Free disposal, monotonicity and equilibrium

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Abstract

This paper studies the effect of free disposal on the existence of Walrasian equilibrium for exchange economies with indivisible objects. It is shown that allowing an agent to enjoy free disposal has the same effect for generating an equilibrium (or eliminating existing equilibria) as allowing every agent to enjoy free disposal. A new equilibrium existence theorem is given to show how this observation can enhance the existence results by Kelso and Crawford (1982) and Sun and Yang (2006).

Keywords: Indivisibility; equilibrium; free disposal; monotonic cover.

1 Introduction

This paper studies the effect of free disposal on the existence of Walrasian equilibrium for an exchange economy with indivisible objects. The assumption of free disposal is often applied to ensure that each agent's preferences satisfy monotonicity.[1] The intuition behind this argument is that when an agent is allowed to discard unwanted objects for free, adding objects to the agent's bundle never makes the agent worse off. In this paper, we use the notion of monotonic cover to formulate the effect of free disposal on agents' preferences. Namely, we assume that when free disposal is available to an agent, the agent's original utility function would be replaced by its monotonic cover.

Free disposal not only changes agents' preferences, but it also possibly affects the existence of Walrasian equilibrium. The motivation of the paper is to investigate conditions under which the existence of equilibrium can be free from the effect of free disposal. One of our main results (Theorem 1) shows that free disposal has

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[1] See, for example, Bikhchandani and Mamer (1997), Ma (1998), Gul and Stacchetti (1999), and Fujishige and Yang (2003) among others.

no effect on the existence of equilibrium if there exists an agent whose preferences satisfy monotonicity. One interpretation for this observation is that allowing an agent to enjoy free disposal has the same effect for generating an equilibrium (or eliminating existing equilibria) as allowing every agent to enjoy free disposal. Thus, when the effect of free disposal is helpful to yield an equilibrium for an economy, it is sufficient to offer free disposal to some agent.

Moreover, in sight of Theorem 1, we note that each equilibrium existence result can be correspondingly extended to yield a new existence theorem with the aid of the notion of monotonic cover. To clarify the impact of this observation, we recall two conditions on preferences, namely, the *gross substitutes* (GS) condition (Kelso and Crawford, 1982) and the *gross substitutes and complements* (GSC) condition (Sun and Yang, 2006), each of which can ensure the existence of Walrasian equilibrium, and show how these two conditions can be used to generate new existence results.

Finally, we note that the effect of free disposal cannot destroy the gross substitutability of preferences. Namely, the monotonic cover of a utility function satisfying the GS condition cannot fail the GS condition. This result means that when the existence of Walrasian equilibrium is guaranteed by the GS condition, it is immune to the effect of free disposal.

The rest of the paper is organized as follows. Section 2 gives the model and fundamental definitions. Section 3 establishes the main theorems and Section 4 concludes.

2 Preliminaries

Consider an exchange economy with a finite set $N = \{1, \ldots, n\}$ of agents and a finite set $\Omega = \{a_1, \ldots, a_m\}$ of indivisible objects, and a perfectly divisible good called money. Each agent $i \in N$ has quasi-linear preferences, namely, i's utility equals $u_i(A) - c$ from consuming a bundle $A \subseteq \Omega$ in return for payment c, where $u_i : 2^\Omega \to \mathbb{R}$ with $u_i(\emptyset)$ denotes the utility function of agent i. Moreover, we assume that each agent i is initially endowed with a sufficient amount of money $M_i > u_i(A)$ for all $A \subseteq \Omega$. Under these assumptions, each agent will not be subject to any budget constraint, and hence the initial endowment of objects to the agents is irrelevant for the efficient allocations and their supporting prices. Thus, we choose to leave the initial endowment of objects unspecified, and represent this exchange economy by $E = \left(\Omega; (u_i)_{i\in N}\right)$.

A *price vector* $p = (p_a)_{a\in\Omega} \in \mathbb{R}^\Omega$ assigns a price for each object a in Ω. For any bundle $A \subseteq \Omega$, let $p(A)$ be a shorthand for $\sum_{a\in A} p_a$, and for each object $a \in \Omega$, let $e^a \in \mathbb{R}^\Omega$ denote the characteristic vector whose i-th coordinate is 1 if $a = a_i$ and 0 otherwise.

For each agent i with utility function u_i, the *demand correspondence* $D_{u_i} : \mathbb{R}^\Omega \to 2^\Omega$ is defined by

$$D_{u_i}(p) := \arg\max_{A\subseteq\Omega} U_i(A, p),$$

where $U_i(A,p) := u_i(A) - p(A)$ denotes the utility of consuming the bundle A at price level p.

An *allocation* of objects for the economy $E = \left(\Omega; (u_i)_{i\in N}\right)$ is a partition of Ω, i.e., a set of mutually exclusive bundles $\mathbf{X} = (X_1, \ldots, X_n)$ that exhaust Ω, where X_i represents the set of object consumed by agent i under the allocation $\mathbf{X}$.

A *Walrasian equilibrium* for the economy $E = \left(\Omega; (u_i)_{i\in N}\right)$ is a pair $(\mathbf{X}, p)$, where $\mathbf{X} = (X_1, \ldots, X_n)$ is an allocation and $p \in \mathbb{R}^{\Omega}$ is a price vector such that for each agent $i \in N$, $u_i(X_i) - p(X_i) \geq u_i(A) - p(A)$ for each bundle $A \subseteq \Omega$, i.e., $X_i \in D_{u_i}(p)$. In that case, $\mathbf{X}$ is called an equilibrium allocation and p is called an equilibrium price.

The utility function $u_i : 2^{\Omega} \to \mathbb{R}$ is called *monotone* if for all $B \subseteq A \subseteq \Omega$, $u_i(B) \leq u_i(A)$. The *monotonic cover* of u_i is the utility function $\hat{u}_i : 2^{\Omega} \to \mathbb{R}$ given by $\hat{u}_i(A) = \max_{B\subseteq A} u_i(B)$ for each $A \subseteq \Omega$. Note that u_i is monotone if and only if $u_i = \hat{u}_i$. The *monotonic cover* of an economy $E = \left(\Omega; (u_i)_{i\in N}\right)$ is defined to be $\hat{E} = \left(\Omega; (\hat{u}_i)_{i\in N}\right)$. In case $E = \hat{E}$, we call E an *economy with free disposal.* Let $\mathcal{E}$ denote the class of economies in which there exists at least one agent whose utility function is monotone, and let $\hat{\mathcal{E}}$ denote the class of economies with free disposal. Clearly, we have $\hat{\mathcal{E}} \subseteq \mathcal{E}$.

An interpretation for the relation between a utility function and its monotonic cover is that once discarding unwanted objects becomes costless for an agent i, i's original utility function u_i would be replaced by its monotonic cover $\hat{u}_i$. Moreover, free disposal not only changes utility functions of agents in an economy, but also possibly affects the existence of Walrasian equilibrium. To illustrate these phenomenons, we consider the following two economies.

The first economy $E_1 = \left(\Omega_1; \left(u_i^1\right)_{i\in N_1}\right)$ with $\Omega_1 = \{a, b, c\}$ and $N_1 = \{1, 2\}$ is given by

$$u_1^1(A) = \begin{cases} 6, & \text{if } A = \{a,b,c\}, \\ 5, & \text{if } A = \{a\}, \\ 1, & \text{if } A = \{c\}, \\ 0, & \text{otherwise,} \end{cases} \qquad u_2^1(A) = \begin{cases} 7, & \text{if } A = \{a,b\}, \\ 5, & \text{if } A = \{a\} \text{ or } A = \{b\}, \\ 0, & \text{otherwise,} \end{cases}$$

and the second economy $E_2 = \left(\Omega_2; (u_i)_{i\in N_2}\right)$ with $\Omega_2 = \{a, b, c, a'\}$ and $N_2 = \{1, 2, 3\}$ is given by

$$u_1^2(A) = \begin{cases} 9, & \text{if } A = \{a,b\}, \\ 8, & \text{if } A = \{a,b,a'\}, \\ -6, & \text{if } A = \{a'\}, \\ 0, & \text{otherwise.} \end{cases} \quad u_2^2(A) = u_3^2(A) = \begin{cases} 9, & \text{if } A = \{a,c\} \text{ or } \{b,c\}, \\ 4, & \text{if } A = \{c\}, \\ -6, & \text{if } a' \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Note that E_1 has no Walrasian equilibrium, but $\hat{E}_1$ has a Walrasian equilibrium $\left(\mathbf{X}^1, p^1\right)$ with $X_1^1 = \{a, c\}$, $X_2^1 = \{b\}$, $p_a^1 = p_b^1 = 4$ and $p_c^1 = 0$. This means that allowing all the agents in the economy E_1 to enjoy free disposal is helpful to yield an equilibrium. On the other hand, the second economy E_2 illustrates that free disposal might destroy existing equilibria: E_2 has a Walrasian equilibrium $\left(\mathbf{X}^2, p^2\right)$ with $X_1^2 = \{a, b, a'\}$, $X_2^2 = \{c\}$, $X_3^2 = \emptyset$, $p_a^2 = p_b^2 = 6$, $p_c^2 = 4$ and $p_{a'}^2 = -5$, while $\hat{E}_2$ has no Walrasian equilibrium.

3 Existence of Walrasian equilibrium

The examples given at the end of Section 2 show that the existence of Walrasian equilibrium can be significantly affected by the free disposal condition. A natural question is under which conditions free disposal has no effect on the existence of equilibrium. The following result sheds some light on this issue by showing that if there exists an agent whose utility function is monotone, then the existence of equilibrium is free from the effect of free disposal.

Theorem 1 *Let $E = \left(\Omega; (u_i)_{i \in N}\right)$ be an economy with an agent $j \in N$ whose utility function u_j is monotone, i.e., $E \in \mathcal{E}$.*

(a) *Each equilibrium allocation for E is an equilibrium allocation for $\hat{E}$.*

(b) *Each equilibrium price vector p for $\hat{E}$ is an equilibrium price vector for E.*

(c) *E has a Walrasian equilibrium if and only if $\hat{E}$ has a Walrasian equilibrium.*

The proof of Theorem 1 requires the following lemma.

Lemma 2 *Let $E = \left(\Omega; (u_i)_{i \in N}\right)$ be an economy and let $j \in N$ be an agent whose utility function u_j is monotone. If $(\mathbf{X}, p)$ is a Walrasian equilibrium for E, then $\{a \in \Omega : p_a < 0\} \subseteq X_j$.*

Proof. Suppose that there exists $a \in \Omega \backslash X_j$ such that $p_a < 0$. Since u_j is monotone, we have

$$u_j\left(X_j \cup \{a\}\right) - p\left(X_j \cup \{a\}\right) \geq u_j\left(X_j\right) - p\left(X_j\right) - p_a > u_j\left(X_j\right) - p\left(X_j\right),$$

violating the assumption that $(\mathbf{X}, p)$ is a Walrasian equilibrium for E. ■

We are now ready to prove Theorem 1.

Proof of Theorem 1. (a) Assume that $(\mathbf{X}, p)$ is a Walrasian equilibrium for E. Let $p' \in \mathbb{R}_+^{\Omega}$ be the price vector given by

$$p_a^{'} = \begin{cases} p_a, & \text{if } p_a \geq 0, \\ 0, & \text{if } p_a < 0. \end{cases}$$

Clearly, $p' \geq p$. We are going to show that $(\mathbf{X}, p')$ is a Walrasian equilibrium for $\hat{E}$.

We first show that $(\mathbf{X}, p')$ is a Walrasian equilibrium for E. Let $\bar{A} = \{a \in \Omega : p_a < 0\}$. In case $\bar{A} = \emptyset$, then $p' = p$ and we have done. In case $\bar{A} \neq \emptyset$, by Lemma 2, we have $\bar{A} \subseteq X_j$. It follows that for any bundle $A \subseteq \Omega$,

$$U_j\left(X_j, p'\right) = U_j\left(X_j, p\right) + p\left(\bar{A}\right) \geq U_j\left(A \cup \bar{A}, p\right) + p\left(\bar{A}\right) = U_j\left(A \cup \bar{A}, p'\right) \geq U_j\left(A, p'\right)$$

and for each agent $i \in N$ with $i \neq j$,

$$U_i\left(X_i, p'\right) = U_i\left(X_i, p\right) \geq U_i\left(A, p\right) \geq U_i\left(A, p'\right).$$

We then verify that

$$\hat{u}_i\left(X_i\right) = u_i\left(X_i\right) \text{ for each agent } i \in N. \tag{1}$$

Suppose, to the contrary, that $\hat{u}_i\left(X_i\right) > u_i\left(X_i\right)$ for some agent $i \in N$ with $i \neq j$. Then there exists a proper subset B of X_i such that $\hat{u}_i\left(X_i\right) = u_i\left(B\right) = \hat{u}_i\left(B\right)$. This implies $u_i\left(B\right) - p\left(B\right) > u_i\left(X_i\right) - p\left(B\right) \geq u_i\left(X_i\right) - p\left(X_i\right)$, violating the assumption that $(\mathbf{X}, p)$ is a Walrasian equilibrium for E.

Finally, suppose on the contrary that $(\mathbf{X}, p')$ is not a Walrasian equilibrium for $\hat{E}$. Then there exists an agent i with $i \neq j$ such that $\hat{u}_i\left(X_i\right) - p'\left(X_i\right) < \hat{u}_i\left(T\right) - p'\left(T\right)$ for some bundle $T \subseteq \Omega$. Since $(\mathbf{X}, p')$ is a Walrasian equilibrium for E, together with (1), we have

$$u_i\left(T\right) - p'\left(T\right) \leq u_i\left(X_i\right) - p'\left(X_i\right) = \hat{u}_i\left(X_i\right) - p'\left(X_i\right) < \hat{u}_i\left(T\right) - p'\left(T\right), \tag{2}$$

and hence $u_i\left(T\right) < \hat{u}_i\left(T\right)$. This implies that there exists some proper subset C of T such that $\hat{u}_i\left(T\right) = u_i\left(C\right)$. Combining with (2), we have

$$u_i\left(X_i\right) - p'\left(X_i\right) < u_i\left(C\right) - p'\left(T\right) \leq u_i\left(C\right) - p'\left(C\right),$$

violating the fact that $(\mathbf{X}, p')$ is a Walrasian equilibrium for E.

(b) Assume that $(\mathbf{X}, p)$ is a Walrasian equilibrium for $\hat{E}$. Clearly, $p \geq 0$. We are going to show that there exists a Walrasian equilibrium $(\mathbf{Y}, p)$ for E such that $Y_i \subseteq X_i$ and $\hat{u}_i\left(X_i\right) = u_i\left(Y_i\right) = \hat{u}_i\left(Y_i\right)$ for each agent i with $i \neq j$, and $Y_j = \left(\cup_{i \in N}\left(X_i \backslash Y_i\right)\right) \cup X_j$. Let i be an agent with $i \neq j$. We consider two cases.

Case I. $u_i\left(X_i\right) = \hat{u}_i\left(X_i\right)$. Let $Y_i = X_i$. Then for any bundle $A \subseteq \Omega$,

$$u_i\left(Y_i\right) - p\left(Y_i\right) = \hat{u}_i\left(X_i\right) - p\left(X_i\right) \geq \hat{u}_i\left(A\right) - p\left(A\right) \geq u_i\left(A\right) - p\left(A\right). \tag{3}$$

Case II. $u_i\left(X_i\right) < \hat{u}_i\left(X_i\right)$. Then there exists a proper subset Y_i of X_i such that $\hat{u}_i\left(X_i\right) = u_i\left(Y_i\right) = \hat{u}_i\left(Y_i\right)$. This implies

$$\hat{u}_i\left(X_i\right) - p\left(X_i\right) \geq \hat{u}_i\left(Y_i\right) - p\left(Y_i\right) = \hat{u}_i\left(X_i\right) - p\left(Y_i\right),$$

and hence $p_a = 0$ for all $a \in X_i \backslash Y_i$. It follows that (3) holds for any bundle $A \subseteq \Omega$.

Let $Y_j = \left(\cup_{i \in N}\left(X_i \backslash Y_i\right)\right) \cup X_j$. Since u_j is monotone, the combination of Cases I and II implies that for any bundle $A \subseteq \Omega$,

$$u_j\left(Y_j\right) - p\left(Y_j\right) = \hat{u}_i\left(Y_j\right) - p\left(X_j\right) \geq \hat{u}_j\left(X_j\right) - p\left(X_j\right) \geq \hat{u}_i\left(A\right) - p\left(A\right) \geq u_i\left(A\right) - p\left(A\right).$$

The result of (c) is an immediate consequence of the combination of (a) and (b). This completes the proof. ■

Theorem 1 has a number of significant consequences. First, the result of Theorem 1 (c) can be rephrased to illustrate that allowing *an* agent to enjoy free disposal has the same effect for generating an equilibrium (or eliminating existing equilibria) as allowing *every* agent to enjoy free disposal. Thus, when the effect of free disposal is helpful to yield an equilibrium for an economy, e.g., the economy E_1 given at the end of Section 2, it is sufficient to offer free disposal to some agent.

Theorem 3 *Let $E = \left(\Omega; (u_i)_{i \in N}\right)$ be an arbitrary economy. For any agent $j \in N$, the economy $E' = (\Omega; u_1, \ldots, \hat{u}_j, \ldots, u_n)$ has a Walrasian equilibrium if and only if $\hat{E}$ has a Walrasian equilibrium.*

Second, Theorem 1 establishes useful links between economies that share the same monotonic cover. Namely, for any two economies E_1 and E_2 in $\mathcal{E}$ such that $\hat{E}_1 = \hat{E}_2$, E_1 has a Walrasian equilibrium if and only if E_2 has a Walrasian equilibrium.

Third, Theorem 1 indicates that $\hat{\mathcal{E}}$, the class of economies with free disposal, plays a central role in analyzing the existence problem of equilibrium in the sense that each equilibrium existence theorem for economies in $\hat{\mathcal{E}}$ has a natural corresponding extension for economies in $\mathcal{E}$. To clarify the point, we shall recall two important conditions on utility functions, namely, the gross substitutes condition (Kelso and Crawford, 1982) and the gross substitutes and complements condition (Sun and Yang, 2006), each of which can guarantee the existence of Walrasian equilibrium, and discuss how these results can be extended to generate new existence theorems.

The utility function u_i satisfies the *gross substitutes* (GS) condition if for any two price vectors p and q with $q \geq p$, and any bundle $A \in D_{u_i}(p)$, there exists $B \in D_{u_i}(q)$ such that $\{a \in \Omega : q_a = p_a\} \subseteq B$. Thus, the GS condition ensures that the demand for an object does not decrease when prices of some other objects increase. Theorem 2 of Kelso and Crawford (1982, p. 1490) shows that if each agent's utility function satisfies the GS condition, then there exists a Walrasian equilibrium.

In contrast to Kelso and Crawford (1982), Sun and Yang (2006) study an economy $E = \left(\Omega; (u_i)_{i \in N}\right)$ in which all the objects in Ω can be divided into two groups S_1 and S_2, and show that if objects in the same group are substitutes and objects across these two groups are complements, then the economy has a Walrasian equilibrium. Formally, the utility function u_i satisfies the *gross substitutes and complements* (GSC) condition if for any price vector $p \in \mathbb{R}^{\Omega}$, $a \in S_j$, $\delta \geq 0$, and $A \in D_{u_i}(p)$, there exists $B \in D_{u_i}(p + \delta e^a)$ such that $[A \cap S_j] \setminus \{a\} \subseteq B \subseteq (A \cup S_j)$. When $S_1 = \emptyset$ or $S_2 = \emptyset$, the GSC condition reduces to the GS condition. However, it should be noted that when $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$, the GSC condition is logically independent from the GS condition. Theorem 3.1 of Sun and Yang (2006, p 1388) shows that if each agent's utility function satisfies the GSC condition, then there exists a Walrasian equilibrium.

The result of Theorem 1, together with Kelso and Crawford's Theorem 2 and Sun and Yang's Theorem 3.1, can yield new equilibrium existence results to cover economies in which agents' utility functions may fail the GS (or GSC) conditions.

Theorem 4 *Let $E = \big(\Omega; (u_i)_{i \in N}\big)$ be an economy with an agent $j \in N$ whose utility function u_j is monotone, i.e., $E \in \mathcal{E}$.*

(a) *If the monotonic cover $\hat{u}_i$ of each agent i's utility function satisfies the GS condition, then E has a Walrasian equilibrium.*

(b) *If the monotonic cover $\hat{u}_i$ of each agent i's utility function satisfies the GSC condition, then E has a Walrasian equilibrium.*

Proof. Assume that $\hat{u}_i$ satisfies the GS (respectively GSC) condition for each $i \in N$. Then Kelso and Crawford's Theorem 2 (respectively Sun and Yang's Theorem 3.1) implies that the economy $\hat{E} = \big(\Omega; (\hat{u}_i)_{i \in N}\big)$ has a Walrasian equilibrium. Since $E \in \mathcal{E}$, we obtain the desired result by Theorem 1. ■

Finally, we recall a non-existence result by Gul and Stacchetti (1999) and study its implications. Gul and Stacchetti focus on economies with free disposal, and prove that for economies in $\hat{\mathcal{E}}$, the class of utility functions satisfying the GS condition is a largest set for which the existence of Walrasian equilibrium is guaranteed. More precisely, Theorem 2 of Gul and Stacchetti (1999, p. 103) shows that for any agent 1 with a monotone utility function $u_1 : 2^\Omega \to \mathbb{R}$ that violates the GS condition, there exists a finite class of utility functions $\{u_2, \ldots, u_n\}$ such that $E = \big(\Omega; (u_i)_{i \in N}\big) \in \hat{\mathcal{E}}$ and u_i satisfies the GS condition for $i \neq 1$, but there does not exist any Walrasian equilibrium. In some sense, this non-existence theorem can be considered as a converse to Kelso and Crawford's existence result. A natural question is whether this non-existence result still holds for economies in $\mathcal{E}$. In the following result, we answer the question in the negative.

Proposition 5 *There exists a utility function $u_1 : 2^\Omega \to \mathbb{R}$ that violates the GS condition, but for any economy $E = \big(\Omega; (u_i)_{i \in N}\big) \in \mathcal{E}$ in which u_i satisfying the GS condition for $i \neq 1$, there exists a Walrasian equilibrium.*

The proof of Proposition 5 relies on the following lemma, which shows that the effect of free disposal cannot destroy the gross substitutability of a utility function.

Lemma 6 *If the utility function $u_i : 2^\Omega \to \mathbb{R}$ satisfies the GS condition, then the monotonic cover $\hat{u}_i$ of u_i satisfies the GS condition as well.*

Proof. Let $u_i : 2^\Omega \to \mathbb{R}$ be a utility function that satisfies the GS condition. Suppose on the contrary that $\hat{u}_i$ does not satisfies the GS condition. By Theorem 2 of Gul and Stacchetti, there exists an economy $E = (\Omega; u_1, \ldots, \hat{u}_i, \ldots, u_n) \in \hat{\mathcal{E}}$ that has no Walrasian equilibrium but u_k satisfies both the GS condition and monotonicity for $k \neq i$. Together with Theorem 1, we have that the economy $E = (\Omega; u_1, \ldots, u_i, \ldots, u_n)$ has no Walrasian equilibrium, contradicting to Theorem 2 of Kelso and Crawford. ■

We are now ready to prove Proposition 5.

Proof of Proposition 5. Let $\Omega = \{a, b, c\}$. Consider the utility function $u_1 : 2^\Omega \to \mathbb{R}$ given by

$$u_1(A) = \begin{cases} 1, & \text{if } A = \{a\} \text{ or } A = \{a, b, c\}, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, u_1 fails the GS condition while $\hat{u}_1$ satisfies the GS condition. Let $E = \left(\Omega; (u_i)_{i \in N}\right)$ be an economy in $\mathcal{E}$ such that u_i satisfying the GS condition for $i \neq 1$. The result of Lemma 6 implies that $\hat{u}_i$ satisfies the GS condition for $i \neq 1$. Combining Theorem 1 and Theorem 2 of Kelso and Crawford (1982), we obtain that the economy E, as well as its monotonic cover $\hat{E}$, has a Walrasian equilibrium. ■

We close this section with another implication of Lemma 6. Namely, when the existence of Walrasian equilibrium is ensured by the GS condition, it is free from the effect of free disposal.

4 Concluding remarks

This paper contributes to the literature on the existence of Walrasian equilibrium by analyzing the effect of free disposal. We use the notion of monotonic cover to embody the effect of free disposal and to extend existing results, including the works of Kelso and Crawford (1982), Sun and Yang (2006), and Gul and Stacchetti (1999). Most of our results focus on the existence of equilibrium. It might be interesting to study the effect of free disposal on the structure of equilibrium allocations as well as the structure of equilibrium payoff vectors following the line indicated by the results of Theorem 1 (a) and (b).

References

[1] BIKHCHANDANI, S., AND J. W. MAMER (1997): "Competitive Equilibrium in an Exchange Economy with Indivisibilities," *Journal of Economic Theory*, 74, 385-413.

[2] FUJISHIGE, S., AND Z. YANG (2003): "A Note on Kelso and Crawford's Gross Substitutes Condition," *Mathematics of Operations Research*, 28, 463-469.

[3] GUL, F., AND E. STACCHETTI (1999): "Walrasian Equilibrium with Gross Substitutes," *Journal of Economic Theory*, 87, 95-124.

[4] KELSO, A., AND V.P. CRAWFORD (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 50, 1483-1504.

[5] MA, J. (1998): "Competitive Equilibrium with Indivisibilities," *Journal of Economic Theory*, 82, 458-468.

[6] SUN, N., AND Z. YANG (2006): "Equilibria and Indivisibilities: Gross Substitutes and Complements," *Econometrica,* 74, 1385-1402.