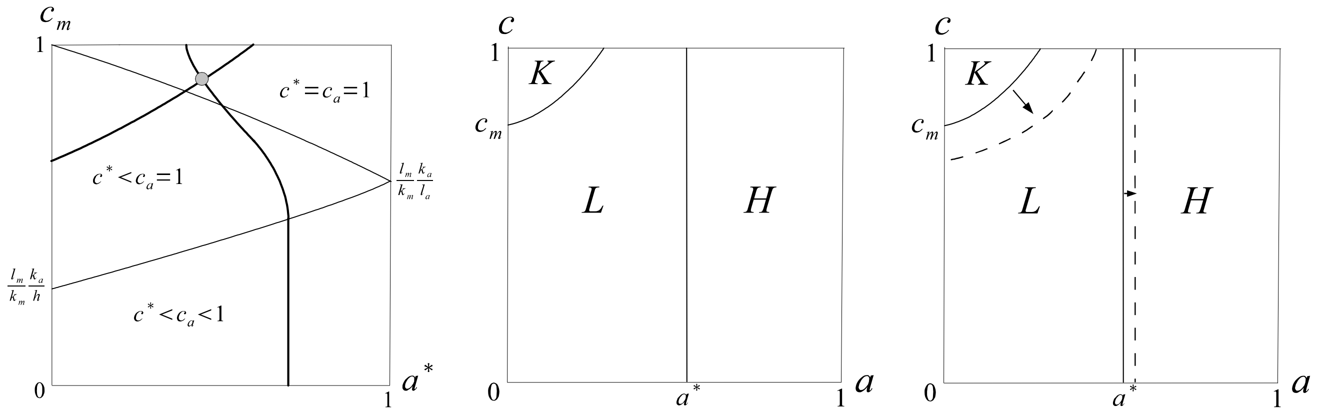
ability of machines is very low, hiring machines is not profitable at all and thus all tasks are performed by humans, i.e. $c_m = 1$. Figure 6 (a) illustrates an example of the determination of equilibrium c_m and a^* in this case. Equilibrium a^* is determined at the intersection of (HL) with $c_m = 1$. Figure 6 (b) illustrates the corresponding task assignment on the (a, c) plane, which shows that unskilled (skilled) workers perform all tasks with $a < (>) a^*$.

When k_m becomes high enough that (P) is located at the right side of (HL) at $c_m = 1$, the two loci intersect and thus machines begin to be used, i.e. $c_m < 1$. Note that k_a is not important for the first step of mechanization, because mechanization starts from the most manual tasks in which analytical ability is of no use. Because of low machine productivities, they perform only highly manual and easy-to-codify tasks that were previously performed by unskilled workers, i.e. $c^* = c_a = 1$ holds. Figure 7 (a) and (b) respectively illustrate the determination of equilibrium c_m and a^* and task assignment. Figure 7 (c) presents the effect of small increases in k_m and k_a on the task assignment. Since machines come to perform a greater portion of highly manual and easy-to-codify tasks, a^* increases and $c_l(a)$ decreases, that is, workers shift to more analytical and, for unskilled workers, harder-to-routinize tasks.

As k_m and k_a grow over time, mechanization spreads to relatively analytical tasks as well, and eventually, machines come to perform highly analytical tasks, those previously performed by skilled workers. Figure 8 (a) and (b) respectively illustrate the determination of equilibrium c_m and a^* and task assignment when $c_m < c^* < c_a = 1$. Machines perform some tasks with $a > a^*$ but not the most analytical ones, i.e. $c^* < c_a = 1$. Productivity growth lowers $c_h(a)$ as well as $c_l(a)$ (and raises a^*), thus skilled workers too shift to more difficult-to-codify tasks (Figure 8 (c)).

Finally, the economy reaches the case $c_m < c^* < c_a < 1$, which is illustrated in Figure 9. Machines perform a portion of the most analytical tasks, i.e. $c_a < 1$. Unlike the previous cases, productivity growth affects two type of workers equally and thus a^* does not change, while $c_h(a)$ and $c_l(a)$ decrease and thus workers shift to more difficult-to-codify tasks.

To summarize, when manual and analytical abilities of machines with $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ improve proportionately over time, mechanization starts from highly manual and easy-to-codify

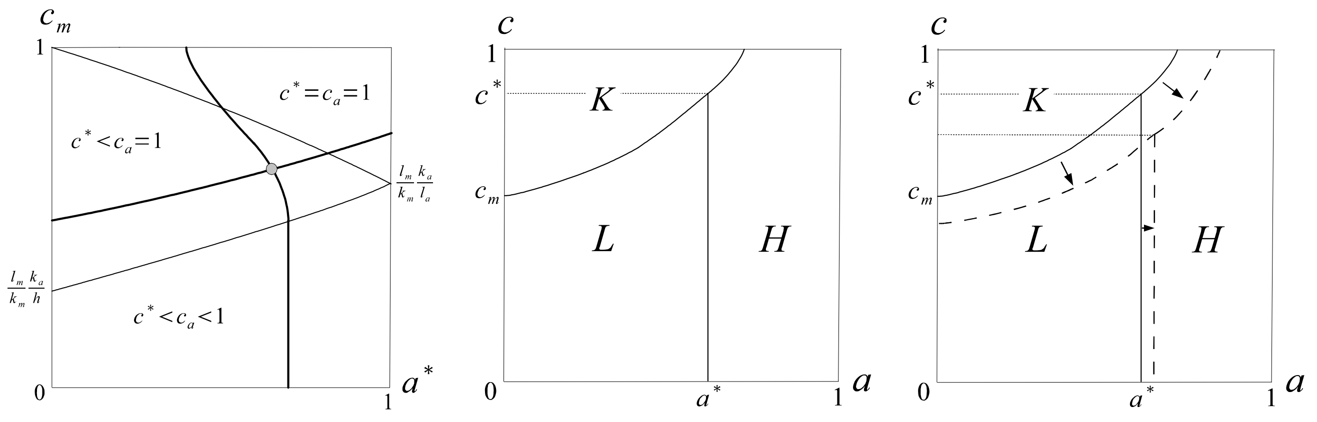


(a) Equilibrium

(b) Task assignment

(c) Effect of productivity growth with constant $\frac{k_a}{k_m}$ on task assignment

Figure 7: Equilibrium, task assignment, and the effect of productivity growth with constant $\frac{k_a}{k_m}$ when $c_m < c^* = c_a = 1$

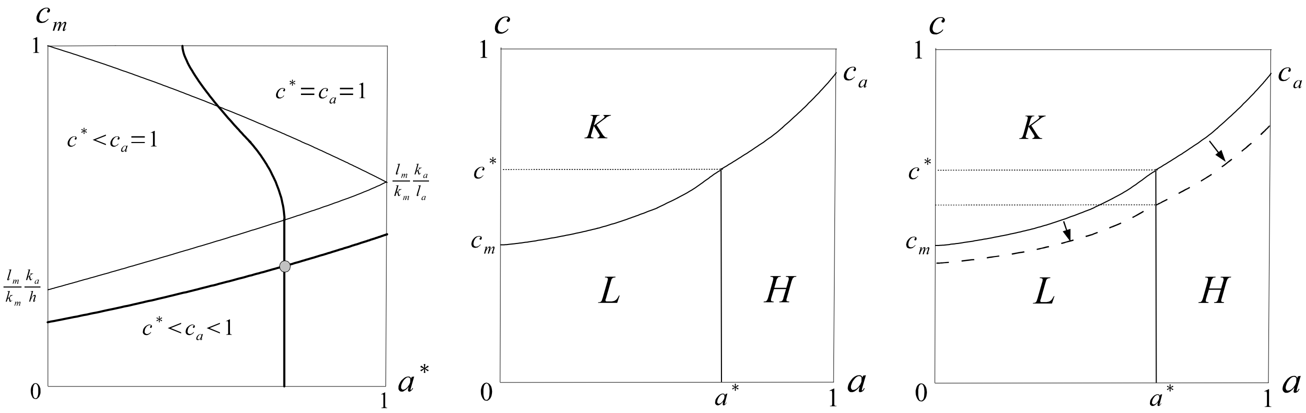


(a) Equilibrium

(b) Task assignment

(c) Effect of productivity growth with constant $\frac{k_a}{k_m}$ on task assignment

Figure 8: Equilibrium, task assignment, and the effect of productivity growth with constant $\frac{k_a}{k_m}$ when $c_m < c^* < c_a = 1$



(a) Equilibrium (b) Task assignment (c) Effect of productivity growth with constant $\frac{k_a}{k_m}$ on task assignment

Figure 9: Equilibrium, task assignment, and the effect of productivity growth with constant $\frac{k_a}{k_m}$ when $c_m < c^* < c_a < 1$

tasks and gradually spreads to more analytical and harder-to-routinize tasks. Eventually, machines come to perform highly analytical tasks, those previously performed by skilled workers. Accordingly, workers shift to tasks that are more difficult to codify and, except at the final stage, more analytical.

The dynamics of task assignment accord with long-run trends of mechanization and of shifts in tasks performed by humans (except job polarization after the 1990s) detailed in the introduction, which is summarized as: initially, mechanization proceeded in tasks intensive in manual labor, while mechanization of tasks intensive in analytical labor started during the Second Industrial Revolution and has progressed on a large scale in the post World War II era, especially since the 1970s, because of the rapid growth of IT technologies; humans shifted from manual tasks to analytical tasks until about the 1960s, whereas, thereafter, they have shifted away from routine analytical tasks as well as routine manual tasks toward non-routine manual tasks in services as well as non-routine analytical tasks.[15]

Effects of the productivity growth on earnings, earnings inequality, and aggregate output are examined in the next proposition.

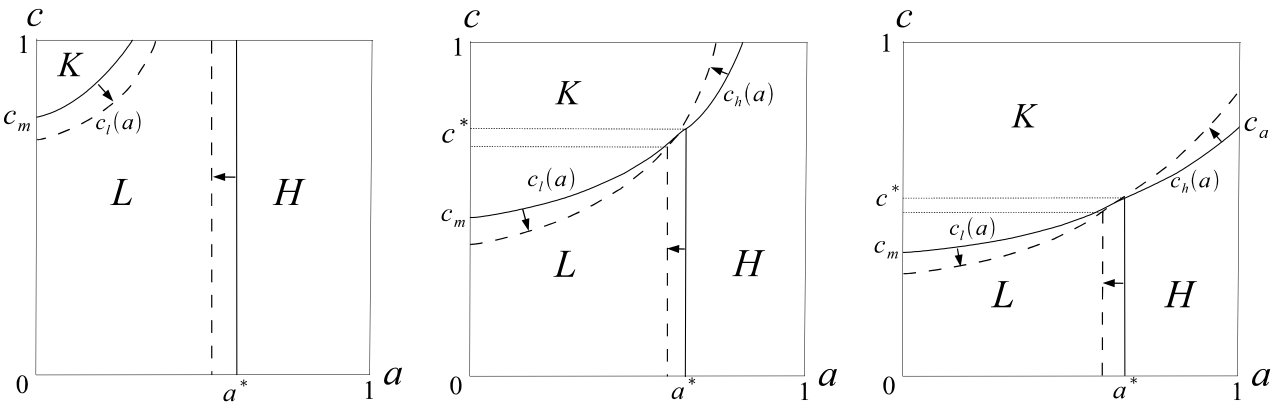
Proposition 2 *Suppose that k_m and k_a satisfying $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ grow proportionately over time when $c_m < 1$.*

(*i*) *Earnings of skilled workers increase over time. When $c^* < c_a < 1$, earnings of unskilled workers too increase.*

(*ii*) *Earnings inequality, $\frac{w_h}{w_l}$, rises over time when $c_a = 1$ and is time-invariant when $c_a < 1$.*

---

[15] Acemoglu and Autor (2011) document that the employment share of service occupations, which is intensive in non-routine manual tasks, continued to rise between 1959−2007 and the rise is large after the 1990s, while Autor, Levy, and Murnane (2003) find that the share of non-routine manual tasks in total tasks performed by humans continued to fall in the U.S. economy between 1960 and 1998 (although the fall is moderate in the 1990s). A likely reason of the decrease in the share is a large fall in the employment share of production occupations, which is intensive in non-routine as well as routine manual tasks.



(a) when $c_m < c^* = c_a = 1$ (b) when $c_m < c^* < c_a = 1$ (c) when $c_m < c^* < c_a < 1$

Figure 10: Effect of an increase in $\frac{N_h}{N_l}$ on task assignment when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$

(*iii*) *The output of the final good, Y, increases over time.*

The proposition shows that, while skilled workers *always* benefit from mechanization, the effect on earnings of unskilled workers is ambiguous when mechanization mainly affects them, i.e. when $c_a = 1$, and the effect turns positive when $c_a < 1$. Mechanization worsens earnings inequality, $\frac{w_h}{w_l}$, when $c_a = 1$, while it has no effect when $c_a < 1$. The output of the final good *always* increases, even if $l_a < h < l_m$ and thus workers' productivities, $A_h(a)$ and $A_l(a)$, fall as they shift to more analytical tasks.

So far, the proportion of skilled workers to unskilled workers, $\frac{N_h}{N_l}$, is held constant, which has increased over time in real economy. Thus, the next proposition examines effects of the growth of $\frac{N_h}{N_l}$ under constant machine qualities.

Proposition 3 *Suppose that $\frac{N_h}{N_l}$ grows over time when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ and $c_m < 1$.*

(*i*) *c_m, a^*, c^* (when $c^* < 1$), and $c_l(a)$ decrease, while c_a (when $c_a < 1$) and $c_h(a)$ (when $c^* < 1$) increase over time.*

(*ii*) *w_l (w_h) rises (falls) and earnings inequality, $\frac{w_h}{w_l}$, shrinks over time.*

(*iii*) *Y increases over time under constant $N_h + N_l$.*

Figure 10 illustrates the effect of an increase in $\frac{N_h}{N_l}$ on task assignment. Since skilled workers become abundant relative to unskilled workers, they take over a portion of tasks previously performed by unskilled workers, i.e. a^* decreases. Further, earnings of unskilled workers rise and those of skilled workers fall, thus some tasks previously performed by unskilled workers are mechanized, i.e. $c_l(a)$ decreases, while, when $c^* < 1$, skilled workers take over some tasks performed by machines before, i.e. $c_h(a)$ increases. That is, skilled workers shift to more manual tasks, and unskilled workers shift to harder-to-routinize tasks. The output of the final good increases even when the total population is constant, mainly because skilled workers are more productive than unskilled workers at any tasks with $a > 0$.

By combining the results on effects of an increase in $\frac{N_h}{N_l}$ with those of the productivity growth, the model can explain long-run trends of earnings and earnings inequality until the 1970s (except the wartime 1940s) detailed in the introduction, which is: in the early stage of industrialization when the growth of the relative supply of skilled workers was slow, earnings of unskilled workers grew very moderately and earnings inequality rose; in later periods when the relative supply of skilled workers grew faster, unskilled workers benefited more from mechanization, while, as before, the rising inequality was the norm in economies with lightly regulated labor markets (such as the U.S.), except in periods of rapid growth of the education level of the population and in the 1940s, when the inequality fell.[16]

The model, however, fails to capture the trends after the 1980s, which is: earnings of unskilled workers stagnated and those of skilled workers rose until the mid 1990s in the U.S.;[17] the inequality rose greatly in the 1980s, and wage polarization has proceeded since the 1990s in economies including the U.S. By contrast, the model predicts that earnings of unskilled workers increase and the inequality shrinks when highly analytical tasks are affected by mechanization, i.e. when $c_a < 1$, and the relative supply of skilled workers rises.

5 Mechanization with time-varying $\frac{k_a}{k_m}$

The previous section has examined the case in which k_m and k_a grow proportionately. This special case has been taken up first for analytical simplicity. However, the assumption of the proportionate growth is rather restrictive, because, according to the trend of mechanization described in the introduction, the growth of k_m was apparently faster than that of k_a before World War II, while k_a seems to have grown faster than k_m most recently.[18]

This section examines the general case in which they may grow at different rates. This case is much more difficult to analyze because, as shown in Lemmas 1–3, a change in $\frac{k_a}{k_m}$ shifts the graph of (HL) as well as that of (P) (see Figures 3 (b) and 4).

Starting from the situation where $\frac{k_a}{k_m} < \frac{l_a}{l_m} (< \frac{h}{l_m})$ holds, if k_a keeps growing faster than k_m, i.e. the rapid growth of IT technologies is long-lasting, $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$, then $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$ come to be satisfied. That is, comparative advantages of machines to two type of workers change over time. When $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$, $c^* < 1$ holds, and when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$, $c_a < c^* < 1$ holds from $c^* = \min\left\{\frac{k_m}{l_m}\frac{A_l(a^*)}{A_k(a^*)}c_m, 1\right\}$ and $c_a = \min\left\{\frac{h}{k_a}\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m, 1\right\}$ (see Figure 11).

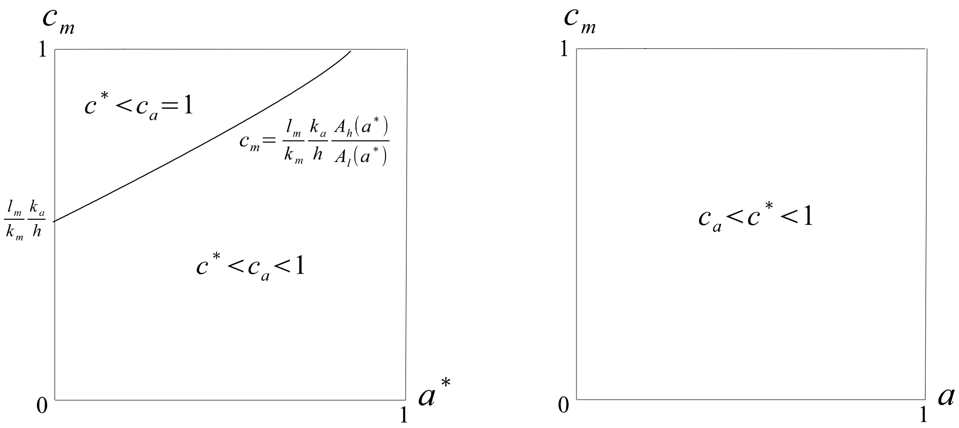
Figure 12 illustrates $c_l(a)$ and $c_h(a)$ and thus task assignment on the (a, c) space when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ (the figure is drawn assuming $c_a < 1$) and when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$. Unlike the original case $\frac{k_a}{k_m} < \frac{l_a}{l_m} (< \frac{h}{l_m})$, $c_l(a)$ is downward sloping and, when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$, $c_h(c)$ too is downward sloping. Hence, when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$, for given c, machines tend to perform tasks

---

[16] Combined effects of an increase in $\frac{N_h}{N_l}$ and improvements of machine qualities on task assignment accord with the trend of task shifts in real economy when $c^* = 1$. When $c^* < 1$, they are consistent with the fact, unless the negative effect of an increase in $\frac{N_h}{N_l}$ on $c_h(a)$ is strong (see Figure 10).

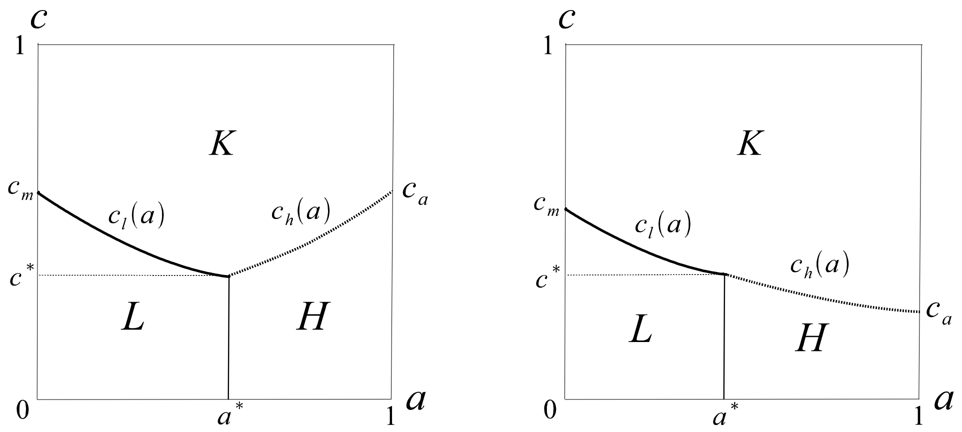
[17] According to Acemoglu and Autor (2011), real wages of full-time male workers without college degrees are lower in 1995 than in 1980, while wages of those with more than college education are higher. As for female workers, real wages rose during the period except for high school dropouts, but the rise was moderate for those without college degrees.

[18] Note that k_a seems to have been positive even before the Industrial Revolution: various machines had automatic control systems whose major examples are float valve regulators used in ancient Greece and in the medieval Arab world to control devices such as water clocks, oil lamps, and the level of water in tanks, and temperature regulators of furnaces invented in early 17th century Europe.



(a) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ (b) when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$

Figure 11: c^* and c_a on the (a^*, c_m) space when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$



(a) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ (b) when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$

Figure 12: $c_l(a)$ and $c_h(a)$ when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ ($c_a < 1$ is assumed) and when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$

with *intermediate* a and the proportion of tasks performed by machines is *highest at* $a = a^*$. When $\frac{k_a}{k_m} > \frac{h}{l_m}(> \frac{l_a}{l_m})$, for given c, machines tend to perform relatively *analytical* tasks and the proportion of tasks performed by machines *increases* with a.

Now, effects of changes in k_m and k_a on task assignment, earnings, earnings inequality, and output are examined. Since results are different depending on the shape of (HL) (note Lemmas 1−3), they are presented in three separate propositions.[19,20] The next proposition analyzes the case $c_m \geq \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} \Leftrightarrow c^* = c_a = 1$, which arises only when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$.

Proposition 4 *When* $c_m \geq \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} \Leftrightarrow c^* = c_a = 1$ *(possible only when* $\frac{k_a}{k_m} < \frac{l_a}{l_m}$*),*

(i) c_m *decreases and* a^* *increases with* k_m *and* k_a ($\lim_{c_m \to 1} \frac{da^*}{dk_m} = \lim_{c_m \to 1} \frac{da^*}{dk_a} = 0$).

(ii) $c_l(a)$ *decreases with* k_m *and* k_a*.*

(iii) w_h, $\frac{w_h}{w_l}$*, and* Y *increase with* k_m *and* k_a*.* w_l *increases with* k_a*.*

The only difference from the constant $\frac{k_a}{k_m}$ case is that w_l increases when k_a rises with k_m unchanged. As before, with improved machine qualities, c_m and $c_l(a)$ decrease and a^* increases, that is, workers shift to more analytical and, for unskilled workers, harder-to-codify tasks, and earnings of skilled workers, earnings inequality $\frac{w_h}{w_l}$, and output rise.

The next proposition examines the case $c_m \in \left[\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}, \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}\right) \Leftrightarrow c^* < c_a = 1$, which is possible only when $\frac{k_a}{k_m} < \frac{h}{l_m}$.

Proposition 5 *When* $c_m \in \left[\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}, \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}\right) \Leftrightarrow c^* < c_a = 1$ *(possible only when* $\frac{k_a}{k_m} < \frac{h}{l_m}$*),*

(i) c_m *decreases with* k_m *and* k_a*.* a^* *increases when* $\frac{k_a}{k_m}$ *non-increases.*

(ii) $c_l(a)$ *and* $c_h(a)$ *decrease with* k_m *and* k_a*.*

(iii) w_h *and* Y *increase with* k_m *and* k_a*, while* w_l *increases with* k_a*.* $\frac{w_h}{w_l}$ *increases when* $\frac{k_a}{k_m}$ *non-increases.*

There are several differences from the constant $\frac{k_a}{k_m}$ case. First, effects of productivity growth with increasing $\frac{k_a}{k_m}$ on a^* and earnings inequality are ambiguous, and w_l increases with k_a. Second, although $c_l(a)$ (thus c_m) and $c_h(a)$ decrease as in the original case and thus workers shift to harder-to-routinize tasks, workers may not shift to more analytical tasks when a^* decreases (possible only when $\frac{k_a}{k_m}$ increases) and when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ (see Figure 12 (a)). Remaining results are same as before, that is, workers shift to more analytical and harder-to-codify tasks and earnings inequality rises when $\frac{k_a}{k_m}$ non-increases (the shift of unskilled workers to more analytical tasks is true when $\frac{k_a}{k_m} \leq \frac{l_a}{l_m}$ too holds), and earnings of skilled workers and output rise.

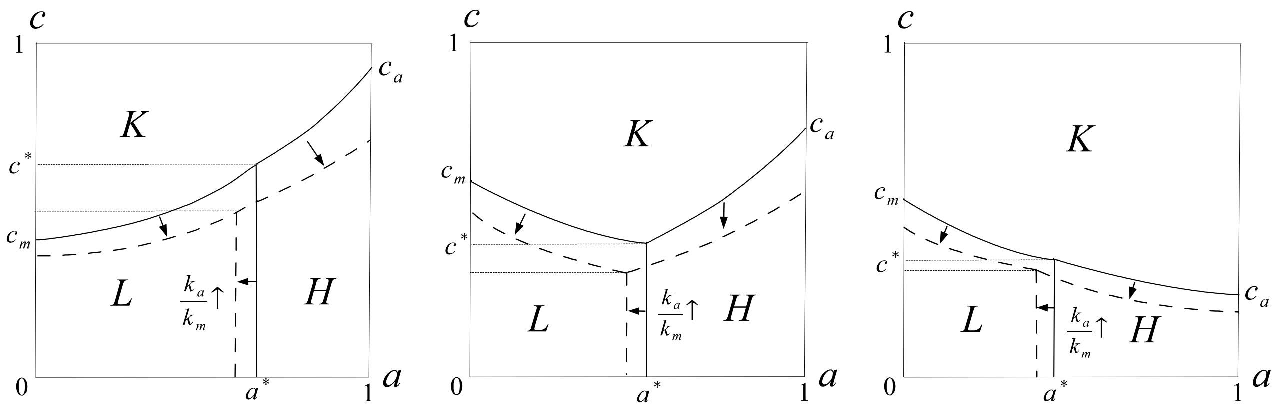
Proposition 6 examines the case $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow c^*, c_a < 1$ ($c^* < (>) c_a$ when $\frac{k_a}{k_m} < (>) \frac{h}{l_m}$).

Proposition 6 *When* $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow c^*, c_a < 1$ *(*$c^* < (>) c_a$ *when* $\frac{k_a}{k_m} < (>) \frac{h}{l_m}$*),*

(i) c_m *and* c_a *decrease with* k_m *and* k_a*, and* a^* *decreases with* $\frac{k_a}{k_m}$*.*

(ii) $c_l(a)$ *and* $c_h(a)$ *decrease with* k_m *and* k_a*.*

(iii) w_h *and* Y *increase with* k_m *and* k_a*, while* w_l *increases when* $\frac{k_a}{k_m}$ *non-decreases.* $\frac{w_h}{w_l}$ *decreases with* $\frac{k_a}{k_m}$*.*



(a) when $\frac{k_a}{k_m} < \frac{l_a}{l_m} (< \frac{h}{l_m})$ (b) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ (c) when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$

Figure 13: Effect of productivity growth with increasing $\frac{k_a}{k_m}$ when $c_m < c^* < c_a < 1$

Unlike the constant $\frac{k_a}{k_m}$ case, a^* and thus $\frac{w_h}{w_l}$ decrease with $\frac{k_a}{k_m}$, and the effect on w_l is ambiguous when $\frac{k_a}{k_m}$ decreases. As for task assignment, while $c_l(a)$ (thus c_m) and $c_h(a)$ decrease as in the original case (thus workers shift to harder-to-routinize tasks), tasks performed by humans change in the skill dimension as well. In particular, when $\frac{k_a}{k_m}$ rises (falls), that is, when productivity growth is such that comparative advantages of machines to humans in analytical (manual) tasks increase, unskilled workers shift to more *manual* (analytical) tasks under $\frac{k_a}{k_m} > (<) \frac{l_a}{l_m}$, and skilled workers too shift to such tasks under $\frac{k_a}{k_m} > (<) \frac{h}{l_m}$.[21] Figure 13 illustrates the effect of productivity growth with increasing $\frac{k_a}{k_m}$ on task assignment. Earnings of skilled workers and output rise as before.

Finally, Proposition 7 examines effects of an increase in $\frac{N_h}{N_l}$ when $\frac{k_a}{k_m} \geq \frac{l_a}{l_m}$ is allowed.

Proposition 7 *Suppose that $\frac{N_h}{N_l}$ grows over time when $c_m < 1$.*

(i) c_m, a^*, and $c_l(a)$ decrease, while c_a (when $c_a < 1$) and $c_h(a)$ (when $c^* < 1$) increase over time. c^* (when $c^* < 1$) falls (rises) when $\frac{k_a}{k_m} \leq \frac{l_a}{l_m}$ ($\frac{k_a}{k_m} \geq \frac{h}{l_m}$).

(ii) w_l (w_h) rises (falls) and $\frac{w_h}{w_l}$ shrinks over time.

(iii) Y increases over time under constant $N_h + N_l$.

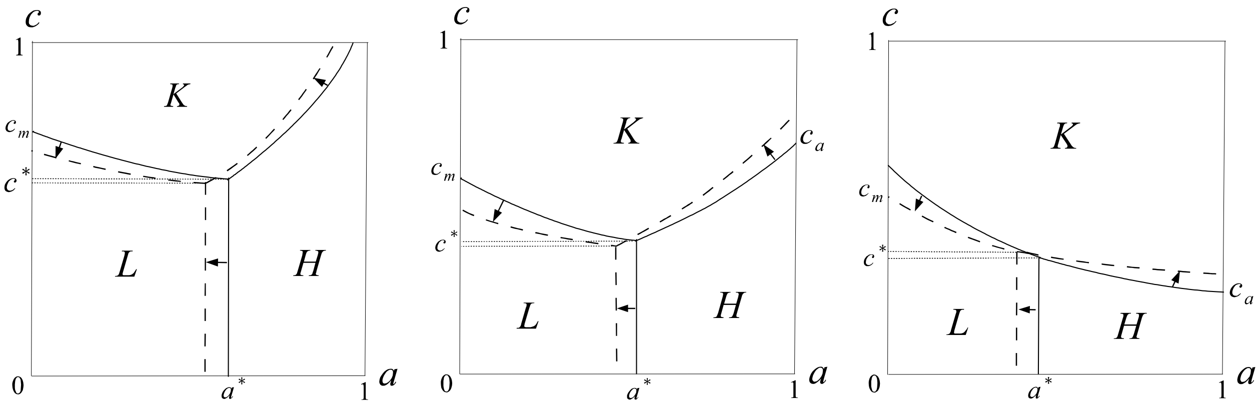
Figure 14 illustrates the effect of an increase in $\frac{N_h}{N_l}$ on task assignment when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and when $\frac{k_a}{k_m} > \frac{h}{l_m}$. (Note that $c^* = c_a = 1$ does not arise in these cases and $c^* < c_a = 1$ does not arise when $\frac{k_a}{k_m} > \frac{h}{l_m}$.) As in the original case of $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, skilled workers take over some tasks

---

[19] When $\frac{k_a}{k_m} > \frac{l_a}{l_m}$, $c_m = 1$ is possible with c^* or $c_a < 1$. However, such situation −the most manual and easy-to-codify task is not mechanized while some of other tasks are − is unrealistic and thus is not examined.

[20] As mentioned in the introduction, proofs of these propositions and Proposition 7 are very lengthy and thus are posted on the author's web site.

[21] When $\frac{k_a}{k_m}$ rises (falls) under $\frac{k_a}{k_m} < (>) \frac{l_a}{l_m}$, unskilled workers shift to more manual (analytical) tasks at low c. The same is true for skilled workers under $\frac{k_a}{k_m} < (>) \frac{h}{l_m}$. At high c, when $\frac{k_a}{k_m}$ rises (falls) under $\frac{k_a}{k_m} < \frac{h}{l_m}$ ($\frac{k_a}{k_m} > \frac{l_a}{l_m}$), skilled (unskilled) workers shift to more analytical (manual) tasks. (See Figure 13.)



(a) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and $c_m < c^* < c_a = 1$

(b) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and $c_m < c^* < c_a < 1$

(c) when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$ and $c_m < c^* < c_a < 1$

Figure 14: Effect of an increase in $\frac{N_h}{N_l}$ on task assignment when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$

previously performed by unskilled workers, i.e. a^* decreases, and machines (skilled workers) come to perform a portion of tasks performed by unskilled workers (machines) before, i.e. $c_l(a)$ decreases ($c_h(a)$ increases). However, unlike before, $c_l(a)$ is downward-sloping on the (a, c) plane, and, when $\frac{k_a}{k_m} > \frac{h}{l_m}$, $c_h(a)$ too is downward-sloping. Thus, unskilled workers shift to harder-to-routinize *and* more manual tasks and, when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$, skilled workers shift to more manual tasks (see the figure). As in the original case, earnings of unskilled (skilled) workers rise (fall), earnings inequality shrinks, and output increases.

Based on the propositions, it is examined whether the model with general productivity growth can explain long-run trends of task shifts, earnings, and earnings inequality in real economy. Since the proportion of tasks performed by machines seems to have been and be higher in more manual tasks (consider, for example, the low proportion in non-routine analytical tasks mainly associated with management, professional, and technical jobs), it would be plausible to suppose that $\frac{k_a}{k_m} < \frac{l_a}{l_m} (< \frac{h}{l_m})$ has continued to hold until now, although it may change in future (thus $c_l(a)$ and $c_h(a)$ are downward-sloping on the (a, c) plane). Judging from the history of mechanization and task shifts described in the introduction, k_m seems to have grown faster than k_a in most periods of time until around the early 1990s, after which the growth of k_a appears to be faster due to the rapid growth of practical applications of IT technologies.[22,23] Thus, suppose that $\frac{k_a}{k_m}$ falls over time when $c_a = 1$, while, when $c_a < 1$, $\frac{k_a}{k_m}$ falls initially, then rises over time.

---

[22] It is true that several components of the "composite" analytical ability k_a, such as simple calculation, seems to have risen faster than the "composite" manual ability k_m since earlier periods, but remaining components seem to have grown slowly until recently.

[23] The supposed turning point would be not be far off the mark considering that a decrease in the employment share of production occupations, which are intensive in manual tasks, is greatest in the 1980s and slowed down considerably after the 1990s, while a decrease in the share of clerical occupations accelerated after the 1990s, according to Acemoglu and Autor (2011).

First, the dynamics of earnings and earnings inequality are examined. Since the result when $c^* = c_a = 1$ is almost the same as the constant $\frac{k_a}{k_m}$ case, the model is consistent with the actual trends in the early stage of mechanization. It accords with the trends in the intermediate stage (when $c^* = c_a < 1$) as well (except a sharp decline of the inequality in the wartime 1940s), because the result is same as before when $\frac{k_a}{k_m}$ falls. Further, unlike the constant $\frac{k_a}{k_m}$ case, the model could explain stagnant earnings of unskilled workers in the 1980s and the early 1990s and the large inequality rise in the 1980s, because the effect of productivity growth with decreasing $\frac{k_a}{k_m}$ on their earnings is ambiguous and that on the inequality is positive when $c^* < c_a < 1$ (and the growth of $\frac{N_h}{N_l}$ slowed down during the period). When $\frac{k_a}{k_m}$ rises under $c^* < c_a < 1$, earnings of unskilled workers too grow, which is consistent with the development in the 1990s and the early 2000s.[24] Although the model with two types of workers cannot explain wage polarization after the 1990s, the falling inequality predicted by the model may capture a part of the development, the shrinking inequality between low-skill and middle-skill workers (most recently, mildly high-skill workers as well).

As for the dynamics of task shifts, the result under $c^* = c_a = 1$ is same as the constant $\frac{k_a}{k_m}$ case, and so is the result under $c^* < c_a = 1$ when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ and $\frac{k_a}{k_m}$ falls: $c_l(a)$ and $c_h(a)$ decrease, and a^* increase over time, unless $\frac{N_h}{N_l}$ grows very strongly for $c_h(a)$ and a^*. Hence, the dynamics accord with the long-run trend until recently, i.e. workers shift to more analytical and harder-to-routinize tasks over time. By contrast, when $c^* < c_a < 1$, while $c_l(a)$ and $c_h(a)$ decrease over time (unless the growth of $\frac{N_h}{N_l}$ is very strong for $c_h(a)$) as before, unlike the constant $\frac{k_a}{k_m}$ case, a^* increases (decreases) when $\frac{k_a}{k_m}$ falls (rises). Hence, workers shift to more analytical and harder-to-routinize tasks while $\frac{k_a}{k_m}$ falls, whereas after $\frac{k_a}{k_m}$ starts to rise, they shift to harder-to-codify tasks overall and shift to more *manual* tasks at low c (footnote 21), which may be consistent with the fact that the shift to non-routine manual tasks in services increased after the 1990s (footnote 15), although the model cannot explain recent job polarization.

To summarize, the model with realistic productivity growth could explain long-run trends of task shifts, earnings, and earnings inequality, except job and wage polarization after the 1990s and a sharp decline of the inequality in the wartime 1940s.

If the rapid progress of IT technologies continues and $\frac{k_a}{k_m}$ keeps rising, comparative advantages of machines to two type of workers could change over time, i.e. first, from $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ to $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$, then to $\frac{k_a}{k_m} > \frac{h}{l_m}$. The model predicts what will happen to task assignment, earnings, and earnings inequality under such situations. As before, both types of workers shift to tasks that are more difficult to routinize (unless $\frac{N_h}{N_l}$ rises greatly, which is very unlikely). By contrast, unlike before, unskilled workers shift to more *manual* tasks, and, when $\frac{k_a}{k_m} > \frac{h}{l_m}$ (and $\frac{N_h}{N_l}$ does not grow strongly), skilled workers too shift to such tasks (see Figures 13 and 14). That is, workers will shift to relatively manual and difficult-to-codify tasks: the growth of service occupations such as personal care and protective service may continue into

---

[24] According to Acemoglu and Autor (2011), real wages of full-time workers of all education groups exhibited sound growth in the late 1990s and in the early 2000s in the U.S. Earnings growth of low education groups are stronger for females, probably because a higher proportion of them are in growing service occupations. After around 2004, however, earnings of all groups except male workers with post-college education have stagnated.

the future. Earnings of unskilled workers as well as those of skilled workers will rise, and earnings inequality will shrink over time, although the analysis based on the model with two types of workers would not capture the total picture, considering the recent widening inequality between mildly and extremely high-skill workers.

6 Conclusion

Since the Industrial Revolution, mechanization has strongly affected types of tasks humans perform, relative demands for workers of different skill levels, their earnings, earnings inequality, and aggregate output. This paper has developed a Ricardian model of task assignment and examined how improvements of qualities of machines and an increase in the relative supply of skilled workers affect these variables. The analysis has shown that tasks and workers strongly affected by the productivity growth change over time. The model can capture long-run trends of these variables in real economy except job and wage polarization after the 1990s and a sharp decline of the inequality in the wartime 1940s. The model has also been employed to examine possible future trends of these variables when the rapid growth of IT technologies continues.

Several extensions of the model would be fruitful. First, in order to understand recent job and wage polarization and future trends of the variables, the model with more than two type of workers, who are different in levels of analytical ability or ability in non-routine tasks, could be developed. Second, some empirical works find that international trade and offshoring have important effects on earnings inequality,[25] thus it may be interesting to examine effects of these factors and productivity growth jointly.

References

[1] Acemoglu, D. and D. Autor (2011), "Skills, Tasks and Technologies: Implications for Employment and Earnings," in O. Ashenfelter and D. Card, eds., *Handbook of Labor Economics* Volume IV, Part B, Amsterdam: Elsevier, Chapter 12, 1043−1171.

[2] Acemoglu, D. and F. Zilibotti (2001), "Productivity Differences," *Quarterly Journal of Economics* 116, 563−606.

[3] Autor, D., F. Levy and R. Murnane (2002), "Upstairs, Downstairs: Computers and Skills on Two Floors of a Large Bank," *Industrial and Labor Relations Review* 55 (3), 432−447.

[4] Autor, D., F. Levy and R. Murnane (2003), "The Skill Content of Recent Technological Change: An Empirical Exploration," *Quarterly Journal of Economics* 116 (4).

[5] Autor, D., L. Katz, and M. Kearney (2006), "The Polarization of the U.S. Labor Market," *American Economic Review Papers and Proceedings* 96 (2), 189−194.

---

[25] For example, Firpo, Fortin, and Lemieux (2011) find that the effect of offshoring on wage inequality is strong in the 2000s for the U.S. economy.

[6] Bartel, A., C. Ichniowski, and K. Shaw (2007), "How Does Information Technology Affect Productivity? Plant-Level Comparisons of Product Innovation, Process Improvement, and Worker Skills," *Quarterly Journal of Economics* 122(4), 1721−1758.

[7] Costinot, A. and J. Vogel (2010), "Matching and Inequality in the World Economy," *Journal of Political Economy* 118 (4), 747−786.

[8] Feinstein, C. H. (1998), "Pessimism Perpetuated: Real Wages and the Standard of Living in Britain during and after the Industrial Revolution," *Journal of Economic History* 58(3), 625−658.

[9] Firpo, S., N. Fortin, and T. Lemieux (2011), "Occupational Tasks and Changes in the Wage Structure," mimeo, University of British Columbia.

[10] Givon, D. (2006), "Factor Replacement versus Factor Substitution, Mechanization and Asymptotic Harrod Neutrality," mimeo, Hebrew University.

[11] Garicano, L. and E. Rossi-Hansberg (2006), "Organization and Inequality in a Knowledge Economy," *Quarterly Journal of Economics* 121 (4), 1383−1435.

[12] Goldin, C. and L. F. Katz (1998), "The Origins of Technology-Skill Complementarity," *Quarterly Journal of Economics*, 113 (3), 693−732.

[13] Goldin, C. and L. F. Katz (1999), "The Returns to Skill in the United States across the Twentieth Century," NBER Working Paper No. 7126.

[14] Goos, M. and A. Manning (2003), "Lousy and Lovely Jobs: The Rising Polarization of Work in Britain," LSE Center for Economic Performance Discussion Papers 0604.

[15] Goos, M., A. Manning, and A. Salomons (2010), "Explaining Job Polarization in Europe: The Roles of Technological Change, Globalization and Institutions," CEP Discussion Paper 1026.

[16] Grossman, G. M. and E. Rossi-Hansberg (2008), "Trading Tasks: A Simple Theory of Offshoring," *American Economic Review* 98 (5), 1978-97.

[17] Landes, D. (2003), *The Unbound Prometheus: Technical Change and Industrial Development in Western Europe from 1750 to the Present* (2nd ed.), New York: Cambridge University Press.

[18] Mokyr, J. ed. (1985), *The Economics of the Industrial Revolution*, Totowa, N.J.: Rowman and Allanheld.

[19] Mokyr, J. (1999), "The Second Industrial Revolution, 1870−1914" in V. Castronovo, ed., *Storia dell'economia Mondiale*, Rome: Laterza Publishing, pp. 219−245.

[20] OECD (2008), *Growing Unequal? Income Distribution and Poverty in OECD Countries*, Paris: OECD.

[21] Peretto, P. and J. Seater (2008), "Factor-Eliminating Technical Change," mimeo, Duke University.

[22] Sampson, T. (2011), "Assignment Reversals: Trade, Skill Allocation and Wage Inequality," mimeo, Harvard University.

[23] Sattinger, M. (1993), “Assignment Models of the Distribution of Earnings,” Journal of Economic Literature 31, 831−80.

[24] Zeira, J. (1998), ”Workers, Machines, and Economic Growth,” *Quarterly Journal of Economics* 113, 1091−1117.

[25] Zeira, J. (2006), ”Machines as Engines of Growth,” mimeo, Hebrew University.

Appendix: Proofs of Lemmas and Propositions

Proof of Lemma 1. [Derivation of the LHS of the equation]: When $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ and thus $c_m < \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} \Leftrightarrow c^* = c_l(a^*) < 1$, the LHS of (HL) equals $\frac{N_h}{N_l}$ times

$$\int_0^{a^*}\int_0^{c_l(a)}\frac{1}{A_l(a)}dcda = \int_0^{a^*}\frac{c_l(a)}{A_l(a)}da = \frac{k_m}{l_m}c_m\int_0^{a^*}\frac{da}{A_k(a)}. \tag{23}$$

Hence, when $\frac{k_a}{k_m} \neq 1$, the LHS of (HL) equals

$$\frac{N_h}{N_l}\frac{k_m}{l_m}\frac{c_m}{k_m - k_a}\ln\left(\frac{k_m}{A_k(a^*)}\right). \tag{24}$$

Applying l’Hôpital’s rule to the above equation, the LHS of (HL) when $\frac{k_a}{k_m} = 1$ equals

$$\begin{aligned} -\frac{N_h}{N_l}\frac{1}{l_m}\frac{c_m}{\lim_{\frac{k_a}{k_m}\to 1}(1-\frac{k_a}{k_m})}\lim_{\frac{k_a}{k_m}\to 1}\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right) &= \frac{N_h}{N_l}\frac{c_m}{l_m}\lim_{\frac{k_a}{k_m}\to 1}\left(\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}\right) \\ &= \frac{N_h}{N_l}\frac{c_m a^*}{l_m}. \end{aligned} \tag{25}$$

[Derivation of the RHS of the equation]: When $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow c_a = c_h(1) < 1$, the RHS of (HL) is expressed as

$$\int_{a^*}^1\int_0^{c_h(a)}\frac{1}{A_h(a)}dcda = \int_{a^*}^1\frac{c_h(a)}{A_h(a)}da = \frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\int_{a^*}^1\frac{da}{A_k(a)}. \tag{26}$$

Hence, when $\frac{k_a}{k_m} \neq 1$, the RHS of (HL) equals

$$\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m - k_a}\ln\left(\frac{A_k(a^*)}{k_a}\right). \tag{27}$$

By applying l’Hôpital’s rule to the above equation, the LHS of (HL) when $\frac{k_a}{k_m} = 1$ equals

$$\begin{aligned} \frac{A_l(a^*)}{A_h(a^*)}\frac{1}{l_m}\frac{c_m}{\lim_{\frac{k_a}{k_m}\to 1}(1-\frac{k_a}{k_m})}\lim_{\frac{k_a}{k_m}\to 1}\ln\left[a^*+(1-a^*)\frac{k_m}{k_a}\right] &= -\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{l_m}\lim_{\frac{k_a}{k_m}\to 1}\left(\frac{-(1-a^*)(\frac{k_a}{k_m})^{-2}}{a^*+(1-a^*)\frac{k_m}{k_a}}\right) \\ &= \frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{l_m}(1-a^*). \end{aligned} \tag{28}$$

[Relations of a^* satisfying the equation with $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$]: Clearly, a^* satisfying the equation decreases with $\frac{N_h}{N_l}$. Noting that, from (24) and (27), (HL) when $\frac{k_a}{k_m} \neq 1$ can be expressed as

$$\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\left[-\frac{N_h}{N_l}\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right)-\frac{A_l(a^*)}{A_h(a^*)}\ln\left(a^*+(1-a^*)\frac{k_m}{k_a}\right)\right]=0, \tag{29}$$

the derivative of the above equation with respect to $\frac{k_a}{k_m}$ equals

$$\begin{aligned}&\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\left(-\frac{N_h}{N_l}\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}-\frac{A_l(a^*)}{A_h(a^*)}\frac{-(1-a^*)(\frac{k_a}{k_m})^{-2}}{a^*+(1-a^*)\frac{k_m}{k_a}}\right)\\&=\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\frac{k_m}{A_k(a^*)}\left[-\frac{N_h}{N_l}a^*+\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)\frac{k_m}{k_a}\right],\end{aligned} \tag{30}$$

where the expression inside the large bracket can be rewritten as

$$\begin{aligned}&-\frac{N_h}{N_l}a^*+\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)\frac{k_m}{k_a}=\left[\ln\left(\frac{A_k(a^*)}{k_a}\right)\right]^{-1}\frac{N_h}{N_l}\left[-a^*\ln\left(a^*+(1-a^*)\frac{k_m}{k_a}\right)-(1-a^*)\frac{k_m}{k_a}\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right)\right]\\&=\left[\ln\left(\frac{A_k(a^*)}{k_a}\right)\right]^{-1}\frac{N_h}{N_l}\frac{k_m}{k_a}\left[a^*\frac{k_a}{k_m}\ln\left(\frac{k_a}{k_m}\right)-\left(a^*\frac{k_a}{k_m}+1-a^*\right)\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right)\right].\end{aligned} \tag{31}$$

The expression inside the large bracket of the above equation is positive, because the expression equals 0 at $\frac{k_a}{k_m}=1$ and its derivative with respect to $\frac{k_a}{k_m}$ equals

$$a^*\left[\ln\left(\frac{k_a}{k_m}\right)-\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right)\right], \tag{32}$$

which is negative (positive) for $\frac{k_a}{k_m} < (>) 1$. Thus, noting that $\ln\left(\frac{A_k(a^*)}{k_a}\right) > (<) 0$ for $\frac{k_a}{k_m} < (>) 1$, (30) is positive. The derivative of (29) with respect to a^* is positive from $\partial\frac{A_l(a^*)}{A_h(a^*)}/\partial a^* < 0$. Hence, a^* satisfying (17) decreases with $\frac{k_a}{k_m}$ when $\frac{k_a}{k_m} \neq 1$. When $\frac{k_a}{k_m} \to 1$, (30) equals

$$\begin{aligned}&\lim_{\frac{k_a}{k_m}\to 1}\left\{\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{1}{a^*\frac{k_a}{k_m}+1-a^*}\left[-\frac{N_h}{N_l}a^*+\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)\frac{k_m}{k_a}\right]\right\}\\&=-\frac{c_m}{l_m}\lim_{\frac{k_a}{k_m}\to 1}\left\{\frac{-\left(a^*\frac{k_a}{k_m}+1-a^*\right)\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)(\frac{k_a}{k_m})^{-2}-\left(-\frac{N_h}{N_l}a^*+\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)\frac{k_m}{k_a}\right)a^*}{\left(a^*\frac{k_a}{k_m}+1-a^*\right)^2}\right\}\\&=\frac{c_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}(1-a^*) > 0.\end{aligned} \tag{33}$$

where (18) is used to derived the last equality. Hence, the same result holds when $\frac{k_a}{k_m}=1$ as well. ■

Proof of Lemma 2. [Derivation of the equation]: Since $c^* < 1$, the LHS of (HL) equals (24) (when $\frac{k_a}{k_m} \neq 1$) and (25) (when $\frac{k_a}{k_m}=1$) in the proof of Lemma 1.

The RHS of (HL) when $c_a = 1 \Leftrightarrow c_h(1) \geq 1$, $c^* < 1 \Leftrightarrow c_h(a^*) < 1$, and $\frac{k_a}{k_m} \neq 1$ is expressed as

$$\begin{aligned}\int_{a^*}^{c_h^{-1}(1)}\int_0^{c_h(a)}\frac{dcda}{A_h(a)}+\int_{c_h^{-1}(1)}^1\int_0^1\frac{dcda}{A_h(a)}&=\int_{a^*}^{c_h^{-1}(1)}\frac{c_h(a)}{A_h(a)}da+\int_{c_h^{-1}(1)}^1\frac{da}{A_h(a)}\\&=\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\int_{a^*}^{c_h^{-1}(1)}\frac{da}{A_k(a)}+\int_{c_h^{-1}(1)}^1\frac{da}{A_h(a)}\\&=\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m-k_a}\ln\left(\frac{A_k(a^*)}{A_k(c_h^{-1}(1))}\right)+\frac{1}{h-l_m}\ln\left(\frac{h}{A_h(c_h^{-1}(1))}\right),\end{aligned}\tag{34}$$

where $c_h^{-1}(1)$, i.e. the value of a when $c_h(a)=1$, equals, from (1) and (3),

$$\begin{aligned}\frac{A_h(a)}{A_k(a)}=\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{1}{c_m}&\Leftrightarrow a(h-l_m)+l_m=\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{1}{c_m}[-a(k_m-k_a)+k_m]\\&\Leftrightarrow a=\frac{l_m\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}.\end{aligned}\tag{35}$$

Hence, from (34) and

$$\begin{aligned}A_k(c_h^{-1}(1))&=\frac{-l_m\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)(k_m-k_a)+k_m\left[(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\\&=\frac{(hk_m-l_mk_a)c_m}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m},\end{aligned}\tag{36}$$

$$\begin{aligned}A_h(c_h^{-1}(1))&=\frac{l_m\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)(h-l_m)+l_m\left[(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\\&=\frac{\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m},\end{aligned}\tag{37}$$

the RHS of (HL) when $\frac{k_a}{k_m}\neq 1$, equals

$$\frac{1}{h-l_m}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}h\right]+\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m-k_a}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right].\tag{38}$$

By applying l'Hôpital's rule to the above equation, the RHS when $\frac{k_a}{k_m}=1$ equals

$$
\begin{aligned}
&\frac{1}{h-l_m}\lim_{\frac{k_a}{k_m}\to 1}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(h-l_m\frac{k_a}{k_m})}h\right]+\frac{\frac{1}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m}{\lim_{\frac{k_a}{k_m}\to 1}(1-\frac{k_a}{k_m})}\lim_{\frac{k_a}{k_m}\to 1}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(h-l_m\frac{k_a}{k_m})c_m}{a^*\frac{k_a}{k_m}+(1-a^*)}}\right]\\
&=\frac{1}{h-l_m}\ln\left[\frac{h}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\right]-\frac{1}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\lim_{\frac{k_a}{k_m}\to 1}\left[\frac{a^*}{a^*\frac{k_a}{k_m}+(1-a^*)}+\frac{l_m}{h-l_m\frac{k_a}{k_m}}-\frac{l_m\frac{A_h(a^*)}{A_l(a^*)}}{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\right]\\
&=\frac{1}{h-l_m}\ln\left[\frac{h}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\right]-\frac{1}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\left[\frac{A_h(a^*)}{h-l_m}-\frac{l_m\frac{A_h(a^*)}{A_l(a^*)}}{(h-l_m)c_m}\right]\\
&=\frac{1}{h-l_m}\left\{\ln\left[\frac{h}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\right]-\frac{A_l(a^*)}{l_m}c_m+1\right\}. \qquad (39)
\end{aligned}
$$

[Relations of a^* satisfying the equation with $\frac{N_h}{N_l}$ and c_m]: When $\frac{k_a}{k_m}\neq 1$, the derivative of the LHS−RHS of (19) with respect to a^* equals

$$
\begin{aligned}
&\frac{N_h}{N_l}\frac{k_m}{l_m}c_m\frac{1}{A_k(a^*)}+\frac{1}{h-l_m}\left[\frac{1}{\frac{A_h(a^*)}{A_l(a^*)}}-\frac{(k_m-k_a)\frac{l_m}{k_m}}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\right]\frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}+\frac{k_m}{l_m}c_m\frac{A_l(a^*)}{A_h(a^*)}\frac{1}{A_k(a^*)}\\
&\qquad-\frac{c_m\frac{A_l(a^*)}{A_h(a^*)}}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}-\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\\
&=c_m\left[\begin{array}{c}\frac{N_h}{N_l}\frac{k_m}{l_m}\frac{1}{A_k(a^*)}+\frac{\frac{A_l(a^*)}{A_h(a^*)}}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}+\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{1}{A_k(a^*)}\\
-\frac{\frac{A_l(a^*)}{A_h(a^*)}}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}-\frac{k_m}{l_m}\frac{1}{k_m-k_a}\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\end{array}\right]\\
&=\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\left\{\left(\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right)\frac{k_m-k_a}{A_k(a^*)}-\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[1+\frac{(k_m-k_a)\frac{A_h(a^*)}{A_k(a^*)}\left[\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}-c_m\right]}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\right\}>0, \qquad (40)
\end{aligned}
$$

where the last equality is derived by using

$$
\begin{aligned}
\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}&=\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m-\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}+\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\\
&=1+\frac{(k_m-k_a)\frac{A_h(a^*)}{A_k(a^*)}\left[\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}-c_m\right]}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}>(<)1 \text{ when } \frac{k_a}{k_m}<(>)1 \quad (\because c_m<\tfrac{l_m}{k_m}\tfrac{A_k(a^*)}{A_l(a^*)}). \qquad (41)
\end{aligned}
$$

The derivative of the LHS-RHS of (19) with respect to c_m when $\frac{k_a}{k_m}\neq 1$ equals

$$
\frac{1}{(h-l_m)c_m}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}h\right]-\frac{1+\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\frac{A_l(a^*)}{A_h(a^*)}(h-l_m)}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}+\frac{k_m}{l_m}\frac{1}{k_m-k_a}\frac{A_l(a^*)}{A_h(a^*)} \qquad (42)
$$

$$
=\frac{1}{(h-l_m)c_m}\ln\left[1+\frac{(h-l_m)hk_m\left[c_m-\frac{l_m}{h}\frac{k_a}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\right]}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}\right]\geq 0 \quad (\because c_m\geq\tfrac{l_m}{k_m}\tfrac{k_a}{h}\tfrac{A_h(a^*)}{A_l(a^*)}), \qquad (43)
$$

where the last equality is derived by using

$$\frac{(k_m-k_a)l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)k_mc_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}h=\frac{\left[(k_m-k_a)l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)k_mc_m\right]h-l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)+l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}$$

$$=1+\frac{(h-l_m)hk_m\left[c_m-\frac{l_m}{h}\frac{k_a}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\right]}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}. \tag{44}$$

Hence, when $\frac{k_a}{k_m}\neq 1$, a^* satisfying (19) decreases with $\frac{N_h}{N_l}$ and c_m ($\frac{\partial a^*}{\partial c_m}=0$ at $c_m=\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$).

The corresponding derivatives when $\frac{k_a}{k_m}\to 1$ are

$$a^*:\lim_{\frac{k_a}{k_m}\to 1}\left(\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\left\{\left(\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right)\frac{1-\frac{k_a}{k_m}}{a^*\frac{k_a}{k_m}+1-a^*}-\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[1+\frac{(1-\frac{k_a}{k_m})A_h(a^*)\left[(a^*\frac{k_a}{k_m}+1-a^*)\frac{l_m}{A_l(a^*)}-c_m\right]}{(h-l_m\frac{k_a}{k_m})c_m}\right]\right\}\right)$$

$$=-\frac{c_m}{l_m}\lim_{\frac{k_a}{k_m}\to 1}\left\{\begin{aligned}&\left(\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right)\frac{-(a^*\frac{k_a}{k_m}+1-a^*)-(1-\frac{k_a}{k_m})(1-a^*)}{(a^*\frac{k_a}{k_m}+1-a^*)^2}-\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\left[1+\frac{(1-\frac{k_a}{k_m})A_h(a^*)\left[(a^*\frac{k_a}{k_m}+1-a^*)\frac{l_m}{A_l(a^*)}-c_m\right]}{(h-l_m\frac{k_a}{k_m})c_m}\right]^{-1}\\&\times\frac{(h-l_m\frac{k_a}{k_m})A_h(a^*)\left[-\left((a^*\frac{k_a}{k_m}+1-a^*)\frac{l_m}{A_l(a^*)}-c_m\right)+(1-\frac{k_a}{k_m})\frac{(1-a^*)l_m}{A_l(a^*)}\right]+l_m(1-\frac{k_a}{k_m})A_h(a^*)\left[(a^*\frac{k_a}{k_m}+1-a^*)\frac{l_m}{A_l(a^*)}-c_m\right]}{(h-l_m\frac{k_a}{k_m})^2c_m}\end{aligned}\right\}$$

$$=\frac{c_m}{l_m}\left\{\left(\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right)-\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\frac{A_h(a^*)\left(\frac{l_m}{A_l(a^*)}-c_m\right)}{(h-l_m)c_m}\right\}>0, \tag{45}$$

$$c_m:\frac{1}{(h-l_m)c_m}\ln\left[1+\frac{(h-l_m)h\left[c_m-\frac{l_m}{h}\frac{A_h(a^*)}{A_l(a^*)}\right]}{l_m\frac{A_h(a^*)}{A_l(a^*)}(h-l_m)}\right]\geq 0. \tag{46}$$

Therefore, the same results hold when $\frac{k_a}{k_m}=1$ as well.

[Relations of a^* satisfying the equation with $\frac{k_a}{k_m}$]: Since (19) can be expressed as

$$-\frac{N_h}{N_l}\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right) \tag{47}$$

$$=\frac{1}{h-l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(h-l_m\frac{k_a}{k_m})}h\right]+\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{A_l(a^*)}{A_h(a^*)}\ln\left[\frac{a^*\frac{k_a}{k_m}+1-a^*}{h-l_m\frac{k_a}{k_m}}\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{c_m}\right],$$

the derivative of the LHS−RHS of (19) with respect to $\frac{k_a}{k_m}$ when $\frac{k_a}{k_m}\neq 1$ equals

$$-\frac{N_h}{N_l}\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\left[\frac{\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right)}{1-\frac{k_a}{k_m}}+\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}\right]-\frac{1}{l_m}\frac{c_m}{(1-\frac{k_a}{k_m})^2}\frac{A_l(a^*)}{A_h(a^*)}\ln\left[\frac{a^*\frac{k_a}{k_m}+1-a^*}{h-l_m\frac{k_a}{k_m}}\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{c_m}\right]$$

$$+\frac{l_m}{h-l_m}\left[\frac{\frac{A_h(a^*)}{A_l(a^*)}}{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}-\frac{1}{h-l_m\frac{k_a}{k_m}}\right]-\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{A_l(a^*)}{A_h(a^*)}\left[\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}+\frac{l_m}{h-l_m\frac{k_a}{k_m}}-\frac{l_m\frac{A_h(a^*)}{A_l(a^*)}}{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\right]$$

$$=\frac{1}{(h-l_m)(1-\frac{k_a}{k_m})}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(h-l_m\frac{k_a}{k_m})}h\right]-\frac{N_h}{N_l}\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}+\frac{l_m\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)}{(h-l_m\frac{k_a}{k_m})\left[(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]}$$

$$-\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{A_l(a^*)}{A_h(a^*)}\left[\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}-\frac{l_m(h-l_m)\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)}{(h-l_m\frac{k_a}{k_m})\left[(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]}\right]$$

$$=\frac{k_m}{k_m-k_a}\left\{-\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{k_m}{l_m}\frac{a^*}{A_k(a^*)}c_m+\frac{k_m\left(1-c_m\frac{A_l(a^*)}{A_h(a^*)}\right)}{hk_m-l_mk_a}+\frac{1}{h-l_m}\ln\left[\frac{(k_m-k_a)l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)k_mc_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}h\right]\right\}. \tag{48}$$

Since the derivative on (HL) is examined, by substituting (19) into the above equation

$$\frac{k_m}{k_m-k_a}\left\{\begin{array}{c}-\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{k_m}{l_m}\frac{a^*}{A_k(a^*)}c_m+\frac{k_m\left(1-c_m\frac{A_l(a^*)}{A_h(a^*)}\right)}{hk_m-l_mk_a}+\frac{N_h}{N_l}\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\ln\left(\frac{k_m}{A_k(a^*)}\right)\\ -\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m-k_a}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\end{array}\right\}$$

$$=\frac{k_mc_m}{(k_m-k_a)^2}\frac{k_m}{l_m}\left\{\begin{array}{c}\frac{N_h}{N_l}\left[\ln\left(\frac{k_m}{A_k(a^*)}\right)+1-\frac{k_m}{A_k(a^*)}\right]\\ -\frac{A_l(a^*)}{A_h(a^*)}\left[(k_m-k_a)\frac{A_h(a^*)}{A_l(a^*)}\frac{1}{c_m}\frac{A_l(a^*)k_mc_m-l_mA_k(a^*)}{A_k(a^*)(hk_m-l_mk_a)}+\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\right]\end{array}\right\}. \quad (49)$$

The above expression is positive at $c_m=\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ from (30) in the proof of Lemma 1 and is negative at $c_m=\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$ from (60) in the proof of Lemma 3. Further, the derivative of the expression inside the big bracket of the above equation with respect to c_m equals

$$-(k_m-k_a)\frac{1}{c_m^2}\frac{l_m}{hk_m-l_mk_a}-\frac{A_l(a^*)}{A_h(a^*)}\left[\frac{h-l_m}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}-\frac{1}{c_m}\right]$$

$$=\frac{l_m}{k_m}\frac{k_m-k_a}{c_m}\left[-\frac{1}{c_m}\frac{k_m}{hk_m-l_mk_a}+\frac{1}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\right]=-\frac{l_m^2}{k_m}\frac{1}{c_m^2}\frac{(k_m-k_a)^2\left[\frac{A_h(a^*)}{A_l(a^*)}-c_m\right]}{(hk_m-l_mk_a)\left[(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]}, \quad (50)$$

which is negative for $c_m\in\left[\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)},\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}\right]$ from $\frac{A_h(a^*)}{A_l(a^*)}-c_m\geq\frac{A_h(a^*)k_m-l_mA_k(a^*)}{A_l(a^*)k_m}=\frac{(hk_m-l_mk_a)a^*}{A_l(a^*)k_m}>0$ $\left(\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}>\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}\Leftrightarrow\frac{k_a}{k_m}<\frac{h}{l_m}\right)$. Hence, there exists a unique $c_m\in\left(\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)},\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}\right)$ such that (48) is positive (negative) for smaller (greater) c_m.

When $\frac{k_a}{k_m}\to 1$, (48) equals

$$\lim_{\frac{k_a}{k_m}\to 1}\frac{1}{1-\frac{k_a}{k_m}}\left\{-\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{1}{l_m}\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}c_m+\frac{1-c_m\frac{A_l(a^*)}{A_h(a^*)}}{h-l_m\frac{k_a}{k_m}}+\frac{1}{h-l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(h-l_m\frac{k_a}{k_m})}h\right]\right\}$$

$$=-\lim_{\frac{k_a}{k_m}\to 1}\left\{\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{1}{l_m}\frac{a^{*2}c_m}{(a^*\frac{k_a}{k_m}+1-a^*)^2}+\frac{l_m\left(1-c_m\frac{A_l(a^*)}{A_h(a^*)}\right)}{(h-l_m\frac{k_a}{k_m})^2}+\frac{1}{h-l_m}\left[\frac{-l_m\frac{A_h(a^*)}{A_l(a^*)}}{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}+\frac{l_m}{h-l_m\frac{k_a}{k_m}}\right]\right\}$$

$$=-\left\{\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{a^{*2}c_m}{l_m}-\frac{l_m\left(1-c_m\frac{A_l(a^*)}{A_h(a^*)}\right)\left(\frac{1}{c_m}\frac{A_h(a^*)}{A_l(a^*)}-1\right)}{(h-l_m)^2}\right\}. \quad (51)$$

The above expression is positive at $c_m=\frac{l_m}{h}\frac{A_h(a^*)}{A_l(a^*)}$ from (33) in the proof of Lemma 1 and is negative at $c_m=\frac{l_m}{A_l(a^*)}$ from (63) in the proof of Lemma 3. Further, the derivative of the expression with respect to c_m is negative. Hence, the same result holds when $\frac{k_a}{k_m}=1$ as well.

■

Proof of Lemma 3. [Derivation of the equation]: The LHS of (HL) when $c^*=1\Leftrightarrow c_l(a^*)\geq 1$ and $\frac{k_a}{k_m}\neq 1$ equals $\frac{N_h}{N_l}$ times

$$\begin{aligned}\int_0^{c_l^{-1}(1)}\int_0^{c_l(a)}\frac{dcda}{A_l(a)}+\int_{c_l^{-1}(1)}^{a^*}\int_0^1\frac{dcda}{A_l(a)} &= \int_0^{c_l^{-1}(1)}\frac{c_l(a)}{A_l(a)}da+\int_{c_l^{-1}(1)}^{a^*}\frac{da}{A_l(a)}\\ &= \frac{k_m}{l_m}c_m\int_0^{c_l^{-1}(1)}\frac{da}{A_k(a)}+\int_{c_l^{-1}(1)}^{a^*}\frac{da}{A_l(a)}\\ &= \frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\ln\left(\frac{k_m}{A_k(c_l^{-1}(1))}\right)+\frac{1}{l_m-l_a}\ln\left(\frac{A_l(c_l^{-1}(1))}{A_l(a^*)}\right),\end{aligned} \tag{52}$$

where the value of $c_l^{-1}(1)$, i.e. a when $c_l(a)=1$, equals, from (2) and (3),

$$\begin{aligned}\frac{A_l(a)}{A_k(a)}=\frac{l_m}{k_m}\frac{1}{c_m} &\Leftrightarrow -a(l_m-l_a)+l_m=\frac{l_m}{k_m}\frac{1}{c_m}[-a(k_m-k_a)+k_m]\\ &\Leftrightarrow a=\frac{l_m(1-c_m)}{(k_m-k_a)\frac{l_m}{k_m}-(l_m-l_a)c_m}.\end{aligned} \tag{53}$$

Hence, from (52) and

$$\begin{aligned}A_k(c_l^{-1}(1)) &= \frac{-l_m(1-c_m)(k_m-k_a)+k_m\left[(k_m-k_a)\frac{l_m}{k_m}-(l_m-l_a)c_m\right]}{(k_m-k_a)\frac{l_m}{k_m}-(l_m-l_a)c_m}\\ &= \frac{(l_ak_m-l_mk_a)c_m}{(k_m-k_a)\frac{l_m}{k_m}-(l_m-l_a)c_m},\end{aligned} \tag{54}$$

$$\begin{aligned}A_l(c_l^{-1}(1)) &= \frac{-l_m(1-c_m)(l_m-l_a)+l_m\left[(k_m-k_a)\frac{l_m}{k_m}-(l_m-l_a)c_m\right]}{(k_m-k_a)\frac{l_m}{k_m}-(l_m-l_a)c_m}\\ &= \frac{\frac{l_m}{k_m}(l_ak_m-l_mk_a)}{(k_m-k_a)\frac{l_m}{k_m}-(l_m-l_a)c_m},\end{aligned} \tag{55}$$

the LHS of (HL) when $\frac{k_a}{k_m}\neq 1$ equals

$$\frac{N_h}{N_l}\left\{\frac{1}{l_m-l_a}\ln\left[\frac{l_ak_m-l_mk_a}{(k_m-k_a)l_m-(l_m-l_a)k_mc_m}\frac{l_m}{A_l(a^*)}\right]+\frac{k_mc_m}{(k_m-k_a)l_m}\ln\left[\frac{(k_m-k_a)l_m-(l_m-l_a)k_mc_m}{(l_ak_m-l_mk_a)c_m}\right]\right\}. \tag{56}$$

Applying l'Hôpital's rule to the above equation, the LHS of (HL) when $\frac{k_a}{k_m}=1$ equals

$$\begin{aligned}&\frac{N_h}{N_l}\left\{\frac{1}{l_m-l_a}\lim_{\frac{k_a}{k_m}\to 1}\ln\left[\frac{l_a-l_m\frac{k_a}{k_m}}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}\frac{l_m}{A_l(a^*)}\right]+\frac{c_m}{\lim_{\frac{k_a}{k_m}\to 1}(1-\frac{k_a}{k_m})l_m}\lim_{\frac{k_a}{k_m}\to 1}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]\right\}\\ &=\frac{N_h}{N_l}\left\{\frac{1}{l_m-l_a}\ln\left[\frac{l_m}{c_mA_l(a^*)}\right]+c_m\lim_{\frac{k_a}{k_m}\to 1}\left(\frac{1}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}-\frac{1}{l_a-l_m\frac{k_a}{k_m}}\right)\right\}\\ &=\frac{N_h}{N_l}\frac{1}{l_a-l_m}\left\{\ln\left[\frac{c_mA_l(a^*)}{l_m}\right]+1-c_m\right\}.\end{aligned} \tag{57}$$

[$a^*\in(0,1)$ **for any** c_m]**:** $a^*<1$ is obvious from the equation. Since $c_m\geq\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$, $a^*=0$ is possible only at $c_m=1$. However, at $c_m=1$, the equation becomes $\frac{N_h}{N_l}\frac{1}{l_m-l_a}\ln\left(\frac{l_m}{A_l(a^*)}\right)=\frac{1}{h-l_m}\ln\left(\frac{h}{A_h(a^*)}\right)$ and thus $a^*>0$.

[Relations of a^* satisfying the equation with $\frac{N_h}{N_l}$, c_m, and $\frac{k_a}{k_m}$]: Since the derivative of the LHS−RHS of (21) and (22) with respect to a^* equals $\frac{N_h}{N_l}\frac{1}{A_l(a^*)}+\frac{1}{A_h(a^*)} > 0$, a^* satisfying the equation decreases with $\frac{N_h}{N_l}$.

When $\frac{k_a}{k_m}\neq 1$, a^* satisfying (21) decreases with c_m, because the derivative of the expression inside the large curly bracket of (21) with respect to c_m equals

$$\left(1-\frac{(l_m-l_a)k_mc_m}{(k_m-k_a)l_m}\right)\frac{k_m}{(k_m-k_a)l_m-(l_m-l_a)k_mc_m}-\frac{k_m}{(k_m-k_a)l_m}+\frac{k_m}{(k_m-k_a)l_m}\ln\left[\frac{(k_m-k_a)l_m-(l_m-l_a)k_mc_m}{(l_ak_m-l_mk_a)c_m}\right]$$
$$=\frac{1}{(1-\frac{k_a}{k_m})l_m}\ln\left[1+\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}\right] > 0. \qquad (58)$$

$\lim_{c_m\to 1}\frac{\partial a^*}{\partial c_m}=0$ is clear from the above equation.

Since (21) can be expressed as

$$\frac{N_h}{N_l}\left\{\frac{1}{l_m-l_a}\ln\left[\frac{l_a-l_m\frac{k_a}{k_m}}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}\frac{l_m}{A_l(a^*)}\right]+\frac{c_m}{(1-\frac{k_a}{k_m})l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]\right\}$$
$$=\frac{1}{h-l_m}\ln\left(\frac{h}{A_h(a^*)}\right), \qquad (59)$$

when $\frac{k_a}{k_m}\neq 1$, the derivative of the expression inside the large curly bracket of (21) with respect to $\frac{k_a}{k_m}$ equals

$$\frac{\frac{l_m}{l_m-l_a}-\frac{c_m}{1-\frac{k_a}{k_m}}}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}-\frac{\frac{l_m}{l_m-l_a}-\frac{c_m}{1-\frac{k_a}{k_m}}}{l_a-l_m\frac{k_a}{k_m}}+\frac{c_m}{(1-\frac{k_a}{k_m})^2l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]$$
$$=-\left(\frac{l_m}{l_m-l_a}-\frac{c_m}{1-\frac{k_a}{k_m}}\right)\frac{(l_m-l_a)(1-c_m)}{[(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m](l_a-l_m\frac{k_a}{k_m})}+\frac{c_m}{(1-\frac{k_a}{k_m})^2l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]$$
$$=-\frac{c_m}{(1-\frac{k_a}{k_m})^2l_m}\left(\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}-\ln\left[1+\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}\right]\right) < 0. \qquad (60)$$

The derivative is negative because the expression inside the large parenthesis of (60) equals 0 at $c_m=1$ and, when $\frac{k_a}{k_m} < (>)1$, it increases (decreases) with $\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}$ and thus decreases with c_m. Hence, a^* satisfying (21) increases with $\frac{k_a}{k_m}$ when $\frac{k_a}{k_m}\neq 1$. $\lim_{c_m\to 1}\frac{\partial a^*}{\partial\frac{k_a}{k_m}}=0$ is clear from the above equation.

The corresponding derivatives when $\frac{k_a}{k_m}\to 1$ are

$$c_m:\lim_{\frac{k_a}{k_m}\to 1}\left\{\frac{1}{(1-\frac{k_a}{k_m})l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]\right\}=\frac{-1}{l_m}\lim_{\frac{k_a}{k_m}\to 1}\left[\frac{-l_m}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}+\frac{l_m}{l_a-l_m\frac{k_a}{k_m}}\right]$$
$$=\frac{1}{l_a-l_m}\frac{1-c_m}{c_m} > 0. \qquad (61)$$

$$\frac{k_a}{k_m}:\lim_{\frac{k_a}{k_m}\to 1}\left\{-\frac{c_m}{(1-\frac{k_a}{k_m})^2l_m}\left(\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}-\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]\right)\right\}$$
$$=\lim_{\frac{k_a}{k_m}\to 1}\left\{\frac{c_m}{2(1-\frac{k_a}{k_m})l_m}\left(\frac{1-c_m}{c_m}\frac{-(l_a-l_m\frac{k_a}{k_m})l_m+(1-\frac{k_a}{k_m})l_m^2}{(l_a-l_m\frac{k_a}{k_m})^2}-\left[\frac{-l_m}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}+\frac{l_m}{l_a-l_m\frac{k_a}{k_m}}\right]\right)\right\}$$

$$=\frac{c_m}{2l_m}\left(\frac{1-c_m}{c_m}\lim_{\frac{k_a}{k_m}\to 1}\frac{2l_m^2(l_a-l_m)}{(l_a-l_m\frac{k_a}{k_m})^3}+\lim_{\frac{k_a}{k_m}\to 1}\left[\frac{-l_m^2}{[(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m]^2}+\frac{l_m^2}{(l_a-l_m\frac{k_a}{k_m})^2}\right]\right) \tag{62}$$

$$=\frac{c_m}{2l_m}\frac{l_m^2}{(l_a-l_m)^2}\left[2\frac{1-c_m}{c_m}+\left(1-\frac{1}{c_m^2}\right)\right]=-\frac{1}{2}\frac{l_m}{(l_a-l_m)^2}\frac{(1-c_m)^2}{c_m}<0, \tag{63}$$

where $l_a - l_m > 0$ from $\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} < 1 \Leftrightarrow 1 < \frac{l_a}{l_m}$. Therefore, the same results hold when $\frac{k_a}{k_m}=1$ as well. ■

Proof of Lemma 4. [**Relations of** c_m **satisfying** (P) **with** $a^*, k_m, k_a,$ **and** r]: Derivatives of the LHS of (P) with respect to a^*, c_m, k_m, and k_a equal

$$a^*:\ \frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}\frac{l_m}{k_m}\frac{r}{c_m}\int_{a^*}^{1}\int_0^{\min\{c_h(a),1\}}\frac{dcda}{A_h(a)}>0, \tag{64}$$

$$c_m:\ -\frac{l_m}{k_m}\frac{r}{c_m^2}\left\{\int_0^{a^*}\int_0^{\min\{c_l(a),1\}}\frac{dcda}{A_l(a)}+\frac{A_h(a^*)}{A_l(a^*)}\int_{a^*}^{1}\int_0^{\min\{c_h(a),1\}}\frac{dcda}{A_h(a)}\right\}<0, \tag{65}$$

$$\begin{aligned}k_m:\ &-\frac{1}{k_m}\left\{1-r\left[\int_0^{a^*}\int_{\min\{c_l(a),1\}}^{1}\frac{dcda}{cA_k(a)}+\int_{a^*}^{1}\int_{\min\{c_h(a),1\}}^{1}\frac{dadc}{cA_k(a)}\right]\right\}\\&-r\left[\int_0^{a^*}\int_{\min\{c_l(a),1\}}^{1}\frac{(1-a)dcda}{c(A_k(a))^2}+\int_0^{a^*}\int_{\min\{c_l(a),1\}}^{1}\frac{(1-a)dcda}{c(A_k(a))^2}\right]<0,\end{aligned} \tag{66}$$

$$k_a:\ -r\left[\int_0^{a^*}\int_{\min\{c_l(a),1\}}^{1}\frac{adcda}{c(A_k(a))^2}+\int_0^{a^*}\int_{\min\{c_l(a),1\}}^{1}\frac{adcda}{c(A_k(a))^2}\right]<0, \tag{67}$$

where $c_l(a^*) = c_h(a^*) = c^*$, $\frac{1}{c_l(a)A_k(a)} = \frac{l_m}{k_m}\frac{1}{c_m}\frac{1}{A_l(a)}$, and $\frac{1}{c_h(a)A_k(a)} = \frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{1}{c_m}\frac{1}{A_h(a)}$ are used to derive the equations. The results are straightforward from the equations.

[(**P**) **does not hold at** $c_m = 0$]: Noting that $c_l(a) = \frac{k_m}{l_m}\frac{A_l(a)}{A_k(a)}c_m$ and $c_h(a) = \frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{A_h(a)}{A_k(a)}c_m$, when $c_m \to 0$, the LHS of (P) becomes

$$r\int_0^{a^*}\frac{da}{A_k(a)}+r\int_{a^*}^{1}\frac{da}{A_k(a)}+r\int_0^1\int_0^1\frac{dadc}{cA_k(a)}=r\int_0^1\frac{da}{A_k(a)}-\frac{r}{k_m-k_a}\ln\left(\frac{k_m}{k_a}\right)\lim_{c\to 0}\ln c=+\infty>1. \tag{68}$$

Hence, (P) does not hold at $c_m = 0$. ■

Proof of Proposition 1. At $c_m = 1$, $c_l(a)$, $c_h(a) > 1$ from (13), thus (P) equals

$$\frac{l_m}{k_m}r\int_0^{a^*}\frac{da}{A_l(a)}+\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}r\int_{a^*}^{1}\frac{da}{A_h(a)}=1. \tag{69}$$

When k_m is very small, the LHS of the above equation is strictly greater than 1 for any $a^* \in [0,1]$ (thus, (P) does not hold for any c_m and a^* from Lemma 4), or a^* satisfying the equation is weakly smaller than $a^* \in (0,1)$ satisfying (HL) at $c_m = 1$ ($a^* \in (0,1)$ holds on (HL) from Lemma 3). In such case, there is no $a^* \in (0,1)$ and $c_m < 1$ satisfying both (HL) and (P), and thus machines are not employed, i.e. $c_m = 1$, in equilibrium, where equilibrium a^* is determined from (HL) with $c_m = 1$.

When k_m becomes large enough that a^* satisfying (69) is greater than $a^* \in (0,1)$ satisfying (HL) at $c_m = 1$, an equilibrium with $c_m < 1$ exists from shapes of (HL) and (P). The dynamics of c_m and a^* are straightforward from shapes of the two loci. The dynamics of c^* and c_a are from $c^* = \min\left\{\frac{k_m}{l_m}\frac{A_l(a^*)}{A_k(a^*)}c_m, 1\right\}$, $c_a = \min\left\{\frac{h}{k_a}\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m, 1\right\}$, and the assumptions that $\frac{k_a}{k_m}$ is time-invariant and satisfies $\frac{k_a}{k_m} < \frac{l_a}{l_m}$. The dynamics of $c_l(a)$ and $c_h(a)$ are from those of the other variables. ■

Proof of Proposition 2. (i) When $c_m \geq \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$, earnings of skilled workers increase over time from Propositions 4 (iii) and 5 (iii) below. Earnings of both types of workers increase when $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ from Proposition 6 (iii) below. (ii) is straightforward from Proposition 1 and the earnings equations (eqs. 15 and 16).

(iii) Y decreases with the LHS and the RHS of (HL) from (10). When $c^* = c_a = 1$ and $\frac{k_a}{k_m} \neq 1$, the RHS of (HL) equals $\frac{1}{h-l_m}\ln\left(\frac{h}{A_h(a^*)}\right)$ from Lemma 3, which decreases with the growth of k_m and k_a with constant $\frac{k_a}{k_m}$ from Proposition 1. When $c^* < c_a < 1$ and $\frac{k_a}{k_m} \neq 1$, the RHS of (HL) equals $\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m-k_a}\ln\left(\frac{A_k(a^*)}{k_a}\right)$ from (27) in the proof of Lemma 1, which decreases with the productivity growth from Proposition 1. When $c^* < c_a = 1$ and $\frac{k_a}{k_m} \neq 1$, the derivative of the RHS of (HL) with respect to c_m equals, from (43) in the proof of Lemma 2 and (19),

$$\begin{aligned} &-\frac{1}{(h-l_m)c_m}\ln\left[1+\frac{(h-l_m)hk_m\left[c_m-\frac{l_m}{h}\frac{k_a}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\right]}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}\right]+\frac{N_h}{N_l}\frac{k_m}{l_m}\frac{1}{k_m-k_a}\ln\left(\frac{k_m}{A_k(a^*)}\right) \\ &= \frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{1}{k_m-k_a}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right] > 0, \end{aligned} \tag{70}$$

and the derivative with respect to a^* equals, from (40) in the proof of Lemma 2 ,

$$-\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\left\{\frac{A_l(a^*)}{A_h(a^*)}\frac{k_m-k_a}{A_k(a^*)}-\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[1+\frac{(k_m-k_a)\frac{A_h(a^*)}{A_k(a^*)}\left[\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}-c_m\right]}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\right\} < 0. \tag{71}$$

From signs of the derivatives and Proposition 1, the RHS of (HL) decreases with the productivity growth. Hence, Y increases over time when $\frac{k_a}{k_m} \neq 1$. The result when $\frac{k_a}{k_m} = 1$ can be proved similarly. ■

Proof of Proposition 3. Since an increase in $\frac{N_h}{N_l}$ shifts (HL) to the left on the (a^*, c_m) space from Lemmas 1−3, the result that c_m and a^* decrease is straightforward from Figures 7−9. Then, $w_l = \frac{l_m}{k_m}\frac{r}{c_m}$ rises and $\frac{w_h}{w_l} = \frac{A_h(a^*)}{A_l(a^*)}$ falls. Since $c^* \equiv \min\left\{\frac{k_m}{l_m}\frac{A_l(a^*)}{A_k(a^*)}c_m, 1\right\}$, c^* falls when $c^* < 1$ from $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, $\frac{da^*}{d\frac{N_h}{N_l}} < 0$, and $\frac{dc_m}{d\frac{N_h}{N_l}} < 0$. $c_l(a)$ decreases from $\frac{dc_m}{d\frac{N_h}{N_l}} < 0$. Proofs of the results for $c_h(a)$, c_a, w_h, and Y are in the proof of Proposition 7. ■