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Collateral choice and the fundamental theorem of asset pricing

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Abstract

In the classical quantitative finance literature it is assumed that there is a risk free rate at which hedgers can borrow and lend in the dynamic replication process of financial derivatives. In such a framework, under complete market conditions [1] and absence of arbitrage opportunities, for a given numeraire whose price cannot vanish, prices of self financing portfolios divided by the numeraire behave like a martingales under a unique martingale measure associated with the numeraire. Nevertheless, in the current market environment a high percentage of deals are collateralized due to counperparty credit risk concerns. Depending on the collateral agreement, collateral can be in the form of cash in different currencies, but also in the form of assets (bonds, shares,...). In this paper we explore how the fundamental valuation theorem and the change of numeraire tollkit is reformulated under this new framework.

1 Collateral choice implications in the replication formula

We will assume that the derivative we want to price is written on a particular underlying whose price at time t will be denoted by S_t, that pays a continuous dividend yield q_t and that can be repoed at a rate r_t^S. We will also assume that the derivative is denominated in the same currency (L) as the underlying but, in order to reflect the most generic situation, the derivative is collateralized in an asset denominated in a different currency F.

[1] Derivatives can be replicated with the vanilla instruments available

We will use the following notation:

- S_t: Underlying asset price at time t.
- r_t^S: Repo rate of the underlying asset.
- r_t^C: Repo rate of the collateral asset.
- i_t^L: OIS (overnight index swap rate) rate in currency L.
- i_t^F: OIS rate in currency F.
- b_t: Short term cross currency basis between currencies L and F (to be defined later on).
- q_t: Dividend yield (assumed continuous) of the underlying asset.
- X_t: Spot FX rate expressed in L/F
- C_t: Price of the collateral asset.

We will assume V_t to be the time t derivative's value from the investor standpoint. Assuming that V_t is positive, the hedger would have a positive amount V_t in cash in currency L available as a byproduct of the dynamic replication strategy. Nevertheless V_t should be posted by the hedger to the investor in the form of the collateral asset denominated in currency F. Therefore the hedger will have to buy the collateral asset. By doing so, the hedger will be left with a long position in an asset denominated in currency L. Both the FX risk and the exposure to the collateral asset price changes will have to be hedged by the derivatives hedger. So that the hedger will have to enter into these transactions at a generic time step t:

- Exchange V_t in cash denominated in L for cash denominated in F in the spot FX market.
- With the cash obtained from the FX spot transaction, the hedger will buy the collateral asset spot and sell it forward (with maturity $t + dt$) through a REPO transaction. Under the REPO transaction the hedger will deliver at time t $\frac{V_t}{X_t}$ in cash denominated in F in exchange of collateral asset shares with the same value [2].
- These shares in the collateral asset will be posted as collateral to the investor.
- At time $t + dt$ the investor will give the collateral back (with a value of $\frac{V_t C_{t+dt}}{X_t C_t}$ measured in currency F) to the hedger, who will give it back to the REPO counterparty.
- At time $t + dt$ the hedger will receive $\frac{V_t}{X_t}\left(1 + r_t^C dt\right)$ from the REPO counterparty in cash denominated in F.
- In order to hedge the FX risk of the last amount of cash denominated in F, at time t the hedger should sell this amount forward (with maturity $t+dt$) receiving at time $t + dt$ cash in currency L with a value equal to the amount to be paid in currency F ($\frac{V_t}{X_t}\left(1 + r_t^C dt\right)$) multiplied by the forward FX rate $X_t \frac{\left(1+c_t^L\right)}{\left(1+\left(c_t^F+b_t\right)\right)}$ seen at time t with maturity $t + dt$. We assume that forward rates cannot be inferred by the spot FX rate and the OIS rates in both currencies, so that an adjustment

[2]We assume no haircut in the REPO transaction

needs to be made in the F rate. Notice that this adjustment represents the short term cross currency basis.

Both cash transactions (in currencies L and F) and collateral asset transactions occurring at times t and $t + dt$ are represented in figure 1. Notice that if V_t was negative, the trades will be right the opposite.

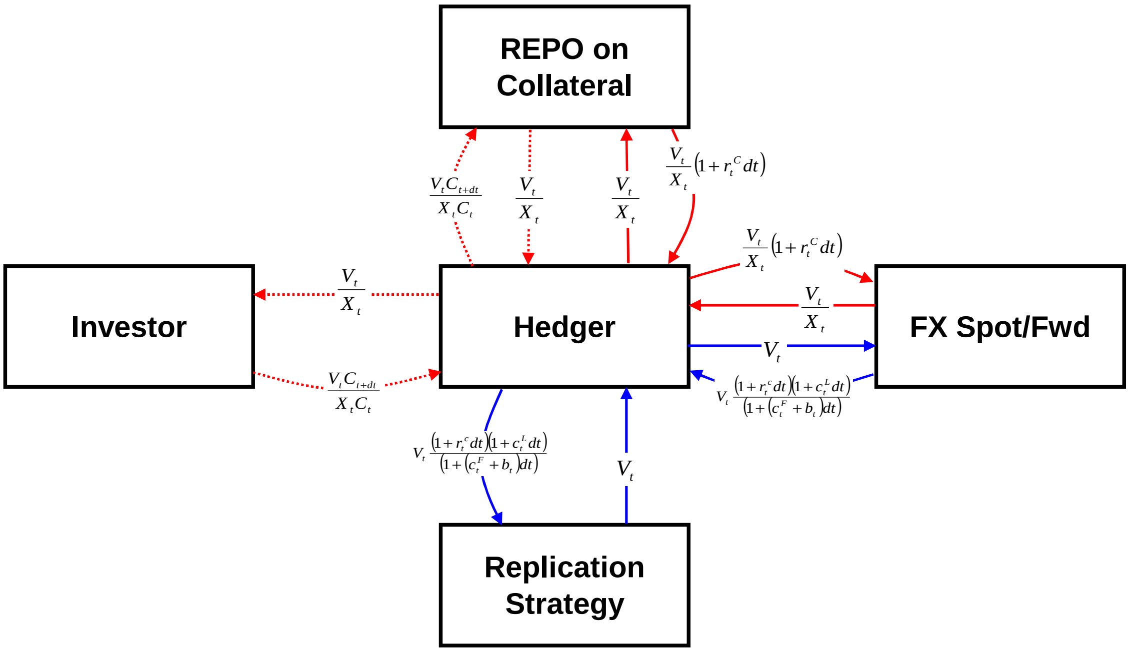

Figure 1: Continuous lines represent cash transactions whereas discontinuous ones represent asset transactions. Blue lines indicate amounts denominated in currency L, whereas red ones represent cash or asset transactions denominated in currency F. Straight lines refer to initial transactions, that take place at time t, and curved lines to final transactions taking place at time $t + dt$.

So that from t to $t + dt$ the value of the amount deposited as collateral experiences a variation equal to:

$$V_t \left(i_t^L + r_t^C - i_t^F - b_t \right) dt \tag{1}$$

Appart from the collateral posting mechanism and its mentioned hedges, the derivatives hedger must also hedge the delta risk associated with the underlying asset. In order to do so, the hedger

- Enters into a REPO transaction under which he receives $\alpha_t S_t$ (assuming α_t shares of the underlying have to be purchased (or sold if negative)) in cash at time t.
- With these proceeds the hedger purchases the delta position in the underlying asset.
- The asset is delivered to the REPO counterparty at time t.
- At time $t + dt$ the REPO counterparty delivers both the asset position (with a value of $\alpha_t S_{t+dt}$) and the dividends paid by it in the differential time interval (with a value of $\alpha_t S_t q_t dt$).

- At time $t+dt$ the hedger delivers cash to the REPO counterparty (with a value of $\alpha_t S_t \left(1 + r_t^S dt\right)$).

Notice that if α_t was negative, the trades will be right the opposite.

So that from t to $t+dt$ the delta hedging generates a change in the hedging portfolio equal to

$$\alpha_t S_{t+dt} + \alpha_t S_t q_t dt - \alpha_t S_t \left(1 + r_t^S dt\right) = \alpha_t dS_t + \alpha_t S_t \left(q_t - r_t^S\right) dt \tag{2}$$

And replication implies that the change in the value of the derivative must be equal to the change in the value of the hedging portfolio. Therefore

$$dV_t = \alpha_t dS_t + \alpha_t S_t \left(q_t - r_t^S\right) dt + V_t \left(i_t^L + r_t^C - i_t^F - b_t\right) dt$$

Being a derivative on S and assuming any other market variable to be non stochastic, V_t must be a function of t and S_t. Assuming that under the real world measure S_t evolves accordingly with the following stochastic differential equation:

$$dS_t = \mu_t^{\mathbb{P}} S_t dt + \sigma_t S_t dW_t^{\mathbb{P}}$$

Which toghether with Itô's Lemma imply

$$\frac{\partial V_t}{\partial t} dt + \frac{\partial V_t}{\partial S_t} dS_t + \frac{1}{2} S_t^2 \sigma_t^2 \frac{\partial^2 V_t}{\partial S_t^2} dt = \alpha_t dS_t + \alpha_t S_t \left(q_t - r_t^S\right) dt + V_t \left(i_t^L + r_t^C - i_t^F - b_t\right) dt$$

In order to be hedged, the term in dS_t must be equal in both sides of the equation, which implies that $\alpha_t = \frac{\partial V_t}{\partial S_t}$, so that

$$\frac{\partial V_t}{\partial t} + S_t \left(r_t^S - q_t\right) \frac{\partial V_t}{\partial S_t} + \frac{1}{2} S_t^2 \sigma_t^2 \frac{\partial^2 V_t}{\partial S_t^2} = \left(i_t^L + r_t^C - i_t^F - b_t\right) V_t \tag{3}$$

is the partial differential equation followed by the derivative price V_t with boundary condition $V_T = V(T, S_T)$.

The solution to (3) is equivalent to calculating the following expected value

$$V_t = E_{\mathbb{Q}} \left[\exp\left(-\int_{u=t}^{T} i_u^L du\right) V_T - \int_{u=t}^{T} \exp\left(-\int_{v=t}^{u} i_v^L dv\right) V_u \left(r_u^C - i_u^F - b_u\right) \middle| \mathcal{F}_t \right] \tag{4}$$

Where $\mathbb{Q}$ is a measure under which S_t has a drift equal to $r_t^S - q_t$.

In order to prove (1) we just have to apply Itô's Lemma to the process $V_t \exp\left(-\int_{u=0}^{t} i_u^L du\right)$ under $\mathbb{Q}$, integrate between t and T and take the expected value conditional to the market information available at time t $\mathcal{F}_t$.

Notice that the second term inside the expected value of the right hand side of equation is an adjustment term to be applied to the price that we would have obtained if collateral was cash in currency L. The reason for this adjustment is due to the exotic (non standard) collateral asset being used. The adjustment has a term that depends on the REPO OIS basis of the collateral asset, that is $r_u^C - i_u^F$, and a second term due to the cross currency basis b_t.

Defining $h_t := i_t^L + r_t^C - i_t^F - b_t$ then the solution to (3) is also equivalent to calculating the following expected value

$$V_t = E_{\mathbb{Q}}\left[\exp\left(-\int_{u=t}^{T} h_u du\right) V_T \middle| \mathcal{F}_t\right] \tag{5}$$

In order to prove (5) we just have to apply Itô's Lemma to the process $V_t \exp\left(-\int_{u=0}^{t} h_u du\right)$ under $\mathbb{Q}$, integrate between t and T and take the expected value conditional to the market information available at time t $\mathcal{F}_t$.

Notice that both (4) and (5) are expected values under the same measure.

2 The spot martingale measure

Assume two different derivatives on the same underlying S with the same payoff function $g(S_T)$ at a future time T. The only difference with the two derivatives is that one of them is collateralized with the underlying asset and the other with a generic collateral.

The time t prices of the two derivatives will be represented by V_t^S (collateralized with the underlying) and by V_t^C (collateralized with a generic collateral asset). Applying the results obtained in the last section, the PDE's to be solved in each case are:

$$\frac{\partial V_t^S}{\partial t} + S_t\left(r_t^S - q_t\right)\frac{\partial V_t^S}{\partial S_t} + \frac{1}{2}S_t^2\sigma_t^2\frac{\partial^2 V_t^S}{\partial S_t^2} = r_t^S V_t^S \tag{6}$$

$$\frac{\partial V_t^C}{\partial t} + S_t\left(r_t^S - q_t\right)\frac{\partial V_t^C}{\partial S_t} + \frac{1}{2}S_t^2\sigma_t^2\frac{\partial^2 V_t^C}{\partial S_t^2} = h_t V_t^C \tag{7}$$

With boundary conditions $V_T^S = V_T^C = g(S_T)$

So that V_t^S and V_t^C are given by the following expected values

$$V_t^S = E_{\mathbb{Q}}\left[\exp\left(-\int_{u=t}^{T} r_u^S du\right) V_T^S \middle| \mathcal{F}_t\right] \tag{8}$$

$$V_t^C = E_{\mathbb{Q}}\left[\exp\left(-\int_{u=t}^{T} h_u du\right) V_T^C \middle| \mathcal{F}_t\right] \tag{9}$$

It is important to notice that

- Both expected values are calculated under the same pricing measure $\mathbb{Q}$ defined as the measure under which the drift of the underlying is $r_t^S - q_t$
- The process $\frac{V_t^S}{\exp\left(\int_{u=0}^{t} r_u^S du\right)}$ is a $\mathbb{Q}$-martingale.
- The process $\frac{V_t^C}{\exp\left(\int_{u=0}^{t} h_u du\right)}$ is a $\mathbb{Q}$-martingale.

It seems that, contrary to the situation reflected in the classical quantitative finance literature, where a martingale measure was associated with a single numeraire, in the presence of collateralization mechanisms under which different collateral assets imply different carries, for a single pricing measure there are several numeraires, as many as different collateral assets are available. It also seems that under the spot martingale measure, in order to obtain martingales, each derivative should be divided by a current account that accrues at a rate equal to the short term carry rate produced by posting (or receiving) the collateral asset and hedging its risks.

In the remaining of the paper we will try to confirm this result under a different pricing measure.

3 The martingale measure associated with the underlying asset

It is well known that the current account is not the only valid numeraire. A self financing portfolio produced by purchasing the underlying asset and reinvesting the dividends paid on it can also be used. Nevertheless, remember that in the last section it seemed that two different current accounts were used to deflate derivatives prices depending on the collateral asset being used. In this section we are about to see that a similar situation will arise when we try to use the underlying asset as numeraire.

We will first deal with V_t^S, that is, the derivative collateralized with the underlying asset.

Replication means that at a generic time interval t the value of the derivative must be equal to the value of the hedging portfolio

$$V_t^S = \alpha_t S_t + \beta_t$$

Where β_t represents the value of collateral less the value of the liability created while purchasing the delta position (or asset if $\alpha_t < 0$).

We define a process $N_t^S := S_t \exp\left(\int_{u=0}^t z_u du\right)$. N_t^S will represent the numeraire used to divide derivatives collateralized in the underlying asset. z_t will be determined by imposing the martingale condition.

If we divide every term of the replication equation by N_t^S

$$\frac{V_t^S}{N_t^S} = \alpha_t \frac{S_t}{N_t^S} + \frac{\beta_t}{N_t^S}$$

Defining $\overline{V}_t^S := \frac{V_t^S}{N_t^S}$

$$\overline{V}_t^S = \alpha_t \exp\left(-\int_{u=0}^t z_u du\right) + \frac{\beta_t}{S_t} \exp\left(-\int_{u=0}^t z_u du\right) \tag{10}$$

And in differential form

$$\frac{\partial \overline{V}_t^S}{\partial t} dt + \frac{\partial \overline{V}_t^S}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 \overline{V}_t^S}{\partial S_t^2} S_t^2 \sigma_t^2 dt =$$
$$\exp\left(-\int_{u=0}^t z_u du\right) \left(-z_t \alpha_t dt + \frac{d\beta_t}{S_t} - z_t \frac{\beta_t}{S_t} dt - \frac{\beta_t}{S_t^2} dS_t + \frac{\beta_t}{S_t} \sigma_t^2 dt\right)$$

$d\beta_t$ represents the change experienced by the replicating portfolio excluding $\alpha_t dS_t$. Therefore, according to section 1

$$d\beta_t = +\alpha_t S_t \left(q_t - r_t^S\right) dt + V_t^S \left(i_t^L + r_t^C - i_t^F - b_t\right) dt$$

Since the derivative is collateralized with the underlying asset, $r_t^C = r_t^S,\ i_t^F = i_t^L,\ b_t = 0$. so that

$$d\beta_t = +\alpha_t S_t \left(q_t - r_t^S\right) dt + r_t^S V_t^S dt$$

$$\frac{\partial \overline{V}_t^S}{\partial t} dt + \frac{\partial \overline{V}_t^S}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 \overline{V}_t^S}{\partial S_t^2} S_t^2 \sigma_t^2 dt =$$
$$\exp\left(-\int_{u=0}^t z_u du\right)\left(-z_t\alpha_t dt + \frac{\alpha_t S_t\left(q_t - r_t^S\right) + r_t^S V_t^S}{S_t} dt - z_t\frac{\beta_t}{S_t}dt - \frac{\beta_t}{S_t^2}dS_t + \frac{\beta_t}{S_t}\sigma_t^2 dt\right) \tag{11}$$

In order to be hedged, the terms in dS_t in both sides of equation (11) should be equal, so that:

$$\beta_t = -S_t^2 \frac{\partial \overline{V}_t^S}{\partial S_t} \tag{12}$$

Which together with (10) imply

$$\alpha_t = \frac{V_t^S}{S_t} + S_t \frac{\partial \overline{V}_t^S}{\partial S_t} \tag{13}$$

Substituting (12) and (13) in (11)

$$\frac{\partial \overline{V}_t^S}{\partial t} + \frac{1}{2}\frac{\partial^2 \overline{V}_t^S}{\partial S_t^2} S_t^2 \sigma_t^2 =$$
$$\exp\left(-\int_{u=0}^t z_u du\right)\left(-z_t\frac{V_t^S}{S_t} - z_t S_t \frac{\partial \overline{V}_t^S}{\partial S_t} + (q_t - r_t^S)\frac{V_t^S}{S_t} + (q_t - r_t^S)S_t\frac{\partial \overline{V}_t^S}{\partial S_t} + r_t^S\frac{V_t^S}{S_t} + z_t S_t\frac{\partial \overline{V}_t^S}{\partial S_t} - \sigma_t^2 S_t \frac{\partial \overline{V}_t^S}{\partial S_t}\right) \tag{14}$$

And canceling terms

$$\frac{\partial \overline{V}_t^S}{\partial t} + (r_t^S - q_t + \sigma_t^2)S_t\frac{\partial \overline{V}_t^S}{\partial S_t} + \frac{1}{2}\frac{\partial^2 \overline{V}_t^S}{\partial S_t^2} S_t^2 \sigma_t^2 = \exp\left(-\int_{u=0}^t z_u du\right)\left((q_t - z_t)\frac{V_t^S}{S_t}\right) \tag{15}$$

For $\overline{V}_t^S$ to be a martingale in a measure under which the drift of S_t is given by $r_t^S - q_t + \sigma_t^2$ (we will call this measure $\mathbb{H}$), z_t must be equal to q_t, so that the right hand side of the equation equals zero. Therefore, the numeraire associated with derivatives collateralized with the underlying asset is

$$N_t^S = S_t \exp\left(\int_{u=0}^t q_u du\right)$$

and the PDE followed by $\overline{V}_t^S$

$$\frac{\partial \overline{V}_t^S}{\partial t} + (r_t^S - q_t + \sigma_t^2)S_t\frac{\partial \overline{V}_t^S}{\partial S_t} + \frac{1}{2}\frac{\partial^2 \overline{V}_t^S}{\partial S_t^2} S_t^2 \sigma_t^2 = 0 \tag{16}$$

N_t^S represents the evolution of a self financing portfolio where dividends paid by the underlying asset are reinvested on it. The fact that this is the only valid numeraire in situations with a dividend paying stock and the fact that this measure implies a drift in the underlying of $r_t - q_t + \sigma_t^2$ was already known in the classical literature, the only difference now is the fact that the REPO rate associated with the underlying r_t^S replaces the risk free rate r_t.

The solution to (12) is

$$\frac{V_t^S}{S_t} = E_{\mathbb{H}}\left[\frac{V_T^S}{S_T \exp\left(\int_{u=t}^T q_u du\right)}\middle|\mathcal{F}_t\right] \tag{17}$$

And as already stated, $\mathbb{H}$ is the measure under which the drift of S_t is given by $r_t^S - q_t + \sigma_t^2$

Now we handle the situation where the derivative is colateralized with a generic asset that implies a carry of h_t. Remember that in section 1 we saw that $h_t = i_t^L + r_t^C - i_t^F - b_t$.

Again, the replication equation is

$$V_t^C = \alpha_t S_t + \beta_t$$

Where V_t^C represents the t value of the derivative collateralized in an asset with short term carry h_t.

We again define a process $N_t^C := S_t \exp\left(\int_{u=0}^t z_u du\right)$. N_t^C will represent the numeraire used to divide derivatives collateralized in the asset C. z_t will again be determined by imposing the martingale condition.

If we divide every term of the replication equation by N_t^C

$$\frac{V_t^C}{N_t^C} = \alpha_t \frac{S_t}{N_t^C} + \frac{\beta_t}{N_t^C}$$

Defining $\widetilde{V}_t^C := \frac{V_t^C}{N_t^C}$

$$\widetilde{V}_t^C = \alpha_t \exp\left(-\int_{u=0}^t z_u du\right) + \frac{\beta_t}{S_t}\exp\left(-\int_{u=0}^t z_u du\right) \tag{18}$$

And in differential form

$$\begin{aligned}&\tfrac{\partial \widetilde{V}_t^C}{\partial t}dt + \tfrac{\partial \widetilde{V}_t^C}{\partial S_t}dS_t + \tfrac{1}{2}\tfrac{\partial^2 \widetilde{V}_t^C}{\partial S_t^2}S_t^2\sigma_t^2 dt = \\ &\exp\left(-\textstyle\int_{u=0}^t z_u du\right)\left(-z_t\alpha_t dt + \tfrac{d\beta_t}{S_t} - z_t\tfrac{\beta_t}{S_t}dt - \tfrac{\beta_t}{S_t^2}dS_t + \tfrac{\beta_t}{S_t}\sigma_t^2 dt\right)\end{aligned}$$

$d\beta_t$ equals

$$d\beta_t = +\alpha_t S_t\left(q_t - r_t^S\right)dt + V_t^S\underbrace{\left(i_t^L + r_t^C - i_t^F - b_t\right)}_{h_t}dt$$

$$\Downarrow$$

$$\begin{aligned}&\tfrac{\partial \widetilde{V}_t^C}{\partial t}dt + \tfrac{\partial \widetilde{V}_t^C}{\partial S_t}dS_t + \tfrac{1}{2}\tfrac{\partial^2 \widetilde{V}_t^C}{\partial S_t^2}S_t^2\sigma_t^2 dt = \\ &\exp\left(-\textstyle\int_{u=0}^t z_u du\right)\left(-z_t\alpha_t dt + \tfrac{\alpha_t S_t\left(q_t - r_t^S\right) + h_t V_t^C}{S_t}dt - z_t\tfrac{\beta_t}{S_t}dt - \tfrac{\beta_t}{S_t^2}dS_t + \tfrac{\beta_t}{S_t}\sigma_t^2 dt\right)\end{aligned} \tag{19}$$

α_t and β_t are again given by

$$\beta_t = -S_t^2 \frac{\partial \widetilde{V}_t^C}{\partial S_t} \quad \alpha_t = \frac{V_t^C}{S_t} + S_t \frac{\partial \widetilde{V}_t^C}{\partial S_t} \tag{20}$$

Substituting (20) in (19)

$$\frac{\partial \widetilde{V}_t^C}{\partial t} + \frac{1}{2}\frac{\partial^2 \widetilde{V}_t^C}{\partial S_t^2} S_t^2 \sigma_t^2 =$$
$$\exp\left(-\int_{u=0}^t z_u du\right)\left(-z_t \frac{V_t^C}{S_t} - z_t S_t \frac{\partial \widetilde{V}_t^C}{\partial S_t} + (q_t - r_t^S)\frac{V_t^C}{S_t} + (q_t - r_t^S) S_t \frac{\partial \widetilde{V}_t^C}{\partial S_t} + h_t \frac{V_t}{S_t} + z_t S_t \frac{\partial \widetilde{V}_t^C}{\partial S_t} - \sigma_t^2 S_t \frac{\partial \widetilde{V}_t^C}{\partial S_t}\right) \tag{21}$$

And canceling terms

$$\frac{\partial \widetilde{V}_t^C}{\partial t} + (r_t^S - q_t + \sigma_t^2) S_t \frac{\partial \widetilde{V}_t^C}{\partial S_t} + \frac{1}{2}\frac{\partial^2 \widetilde{V}_t^C}{\partial S_t^2} S_t^2 \sigma_t^2 = \exp\left(-\int_{u=0}^t z_u du\right)\left((q_t - z_t - r_t^S + h_t)\frac{V_t^C}{S_t}\right) \tag{22}$$

For $\widetilde{V}_t^C$ to be a martingale in $\mathbb{H}$, under which the drift of S_t is given by $r_t^S - q_t + \sigma_t^2$ (same as in the case where the derivative was collateralized with the underlying asset), z_t must be equal to $q_t - r_t^S + h_t$, so that again the right hand side of the equation equals zero. Therefore, the numeraire associated with derivatives collateralized with a generic collateral asset with short term carry h_t is

$$N_t^C = S_t \exp\left(\int_{u=0}^t (q_u + h_u - r_u^S)\, du\right)$$

and the PDE followed by $\overline{V}_t^S$

$$\frac{\partial \widetilde{V}_t^C}{\partial t} + (r_t^S - q_t + \sigma_t^2) S_t \frac{\partial \widetilde{V}_t^C}{\partial S_t} + \frac{1}{2}\frac{\partial^2 \widetilde{V}_t^C}{\partial S_t^2} S_t^2 \sigma_t^2 = 0 \tag{23}$$

N_t^C represents the evolution of a self financing portfolio under which we initially invest in the underlying asset, but we continuously enter into two REPO transactions, the first with S_t and the second with C_t as the underlying assets. Under the first REPO transaction we deliver the position in S_t against cash (and pay an interest r_t^S on it). This cash is converted in currency L and delivered under the second REPO transaction against a position in C_t being received as collateral. Under this second transaction we get a return of r_t^C. Nevertheless, that in order not to incur in FX risk, we will have to sell the amount to be received in F through this second REPO transaction forward. This will leave us with a net carry of $h_t - r_t^S$. Dividends paid by S_t are also reinvested.

The solution of (23) is

$$\frac{V_t^C}{S_t} = E_{\mathbb{H}}\left[\frac{V_T^C}{S_T \exp\left(\int_{u=t}^T (q_u + h_u - r_u^S)\, du\right)} \middle| \mathbb{F}_t\right] \tag{24}$$

And as already stated, $\mathbb{H}$ is the measure under which the drift of S_t is given by $r_t^S - q_t + \sigma_t^2$

To summarize, V_t^S and V_t^C are given by the following expected values

$$\frac{V_t^S}{S_t} = E_{\mathbb{H}}\left[\frac{V_T^S}{S_T \exp\left(\int_{u=t}^T q_u du\right)} \middle| \mathcal{F}_t\right]$$

$$\frac{V_t^C}{S_t} = E_{\mathbb{H}}\left[\frac{V_T^C}{S_T \exp\left(\int_{u=t}^{T}\left(q_u + h_u - r_u^S\right) du\right)}\middle|\mathbb{F}_t\right]$$

It is important to notice that

- Both expected values are calculated under the same pricing measure $\mathbb{H}$ defined as the measure under which the drift of the underlying is $r_t^S - q_t + \sigma_t^2$
- The process $\frac{V_t^S}{S_t \exp\left(\int_{u=0}^{t} q_u du\right)}$ is a $\mathbb{H}$-martingale.
- The process $\frac{V_t^C}{S_t \exp\left(\int_{u=0}^{t}(q_u+h_u-r_u^S)du\right)}$ is a $\mathbb{H}$-martingale.

4 The Radon-Nikodym derivative in the new framework

Taking into account the results obtained in sections 2 and 3, lets first analyze the case of V_t^S (derivative collateralized with the underlying asset)

$$V_t^S = E_{\mathbb{Q}}\left[\frac{V_T^S}{\exp\left(\int_{u=t}^{T} r_u^S du\right)}\middle|\mathcal{F}_t\right] = S_t E_{\mathbb{H}}\left[\frac{V_T^S}{S_T \exp\left(\int_{u=t}^{T} q_u du\right)}\middle|\mathcal{F}_t\right]$$

So that

$$V_t^S = E_{\mathbb{Q}}\left[\underbrace{\frac{V_T^S}{\exp\left(\int_{u=t}^{T} r_u^S du\right)}}_{X_T}\middle|\mathcal{F}_t\right] = E_{\mathbb{H}}\left[\underbrace{\frac{V_T^S}{\exp\left(\int_{u=t}^{T} r_u^S du\right)}}_{X_T}\underbrace{\frac{S_t \exp\left(\int_{u=0}^{t} q_u du\right)}{S_T \exp\left(\int_{u=0}^{T} q_u du\right)}\frac{\exp\left(\int_{u=0}^{T} r_u^S du\right)}{\exp\left(\int_{u=0}^{t} r_u^S du\right)}}_{\frac{d\mathbb{Q}}{d\mathbb{H}}(t,T)}\middle|\mathcal{F}_t\right]$$

So that the Radon-Nikodym derivative takes the expression of the ratio of numeraires at time t times the inverse of the ratio of numeraires at time T. The numeraires we are referring to are the ones used to deflate derivatives collateralized with the underlying.

In the case of V_t^C (derivatives collateralized with C_t), we saw in sections 2 and 3 that

$$V_t^C = E_{\mathbb{Q}}\left[\frac{V_T^C}{\exp\left(\int_{u=t}^{T} h_u du\right)}\middle|\mathcal{F}_t\right] = S_t E_{\mathbb{H}}\left[\frac{V_T^C}{S_T \exp\left(\int_{u=t}^{T}\left(q_u + h_u - r_u^S\right) du\right)}\middle|\mathbb{F}_t\right]$$

So that

$$V_t^S = E_{\mathbb{Q}}\left[\underbrace{\frac{V_T^S}{\exp\left(\int_{u=t}^{T} h_u du\right)}}_{X_T}\middle|\mathcal{F}_t\right] = E_{\mathbb{H}}\left[\underbrace{\frac{V_T^S}{\exp\left(\int_{u=t}^{T} h_u du\right)}}_{X_T}\underbrace{\frac{S_t \exp\left(\int_{u=0}^{t}\left(q_u + h_u - r_u^S\right) du\right)}{S_T \exp\left(\int_{u=0}^{T}\left(q_u + h_u - r_u^S\right) du\right)}\frac{\exp\left(\int_{u=0}^{T} h_u du\right)}{\exp\left(\int_{u=0}^{t} h_u du\right)}}_{\frac{d\mathbb{Q}}{d\mathbb{H}}(t,T)}\middle|\mathcal{F}_t\right]$$

So that the Radon-Nikodym derivative again takes the expression of the ratio of numeraires at time t times the inverse of the ratio of numeraires at time T. The numeraires we are referring to are the ones used to deflate derivatives collateralized with a generic collateral asset C_t.

It is very important to notice that the expression of the Radon-Nikodym derivative is the same in both cases, as we would expect with a single change of measure.

Notice also that under both $\mathbb{Q}$ and $\mathbb{H}$, the ratio of numeraires used to deflate prices of derivatives collateralized with C_t and S_t is given by the same expression

$$\frac{N_t^{C,\mathbb{Q}}}{N_t^{S,\mathbb{Q}}} = \frac{N_t^{C,\mathbb{H}}}{N_t^{S,\mathbb{H}}} = \exp\left(\int_{u=0}^{t} \left(h_u - r_u^S\right) du\right)$$

5 Conclussions

- In a simplified framework under which interest rates, REPO rates and cross currency basis spread are all non stochastic, we have seen that in order to price derivatives that are collateralized, where we could have several collateral assets available, the fundamental theorem of asset pricing needs to be reformulated.
- Contrary to the classical result, under a single pricing measure we have different numeraires, as many as collateral assets are available.
- In order to obtain martingales, each numeraire (associated not only to a pricing measure, but also to a collateral asset) is used to deflate derivatives collateralized in the particular collateral that defines the numeraire.
- The ratio of numeraires associated with different collateral mechanisms is equal under any pricing measure.
- The expression of the Radon-Nikodym derivative is still related with the ratio of numeraires times its inverse at different time instants. This expression does not depend on the particular collateral mechanism being used.

Nevertheless these conclusions have all been obtained under the assumption of non stochastic interest rates, REPO rates and spreads and in a single currency environment. All of these results should be confirmed in the most general situation.

References

[1] Vladimir Piterbarg, Funding Beyond Discounting. Collateral agreements and derivatives pricing. Risk February 2010.

[2] Fujii M and A Takahashi, Choice of collateral currency. Risk January 2011.

[3] Vladimir Piterbarg, Cooking with collateral. Risk August 2012.