European Option General First Order Error Formula

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EUROPEAN OPTION GENERAL FIRST ORDER ERROR FORMULA

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ABSTRACT. We study the value of European security derivatives in the Black-Scholes model when the underlying asset ξ is approximated by random walks $\xi^{(n)}$. We obtain an explicit error formula, up to a term of order $\mathcal{O}(n^{-\frac{3}{2}})$, which is valid for general approximating schemes and general payoff functions. We show how this error formula can be used to find random walks $\xi^{(n)}$ for which option values converge at a speed of $\mathcal{O}(n^{-\frac{3}{2}})$.

1. Introduction

1.1. **Motivation.** The problematic of describing and controlling the error for options evaluated under random walk approximations $\{\xi^{(n)}\}$ of a geometric Brownian motion ξ has attracted the attention of several researchers, such as for instance [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25]. Knowledge and control of the error is of obvious interest when evaluating options through random walk approximations. An explicit error formula, up to a error term of order $n^{-\alpha}$, for some $\alpha > 0$, has allowed an "acceleration" of the speed of convergence to an order of $n^{-\alpha}$ in [13] and [11]. In a broader context, the such error formulae is part of understanding how small modelling errors affect option prices, which is intimately related to the important question of option price robustness.

It is common practice to approximate a real valued functions $f(x)$ by its Taylor expansion, which, for f sufficiently regular, is given around a by

$$f(x) = \sum_{k=0}^{N} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt.$$

The first order term $f^{(1)}(a)$ provides a measure of the sensibility of f to small changes of its parameter x around a.

In the case of an option, its value v depend on the distribution of the underlying ξ, and small random/unknown changes in the distribution of ξ induce a "modelling error" in the pricing. One would like to have an

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analogue to Taylor's expansion for the value $v(\xi)$ of an option, seen as a function of the distribution of the underlying ξ, that would help to *price* and perhaps *hedge* the *modelling error.*

In the case of a *binomial schemes approximations* $\{\xi^{(n)}\}$ of the underlying ξ in the Black-Scholes model, such analogue to Taylor's expansion takes the form

$$v(\xi) = v\left(\xi^{(n)}\right) + \sum_{k=1}^{N} v_k\left(\xi^{(n)}\right) n^{-\frac{k}{2}} + \mathcal{O}(n^{-\frac{N+1}{2}}). \tag{1.1}$$

If $N = 2$ then $n^{-\frac{N}{2}} = n^{-1}$, and explicit formulae for the coefficients $v_1\left(\xi^{(n)}\right)$ and $v_2\left(\xi^{(n)}\right)$ provides what we call a *first order error formula.* In Walsh [25] such *first order error formula* is given for general piecewise $C^{(2)}$ payoffs, but only in the specific case where the binomial scheme is the Cox Ross and Rubinstein scheme applied to the discounted process. In Diener and Diener [4], a first order error formula is provided for *General Binomial Schemes*, but only in the specific case where the payoff is a call option. In Diener and Diener [5], this first order error formula is obtained for digital options. This paper fills the obvious gap: we obtain a first error formula which is valid for both general payoffs *and* for general binomial schemes approximations.

Chang and Palmer [2] showed how knowledge of a first order error formula can be used to obtain schemes for which the error is smooth, that is for which the error has the form $cn^{-1} + o\left(n^{-1}\right)$ for some *constant* c. Korn and Müller [13], developed a optimization procedure to minimize the absolute value of this c. We will show here how, using the error formula obtained in this paper, a slight modification of Korn and Müller [13] optimization procedure allows to reach "accelerations" of the convergence to an order of $\mathcal{O}\left(n^{-1.5}\right)$.

An interest of this paper is that, when the payoff of the option is continuously differentiable, our error formula remains valid for *non-binomial schemes approximations* $\{\xi^{(n)}\}$, as long as $\xi^{(n)}_{\frac{T}{n}}$ satisfy some moment conditions P1-P5 given in section 6. Our error formula is derived from a localization of the error and an expansion of the local errors.

1.2. **Main result.** Throughout this paper we assume that $r > 0$ is the (constant) risk free rate and that $\xi = (\xi, \mathcal{F}, E_x)$ is a geometric Brownian motion with volatility σ and drift r under the risk neutral probability. Here $\mathcal{F}$ is the usual filtration and E_x denote the expectation when $\xi_0 = x$.

For all practical purposes, traders are interested in payoff functions which are piecewise smooth. We consider here payoffs h which are piecewise $C^{(3)}$ and for which

$$\left|h^{(\ell)}(x)\right| \leq Q\left(1 + x^p\right) \text{ for } \ell = 0, ..., 3 \text{ and every } x \geq 0 \tag{1.2}$$

for some integer $p \geq 1$ and some real number Q. By piecewise $C^{(3)}$, we mean that there exists a partition $0 < K_1 < ... < \mathsf{K}_N < \infty$ of $[0, \infty)$ and $N + 1$

functions $h_0, ..., h_N \in C^{(3)}$ such that

$$h = h_0 1_{[0,K_1)} + h_1 1_{[K_1,K_2)} + ... + h_N 1_{[K_N,\infty)}.$$

We denote this class of payoffs by $\mathcal{K}_p^{(3)}$. We put a norm $\| \|_p^{(3)}$ on $\mathcal{K}_p^{(3)}$ corresponding to the smallest value of Q for which (1.2) holds. For any integer $m \geq 0$, we define $\mathcal{K}_p^{(m)}$ and $\| \|_p^{(m)}$ analogously.

We want to provide a *first order error formula* when ξ is approximated by binomial schemes $\{\xi^{(n)}\}_{n=1}^{\infty}$ where $\xi^{(n)}$ is a random walk which, at every positive time t in $\frac{T}{n}\mathbb{N}$, has a probability p_n of jumping from its current state $\xi_t^{(n)}$ to the state $\xi_t^{(n)} u_n$, and a probability $1 - p_n$ of jumping to the state $\xi_t^{(n)} d_n$. Risk neutrality requires that

$$p_n \overset{def}{=} \frac{\exp(r\frac{T}{n}) - d_n}{u_n - d_n},$$

and with

$$u_n \overset{def}{=} \exp\left(\sigma\sqrt{\frac{T}{n}} + \lambda\sigma^2\frac{T}{n} + \mu_n\frac{2\sigma}{T}\left(\frac{T}{n}\right)^{\frac{3}{2}}\right),$$

$$d_n \overset{def}{=} \exp\left(-\sigma\sqrt{\frac{T}{n}} + \lambda\sigma^2\frac{T}{n} + \mu_n\frac{2\sigma}{T}\left(\frac{T}{n}\right)^{\frac{3}{2}}\right),$$

where $|\mu_n| \leq \mathcal{L}$ for some one $\mathcal{L}$, one gets fairly general binomial schemes, analogue to those considered in [2] and [13]. We will refer to these schemes as the *flexible CRR scheme*. Because we always assume that $\xi_0^{(n)} = \xi_0$, E_x also denote the expectation when $\xi_0^{(n)} = x$.

Now if h belongs to $\mathcal{K}_p^{(3)}$, then h can be split into a linear combination of digital options and call options plus a function which is continuously differentiable and in $\mathcal{K}_p^{(3)}$. Indeed it is easy to see that

$$h(x) = g(x) + \sum_{\ell=1}^{N} \Delta h(\mathsf{K}_\ell) 1_{[\mathsf{K}_\ell,\infty)}(x) + \sum_{\ell=1}^{N} \Delta h'(\mathsf{K}_\ell) \max(x - \mathsf{K}_\ell, 0) \tag{1.3}$$

where g is $C^{(1)}$ and belongs to $\mathcal{K}_p^{(3)}$. Since error formulae for digital and call options are already known, thanks to [4], [5] and [2], the contribution of this paper is to find the error formula for the $C^{(1)}$ part of h. For the sake of simplicity, we will restrict our exposition to continuous payoff functions h. Given that $\xi_0 = x$, we denote by $Err_T^n(h)(x)$ the error, under the Black-Scholes model, resulting from pricing with a *flexible CRR scheme* $\{\xi_T^{(n)}\}$ a European option with payoff h and maturity T. In other words

$$Err_T^n(h)(x) \overset{def}{=} e^{-rT} E_x(h(\xi)) - e^{-rT} E_x(h(\xi_T^{(n)})).$$

Let $\mathcal{C}_K(z) = \max(z-K,0)$ denotes the payoff of European call option, and set $\mathfrak{d}_1 = \frac{\left(\ln\left(\frac{x}{\mathsf{K}}\right)+\left(r+\frac{1}{2}\sigma^2\right)T\right)}{\sigma\sqrt{T}}$, and $\mathfrak{d}_2 = \mathfrak{d}_1 - \sigma\sqrt{T}$. The following is due to [4] and [2].

Theorem 1 (Call Option First Order Error Formula)**.** *Let* $\{\xi^{(n)}\}$ *be a* flexible CRR scheme. *For every* $x>0$ *the error* $Err_T^n(\mathcal{C}_K)(x)$ *satisfies*

$$Err_T^n(\mathcal{C}_K)(x) = \Lambda_T^n(K,x) + \mathcal{O}\left(n^{-1.5}\right)$$

where $\Lambda_T^n := \Lambda_T^n(K,x)$ *is given by*

(1.4)

$$\Lambda_T^n = \frac{xe^{-0.5\mathfrak{d}_1^2}}{24\sigma\sqrt{2\pi T}}(A+B),$$

$$A = \sigma^2T\left(6+\mathfrak{d}_1^2+\mathfrak{d}_2^2\right) + 12T^2\left(r-\lambda\sigma^2\right)^2 - 4T\left(\mathfrak{d}_1^2-\mathfrak{d}_2^2\right)\left(r-\lambda\sigma^2\right), \tag{1.5}$$

$$B = 48\sigma^2T\left(\mathfrak{f}^{(n)}\right)\left(1+\mathfrak{f}^{(n)}\right), \tag{1.6}$$

(1.7)

$$\mathfrak{f}^{(n)} = frac\left(\frac{1}{2}\left(\ln x - \ln K - \sigma\sqrt{T}\sqrt{n} + \lambda\sigma^2T\right)\frac{\sqrt{n}}{\sigma\sqrt{T}} + \mu_n\right).$$

The following theorem is the main result of this paper. Given a continuous payoff h in $\mathcal{K}_p^{(3)}$, it provides a formula for the error $Err_T^n(h)(x)$.

Theorem 2 (General First Order Error Formula)**.** *Let* $\{\xi^{(n)}\}$ *be a* flexible CRR scheme *and let* $p\geq 1$. *For every continuous* h *in* $\mathcal{K}_p^{(3)}$, *if* $0<\mathsf{K}_1<\ldots<\mathsf{K}_N<\infty$ *defines a partition of* $[0,\infty)$ *for which* h *is* $C^{(1)}$ *on the corresponding closed subintervals then for every* $x\geq 0$,

$$Err_T^n(h)(x) = \frac{\Upsilon_T(h,x) + \sum_{\ell=1}^{N}\Delta h'(\mathsf{K}_\ell)\,\Lambda_T^n(\mathsf{K}_\ell,x)}{n} + \mathcal{O}\left(n^{-1.5}\right) \tag{1.8}$$

where

$$\begin{aligned}\Upsilon_T(h,x) = {} & \left(\frac{1}{2}\Delta_2 - \frac{1}{3}\Delta_3 + \frac{1}{4}\Delta_4\right)e^{-rT}E_x\left(\xi_T^2h''(\xi_T)\right)\\ & + \frac{1}{24}\frac{4\Delta_3-5\Delta_4}{\sigma\sqrt{T}}e^{-rT}E_x\left(\xi_T^2h''(\xi_T)\,\eta_T\left(\tfrac{\xi_T}{x}\right)\right)\\ & + \frac{1}{24}\frac{\Delta_4}{T\sigma^2}e^{-rT}E_x\left(\xi_T^2h''(\xi_T)\left(\eta_T^2\left(\tfrac{\xi_T}{x}\right)-1\right)\right),\end{aligned} \tag{1.9}$$

and

$$\eta_T(z) = \frac{\ln(z) - \left(r-\frac{1}{2}\sigma^2\right)T}{\sqrt{T}\sigma}, \tag{1.10}$$

$$\Delta_2 = -\sigma^4T^2\lambda + \lambda^2\sigma^4T^2 + r^2T^2 + rT^2\sigma^2 + \frac{5}{12}\sigma^4T^2 - 2T^2r\sigma^2\lambda,$$

$$\Delta_3 = 2rT^2\sigma^2 - 2\sigma^4T^2\lambda + 2\sigma^4T^2,$$

$$\Delta_4 = 2\sigma^4T^2.$$

Remark 1. *In Theorem 2, if additionally h is $C^{(1)}$, then the error formula (1.8) remains valid for any approximation scheme satisfying properties P1-P5 of section 6. This is due to the fact that only these properties are used in the proofs. They boil down to moments conditions and therefore, our error formula remains valid under these broad assumptions on the moments of the single step random walk jumps $\xi^{(n)}_{\frac{T}{n}}$.*

Let $\{\xi^{(n)}\}$ be a *flexible CRR scheme and* h as in Theorem 2. For simplicity assume that $N = 1$. We show here how a slight modification of Korn and Müller [13] optimization procedure allows to reach accelerations to an order of $\mathcal{O}\left(n^{-1.5}\right)$. For this purposes only, we assume that, in addition to the risk free rate r, the volatility σ, the maturity T and integer p, the payoff h and the current value of the underlying x are also fixed and considered *constants*. In other words, only λ and $\mathfrak{f}^{(n)}$ are seen as variables. A glimpse at (1.8) reveals that $Err^n_T\left(h\right)\left(x\right)$ can be written as

$$Err^n_T\left(h\right)\left(x\right) = \frac{P\left(\lambda\right) + Q\left(\mathfrak{f}^{(n)}\right)}{n} + \mathcal{O}\left(n^{-1.5}\right), \tag{1.11}$$

where $\mathfrak{f}^{(n)} := \mathfrak{f}^{(n)}\left(\lambda, \mu_n\right)$ is given by (1.7), and where for some *constants* a, b, c, d,

$$P\left(\lambda\right) = a\lambda^2 + b\lambda + c,$$
$$Q\left(\mathfrak{f}^{(n)}\right) = d\left(\mathfrak{f}^{(n)}\right)\left(1 + \mathfrak{f}^{(n)}\right),$$

The following is admittedly a slight extension[1] of Korn and Müller [13] optimization procedure which shows how an optimal can be obtained for our general payoff functions:

(1) Choose a constant $\chi_0 = Q\left(\mu_0\right)$ for some $-1 < \mu_0 < 1$,
(2) Choose λ_0 such that

$$\mathfrak{m}_{\chi_0} \overset{def}{=} \inf_{\lambda}\left|P\left(\lambda\right) + \chi_0\right| = \left|P\left(\lambda_0\right) + \chi_0\right| \tag{1.12}$$

(3) Set $\mu_n := \mu_0 - \mathfrak{f}^{(n)}\left(\lambda_0, 0\right)$ and note that

$$\mathfrak{f}^{(n)}(\lambda_0,\, \mu_n) = frac\left(\mathfrak{f}^{(n)}\left(\lambda_0, 0\right) + \mu_n\right) = \mu_0.$$

Under the binomial scheme with parameters $(\lambda_0,\, \mu_n)$, equation (1.11) can be rewritten as

$$Err^n_T\left(h\right)\left(x\right) = \frac{\mathfrak{m}_{\chi_0}}{n} + \mathcal{O}\left(n^{-\frac{3}{2}}\right).$$

When $\mathfrak{m}_{\chi_0} = 0$, the scheme convergence has been accelerated to an order of $\mathcal{O}(n^{-\frac{3}{2}})$, otherwise, the constant $\mathfrak{m}_{\chi_0}$ has been *optimized*.

[1]The differences with [13] are that we allow $\chi_0 \neq 0$ (which is sometimes necessary to reach $\mathfrak{m}_{\chi_0} = 0$ as in the example provided below) and we use the error formula (1.8) to show that the remainer term is of order $O(n^{-\frac{3}{2}})$, as opposed to $o\left(n^{-1}\right)$ in [13].

1.3. **Example and simulations.**

Example 1 (Simulation and the error formula)**.** *Consider the* classical *CRR scheme, where* $\lambda = \mu_n = 0$*, and the following payoff function*

$$h(z) = \begin{cases} z^2 & 0 \le z \le \mathsf{K} \\ 1 & \mathsf{K} < z \end{cases},$$

which is continuous and belongs to $\mathcal{K}^{(3)}$ *with* $\mathsf{K} = 1$*. It is easy (with* Maple*) to calculate that*

$$\Upsilon_T(h, x) = \left(\frac{1}{2}\Delta_2 - \frac{1}{3}\Delta_3 + \frac{1}{4}\Delta_4\right)\mathcal{A} + \frac{1}{24}\frac{4\Delta_3 - 5\Delta_4}{\sigma\sqrt{T}}\mathcal{B} + \frac{1}{24}\frac{\Delta_4}{T\sigma^2}\mathcal{C},$$

where

$$\mathcal{A} = x^2 e^{T\left(r-\sigma^2\right)}\left(c + e^{\frac{1}{2}a^2}\right),$$

$$\mathcal{B} = x^2 e^{T\left(r-\sigma^2\right)}\left(\frac{-\sqrt{2}d + ac\sqrt{\pi} + a\sqrt{\pi}e^{\frac{1}{2}a^2}}{\sqrt{\pi}}\right),$$

$$\mathcal{C} = x^2 e^{T\left(r-\sigma^2\right)}\left(\frac{-b\sqrt{2}d - a\sqrt{2}d + a^2\sqrt{\pi}c + e^{\frac{1}{2}a^2}a^2\sqrt{\pi}}{\sqrt{\pi}}\right),$$

and $a = 2\sigma\sqrt{T}$*,* $b = \eta_T(\frac{\mathsf{K}}{x})$*,* $c = e^{\frac{1}{2}a^2}\operatorname{erf}(\frac{b-a}{\sqrt{2}})$*,* $d = \exp\left(-\frac{1}{2}b(-2a+b)\right)$*. In figure 1, we set* $r = 0.08$*,* $\sigma = 0.5$*,* $T = 1$ *and* $x = 1.1$ *and, in accordance with Theorem 2,* $n^{1.5}(Err_T^n h(x) - \frac{\Upsilon_T(h)(x)}{n} - \Delta h'(\mathsf{K}_1)\frac{\Lambda^n_{\mathsf{K}_1}(x)}{n})$ *is bounded.*

Example 2 (Optimal Scheme)**.** *We use the same payoff function* h *of example 1, as well as* $r = 0.08$*,* $\sigma = 0.5$*,* $T = 1$ *and* $x = 1.1$*. Hence, everything is fixed except* λ *and* μ_n*. Reusing the formula of example 1, we calculate that the* General First Order Error Formula*, can be rewritten as*

$$Err_T^n(h)(x) = \frac{a\lambda^2 + b\lambda + c + d\left(\mathfrak{f}^{(n)}\right)\left(1 + \mathfrak{f}^{(n)}\right)}{n} + \mathcal{O}\left(n^{-1.5}\right),$$

where

$$a = -0.031544554932975475877, \qquad b = 0.015054127355591099077,$$
$$c = 0.084196334462544764572, \qquad d = -0.73282116693588932807.$$

Choose $\chi_0 = -\frac{d}{4}$*, and note that* $\chi_0 = Q(\mu_0)$ *with* $\mu_0 = -0.5$*. Note also that in equation (1.12),* $\mathfrak{m}_{\chi_0} = 0$ *is achieved with* $\lambda_0 = 1.8896959961364908175$*. Letting* $\mu_n = \mu_0 - \mathfrak{f}^{(n)}(\lambda_0, 0)$*, the flexible scheme* $\xi^{(n)}$ *with parameters* λ_0 *and* μ_n *satisfies* $Err_T^n(h)(x) = \mathcal{O}\left(n^{-1.5}\right)$*. The convergence is illustrated in figure 2.*

1.4. **Settings and notation.** The following contains some assumptions and notation used throughout the remaining of this paper.

- **Constants r, σ, T, p and $\mathcal{L}$:** we study the convergence of options with payoffs h in $\mathcal{K}_p^{(3)}$, where $p \ge 1$ is some integer, when the geometric Brownian motion is approximated by *flexible binomial schemes* $\{\xi^{(n)}\}$, which depend on a parameters λ and μ_n. We suppose that

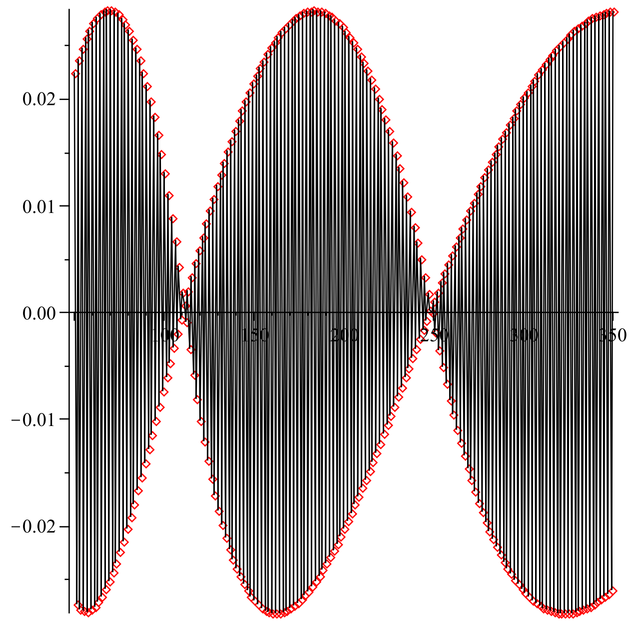

FIGURE 1. The quantity $n^{1.5}(Err_T^n h(x) - \frac{\Upsilon_T(h)(x)}{n} - \Delta h'(\mathsf{K}_1)\frac{\Lambda^n_{\mathsf{K}_1}(x)}{n})$ oscillates rapidly but remains bounded.

$|\mu_n| \leq \mathcal{L}$, for some $\mathcal{L}$. Parameters λ, r, σ, T, p and $\mathcal{L}$ are fixed throughout this paper and expressions in terms of these parameters are considered constants.

Independence of ξ and $\xi^{(n)}$: we assume that ξ and $\xi^{(n)}$ are independent.

Time steps t_m: given n, t_m denotes the m^{th} time step, or in other words, $t_m = m\frac{T}{n}$.

Discounted expectations $\mathcal{E}$ and $\mathcal{E}^n$: for every $t, x \geq 0$ and measurable functions h we denote $\mathcal{E}_t h(x) \stackrel{def}{=} e^{-rt}E_x(h(\xi_t))$ and similarly, $\mathcal{E}^n_t h(x) \stackrel{def}{=} e^{-rt}E_x(h(\xi_t^{(n)}))$. Note that $\mathcal{E}$ and $\mathcal{E}^n$ simply denote the discounted expectation. They are semigroup operator: $\mathcal{E}_{t+s}h = \mathcal{E}_t\mathcal{E}_s h$ and $\mathcal{E}^n_{t+s}h = \mathcal{E}^n_t\mathcal{E}^n_s h$. Because ξ and $\xi^{(n)}$ are independent, $\mathcal{E}$ and $\mathcal{E}^n$ commute: $\mathcal{E}^n_t\mathcal{E}_s h = \mathcal{E}_s\mathcal{E}^n_t h$.

The error Err^n: we denote $Err^n_t(h)(x) \stackrel{def}{=} \mathcal{E}_t h(x) - \mathcal{E}^n_t h(x)$. Whenever possible, we write $Err^n_t f(x)$ instead of $Err^n_t(f)(x)$. Note that operator Err^n commutes with $\mathcal{E}$ and $\mathcal{E}^n$ and therefore with itself.

The identity function I and the symbols $\delta_k^{(n)}$ and $\Delta_k^{(n)}$: the letter I denotes the identity operator: $I(z) \stackrel{def}{=} z$, for every z. *Among*

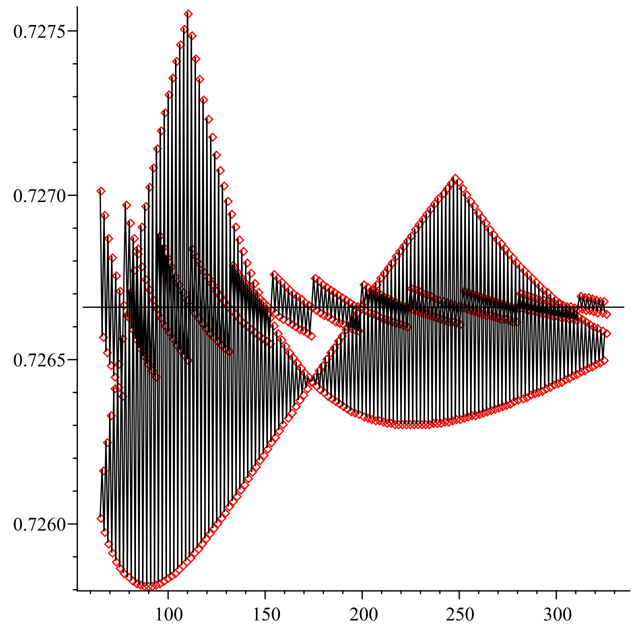

FIGURE 2. The value of the option as a function of n for both the *optimal scheme* and the *classical CRR scheme* of example 2. The horizontal line is the value of the option in the Black-Scholes model.

others, this allows to define expressions such as

$$Err^n_{\frac{T}{n}}\left(\int_1^I g(u)\,du\right)(x) = e^{-r\frac{T}{n}} E_x\left(\int_1^{\xi_{\frac{T}{n}}} g(u)\,du - \int_1^{\xi^{(n)}_{\frac{T}{n}}} g(u)\,du\right),$$

and for any integer $k \geq 0$,

$$\delta_k^{(n)} \overset{def}{=} \mathcal{E}^n_{\frac{T}{n}}\left(|I-1|^k\right)(1) = e^{-r\frac{T}{n}} E_1(|\xi^{(n)}_{\frac{T}{n}} - 1|^k),$$

$$\Delta_k^{(n)} \overset{def}{=} Err^n_{\frac{T}{n}}\left((I-1)^k\right)(1) = e^{-r\frac{T}{n}} E_1((\xi_{\frac{T}{n}} - 1)^k - (\xi^{(n)}_{\frac{T}{n}} - 1)^k).$$

Note that $\Delta_1^{(n)} = 0$ because both ξ and $\xi^{(n)}$ are risk neutral.

A function χ_p on $K_p^{(3)}$: given h in $\mathcal{K}_p^{(3)}$ and a partition $0 < K_1 < \ldots < \mathsf{K}_N < \infty$ of $[0,\infty)$ for which h is $C^{(3)}$ when restricted to the closed intervals defined by this partition, χ_p is defined by:

$$\chi_p(h) \overset{def}{=} \|h\|_p^{(3)} + \sum_{\ell=1}^{N}\sum_{j=0}^{2}\sum_{k=0}^{2} (\mathsf{K}_\ell)^j \left|\Delta h^{(k)}(\mathsf{K}_\ell)\right|. \tag{1.13}$$

1.5. **Proof of the main result.**

Proof. of Theorem 2 Using (1.3) to split the payoff function h into the sum of call options and a continuously differentiable function g in $\mathcal{K}_p^{(3)}$ one obviously gets

$$Err_T^n\,(h)\,(x) = \sum_{\ell=1}^{N} \Delta h'\,(\mathsf{K}_\ell)\,\Lambda_T^n(\mathsf{K}_\ell, x) + Err_T^n\,(g)\,(x)\,.$$

where $\Lambda_T^n(\mathsf{K}, x) = Err_T^n\,(\max{(I-\mathsf{K}, 0)})\,(x)$ is the error for a call option with strike K. Because smooth functions are undoubtedly easier to deal with, we replace g by $\mathcal{E}_{\frac{T}{n}}g$, the option itself evaluated over one single time step. This provides a new *smoothed payoff* which is infinitely differentiable. Obviously this smoothing of g splits the error, $Err_T^n g$, into the sum of two terms: $Err_T^n(g-\mathcal{E}_{\frac{T}{n}}g)$, the error coming from the *payoff smoothing* procedure itself, and $Err_T^n\mathcal{E}_{\frac{T}{n}}g$, the error of the *smoothed payoff.* The fact that the *payoff smoothing error* is negligible, that is of order $\mathcal{O}(n^{-\frac{3}{2}})$, is what Proposition 2 says. As for the *smoothed payoff error*, Theorem 4 says that

$$Err_T^n\left(\mathcal{E}_{\frac{T}{n}}g\right)(x) = \frac{1}{n}\sum_{k=2}^{4}\frac{\Delta_k}{k!}x^k\frac{\partial^k}{\partial x^k}\mathcal{E}_T g\,(x) + \mathcal{O}(n^{-\frac{3}{2}}). \tag{1.14}$$

Using the *representation formulae for the derivatives* $\frac{\partial^k}{\partial x^k}\mathcal{E}_T g\,(x)$, Theorem 5, it is tedious but otherwise completely trivial to rewrites the above as

$$Err_T^n\left(\mathcal{E}_{\frac{T}{n}}g\right)(x) = \Upsilon_T\,(g)\,(x) + \mathcal{O}(n^{-\frac{3}{2}}).$$

Since $g'' = h''$ then $\Upsilon_T\,(g)\,(x) = \Upsilon_T\,(h)\,(x)$, which completes the proof. □

1.6. **Outline of the paper.** To summarize the proof of Theorem 2, finding the *first order error formula* for h in $C^{(0)} \cap \mathcal{K}_p^{(3)}$, hinges around establishing equation (1.14) when g belongs to $C^{(1)} \cap \mathcal{K}_p^{(3)}$, and to use the *representation formulae for the derivatives of European options.*

This paper exhibits how such a formula comes naturally —and in great generality— from a *localization formula* and an *expansion formula* of these *local errors*, used in conjunction with our *representation formulae. Local errors* refer here to errors when the maturity is $\frac{T}{n}$, and *error localization* refers to expressing an error as a sum of *(discounted expected) local errors* and a sum of *errors of local errors.*

We now outline how (1.14) is obtained. Thanks to the *localization formula* (Theorem 3),

$$Err_T^n\mathcal{E}_{\frac{T}{n}}g = \sum_{j=0}^{n-1}\mathcal{E}_{T-t_{j+1}}\left(Err_{\frac{T}{n}}^n\mathcal{E}_{t_j}\mathcal{E}_{\frac{T}{n}}g\right) - \sum_{j=0}^{n-1}Err_{T-t_{j+1}}^n\left(Err_{\frac{T}{n}}^n\mathcal{E}_{t_j}\mathcal{E}_{\frac{T}{n}}g\right).$$

Thus, because Err^n and $\mathcal{E}$ commute, and because $\mathcal{E}$ is a semigroup,

$$Err^n_T\mathcal{E}_T g = \sum_{j=1}^{n} Err^n_{\frac{T}{n}}\mathcal{E}_T g - \sum_{j=1}^{n} Err^n_{T-t_j}(Err^n_{\frac{T}{n}}\mathcal{E}_{t_j} g). \tag{1.15}$$

To avoiding technicalities, let us *temporarily* ignore for this outline, that local errors $Err^n_{\frac{T}{n}}\mathcal{E}_{t_j} g\,(x)$ depend on the payoff $\mathcal{E}_{t_j} g$ and on the initial value x of the underlying, and that the $\mathcal{O}$ terms are not uniform in the payoff and the initial value of the underlying. Then, as pointed out in Remark 3, we can rewrite our *local error expansion formula* for $Err^n_{\frac{T}{n}}\mathcal{E}_{t_j} g$ as

$$Err^n_{\frac{T}{n}}\mathcal{E}_{t_j} g = \frac{1}{n^2}\sum_{k=2}^{4}\frac{\Delta_k}{k!}x^k\frac{\partial^k}{\partial x^k}\mathcal{E}_{t_j} g + \mathcal{O}(n^{-\frac{5}{2}}),$$

from which we obtain the following *error localization expansion formula* for $Err^n_T\mathcal{E}_{\frac{T}{n}} g$,

$$\begin{aligned} Err^n_T\mathcal{E}_{\frac{T}{n}} g = {}& \frac{1}{n}\textstyle\sum_{k=2}^{4}\frac{\Delta_k}{k!}x^k\frac{\partial^k}{\partial x^k}\mathcal{E}_T g \\ & - \sum_{k=2}^{4}\sum_{j=1}^{n}\frac{\Delta_k}{k!n^2}Err^n_{T-t_j}(I^k\frac{\partial^k}{\partial x^k}\mathcal{E}_{t_j} g) + \mathcal{O}(n^{-\frac{3}{2}}). \end{aligned}$$

Therefore if

$$Err^n_{T-t_j}(I^k\frac{\partial^k}{\partial x^k}\mathcal{E}_{t_j} g) = \sqrt{t_j}^{-(k-1)}\mathcal{O}(n^{-1}), \tag{1.16}$$

simple calculations give, as wanted,

$$Err^n_T\mathcal{E}_{\frac{T}{n}} g = \frac{1}{n}\textstyle\sum_{k=2}^{4}\frac{\Delta_k}{k!}x^k\frac{\partial^k}{\partial x^k}\mathcal{E}_T g + \mathcal{O}(n^{-\frac{3}{2}}).$$

To prove (1.16), we use the *European option derivative representation formulae* which shows that, for $s > 0$ and $k = 2, 3, 4$,

$$I^k\frac{\partial^k}{\partial x^k}\mathcal{E}_s g = \sum_{\ell=0}^{k-2} a_{k,\ell}\, s^{-\frac{\ell}{2}}\,\mathfrak{E}^{(\ell)}_s\left(I^2 h''\right),$$

for some constants $a_{k,\ell}$ and some nice and smooth functions $\mathfrak{E}^{(\ell)}_s\left(I^2 h''\right)$. In particular, if t_j is a *long maturity*, meaning that $\frac{T}{2} \leq t_j \leq T$, then the functions $\mathfrak{E}^{(\ell)}_{t_j}\left(I^2 h''\right)$ are as smooth as it can be, yielding the equation $Err^n_{T-t_j}(\mathfrak{E}^{(\ell)}_{t_j}\left(I^2 h''\right)) = \mathcal{O}(n^{-1})$, and therefore

$$\sum_{k=2}^{4}\sum_{\frac{T}{2}\leq t_j\leq T}\frac{\Delta_k}{k!n^2}Err^n_{T-t_j}(I^k\frac{\partial^k}{\partial x^k}\mathcal{E}_{t_j} g) = \mathcal{O}(n^{-2}).$$

Now if t_j is a *short maturity*, that is when $0 < t_j < \frac{T}{2}$, then $Err^n_{T-t_j}$ is a long maturity error and, using an *extension of Berry-Esseen theorem*, we show that $Err^n_{T-t_j}(\mathfrak{E}^{(\ell)}_{t_j}\left(I^2h''\right)) = \mathcal{O}(n^{-\frac{1}{2}})$, from which (1.16) can be derived (see Lemma 4).

To complete the outline of this paper, let us mention that section 2 gathers the results about *localization*, section 3 deals with the *smoothed payoff error* while section 4 establishes our *representation formulae for the derivatives of European options*. The fact that *payoff smoothing error* are negligible is proved in section 5. The appendix contains auxiliary results including a list of simple properties, P1-P5, enjoyed by all *flexible CRR schemes.*

Remark 2 (On the $\mathcal{O}$ notation)**.** *In the* remaining *of the paper, unless otherwise mentioned, the $\mathcal{O}$ notation is* uniform. *By this we mean that if A, B and $C \geq 0$ are real valued, then the expression $A = B + C\mathcal{O}\left(n^{-1}\right)$ means that there exists a constant Q, which may depend only on our parameters r, σ, T, p and $\mathcal{L}$, such that $|A - B| \leq CQn^{-1}$.*

2. Local errors and error localization

Theorem 3 (Error localization formula)**.** *Let $n, m \geq 1$ be some integers and let h be a polynomially bounded function. Then,*

$$Err^n_{t_m}h = \sum_{j=0}^{m-1}\mathcal{E}_{t_m-t_{j+1}}\left(Err^n_{\frac{T}{n}}\mathcal{E}_{t_j}h\right) - \sum_{j=0}^{m-1}Err^n_{t_m-t_{j+1}}\left(Err^n_{\frac{T}{n}}\mathcal{E}_{t_j}h\right). \tag{2.1}$$

Proof. First we show that

$$Err^n_{t_m}h = \sum_{j=0}^{m-1}\mathcal{E}^n_{t_m-t_{j+1}}\left(Err^n_{\frac{T}{n}}\mathcal{E}_{t_j}h\right). \tag{2.2}$$

Note that, the rhs sum being telescopic,

$$\mathcal{E}_{t_m}h - \mathcal{E}^n_{t_m}h = \sum_{j=0}^{m-1}\left(\mathcal{E}^n_{t_m-t_{j+1}}\mathcal{E}_{t_{j+1}}h - \mathcal{E}^n_{t_m-t_j}\mathcal{E}_{t_j}h\right). \tag{2.3}$$

Also, because $\mathcal{E}_t$ and $\mathcal{E}^n_t$ are semigroup operators,

$$\begin{aligned}\mathcal{E}^n_{t_m-t_{j+1}}\mathcal{E}_{t_{j+1}}h &= \mathcal{E}^n_{t_m-t_{j+1}}\mathcal{E}_{\frac{T}{n}}\mathcal{E}_{t_j}h,\\ \mathcal{E}^n_{t_m-t_j}\mathcal{E}_{t_j}h &= \mathcal{E}^n_{t_m-t_{j+1}}\mathcal{E}^n_{\frac{T}{n}}\mathcal{E}_{t_j}h.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{E}^n_{t_m-t_{j+1}}\mathcal{E}_{t_{j+1}}h - \mathcal{E}^n_{t_m-t_j}\mathcal{E}_{t_j}h &= \mathcal{E}^n_{t_m-t_{j+1}}\left(\left(\mathcal{E}_{\frac{T}{n}} - \mathcal{E}^n_{\frac{T}{n}}\right)\mathcal{E}_{t_j}h\right)\\ &= \mathcal{E}^n_{t_m-t_{j+1}}\left(Err^n_{\frac{T}{n}}\mathcal{E}_{t_j}h\right).\end{aligned}$$

Substituting this in (2.3) gives (2.2). Obviously

$$\mathcal{E}^n_{t_m-t_{j+1}}\left(Err^n_{\frac{T}{n}}\mathcal{E}_{t_j}h\right) = \mathcal{E}_{t_m-t_{j+1}}\left(Err^n_{\frac{T}{n}}\mathcal{E}_{t_j}h\right) - Err^n_{t_m-t_{j+1}}\left(Err^n_{\frac{T}{n}}\mathcal{E}_{t_j}h\right),$$

so (2.1) follows form (2.2) in the most trivial manner. $\square$

Local errors are rich in ways they can be analyzed, including a simple Taylor expansion as in Lemma 1 below, where the expression $\sum_{\ell=2}^{N}$ is understood to vanish in the case $N<2$.

Lemma 1 (Local error expansion formula)**.** *For every integer $N\geq 0$, $p\geq 1$, $g\in C^{(N)}\cap\mathcal{K}_p^{(N+1)}$ and $x\geq 0$,*

$$Err_{\frac{T}{n}}^{n}g\left(x\right)=\sum_{k=2}^{N}\frac{\Delta_k^{(n)}x^kg^{(k)}(x)}{k!}+\mathcal{R}_{\frac{T}{n}}^{n,N}\left(I^{N+1}g^{(N+1)}\right)(x)\tag{2.4}$$

where

$$\mathcal{R}_{\frac{T}{n}}^{n,N}\left(g\right)\left(x\right)\overset{def}{=}\frac{1}{N!}Err_{\frac{T}{n}}^{n}\left(\int_1^I\tfrac{g(xu)(I-u)^N}{u^{N+1}}du\right)(1)\,.\tag{2.5}$$

Proof. Recall that the Taylor expansion of $g\left(y\right)$ around x is

$$g\left(y\right)-g(x)=\sum_{k=1}^{N}\frac{g^{(k)}\left(x\right)}{k!}\left(y-x\right)^k+\frac{1}{N!}\int_x^y g^{(N+1)}\left(u\right)\left(y-u\right)^N du.$$

Using the (discounted expected) Taylor expansions of $g(x\xi_{\frac{T}{n}})$ and $g(x\xi_{\frac{T}{n}}^{(n)})$ around x in

$$Err_{\frac{T}{n}}^{n}g\left(x\right)=e^{-r\frac{T}{n}}E_1\left(\left(g(x\xi_{\frac{T}{n}})-g(x)\right)-\left(g(x\xi_{\frac{T}{n}}^{(n)})-g(x)\right)\right),$$

one precisely gets (2.4) after a simple manipulation of the remainder. □

Remark 3 (Order of the remainder $\mathcal{R}_{\frac{T}{n}}^{n,N}$)**.** *If, for some constants $\alpha\geq 0$ and $\beta\geq 1$,*

$$\left|x^{N+1}g^{(N+1)}\left(x\right)\right|\leq\alpha\left(1+x^\beta\right),$$

then

$$\begin{aligned}\left|\mathcal{R}_{\frac{T}{n}}^{n,N}\left(I^{N+1}g^{(N+1)}\right)(x)\right|&\leq\frac{\alpha}{N!}Err_{\frac{T}{n}}^{n}\left(\left|\int_1^I u^{-(N+1)}\left(I-u\right)^N du\right|\right)(1)\\&\quad+\frac{\alpha x^\beta}{N!}Err_{\frac{T}{n}}^{n}\left(\left|\int_1^I u^{\beta-(N+1)}\left(I-u\right)^N du\right|\right)(1)\,,\end{aligned}$$

and, thanks to property P5,

$$\left|\mathcal{R}_{\frac{T}{n}}^{n,N}\left(I^{N+1}g^{(N+1)}\right)(x)\right|=\alpha\left(1+x^\beta\right)\mathcal{O}\left(n^{-\frac{N+1}{2}}\right).$$

Now a glimpse at the error localization formula reveals that we deal with local errors $Err_{\frac{T}{n}}^{n}\mathcal{E}_sg$ where the payoff has the form $\mathcal{E}_sg$ for some time step s. As pointed out in section 1.6, we are interested in the case $s>0$, and g in $C^{(1)}\cap\mathcal{K}_p^{(3)}$. But in this case, the European option derivative representation formulae *(see Remark 6) guarantees that*

$$\left|I^5\frac{\partial^5}{\partial x^5}\mathcal{E}_sg\left(x\right)\right|\leq\|g\|_p^{(2)}\left(\sqrt{s}\right)^{-3}\left(1+x^{p+2}\right)$$

and thus

$$\left|\mathcal{R}^{n,4}_{\frac{T}{n}}\left(I^5\frac{\partial^5}{\partial x^5}\mathcal{E}_s g\right)(x)\right| = \|g\|^{(2)}_p \left(\sqrt{s}\right)^{-3}\left(1+x^{p+2}\right)\mathcal{O}\left(n^{-\frac{5}{2}}\right). \tag{2.6}$$

According to Theorem 3, the error $Err^n_{t_m} g(x)$ can be decomposed into two components which need a separate analysis: (1) the *main term of the error*, denoted $MErr^n_{t_m} g(x)$, which is the sum, for $j = 0, ..., m-1$, of the local errors $\mathcal{E}_{t_m - t_{j+1}}(Err^n_{\frac{T}{n}}\mathcal{E}_{t_j} g)$; (2) the *compounded errors term*, denoted $CErr^n_{t_m} g(x)$, which is the sum, for $j = 0, ..., m-1$, of these *errors of local errors* $Err^n_{t_m - t_{j+1}}(Err^n_{\frac{T}{n}}\mathcal{E}_{t_j} g)$. In other words,

$$MErr^n_{t_m} g(x) \overset{def}{=} \sum_{j=0}^{n-1}\mathcal{E}_{t_m - t_{j+1}}\left(Err^n_{\frac{T}{n}}\mathcal{E}_{t_j} g\right)(x),$$

$$CErr^n_{t_m} g(x) \overset{def}{=} \sum_{j=0}^{n-1} Err^n_{t_m - t_{j+1}}\left(Err^n_{\frac{T}{n}}\mathcal{E}_{t_j} g\right)(x).$$

Of course we want to combine the *error localization formula* and the *local error expansion formula* which gives:

Proposition 1 (Error localization expansion formula)**.** *Let integer $M \geq 0$, let $0 < t_m \leq T$ be the m^{th} time step, and assume that g belongs to $C^{(M)} \cap \mathcal{K}^{(M+1)}_p$. Then, for every integer $N \geq 0$ and for every $x > 0$,*

$$Err^n_{t_m} g(x) = MErr^n_{t_m} g(x) - CErr^n_{t_m} g(x)$$

where

$$MErr^n_{t_m} g(x) = m\sum_{k=2}^{N}\frac{\Delta^{(n)}_k}{k!}x^k\frac{\partial^k}{\partial x^k}\mathcal{E}_{t_{m-1}}(g)(x) + m\mathcal{R}^{n,N}_{\frac{T}{n}}\left(I^{N+1}\frac{\partial^{N+1}}{\partial x^{N+1}}\mathcal{E}_{t_{m-1}} g\right)(x) \tag{2.7}$$

$$\begin{aligned} CErr^n_{t_m} g(x) = {} & \sum_{j=1}^{m-1}\sum_{k=2}^{N}\frac{\Delta^{(n)}_k}{k!}Err^n_{t_m - t_{j+1}}\left(I^k\frac{\partial^k}{\partial x^k}\mathcal{E}_{t_j} g\right)(x) \\ & + \sum_{j=1}^{m-1}\mathcal{R}^{n,N}_{\frac{T}{n}}\left(Err^n_{t_m - t_{j+1}}\left(I^{N+1}\frac{\partial^{N+1}}{\partial x^{N+1}}\mathcal{E}_{t_j} g\right)\right)(x) \\ & + \sum_{k=2}^{M}\frac{\Delta^{(n)}_k}{k!}Err^n_{t_m - t_{j+1}}\left(I^k\frac{\partial^k}{\partial x^k} g\right)(x) \\ & + \mathcal{R}^{n,M}_{\frac{T}{n}}\left(Err^n_{t_{m-1}}\left(I^{M+1}\frac{\partial^{M+1}}{\partial x^{M+1}} g\right)\right)(x) \end{aligned} \tag{2.8}$$

Proof. Note that, for every $s > 0$, $\mathcal{E}_s g$ belongs to $C^{(N)} \cap \mathcal{K}^{(N+1)}_p$, for every integer $N \geq 0$. The result is obtained by a mere combination of the *error* *localization formula* and the *local error expansion formula*, using the facts that for every steps t_ℓ and t_m, every polynomially bounded function ψ and every integer $k \geq 0$, the following holds:

(1) $\mathcal{E}_{t_\ell}$ and $Err^n_{\frac{T}{n}}$ commute,
$$\mathcal{E}_{t_\ell} Err^n_{\frac{T}{n}} \psi = Err^n_{\frac{T}{n}} \mathcal{E}_{t_\ell} \psi,$$

(2) by independence and Fubini's theorem, $Err^n_{t_\ell}$ and $\mathcal{R}^{n,N}_{\frac{T}{n}}$ also commute,
$$Err^n_{t_\ell}(\mathcal{R}^{n,N}_{\frac{T}{n}}(\psi)) = \mathcal{R}^{n,N}_{\frac{T}{n}}\left(Err^n_{t_\ell}(\psi)\right),$$

(3) thanks to Lemma 3,
$$\mathcal{E}_{t_\ell}\left(I^k \frac{\partial^k}{\partial x^k}\mathcal{E}_{t_m}\psi\right) = I^k \frac{\partial^k}{\partial x^k}\mathcal{E}_{t_\ell}\mathcal{E}_{t_m}\psi = I^k \frac{\partial^k}{\partial x^k}\mathcal{E}_{t_\ell + t_m}\psi.$$

□

Remark 4. *We will use the* error localization expansion formula *with $N = 4$. The reason for this is,* in essence*, that if g belongs to $C^{(4)} \cap \mathcal{K}_p^{(5)}$, then each of the remainders —the $\mathcal{R}$-terms— in formula (2.7) and (2.8) are of order $n^{-\frac{5}{2}}$, which makes them collectively of order $n^{-\frac{3}{2}}$, and thus negligible.*

3. The smoothed payoff error

As noted earlier —in section 1.6— we are particularly interested in the error $Err^n_T \mathcal{E}_{\frac{T}{n}} g\,(x)$, which decomposes into
$$Err^n_T \mathcal{E}_{\frac{T}{n}} g\,(x) = MErr^n_T \mathcal{E}_{\frac{T}{n}} g\,(x) - CErr^n_T \mathcal{E}_{\frac{T}{n}} g\,(x).$$

Now thanks to the *European option derivative representation formulae*, Theorem 5, for every integer $k \geq 0$,
$$x^k \frac{\partial^k}{\partial x^k}\mathcal{E}_T g\,(x) = \|g\|_p^{(2)}\left(1 + x^{p+2}\right)\mathcal{O}(n^{-\frac{3}{2}}).$$

Therefore, using property P3, $\Delta_k^{(n)} = \left(\frac{1}{n}\right)^2 \Delta_k + \mathcal{O}(n^{-\frac{5}{2}})$, and Remark 3, one rewrites the *main term of the error* as
$$MErr^n_T \mathcal{E}_{\frac{T}{n}} g\,(x) = \frac{1}{n}\sum_{k=2}^{4} \frac{\Delta_k}{k!} x^k \frac{\partial^k}{\partial x^k}\mathcal{E}_T g\,(x) + \|g\|_p^{(2)}\left(1 + x^{p+2}\right)\mathcal{O}(n^{-\frac{3}{2}}).$$

As for the *compounded errors term*, $CErr^n_T \mathcal{E}_{\frac{T}{n}} g\,(x)$, the *error localization expansion formula* gives

$$\begin{aligned}(3.1)\qquad CErr^n_T \mathcal{E}_{\frac{T}{n}} g\,(x) = &\sum_{j=1}^{n}\sum_{k=2}^{4} \frac{\Delta_k^{(n)}}{k!} Err^n_{T-t_j}\left(I^k \frac{\partial^k}{\partial x^k}\mathcal{E}_{t_j} g\right)(x) \\ &+ \sum_{j=1}^{n} \mathcal{R}^{n,4}_{\frac{T}{n}}\left(Err^n_{T-t_j}\left(I^{N+1}\frac{\partial^{N+1}}{\partial x^{N+1}}\mathcal{E}_{t_j} g\right)\right)(x).\end{aligned}$$

Now it is clear from the *European option derivative representation formulae* that, for every $t_j > 0$, the functions $I^k \frac{\partial^k}{\partial x^k}\mathcal{E}_{t_j} g$ are infinitely differentiable and, together with there derivatives, uniformly bounded over all $x \geq 0$. This suggests —rightly so— that for every *fixed* t_j, $0 < t_j < T$, $Err^n_{T-t_j}(I^k \frac{\partial^k}{\partial x^k}\mathcal{E}_{t_j} g)$ is of order n^{-1}. But unfortunately here t_j is *not fixed* (and so isn't $T - t_j$) but rather takes all positive time steps up to T. Now the greater $T - t_j$ is, the greater the averaging effect is (that's the Berry-Esseen theorem effect). On the other hand, the greater t_j is, the "smaller" $I^k \frac{\partial^k}{\partial x^k}\mathcal{E}_{t_j} g$ is (because as t_j approaches zero, the maximum of function $I^k \frac{\partial^k}{\partial x^k}\mathcal{E}_{t_j} g$ as well as the maximum of each of its derivatives goes to infinity). All in all, thanks to Lemma 4, the correct estimate is, for $0 < t_j < T$,

$$Err^n_{T-t_j}(I^k \frac{\partial^k}{\partial x^k}\mathcal{E}_{t_j} g) = \chi_p(g)\sqrt{t_j}^{-(k-1)}\mathcal{O}\left(n^{-1}\right)\left(1 + x^{p+3}\right), \tag{3.2}$$

where $\varkappa_p$ is defined by (1.13). This obviously cannot work for $t_j = T$, but in the formula (3.1) for $CErr^n_T\mathcal{E}_{\frac{T}{n}} g(x)$, this case accounts for only $\mathcal{O}(n^{-\frac{3}{2}})\,\|g\|^{(2)}_p\left(1 + x^{p+2}\right)$, thanks to remark 3 and property P3. Thus, replacing (3.2) in (3.1) gives

$$\begin{aligned}
CErr^n_T\mathcal{E}_{\frac{T}{n}} g(x) &= \sum_{j=1}^{n-1}\sum_{k=2}^{4}\frac{O\left(n^{-2}\right)}{k!}\chi_p(g)\sqrt{t_j}^{-(k-1)}\mathcal{O}\left(n^{-1}\right)\left(1 + x^{p+3}\right) \\
&+ \sum_{j=1}^{n-1}\chi_p(g)\sqrt{t_j}^{-4}\mathcal{O}\left(n^{-1}\right)\left(1 + x^{p+3}\right)\mathcal{O}(n^{-\frac{5}{2}}) \\
&+ \|g\|^{(2)}_p\left(1 + x^{p+2}\right)\mathcal{O}(n^{-\frac{3}{2}}) \\
&= \chi_p(g)\left(1 + x^{p+3}\right)\mathcal{O}(n^{-\frac{3}{2}}).
\end{aligned}$$

We have proved the following result:

Theorem 4. *For every $g \in C^{(1)} \cap \mathcal{K}^{(3)}_p$ and every $x > 0$,*

$$Err^n_T\mathcal{E}_{\frac{T}{n}} g(x) = \frac{1}{n}\sum_{k=2}^{4}\tfrac{\Delta_k}{k!}x^k\mathcal{E}^{(k)}_T g(x) + \varkappa_p(g)\left(1 + x^{p+3}\right)\mathcal{O}(n^{-\frac{3}{2}}). \tag{3.3}$$

4. European option derivative representation formulae

Let $\phi(z)$ be the pdf of a standard normal random variable and let, as usual, $\phi^{(j)}(z)$ be its j^{th} derivative. To shorten expressions, we denote $\zeta_s(z) \overset{def}{=} e^{\sqrt{s}\sigma z + \left(r - \frac{1}{2}\sigma^2\right)s}$, with which we can write

$$\mathcal{E}_s h(x) = e^{-rs}\int_{-\infty}^{\infty} h\left(x\zeta_s(z)\right)\phi(z)dz.$$

We will show that the derivatives of $\mathcal{E}_s h(x)$ can be expressed as linear combinations of smooth functions $\mathfrak{E}_s^{(j)} h(x)$ of the form

$$\mathfrak{E}_s^{(j)} h(x) \overset{def}{=} e^{-rs} \int_{-\infty}^{\infty} h(x\zeta_s(z)) \phi^{(j)}(z) dz.$$

Not only do we need expressions for the derivatives of $\mathcal{E}_s h(x)$, but it turns out that, actually, we need expressions for the derivatives of $I^k \frac{\partial^k}{\partial x^k} \mathcal{E}_s h$, for integers $k \geq 0$. This motivates the notation

$$\mathcal{E}_s^{(k)} h(x) \overset{def}{=} \frac{\partial^k}{\partial x^k} \mathcal{E}_s h(x),$$

$$\mathcal{E}_s^{\langle k \rangle} h(x) \overset{def}{=} x^k \mathcal{E}_s^{(k)} h(x),$$

and, more generally, for any function ψ in $C^{(k)}$,

$$\psi^{\langle k \rangle}(x) \overset{def}{=} x^k \psi^{(k)}(x).$$

Now let h be any continuous function in $\mathcal{K}^{(1)}$ and let $s > 0$. Integration by parts gives

$$\int_{-\infty}^{\infty} \zeta_s(z) h'(x\zeta_s(z)) \phi^{(j)}(z) dz = \frac{-1}{x\sqrt{s}\sigma} \mathfrak{E}_s^{(j+1)} h(x), \tag{4.1}$$

and since

$$\frac{\partial}{\partial x} \mathfrak{E}_s^{(j)} h(x) = \int_{-\infty}^{\infty} \zeta_s(z) h'(x\zeta_s(z)) \phi^{(j)}(z) dz,$$

then

$$\frac{\partial}{\partial x} \mathfrak{E}_s^{(j)} h(x) = \frac{-1}{x\sqrt{s}\sigma} \mathfrak{E}_s^{(j+1)} h(x). \tag{4.2}$$

Thus, for integers $j, k \geq 0$, repeated differentiation gives that, for some real numbers $\alpha_1,\ldots,\alpha_k$,

$$\tfrac{\partial^k}{\partial x^k} \mathfrak{E}_s^{(j)} h(x) = \sum_{\ell=1}^{k} \frac{\alpha_\ell}{x^k} \left(\frac{-1}{\sqrt{s}\sigma} \right)^\ell \mathfrak{E}_s^{(j+\ell)} h(x). \tag{4.3}$$

Note that equation (4.1) says that for $j \geq 1$,

$$\mathfrak{E}_s^{(j)} h(x) = -\sqrt{s}\sigma e^{-rs} \int_{-\infty}^{\infty} (x\zeta_s(z)) h'(x\zeta_s(z)) \phi^{(j-1)}(z) dz.$$

In other words

$$\mathfrak{E}_s^{(j)} h(x) = -\sqrt{s}\sigma \mathfrak{E}_s^{(j-1)} (Ih')(x). \tag{4.4}$$

Hence if $h \in C^{(1)} \cap \mathcal{K}^{(2)}$, then, for $j \geq 2$, the relation (4.4) can be used a second time, giving

$$\mathfrak{E}_s^{(j-1)} (Ih')(x) = -\sqrt{s}\sigma \mathfrak{E}_s^{(j-2)} \left(I (Ih')' \right)(x),$$

$$\mathfrak{E}_s^{(j)} h(x) = \left(\sqrt{s}\sigma\right)^2 \left(\mathfrak{E}_s^{(j-2)} (Ih')(x) + \mathfrak{E}_s^{(j-2)} (I^2 h'')(x) \right). \tag{4.5}$$

Nothing that

$$\mathcal{E}_s h(x) = \mathfrak{E}_s^{(0)} h(x),$$

we have essentially obtained the following result which can be used to obtain explicit expressions for $\frac{\partial^k}{\partial x^k}\mathcal{E}_s h(x)$ and $\frac{\partial^\ell}{\partial x^\ell}\mathcal{E}_s^{\langle k\rangle} h(x)$, for any value of $k,\ell \geq 0$.

Theorem 5 (European option derivative representation formulae)**.** *If h is a continuous function in $\mathcal{K}^{(1)}$, then for every $j \geq 0$, there exists real numbers for some real numbers $\alpha_1,...,\alpha_k$, such that*

$$x^k \tfrac{\partial^k}{\partial x^k} \mathfrak{E}_s^{(j)} h(x) = \sum_{\ell=1}^{k} \alpha_\ell \sqrt{s}^{-\ell} \mathfrak{E}_s^{(j+\ell)} h(x) \tag{4.6}$$

If additionally $k \geq 1$, there exists real numbers for some real numbers $\alpha_1,...,\alpha_k$, such that

$$x^k \tfrac{\partial^k}{\partial x^k} \mathfrak{E}_s^{(j)} h(x) = \sum_{\ell=0}^{k-1} \alpha_\ell \sqrt{s}^{-\ell} \mathfrak{E}_s^{(j+\ell)} \left(Ih'\right)(x) \tag{4.7}$$

If additionally $k \geq 2$ and $h \in C^{(1)} \cap \mathcal{K}^{(2)}$, there exists real numbers for some real numbers $\alpha_1,...,\alpha_k$ and $\beta_1,...,\beta_k$, such that

$$x^k \tfrac{\partial^k}{\partial x^k} \mathfrak{E}_s^{(j)} h(x) = \sum_{\ell=0}^{k-2} \alpha_\ell \sqrt{s}^{-\ell} \mathfrak{E}_s^{(j+\ell)} \left(I^2 h''\right)(x) + \sum_{\ell=0}^{k-2} \beta_\ell \sqrt{s}^{-\ell} \mathfrak{E}_s^{(j+\ell)} \left(Ih'\right)(x) \tag{4.8}$$

In particular, for $h \in C^{(1)} \cap \mathcal{K}^{(2)}$ we have

$$\begin{aligned}
&x^2 \mathcal{E}_s^{(2)} h(x) = \mathfrak{E}_s^{(0)} \left(I^2 h''\right)(x) \\
&x^3 \mathcal{E}_s^{(3)} h(x) = -2\mathfrak{E}_s^{(0)} \left(I^2 h''\right)(x) - \tfrac{1}{\sigma\sqrt{s}} \mathfrak{E}_s^{(1)} \left(I^2 h''\right)(x), \\
&x^4 \mathcal{E}_s^{(4)} h(x) = 6\mathfrak{E}_s^{(0)} \left(I^2 h''\right)(x) + \tfrac{5}{\sqrt{s}\sigma} \mathfrak{E}_s^{(1)} \left(I^2 h''\right)(x) + \tfrac{1}{s\sigma^2} \mathfrak{E}_s^{(2)} \left(I^2 h''\right)(x)
\end{aligned}$$

Proof. Equations (4.6), (4.7) and (4.8) is the content of the short discussion at the beginning of this section. In order to get expressions for $\frac{\partial^k}{\partial x^k}\mathcal{E}_s h(x) = \frac{\partial^k}{\partial x^k}\mathfrak{E}_s^{(0)}(h)(x)$, $k = 2,3,4$, one first calculate the actual values of α_j, $j = 0,...,4$, such that (4.6) holds, and repeatedly calls (4.4) and (4.5). This is tedious but otherwise trivial. ☐

Remark 5 (Expressing $\mathfrak{E}_s^{(\ell)}$ in terms of $\mathcal{E}_s$)**.** *Recall η_s from (1.10) and note that $\eta_s(\zeta_s(z)) = z$, note that $\frac{\phi^{(\ell)}(z))}{\phi(z))}$ is a polynomial in z. Expressions involving $\mathfrak{E}_s^{(\ell)}\left(I^2 h''\right)(x)$ can also be written in terms of $\mathcal{E}_s$ or the $e^{-rs}E_x$ in* *following manner:*

$$\begin{aligned}\mathfrak{E}_s^{(\ell)}\left(I^2h''\right)(x) &= e^{-rs}\int_{-\infty}^{\infty}\left(x\zeta_s(z)\right)^2 h''\left(x\zeta_s(z)\right)\frac{\phi^{(\ell)}(\eta_s(\frac{x\zeta_s(z)}{x}))}{\phi(\eta_s(\frac{x\zeta_s(z)}{x}))}\phi(z)dz\\ &= \mathcal{E}_s\left(I^2h''\frac{\phi^{(\ell)}(\eta_s\left(\frac{I}{x}\right))}{\phi(\eta_s\left(\frac{I}{x}\right))}\right)(x)\\ &= e^{-rs}E_x\left(\xi_s^2h''\left(\xi_s\right)\frac{\phi^{(\ell)}(\eta_s(\frac{\xi_s}{x}))}{\phi(\eta_s(\frac{\xi_s}{x}))}\right).\end{aligned}$$

In the statement of Theorem 2, we use the latest form.

Remark 6 (Boundedness of $\mathcal{E}_s^{\langle k\rangle}g$ and $x^m\frac{\partial^m}{\partial x^m}\mathcal{E}_s^{\langle k\rangle}g$)**.** *The* error localization expansion formula*, expresses errors $Err_T^n\mathcal{E}_{\frac{T}{n}}g\left(x\right)$, for g in $C^{(1)}\cap\mathcal{K}^{(3)}$, in terms of errors of payoffs of the form $Err_{T-t_j}^n\mathcal{E}_{t_j}^{\langle k\rangle}g(x)$. Use the* error localization expansion formula *to analyze these errors, bringing up errors of the form*

$$Err_{t_i}^n\left(I^\ell\frac{\partial^\ell}{\partial x^\ell}\mathcal{E}_{t_j}^{\langle k\rangle}g\right)(x).$$

With the European option derivative representation formulae*, the form functions $I^\ell\frac{\partial^\ell}{\partial x^\ell}\mathcal{E}_{t_j}^{\langle k\rangle}g$ is extremely simple and easy to deal with. If for instance $k,\ell\geq 0$ and $k+\ell\geq 2$ and $g\in C^{(1)}\cap\mathcal{K}^{(2)}$ then*

$$\begin{aligned}I^\ell\frac{\partial^\ell}{\partial x^\ell}\mathcal{E}_{t_j}^{\langle k\rangle}g &= \sum_{j=0}^{k+\ell-2}\alpha_j\sqrt{s}^{-j}\mathfrak{E}_s^{(j)}\left(I^2g''\right)(x)\\ &\quad+\sum_{j=0}^{k+\ell-2}\beta_j\sqrt{s}^{-j}\mathfrak{E}_s^{(j)}\left(Ig'\right)(x),\end{aligned}$$

for some real numbers α_i,β_i, $i=0,...,k+\ell-2$.

We are preoccupied with the boundedness of functions $I^\ell\frac{\partial^\ell}{\partial x^\ell}\mathcal{E}_{t_j}^{\langle k\rangle}g$. This boundedness immediately follows from the fact that, given real number $a,b\geq 0$ and integer j, there exists a constant Q such that for every function ψ satisfying $\left|\psi\left(x\right)\right|\leq a\left(1+x^b\right)$, and for every for every $x,s>0$,

$$\left|\mathfrak{E}_s^{(j)}\left(\psi\right)(x)\right|\leq Qa\left(1+x^b\right).$$

5. The payoff smoothing error

Recall $\chi_p\left(h\right)$ from (1.13).

Proposition 2. *For every $h\in C^{(1)}\cap\mathcal{K}_p^{(3)}$ and every $x>0$,*

$$Err_T^n\left(h-\mathcal{E}_{\frac{T}{n}}\left(h\right)\right)(x)=\chi_p\left(h\right)\left(1+x^{p+3}\right)\mathcal{O}\left(n^{-1.5}\right)\tag{5.1}$$

Proof. We get from Taylor expansion theorem that

$$\mathcal{E}_{\frac{T}{n}}h(x) = e^{-r\frac{T}{n}}h(x) + xh'(x)\,\mathcal{E}_{\frac{T}{n}}(I-1) + \mathcal{E}_{\frac{T}{n}}\left(\int_1^I (xu)^2 h''(xu)\,\tfrac{(I-u)}{u^2}du\right)(1).$$

Hence, since from risk neutrality, $\mathcal{E}_{\frac{T}{n}}(I-1) = 1-e^{-r\frac{T}{n}} = \mathcal{O}\left(n^{-1}\right)$,

$$h(x) - \mathcal{E}_{\frac{T}{n}}h(x) = (1-e^{-r\frac{T}{n}})\left(h(x) - xh'(x)\right) - \mathcal{E}_{\frac{T}{n}}\left(\int_1^I (xu)^2 h''(xu)\,\tfrac{(I-u)}{u^2}du\right)(1),$$

and therefore

$$Err_T^n\left(h - \mathcal{E}_{\frac{T}{n}}(h)\right)(x) = \mathcal{O}\left(n^{-1}\right)Err_T^n\left(h - Ih'\right)(x) - \mathcal{E}_{\frac{T}{n}}\left(\int_1^I Err_T^n\left(I^2h''\right)(xu)\,\tfrac{(I-u)}{u^2}du\right)(1). \tag{5.2}$$

But $h - Ih' \in C^{(0)} \cap \mathcal{K}^{(2)}_{p+2}$ and $I^2h'' \in \mathcal{K}^{(1)}_{p+2}$. Hence, from Theorem 6,

$$\begin{aligned}\left|Err_T^n\left(h - Ih'\right)(x)\right| &\leq \mathcal{O}\left(n^{-0.5}\right)\chi_p(h)\left(1 + (x)^{p+3}\right),\\ \left|Err_T^n\left(I^2h''\right)(xu)\right| &\leq \mathcal{O}\left(n^{-0.5}\right)\chi_p(h)\left(1 + (xu)^{p+3}\right).\end{aligned} \tag{5.3}$$

Now

$$\left|\int_1^I \left(1 + (xu)^{p+3}\right)\tfrac{(I-u)}{u^2}du\right| = \int_1^I \left(1 + (xu)^{p+3}\right)\tfrac{(I-u)}{u^2}du,$$

and since, thanks to P5,

$$\mathcal{E}_{\frac{T}{n}}\left(\int_1^I \left(1 + (xu)^{p+3}\right)\tfrac{(I-u)}{u^2}du\right) = \mathcal{O}\left(n^{-1}\right)\left(1 + x^{p+3}\right),$$

we obtain

$$\mathcal{E}_{\frac{T}{n}}\left(\int_1^I Err_T^n\left(I^2h''\right)(xu)\,\tfrac{(I-u)}{u^2}du\right)(1) = \mathcal{O}\left(n^{-\frac{3}{2}}\right)\chi_p(h)\left(1 + x^{p+3}\right). \tag{5.4}$$

□

6. Auxiliary results

We list here basic properties satisfied by all flexible CRR Schemes. Here $F_{\xi_t^{(n)}}$ and F_{ξ_t} denote the cumulative distribution functions of $\xi_t^{(n)}$ and ξ_t, with $\xi_0^{(n)} = \xi_0$. Proofs are left to the reader.

Lemma 2 (Properties of $\{\xi^{(n)}\}$). *For every flexible CRR Schemes $\{\xi^{(n)}\}_{n=1}^{\infty}$ the following hold*

P1 (Berry-Esseen): *There exists a constant Q such that for every $t \in \frac{T}{n}\mathbb{N}$, with $\frac{T}{2} \leq t \leq T$,*

$$\sup_z \left| F_{\xi_t^{(n)}}(z) - F_{\xi_t}(z) \right| \leq Q n^{-\frac{1}{2}}.$$

P2 (local estimate of the distance to 1): *For integers $k \geq 0$,*

$$\delta_k^{(n)} \overset{def}{=} \mathcal{E}_{\frac{T}{n}}^n \left(|I-1|^k \right)(1) = \mathcal{O}\left(n^{-\frac{k}{2}} \right).$$

P3 (local error of the difference to 1): *For integers $k = 2, 3, 4$, there exists Δ_k such that*

$$\Delta_k^{(n)} \overset{def}{=} Err_{\frac{T}{n}}^n \left((I-1)^k \right)(1) = \tfrac{\Delta_k}{n^2} + \mathcal{O}(n^{-\frac{5}{2}}).$$

P4 (local and global estimates for log and power functions):

$$\mathcal{E}_{\frac{T}{n}}^n \left(|\ln(I)| \right)(1) = \mathcal{O}\left(n^{-\frac{1}{2}} \right).$$

Furthermore, for every fixed *real number γ,*

$$\mathcal{E}_{\frac{T}{n}}^n \left(I^\gamma \right) = \mathcal{E}_{\frac{T}{n}} \left(I^\gamma \right) + \mathcal{O}\left(n^{-2} \right)$$

and (consequently)

$$\mathcal{E}_{\frac{T}{n}}^n \left(|I^\gamma - 1| \right)(1) = \mathcal{O}\left(n^{-\frac{1}{2}} \right),$$

$$\max_{j=0,\ldots,n} \left| \mathcal{E}_{\frac{jT}{n}}^n \left(I^\gamma \right)(x) - \mathcal{E}_{\frac{jT}{n}} \left(I^\gamma \right)(x) \right| = x^\gamma \mathcal{O}\left(n^{-1} \right),$$

$$\max_{j=0,\ldots,n} \left| \mathcal{E}_{\frac{jT}{n}}^n \left(I^\gamma \right)(x) \right| = x^\gamma \mathcal{O}(1).$$

P5 (Remainder related local estimate): *For integer β and integer $N \geq 0$,*

$$\mathcal{E}_{\frac{T}{n}}^n \left(\left| \int_1^I u^\beta (I-u)^N \, du \right| \right)(1) = \mathcal{O}(n^{-\frac{N+1}{2}}).$$

Remark 7. *All the properties P1-P5 remain valid if $\xi^{(n)}$ is replaced by ξ.*

Recall the notation of section 4. The following lemma is a practical and simple result that we used on few occasions.

Lemma 3. *Let integer $k \geq 0$. For every $h \in C^{(k-1)} \cap \mathcal{K}^{(k)}$, $s > 0$ and $x \geq 0$,*

$$x^k \frac{d^k}{dx^k} \mathcal{E}_s h(x) = \mathcal{E}_s \left(I^k \frac{d^k}{dx^k} h \right)(x) \tag{6.1}$$

Proof. Since $\mathcal{E}_s h(x) = e^{-rs}\int_{-\infty}^{\infty} h(x\zeta_s(z))\phi(z)dz$, one writes

$$\begin{aligned}
x^k (\mathcal{E}_s h)^{(k)}(x) &= x^k e^{-rs}\frac{d^k}{dx^k}\int_{-\infty}^{\infty} h(x\zeta_s(z))\phi(z)dz \\
&= x^k e^{-rs}\int_{-\infty}^{\infty} (\zeta_s(z))^k h^{(k)}(x\zeta_s(z))\phi(z)dz \\
&= e^{-rs}\int_{-\infty}^{\infty} (x\zeta_s(z))^k h^{(k)}(x\zeta_s(z))\phi(z)dz \\
&= \mathcal{E}_s\left(I^k h^{(k)}\right)(x).
\end{aligned}$$

□

The result below extends Berry-Esseen property P1.

Theorem 6 (Berry-Esseen extension)**.** *If $\varphi \in \mathcal{K}_p^{(1)}$ then,*

$$\max_{\frac{T}{2}\le t_m\le T}\left|Err_{t_m}^n\varphi(x)\right| \le (\|\varphi\|_p^{(1)} + \sum|\Delta\varphi|)\left(1+x^{p+1}\right)\mathcal{O}(n^{-\frac{1}{2}}). \tag{6.2}$$

Proof. If φ belongs to $\mathcal{K}_p^{(1)}$ but is not continuous, then it can be decomposed into a (finite) sum of piecewise constant functions and a continuous function φ_* in $\mathcal{K}_p^{(1)}$. Note that $\|\varphi_*\|_p^{(1)} \le \|\varphi\|_p^{(1)} + \sum|\Delta\varphi|$. Since the convergence of option value occurs at a rate of a least $n^{-\frac{1}{2}}$ when the payoff is piecewise constant, thanks to property P1, we can assume, without loss of generality, that $\varphi \in C^{(0)} \cap \mathcal{K}_p^{(1)}$. Recall that, thanks to the *European option derivative representation formulae*, there exists a constant Q such that for $0 < s \le T$ and for $k = 2, ..., 5$,

$$\left|\mathcal{E}_s^{\langle k\rangle}(\varphi)(x)\right| \le Q\frac{\|\varphi\|_p^{(1)}}{\sqrt{s}^{k-1}}\left(1+x^{p+1}\right).$$

Let time step $t_m \in \left[\frac{T}{2}, T\right]$. Substituting the above estimate in the *error localization expansion formula* (with $M = 0$ and $N = 4$) we get, thanks to Remark 3 and property P3,

$$\begin{aligned}
MErr_{t_m}^n\varphi(x) &= m\sum_{k=2}^{4}\frac{\mathcal{O}\left(n^{-2}\right)}{k!}\frac{\|\varphi\|_p^{(1)}}{\sqrt{t_m}^{k-1}}\left(1+x^{p+1}\right) \\
&\quad + m\frac{\|\varphi\|_p^{(1)}}{\sqrt{t_m}^{k-1}}\left(1+x^{p+1}\right)\mathcal{O}(n^{-\frac{5}{2}}), \\
&= \|\varphi\|_p^{(1)}\left(1+x^{p+1}\right)\mathcal{O}\left(n^{-1}\right).
\end{aligned}$$

Using additionally the fact that $Err^n_{t_\ell}$ and $\mathcal{R}^{n,N}_{\frac{T}{n}}$ commute, we similarly get

$$\begin{aligned}
CErr^n_{t_m}\varphi\left(x\right) &= \sum_{j=1}^{m-1}\sum_{k=2}^{4}\frac{\mathcal{O}\left(n^{-2}\right)}{k!}\frac{\|\varphi\|_p^{(1)}}{\sqrt{t_j}^{k-1}}\left(1+x^{p+1}\right)\\
&+\sum_{j=1}^{m-1}\frac{\|\varphi\|_p^{(1)}}{\sqrt{t_j}^{4}}\left(1+x^{p+1}\right)\mathcal{O}(n^{-\frac{5}{2}})\\
&+\|\varphi\|_p^{(1)}\left(1+x^{p+1}\right)\mathcal{O}(n^{-\frac{1}{2}})\\
&=\|\varphi\|_p^{(1)}\left(1+x^{p+1}\right)\mathcal{O}(n^{-\frac{1}{2}}).
\end{aligned}$$

□

The result below is used in the proof of Theorem 4.

Lemma 4. *Let integer $k \geq 2$ and $g \in C^{(1)} \cap \mathcal{K}_p^{(3)}$. Then,*

$$\left|Err^n_{T-t_m}(\mathcal{E}^{\langle k\rangle}_{t_m}g)\left(x\right)\right| \leq \tfrac{\chi_p(g)}{\sqrt{t_m}^{k-1}}\mathcal{O}\left(n^{-1}\right)\left(1+x^{p+3}\right), \tag{6.3}$$

for all time step $0 < t_m < T$ and $x \geq 0$.

Proof. Using the fact that, for every $s, t \geq 0$, thanks to Lemma 3,

$$\mathcal{E}_s\mathcal{E}^{\langle k\rangle}_t g\left(x\right) = \mathcal{E}^{\langle k\rangle}_s\mathcal{E}_t g\left(x\right) = \mathcal{E}^{\langle k\rangle}_{s+t}g\left(x\right),$$

the *error localization expansion formula* (with $M = N = 4$) gives

$$\begin{aligned}
MErr^n_{T-t_m}\mathcal{E}^{\langle k\rangle}_{t_m}g\left(x\right) &= m\sum_{\ell=2}^{4}\frac{\Delta_\ell^{(n)}}{\ell!}x^\ell\frac{\partial^\ell}{\partial x^\ell}\mathcal{E}^{\langle k\rangle}_{T-t_1}g\left(x\right)\\
&+m\mathcal{R}^{n,4}_{\frac{T}{n}}\left(I^5\frac{\partial^5}{\partial x^5}\mathcal{E}^{\langle k\rangle}_{T-t_1}g\right)\left(x\right)
\end{aligned}$$

(6.4)

$$\begin{aligned}
CErr^n_{T-t_m}\mathcal{E}^{\langle k\rangle}_{t_m}g\left(x\right) &= \sum_{j=0}^{m-1}\sum_{\ell=2}^{4}\frac{\Delta_\ell^{(n)}}{\ell!}Err^n_{T-t_m-t_{j+1}}\left(x^\ell\frac{\partial^\ell}{\partial x^\ell}\mathcal{E}^{\langle k\rangle}_{t_j+t_m}g\right)\left(x\right)\\
&+\sum_{j=0}^{m-1}\mathcal{R}^{n,4}_{\frac{T}{n}}\left(Err^n_{T-t_m-t_{j+1}}\left(x^5\frac{\partial^5}{\partial x^5}\mathcal{E}^{\langle k\rangle}_{t_j+t_m}g\right)\right)\left(x\right).
\end{aligned}$$

Recall that, thanks to the *European option derivative representation formulae*, there exists a constant Q such that for $k, \ell = 2, ..., 5$,

$$\left|x^\ell\frac{\partial^\ell}{\partial x^\ell}\mathcal{E}^{\langle k\rangle}_s g\right| \leq Q\frac{\|g\|_p^{(2)}}{\sqrt{s}^{(k+\ell-2)}}\left(1+x^{p+1}\right).$$

Thanks to Remark 3 and property P3, replacing this estimate in the above formulae one gets that, not only

$$\max_{0<t_m<T}\left|MErr^n_{T-t_m}\mathcal{E}^{\langle k\rangle}_{t_m}g\left(x\right)\right| = \|g\|_p^{(2)}\left(1+x^{p+2}\right)\mathcal{O}\left(n^{-1}\right),$$

but also, that the contribution in $\left|CErr^n_{T-t_m}\mathcal{E}^{\langle k\rangle}_{t_m}g\left(x\right)\right|$ of those t_j's for which $\frac{T}{2}\leq t_j+t_m\leq T$, amounts to $\|g\|^{(2)}_p\left(1+x^{p+2}\right)\mathcal{O}\left(n^{-1}\right)$. Hence we are left to consider the t_j's for which $0<t_j+t_m<\frac{T}{2}$, in which case

$$\frac{T}{2}\leq T-t_m-t_{j+1}<T.$$

Let us write $s:=s(j,m,n)=t_j+t_m$ and $t:=t(j,m,n)=T-t_m-t_{j+1}$. Recall that, thanks to the *European option derivative representation formulae*, $x^\ell\frac{\partial^\ell}{\partial x^\ell}\mathcal{E}^{\langle k\rangle}_s g\left(x\right)$ is a linear combination of terms of the form $s^{-\frac{j}{2}}\mathfrak{E}^{(j)}_s\left(Ig'\right)(x)$ and $s^{-\frac{j}{2}}\mathfrak{E}^{(j)}_s\left(I^2g''\right)(x)$ with $j\in\{0,..,k+\ell-2\}$. Now

$$\mathfrak{E}^{(j)}_s\left(I^2g''\right)(x)=\mathcal{E}_s\left(\left(xI\right)^2g''\left(xI\right)\frac{\phi^{(j)}(\eta_s(I))}{\phi(\eta_s(I))}\right)(1)$$

so that

$$Err^n_t(\mathfrak{E}^{(j)}_s\left(I^2g''\right))(x)=\mathfrak{E}^{(j)}_s\left(Err^n_t(I^2g'')\right)(x)\,,$$

and (recall $\frac{T}{2}\leq t<T$) Berry-Esseen extension Theorem 6 yields

$$\left|Err^n_t(I^2g'')(x)\right|=\chi_p\left(g\right)\mathcal{O}(n^{-\frac{1}{2}})\left(1+x^{p+3}\right),$$

and since the same is true for the terms $\mathfrak{E}^{(j)}_s\left(Ig'\right)(x)$, it follows that

$$Err^n_t(I^\ell\frac{\partial^\ell}{\partial x^\ell}\mathcal{E}^{\langle k\rangle}_s g)(x)=\sqrt{s}^{-(k+\ell-2)}\chi_p\left(g\right)\mathcal{O}(n^{-\frac{1}{2}})\left(1+x^{p+3}\right).\tag{6.5}$$

Noticing that, with constant $\mathcal{Q}=2\left(1+T\right)^4$,

$$\|g\|^{(2)}_p\left(1+x^{p+2}\right)\leq\mathcal{Q}\sqrt{s}^{-(k+\ell-2)}\chi_p\left(g\right)\left(1+x^{p+3}\right),$$

for every $s,x>0$, and $k,\ell=2,...,5$, we can replace the estimate (6.5) in the formula (6.4) for $CErr^n_{T-t_m}\mathcal{E}^{\langle k\rangle}_{t_m}g\left(x\right)$, obtaining

$$\begin{aligned}&\left|CErr^n_{T-t_m}\mathcal{E}^{\langle k\rangle}_{t_m}g\left(x\right)\right|\\&=\sum_{j=0}^{n-m-1}\sum_{\ell=2}^{4}\left(\frac{\mathcal{O}\left(n^{-2.5}\right)}{\sqrt{t_m+t_j}^{k+\ell-2}}+\frac{\mathcal{O}\left(n^{-3}\right)}{\sqrt{t_m+t_j}^{k+3}}\right)\chi_p\left(g\right)\left(1+x^{p+3}\right),\end{aligned}$$

from which one easily gets

$$\left|CErr^n_{T-t_m}\mathcal{E}^{\langle k\rangle}_{t_m}g\left(x\right)\right|=\frac{\mathcal{O}\left(n^{-1}\right)}{\sqrt{t_m}^{k-1}}\chi_p\left(g\right)\left(1+x^{p+3}\right).$$

□

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