Uniform boundedness of feasible per capita output streams under convex technology and non-stationary labor

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UNIFORM BOUNDEDNESS OF FEASIBLE PER CAPITA OUTPUT STREAMS UNDER CONVEX TECHNOLOGY AND NON-STATIONARY LABOR

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SUMMARY. This paper shows that under classical assumptions on technological mapping and presence of an indispensable production factor there is a bound on long-term *per capita* production. The bound does not depend on initial state of economy. It is shown that all feasible processes converge uniformly over every bounded set of initial inputs p.c. to some set (dependent on technology).
KEYWORDS: boundedness, convergence, multi-sector growth model.
JEL CLASSIFICATION NUMBERS: C61, O41.

1. INTRODUCTION

In mathematical economics literature the following result is known: under a fixed (positive) growth rate of labor and labor indispensability set of all feasible output per capita (p.c.) starting from a given initial state is compact in the product topology. Moreover, there exists a real number $B(y_0) > 0$, such that for every feasible production sequence $\{y_t\}_{t=0}^{\infty}$ starting form y_0 it holds: $\|y_t\| \leq B(y_0)$, $t = 0, 1, \ldots$.[1] It is possible to strengthen this property. It appears that - whatever initial state of the economy is - in long term the maximal level of production (p.c.) can not exceed some number which characterizes the technology.[2] These properties justify often used assumption (in the context of multiproduct growth models without an *explicit* indispensable factor of production) that the technological space of input-outputs vectors is a compact set or - in a weaker form - that if inputs are above a fixed level then there is not possible to produce greater outputs.[3]

In the paper we analyze a model, in which there is *explicit* labor, though

[1] [2, 9].

[2] Formally: there exists a real number $B' > 0$, such that for any initial state y_0 there exists a period T_{y_0} s.t. $\forall\, t \geq T_{y_0}$ hold inequalities $\|y_t\| < B'$, where y_t is p.c. output in period t in a feasible process starting from y_0 (see theorem 2) below.

[3] See [1, 3, 7].

- which is not a standard approach - we allow to vary its period-to-period level within some interval (depending on initial state of labor).[4] We shall characterize technological mapping with two numbers that describe long term production possibilities in the economy. We shall point at some subsets (dependent on the two mentioned numbers) toward which converge all feasible processes (independently of their initial state). Moreover, we shall evaluate the speed of convergence to one of these sets.[5]
The next section presents notation. Then we introduce the technological mapping. Section 4 presents the model. Part 5 contains the main results of the paper - theorems 2 and 3. The last section is devoted for illustration of introduced definitions.

2. Notation

By $\mathbf{R}^n$ we denote the n-dimensional Euclidean space and $\mathbf{R}^n_+$ is its non-negative orthant. Coordinates of $z \in \mathbf{R}^n$ are z_i, $i = 1, \ldots, n$. For vectors z, $z' \in \mathbf{R}^n$ we write $z \geq z'$, if $z_i \geq z'_i$, $i = 1, \ldots, n$; $z \gneq z'$ means $z \geq z'$ and $z \neq z'$; $z > z'$ is equivalent to $z_i > z'_i$, $i = 1, \ldots, n$. $2^{\mathbf{R}^n_+}$ denotes the family of all subsets of $\mathbf{R}^n_+$. The Euclidean norm of a vector $z \in \mathbf{R}^n$ is denoted by $\|z\|$ i.e. $\|z\| = (z_1^2 + \ldots + z_n^2)^{1/2}$. Symbol $y \in \mathbf{R}^n_+$ stands for output of goods (except labor); $x \in \mathbf{R}^n_+$ represents inputs (but labor). By $l \in \mathbf{R}_+$ we denote labor inputs (population). Pair $(l, x) \in \mathbf{R}_+ \times \mathbf{R}^n_+$ is to be understood as follows: $l \in \mathbf{R}_+$ and $x \in \mathbf{R}^n_+$. $[a, b)$ ((a, b), $[a, b]$) stands for left-hand-side closed interval (open, closed) with endpoints a and b, $a, b \in \mathbf{R}$, $a < b$.

3. Technological mapping

Technological mapping is a multifunction $\Gamma : \mathbf{R}_+ \times \mathbf{R}^n_+ \to 2^{\mathbf{R}^n_+}$ which takes inputs (l, x) to all technologically possible outputs y (within a fixed time-period and under a given state of technology). We assume that Γ enjoys the following properties:

(i) If $y \in \Gamma(l, x)$ and $y' \in \Gamma(l', x')$ then $y + y' \in \Gamma(l + l', x + x')$.
(ii) $\forall (l, x) \geq 0 \, \forall \lambda \geq 0$:

$$y \in \Gamma(l, x) \Rightarrow \lambda \in \Gamma(\lambda l, \lambda x).$$

[4] Obviously there are papers where analyzed models allow for non exponential growth of labor force (population), for example - non-stationary models, see [7], but they tended to describe properties of *optimal* (in a way) processes.

[5] More detailed analysis is contained in [4]. It is shown therein, that all feasible processes (under sufficiently strong assumptions) are almost always in a neighborhood of so called quasi-reccurent points. However those results do not imply ours.

(iii) $\forall (l', x') \geq (l, x) \geq 0 : \qquad y \in \Gamma(l, x) \Rightarrow y \in \Gamma(l', x')$.
(iv) $y \in \Gamma(l, x) \wedge y \geq y' \geq 0 \Rightarrow y' \in \Gamma(l, x)$.
(v) $l = 0 \wedge y \in \Gamma(l, x) \Rightarrow y = 0$.
(vi) If $\forall q \in \{1, 2, \ldots\}\, (l^q, x^q) \in \mathbf{R}_+ \times \mathbf{R}^n_+, y^q \in \Gamma(l^q, x^q)$ i $l^q \xrightarrow{q} l$, $x^q \xrightarrow{q} x$, $y^q \xrightarrow{q} y$ then $y \in \Gamma(l, x)$.

The above assumptions are standard. From i, ii, vi we conclude that technological space (production set) $Z := \{(y, l, x) \in \mathbf{R}^n_+ \times \mathbf{R}_+ \times \mathbf{R}^n_+ : y \in \Gamma(l, x)\}$ is a closed convex cone contained in non-negative orthant of $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$. Conditions iii, iv imply that costless waste is possible. Assumption v states that labor input is essential (indispensable) in production process.

4. Dynamics

Le us fix initially available output $\overline{y} \in \mathbf{R}^n_+$ and labor $\bar{l} \in \mathbf{R}_+$. Fix also some real numbers $\beta_u \geq 1, \beta_l \in (0, 1)$ - they will be interpreted as maximal and minimal change-rate of population (labor) within a period, respectively. We assume that time is discrete.
A sequence $\{y_t\}_{t=0}^\infty \subset \mathbf{R}^n_+$ (denoted by y^F) is called $(\bar{l}, \overline{y})$-feasible process,[6] if there exist sequences $\{x_t\}_{t=0}^\infty \subset \mathbf{R}^n_+$, $\{l_t\}_{t=0}^\infty \subset \mathbf{R}_+$ such that it holds

$$\begin{array}{ll} y_{t+1} \in \Gamma(l_t, x_t), & \\ y_t - x_t \geq 0, & t = 0, 1, 2, \ldots, \\ l_{t+1} \in [\beta_l l_t, \beta_u l_t], & \\ y_0 = \overline{y},\ l_0 = \bar{l}. & \end{array} \tag{1}$$

The sequences $\{x_t\}_{t=0}^\infty \subset \mathbf{R}^n_+$, $\{l_t\}_{t=0}^\infty \subset \mathbf{R}_+$ s.t. together with y^F 1 is satisfied are said to be inputs and labor trajectory, respectively, with respect to (w.r.t.) process y^F. Set of all $(\bar{l}, \overline{y})$-feasible processes is denoted by $F(\bar{l}, \overline{y})$.
System (1) is typical description of dynamics in the context of multiproduct economy, though labor dynamics is non-standard.[7] We do not assume a constant growth rate of labor. Instead we take a dependence of labor in the consecutive periods in form of inclusion $l_{t+1} \in [\beta_l l_t, \beta_u l_t], t = 0, 1, \ldots$ (which seems to be 'natural' under stable environment). This inclusion puts limits on period-to-period changes of population, but it does not impose any particular labor trajectory with respect to process y^F, so that it is possible that there exist continuum

[6] If it is unambiguous we simply write that y^F is feasible process.
[7] Compare with [2].

labor trajectories w.r.t. a given y^F. The same can be said about inputs trajectories. Such an approach makes it possible to investigate properties of all feasible production processes without a reference to a particular labor or inputs trajectory.

5. Results

We shall show that $(\bar{l}, \overline{y})$-feasible processes are bounded (in per capita terms). We shall also characterize technological mapping Γ by numbers $M^{\sup}$ and $\overline{\Lambda}$ (to be defined below). Number $M^{\sup}$ is an upper bound on production levels p.c. (measured by norm of production vector) achievable in long-term (independently on initial state). The number $\overline{\Lambda}$ is the minimal level of production inputs p.c. (measured by norm) such that the corresponding outputs p.c. is lower than $\overline{\Lambda}$.[8]

5.1. Uniform convergence of feasible processes.

Suppose that y^F is a feasible process under initial inputs $(\bar{l}, \overline{y}) \geq 0$. By definition of feasibility there exists trajectories of inputs $\{x_t\}_{t=0}^{\infty} \subset \mathbf{R}_+^n$ and labor $\{l_t\}_{t=0}^{\infty} \subset \mathbf{R}_+$ w.r.t. y^F, for which system (1) holds. Since it is true that $y_t \geq x_t$ and $\beta_u^{-1} \leq l_t/l_{t+1} \leq \beta_l^{-1}$, for $t = 0, 1, 2, \ldots$, then by conditions ii, iii:

$$y_{t+1}/l_{t+1} \in \Gamma(\beta_l^{-1}, \beta_l^{-1} y_t/l_t), \quad t = 0, 1 \ldots . \tag{2}$$

For every pair of initial inputs $(\bar{l}, \overline{y}) \in \mathbf{R}_+ \times \mathbf{R}_+^n, \bar{l} > 0$ define a number $M(\bar{l}, \overline{y})$ as follows:

$$M(\bar{l}, \overline{y}) := \sup\{\sup_t \|y_t/l_t\| \mid \{y_t\}_{t=0}^{\infty} \in F(\bar{l}, \overline{y}) \text{ for some trajectories of labor } \{l_t\}_{t=0}^{\infty} \text{ and inputs } \{x_t\}_{t=0}^{\infty} \text{ w.r.t. } y^F\}. \tag{3}$$

If $\bar{l} = 0$, then $M(\bar{l}, \overline{y}) := 0$ for all $\overline{y} \in \mathbf{R}_+^n$. The number $M(\bar{l}, \overline{y})$ is an upper bound on output p.c. that can be achieved by any feasible process starting from $(\bar{l}, \overline{y})$.

Theorem 1. $\forall (\bar{l}, \overline{y}) \in \mathbf{R}_+ \times \mathbf{R}_+^n : M(\bar{l}, \overline{y}) < \infty$.[9]

Proof. Suppose that there exists an initial state $(\bar{l}, \overline{y}) \in \mathbf{R}_+ \times \mathbf{R}_+^n$: $M(\bar{l}, \overline{y}) = \infty$. Then there exists a sequence of feasible processes $\{y^{Fq}\}_{q=1}^{\infty} \in F(\bar{l}, \overline{y})$, $y^{Fq} = \{y_t^q\}_{t=0}^{\infty}$ and labor inputs trajectories $\{l_t^q\}_{t=0}^{\infty}$ with respect to y^{Fq}, $q = 1, 2, \ldots$ such that $\lim_q \sup_t \|y_t^q/l_t^q\| = \infty$, $\sup_t \|y_t^q/l_t^q\| > q$.

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[8] In general, there is no equality between $M^{\sup}$ and $\overline{\Lambda}$, see example 2.

[9] A similar result for a model with constant growth rate of labor is contained in [9, lemma 5.1].

Moreover there exists a sequence of periods $\{t_q\}_{q=1}^{\infty} \subset \{t\}_{t=0}^{\infty}$ for which it holds $\lim_q \|r_q\| = \infty$ and $\|r_q\| \geq \|r'_q\|$, where $r_q = y_{t_q}/l_{t_q}, r'_q = y_{t_q-1}/l_{t_q-1}$, $q = 1, 2, \ldots$. We shall substantiate the last claim. Suppose that we have chosen sequences $\{y^{Fq}\}_{q=1}^{\infty} \in F(\bar{l}, \overline{y})$ and $\{l_t^q\}_{t=0}^{\infty}$ s.t. $\lim_q \sup_t \|y_t^q/l_t^q\| = \infty$, $\sup_t \|y_t^q/l_t^q\| > q$. Define for any $q = 1, 2, \ldots$, number t_q as the earliest period for which inequality $\|y_t^q/l_t^q\| > q$ holds. The the following is true: $\|r'_q\| \leq q$, $q = 1, 2, \ldots$, and $\|r_q\| \geq \|r'_q\|$. By inclusion (2) we get for all $q = 1, 2, \ldots$:

$$r_q \in \Gamma(\beta_l^{-1}, \beta_l^{-1} r'_q),$$

and by assumption ii:

$$r_q/\|r_q\| \in \Gamma(\beta_l^{-1}/\|r_q\|, \beta_l^{-1} r'_q/\|r_q\|).$$

We can assume (choosing a subsequence if necessary) that $\lim_q \beta_l^{-1}/\|r_q\| = 0$ and $\lim_q r_q/\|r_q\| = r$, $\|r\| = 1$. From assumptions v, vi we conclude that $\lim_q \|r'_q\|/\|r_q\| = \infty$ - contradiction with $\|r'_q\|/\|r_q\| \leq 1$ for $q = 1, 2, \ldots$. □

The theorem states that output p.c. in the economy starting from $(\bar{l}, \overline{y})$ can not grow to infinity regardless labor inputs trajectory. Obviously the maximal attained level of production p.c. depends on initial state. Define $\forall \bar{l} > 0 \, \forall \overline{y} \in \mathbf{R}_+^n$:

$$M^{\sup}(\bar{l}, \overline{y}) := \sup\{\limsup_t \|y_t/l_t\| \,|\, \{y_t\}_{t=0}^{\infty} \in F(\bar{l}, \overline{y}) \text{ for some trajectories of labor } \{l_t\}_{t=0}^{\infty}, \text{ and inputs } \{x_t\}_{t=0}^{\infty} \text{ w.r.t. } y^F\}. \quad (4)$$

If $\bar{l} = 0$, then $M^{\sup}(\bar{l}, \overline{y}) := 0$. By formulas (3), (4) it follows $M^{\sup}(\bar{l}, \overline{y}) \leq M(\bar{l}, \overline{y})$ for all $(\bar{l}, \overline{y}) \in \mathbf{R}_+ \times \mathbf{R}_+^n$.
Number $M^{\sup}(\bar{l}, \overline{y})$ is an upper bound on output p.c. in long term.

Theorem 2. $\sup_{(\bar{l}, \overline{y}) \in \mathbf{R}_+ \times \mathbf{R}_+^n} M^{\sup}(\bar{l}, \overline{y}) < \infty$.[10]

Proof. Suppose that the thesis is false. The there exists a sequence $\{(\bar{l}^q, \overline{y}^q)\}_{q=1}^{\infty} \subset \mathbf{R}_+ \times \mathbf{R}_+^n$ s.t. $\lim_q M^{\sup}(\bar{l}^q, \overline{y}^q) = \infty$. Denote $M^q = M^{\sup}(\bar{l}^q, \overline{y}^q)$. By definition of M^q there exists a sequence $y^{Fq} \in F(\bar{l}^q, \overline{y}^q)$, where $y^{Fq} = \{y_t^q\}_{t=0}^{\infty}$, and such a number t_q, that for all $t \geq t_q$ it holds: $M^q + 1/q \geq \|y_t^q/l_t^q\|$, $q = 1, 2, \ldots$, and (w.l.o.g.) for all

[10] Economic interpretation of this theorem is as follows: necessary condition for unlimited growth of production p.c. (consumption p.c.) is technological progress.

$q = 1, 2, \ldots$ there exists another number $t'_q > t_q$ for which hold inequalities $\|y^q_{t'_q}/l^q_{t'_q}\| > M^q - 1/q$ and

$$\frac{\|y^q_{t'_q-1}/l^q_{t'_q-1}\|}{\|y^q_{t'_q}/l^q_{t'_q}\|} \leq \beta_l^{-1}.$$

We shall justify the last inequality. Suppose that there exists an index $q' \in \{1, 2, \ldots\}$ s.t. for all $q \geq q'$ and for all $t' > t_q$ (numbers t_q are chosen as previously): if $\|y^q_{t'}/l^q_{t'}\| > M^q - 1/q$, then the following inequality holds

$$\frac{\|y^q_{t'-1}/l^q_{t'-1}\|}{\|y^q_{t'}/l^q_{t'}\|} > \beta_l^{-1}.$$

Set $r^q = \|y^q_{t'}/l^q_{t'}\|$, $r'_q = \|y^q_{t'-1}/l^q_{t'-1}\|$. For all $q = 1, 2, \ldots$ it holds

$$M^q + 1/q > r'_q \geq \beta_l^{-1} r_q \geq \beta_l^{-1}(M^q - 1/q),$$

which implies: $(1+\beta^{-1})/q > (\beta_l^{-1} - 1)M^q$. But it is contradiction, since $\lim_q M^q = \infty$ and $\beta_l < 1$. So that there exists a sequence $\{y^q_{t'_q}/l^q_{t'_q}\}_{q=0}^{\infty}$, s.t. $\lim_q \|y^q_{t'_q}/l^q_{t'_q}\| = \infty$ and

$$\frac{\|y^q_{t'_q-1}/l^q_{t'_q-1}\|}{\|y^q_{t'_q}/l^q_{t'_q}\|} \leq \beta_l^{-1}.$$

Proceeding further as in the last part of proof of theorem 1 we get contradiction. □

Define

$$M^{\sup} := \sup_{(\bar{l},\bar{y}) \in \mathbf{R}_+ \times \mathbf{R}^n_+} M^{\sup}(\bar{l}, \bar{y}).$$

By theorem 2: $M^{\sup} < \infty$, which means that there is no possibility of keeping production p.c. higher than $M^{\sup} + \epsilon$, $\epsilon > 0$ in an infinite number of periods, i.e. all feasible processes (in per capita terms) are convergent to ϵ-neighborhood of $\{y \in \mathbf{R}^n_+ | \, \|y\| \leq M^{\sup}\}$. But it is not known how fast the processes converge. Is there such a number t_ϵ, hanging on $\epsilon > 0$, starting from which every feasible process (independently from initial state) runs in the above defined ϵ-neighborhood? The next section partially solves this question.

5.2. **Convergence speed of all feasible processes p.c. starting from a bounded initial states set.** We need

Lemma 1. $\exists \lambda \geq 0 \, \forall \alpha \in [\beta_l, \beta_u]$:

$$\forall y \in \mathbf{R}^n_+ \qquad (\|y\| \geq \lambda \, \wedge \, \alpha y' \in \Gamma(1, y)) \Rightarrow \|y'\| < \|y\|.$$

Proof. Suppose: $\forall q = 1, 2, \dots\ \exists \alpha_q \in [\beta_l, \beta_u]\ \exists y_q \in \mathbf{R}^n_+ :\ \|y_q\| \geq q$ and there exists $y'_q \in \mathbf{R}^n_+$ s.t. $\alpha_q y'_q \in \Gamma(1, y_q) \Rightarrow \|y'_q\| \geq \|y_q\|$. W.l.o.g. it holds $\alpha_q y'_q/\|y'_q\| \in \Gamma(1/\|y'_q\|, y_q/\|y'_q\|)$, $q = 1, 2, \dots$ and in the limit $q \to \infty$ (choosing a subsequence if needed) we get $\alpha y' \in \Gamma(0, y)$, where $\|y'\| = 1$, $\alpha \geq \beta_l > 0$ - contradiction with assumption v. □

Let

$$\Lambda := \{\lambda \in \mathbf{R}_+ : \lambda \text{ satisfies the thesis of lemma 1}\}$$

and

$$\overline{\Lambda} := \inf_{\lambda \in \Lambda} \lambda.$$

It is obvious that $\overline{\Lambda}$ is well-defined. The number $\overline{\Lambda}$ is the smallest one, that production level y' stemming from unit input of labor and production inputs y with $\|y\| \leq \overline{\Lambda}$ is not greater (in per capita terms) then initial production inputs independently on labor changes, i.e. $\|y'/l\| \leq \|y\|$ for all $l \in [\beta_l,\ \beta_u]$.

Theorem 3. *Suppose $\emptyset \neq S \subset \mathbf{R}_+ \times \mathbf{R}^n_+$ is a set satisfying the following conditions:*

1. $(l, y) \in S \Rightarrow l > 0$*;*
2. $\{y/l|\,(l, y) \in S\}$ *is bounded.*

Then $\forall \epsilon > 0\, \exists T_S \in \{0, 1, \dots\}\, \forall (\bar{l}, \overline{y}) \in S\, \forall y^F \in F(\bar{l}, \overline{y})\, \forall t > T_S :$

$$\|y_t/l_t\| < \overline{\Lambda} + \epsilon.$$

Proof. Fix S and $\epsilon > 0$ as in the hypothesis. By ii, v, vi it follows $\sup_{(\bar{l},\overline{y}) \in S} M(\bar{l}, \overline{y}) < \infty$, where $M(\bar{l}, \overline{y})$ is given by formula (3). Denote $r = \sup_{(\bar{l},\overline{y}) \in S} M(\bar{l}, \overline{y})$. We know that if $y \in \mathbf{R}^n_+ \wedge \|y\| \leq \overline{\Lambda} + \epsilon$, then $\|y'\| < \overline{\Lambda} + \epsilon$, whenever $\alpha y' \in \Gamma(1, y)$, $\alpha \in [\beta_l, \beta_u]$ (by lemma 1 and assumption iii). If $r \leq \overline{\Lambda} + \epsilon$, then the thesis follows for $T_S = 0$. Suppose $r > \overline{\Lambda} + \epsilon$. Fix $\lambda \geq 0$ and let

$$B_\lambda := \bigcup_{\substack{y \in \mathbf{R}^n_+, \|y\| = \lambda \\ \alpha \in [\beta_u, \beta_l]}} \alpha^{-1} \Gamma(1, y).$$

Since Γ is a continuous mapping,[11] multiplication is a continuous operation, and correspondence $\alpha \mapsto \alpha^{-1}$ is continuous on compact set $[\beta_l, \beta_u]$ and intersection of sphere (of radius λ centered at $0 \in \mathbf{R}^n$) and

[11]Upper semicontinuity follows from closedness of graph of Γ and since it takes bounded sets into bounded sets (which can be easily proved with help of assumptions v and vi). Lower semicontinuity is a consequence of fact that domain (effective) of Γ contains unique extreme point and is closed, and Γ is graph-convex ([6]; see also a theorem and its proof in [5, p. 61]).

$\mathbf{R}^n_+$ is compact, then B_λ compact (see also [8, lemma 4.5]). Moreover, inclusion $y' \in B_\lambda$ holds, if and only if, for some $\alpha \in [\beta_l, \beta_u]$ and $\|y\| = \lambda$ it holds $\alpha y' \in \Gamma(1, y)$. We claim that for any $\lambda \in [\overline{\Lambda}+\epsilon, r]$ there exists a number $\gamma \in [0, 1)$ s.t. $\forall y' \in B_\lambda : \|y'\| \le \gamma\lambda$. In fact, by definition of $\overline{\Lambda}$ for all $y' \in B_\lambda : \|y'\| < \lambda$ and existence of γ follows from compactness of B_λ. Define $\forall \lambda \in [\overline{\Lambda} + \epsilon, r]$ a number $\gamma(\lambda)$ as follows:

$$\gamma(\lambda) := \min\{\gamma \in [0, 1] | \, \forall y' \in B_\lambda \text{ it holds that } \|y\| \le \gamma\}.$$

Let $\overline{\gamma} := \sup_{\lambda \in [\overline{\Lambda}+\epsilon, r]} \gamma(\lambda)$. If there is equality $\overline{\gamma} = 1$, then there exist sequences $\{\lambda_q\}_{q=0}^\infty \subset [\overline{\Lambda} + \epsilon, r]$, $\{y_q\}_{q=0}^\infty \subset \mathbf{R}^n_+$, s.t. $\|y_q\| = \lambda_q$, $y'_q \in B_{\lambda_q}$, $\|y'_q\| = \gamma(\lambda_q)\|y_q\|$ and $\lim_q \lambda_q = \lambda \in [\overline{\Lambda} + \epsilon, r]$, $\lim_q y_q = y, \lim_q y'_q = y'$ and $\lim \gamma(\lambda_q) = 1$. From this we get, by continuity of Γ, that in the limit $q \to \infty$ it holds $\|y'\| = \|y\|$, $y' \in \Gamma(1, y)$, $\|y\| = \lambda$ - but this is not possible by lemma 1, since $\|y\| > \overline{\Lambda}$. So that there exists a number $\overline{\gamma} \in [0, 1)$ s.t. for all $y \in \mathbf{R}^n_+$ that satisfy inclusion $\|y\| \in [\overline{\Lambda} + \epsilon, r]$ it holds:

$$\forall \alpha \in [\beta_l, \beta_u] \, \forall y' \in \alpha^{-1}\Gamma(1, y) : \quad \|y'\| \le \overline{\gamma}\|y\|.$$

Suppose that $(\bar{l}, \overline{y}) \in S$ and $\|\overline{y}/\bar{l}\| > \overline{\Lambda} + \epsilon$. Take any feasible process $y^F \in F(\bar{l}, \overline{y})$ and a labor trajectory $\{l_t\}_{t=0}^\infty$ w.r.t. y^F. If for some $t \in \{0, 1, \ldots\}$ it holds $\|y_t/l_t\| > \overline{\Lambda}+\epsilon$, then $\overline{\gamma}\|y_t/l_t\| \ge \|y_{t+1}/l_{t+1}\|$. So that if $\|y_t/l_t\| > \overline{\Lambda} + \epsilon$, then $\|y_t/l_t\| \le \overline{\gamma}^t\|\overline{y}/\bar{l}\|$, and - since $\forall t \in \{0, 1, \ldots\}$: $\|y_t/l_t\| \le \overline{\gamma}^t r < \infty$ (by definition of r) - then the maximal number of periods t, for which inequality $\|y_t/l_t\| > \overline{\Lambda} + \epsilon$ holds is not greater than $\frac{\ln(\overline{\Lambda}+\epsilon)-\ln r}{\ln \overline{\gamma}}$. This evaluation is valid for all initial states $(\bar{l}, \overline{y}) \in S$, which proves the thesis. □

The theorem implies that if set S of initial inputs p.c. is bounded, then for any number ϵ there exists a period t_ϵ s.t. every output p.c. y_t/l_t achieved in a feasible process starting from S is in ϵ-neighborhood of $\{y \in \mathbf{R}^n_+ | \, \|y\| \le \overline{\Lambda}\}$, for $t \ge t_\epsilon$.
If we apply this theorem to set of initial per capita inputs not greater than $M^{\sup} + \epsilon, \epsilon > 0$ (for $\epsilon \to 0$) we conclude that $\overline{\Lambda} \ge M^{\sup}$.[12]

6. Examples

In example 1 we evaluate numbers $\overline{\Lambda}$, $M^{\sup}$ and show that for validity of theorem 3 the boundedness of initial states p.c. set is necessary. In example 2 it holds that $\overline{\Lambda} > M^{\sup}$.

[12] One can prove that in case of single-good economy ($n = 1$) equality $\overline{\Lambda} = M^{\sup}$ holds - it follows from assumptions iii, iv. In general, there is no equality - see example 2).

Example 1. Define technological mapping $\Gamma : \mathbf{R}_+ \times \mathbf{R}_+ \to 2^{\mathbf{R}_+}$ as follows

$$\Gamma(l,x) := [0, l^\alpha x^{1-\alpha}],$$

where $\alpha \in (0,1)$ is a fixed parameter. It is obvious that Γ satisfies conditions i-vi. Fix $\beta_l \in (0,1)$, $\beta_u \in [1,\infty)$ and let $\bar{l} > 0, \overline{y} \geq 0$. $(\bar{l},\overline{y})-$feasible processes are sequences $\{y_t\}_{t=0}^\infty \subset \mathbf{R}_+$, s.t., there exist labor and input trajectories $\{l_t\}_{t=0}^\infty \subset \mathbf{R}_+$, $\{x_t\}_{t=0}^\infty \subset \mathbf{R}_+$ for which it holds (see (1))

$$\begin{array}{ll} y_{t+1} \in [0, l_t^\alpha x_t^{1-\alpha}], & \\ y_t - x_t \geq 0, & \\ l_{t+1} \in [\beta_l l_t, \beta_u l_t], & t = 0,1,2,\ldots. \\ y_0 = \overline{y},\ l_0 = \bar{l}. & \end{array}$$

By property iv of Γ we get an equivalent definition of feasibility: a sequence $\{y_t\}_{t=0}^\infty \subset \mathbf{R}_+$ is $(\bar{l},\overline{y})-$feasible process if there exists labor trajectory $\{l_t\}_{t=0}^\infty \subset \mathbf{R}_+$ s.t.::

$$\begin{array}{ll} y_{t+1} \in [0, l_t^\alpha y_t^{1-\alpha}], & \\ l_{t+1} \in [\beta_l l_t, \beta_u l_t], & t = 0,1,2,\ldots, \\ y_0 = \overline{y},\ l_0 = \bar{l}. & \end{array} \tag{5}$$

Writing system 5 in p.c. terms we get

$$\begin{array}{ll} \frac{y_{t+1}}{l_{t+1}} \in [0, \frac{l_t}{l_{t+1}}\left(\frac{y_t}{l_t}\right)^{1-\alpha}], & \\ \frac{l_t}{l_{t+1}} \in [\beta_u^{-1}, \beta_l^{-1}], & t = 0,1,2,\ldots. \\ y_0 = \overline{y},\ l_0 = \bar{l}, & \end{array}$$

Obviously, for any initial state $(\bar{l},\overline{y})-$feasible processes for all t satisfy:

$$\frac{y_{t+1}}{l_{t+1}} \leq \beta_l^{-1}\left(\frac{y_t}{l_t}\right)^{1-\alpha},$$

and, by induction (in limit $t \to \infty$) we get

$$\frac{y_{t+1}}{l_{t+1}} \leq \beta_l^{-1}\left(\frac{y_t}{l_t}\right)^{1-\alpha} \leq \ldots \leq \beta_l^{-[1+(1-\alpha)+(1-\alpha)^2+\ldots+(1-\alpha)^t]}\left(\frac{\overline{y}}{\bar{l}}\right)^{(1-\alpha)^{t+1}} \leq \beta_l^{-1/\alpha}.$$

By the above inequality it holds that $M^{\sup} \leq \beta_l^{-1/\alpha}$. In fact the equality $M^{\sup} = \beta_l^{-1/\alpha}$ is valid, since $\{y_t\}_{t=0}^\infty$ is a $(1,\beta_l^{-1})$-feasible process for $y_0 = 1, y_{t+1} = l_t^\alpha y_t^{1-\alpha}$, $l_t = \beta_l^{1/\alpha+t}, t = 0,1,\ldots$ and it holds $\frac{y_t}{l_t} = \beta_l^{-1/\alpha}$, for $t = 0,1,\ldots$.

Moreover $\overline{\Lambda} = M^{\sup}$. If $y = \beta_l^{-1/\alpha}$, then $y \in \beta_l^{-1}\Gamma(1,y) = [0, \beta_l^{-1}y^{1-\alpha}] = [0, \beta_l^{-1/\alpha}]$. On the other hand, if $y > \beta_l^{-1/\alpha}$, then for all $y' \in \beta_l^{-1}\Gamma(1,y) = [0, \beta_l^{-1}y^{1-\alpha}]$ we get $y' \leq \beta_l^{-1} < \beta_l^{-1}y^{1-\alpha}$. Whence we conclude that

$\frac{y'}{y} < 1$ and inequality $\overline{\Lambda} \leq M^{\sup}$ follows. Since it is always true that $\overline{\Lambda} \geq M^{\sup}$, then (in our example) $\overline{\Lambda} = M^{\sup}$.

Now we shall show that 'boundedness assumption' in theorem 3 is necessary for its validity. Take a sequence of initial states defined by $(\bar{l}^q, \overline{y}^q) = (1, q)$, $q = 1, 2, \ldots$ and sequences $y_t^q = q^{(1-\alpha)^t}$, $l_t^q = 1$, $q = 1, 2, \ldots, ,t = 0, 1, \ldots$. Sequences $\{y_t^q\}_{t=0}^{\infty}$ are $(\bar{l}^q, \overline{y}^q)$-feasible, $q = 1, 2, \ldots$, and for all $M > 0$ and all $T \in \{0, 1, \ldots\}$ there exists such an index q that $y_t^q / l_t^q = y_t^q > M$.

Example 2. Define $\Gamma : \mathbf{R}_+ \times \mathbf{R}_+^2 \to 2^{\mathbf{R}_+^2}$ as follows

$$\Gamma(l, x_1, x_2) := [0, l^\alpha (x_1 + x_2)^{1-\alpha}] \times \{0\},$$

where $\alpha \in (0, 1)$. Dynamics p.c. is now given by (see example 1):[13]

$$\begin{array}{ll} \frac{y_1^{t+1}}{l_{t+1}} \in [0, \frac{l_t}{l_{t+1}} \left(\frac{y_t^1 + y_t^2}{l_t} \right)^{1-\alpha}], & t = 0, 1, 2, \ldots, \\ \frac{l_t}{l_{t+1}} \in [\beta_u^{-1}, \beta_l^{-1}], & \\ \frac{y_2^t}{l_t} = 0, & t = 1, 2, \ldots, \\ y_1^0 = \overline{y}_1,\ y_2^0 = \overline{y}_2,\ l_0 = \bar{l}. & \end{array}$$

($y_i^t, i = 1, 2$ denotes production of i-th good in period t). In this case $M^{\sup} = \beta_l^{-1/\alpha}$ (as in example 1), but $\overline{\Lambda} > \beta_l^{-1/\alpha}$. To see this let us take $y = (y_1, y_2) \in \mathbf{R}_+^2$ and $y' = (y_1', y_2') \in \mathbf{R}_+^2$ s.t. $\|y\| = \beta_l^{-1/\alpha}$ and $\gamma y' \in \Gamma(1, y)$, for for a number $\gamma \in [\beta_l, \beta_u]$. It can be shown (using standard optimization techniques) that maximal value of $\|y'\|$ among y'−s satisfying the above conditions is $2^{(1-\alpha)/2} \beta_l^{-1/\alpha}$ and it is achieved for $y' = (2^{(1-\alpha)/2} \beta_l^{-1/\alpha}, 0)$, and $\beta_l y' \in \Gamma(1, y)$, for $y = (2^{-1/2} \beta_l^{-1/\alpha}, 2^{-1/2} \beta_l^{-1/\alpha})$. The following evaluation is true: $\frac{\|y'\|}{\|y\|} = \frac{2^{(1-\alpha)/2} \beta_l^{-1/\alpha}}{\beta_l^{-1/\alpha}} = 2^{(1-\alpha)/2} > 1$. By definition of $\overline{\Lambda}$ we get that $\overline{\Lambda} > \beta_l^{-1/\alpha}$, and therefore $\overline{\Lambda} > M^{\sup}$.

7. Conclusions

In the paper we showed that the set of feasible processes (p.c.) is bounded (theorem 1) and in the 'long period' all feasible processes (p.c.), independently on initial conditions, run (almost always) in a small neighborhood of a specific set (see theorem 2, comments after the theorem and definition of $M^{\sup}$). Moreover we proved that for any bounded set of initial inputs p.c. all the feasible processes (starting from the set) are contained in a small neighborhood of another

[13] We assume that $\bar{l} > 0$.

set which depends solely on the technology; see theorem 3, following comments and definition of $\overline{\Lambda}$) starting from a period (common for all processes). We also presented an example, where there is no equality between $\overline{\Lambda}$ and M^{sup}. All the properties were proved for a model with stationary technology and non-stationary population.[14]

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[14] It seems that theorems 2 and 3 are new. Their proofs are also original.