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Chan, Raymond H. and Clark, Ephraim and Wong, Wing-Keung

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Raymond H. Chan

The Chinese University of Hong Kong

Ephraim Clark

Middlesex Business School

Wing-Keung Wong

Hong Kong Baptist University

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Abstract

This paper studies some properties of stochastic dominance (SD) for risk-averse and risk-seeking investors, especially for the third order SD (TSD). We call the former ascending stochastic dominance (ASD) and the latter descending stochastic dominance (DSD). We first discuss the basic property of ASD and DSD linking the ASD and DSD of the first three orders to expected-utility maximization for risk-averse and risk-seeking investors. Thereafter, we prove that a hierarchy exists in both ASD and DSD relationships and that the higher orders of ASD and DSD cannot be replaced by the lower orders of ASD and DSD. Furthermore, we study conditions in which third order ASD preferences will be 'the opposite of' or 'the same as' their counterpart third order DSD preferences. In addition, we construct examples to illustrate all the properties developed in this paper. The theory developed in this paper provides investors with tools to identify first, second, and third order ASD and DSD prospects and thus they could make wiser choices on their investment decision.

1 Introduction

According to the von Neuman and Morgenstern (1944) expected utility theory, the functions for risk averters and risk seekers are concave and convex respectively, and both are increasing functions. In this context stochastic dominance (SD) theory has generated a rich and growing academic literature. Linking SD theory to the selection rules for risk averters under different restrictions on the utility functions include Quirk and Saposnik (1962), Fishburn (1964), Hanoch and Levy (1969), Whitmore (1970), Hammond (1974) and Tesfatsion (1976). Linking SD theory to the selection rules for risk seekers include Hammond (1974), Meyer (1977), Stoyan (1983), Wong and Li (1999), Anderson (2004), and Wong (2007).

There are numerous developments in theory and applications for stochastic Dominance. Most of them are related up to second order stochastic Dominance. Studying of third order stochastic Dominance (TSD) are relative rare. Here we list some of the study in TSD. For example, Whitmore (1970) first introduces the concept of third order TSD. Bawa (1975) proves that the TSD rule is the optimal rule when comparing uncertain prospects with equal means. He also demonstrates that third-order stochastic dominance implies dominance under mean-lower partial variance rule. Fishburn and Vickson (1978) show that TSD and DARA stochastic dominance are equivalent concepts when the means of the random alternatives are equal to one another. Bawa, et al. (1979) develop algorithm to obtain the second and third order stochastic dominance admissible Sets by using the empirical distribution function for each stock as a surrogate for the true but unknown distribution. Eeckhoudt and Kimball (1992) make the stronger assumption that the distribution of background risk conditional upon a given level of insurable loss deteriorates in the sense of third-order stochastic dominance as the amount of insurable loss increases.

There are some studies on the TSD for both risk averters and risk seekers. For example, Wong and Li (1999) extend the first and second order convex stochastic dominance theory for risk averters developed by Fishburn (1974) to the first three orders for both risk averters and risk seekers. Li and Wong (1999) extend the theory of stochastic dominance and diversification for risk averters developed by Hadar and Russell (1971) and others by including the third order SD and including the theory to examine the preferences for risk

seekers. Wong (2007) further extends the SD theory of the first three orders to compare both return and loss.

There are also some applications of TSD theory or link TSD theory to other theories. For example, Gotoh and Konno (2000) show that portfolios son a significant portion of the efficient frontier generated by mean-lower semi-skewness model are efficient in the sense of third degree stochastic dominance. They also prove that the portfolios generated by mean-variance-skewness model are semi-efficient in the sense of third degree stochastic dominance. Using stock index data for 24 countries over the period 1989-2001, Fong, et al. (2005) show that winner portfolios stochastically dominate loser portfolios at second and third order. By considering second- and third-order stochastic dominance, Gasbarro, et al. (2007) determine whether investors could increase their utility by switching from one fund to another. Zagst and Kraus (2011) derive parameter conditions implying the second- and third-order stochastic dominance of the Constant Proportion Portfolio Insurance strategy. TSD has been promoted as a normative criterion to refine the partial ordering over income distributions (Davies and Hoy, 1994). In addition, Le Breton and Peluso (2009) introduce the concepts of strong and local third-degree stochastic dominance and characterize them in the spirit of the Lorenz characterization of the second-degree stochastic order. Ng (2000) constructs two examples in the third order stochastic dominance. Thorlund-Petersen (2001) develops the necessary and sufficient conditions for third order SD and provides provide a simple set of axioms for convexity of the marginal utility function of income.

This paper studies some properties of SD for risk-averse and risk-seeking investors, especially for TSD. We call the former ascending stochastic dominance (ASD) and the latter descending stochastic dominance (DSD). We first discuss the basic property of ASD and DSD linking the ASD and DSD of the first three orders to expected-utility maximization for risk-averse and risk-seeking investors. Thereafter, we prove that a hierarchy exists in both ASD and DSD relationships and that the higher orders of ASD and DSD cannot be replaced by the lower orders of ASD and DSD. Furthermore, we study conditions in which third order ASD (TASD) preferences will be 'the opposite of' or 'the same as' their counterpart third order DSD (TDSD) preferences. In addition, we provide examples to illustrate each case of ASD and DSD to the first three orders and demonstrate that the

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higher order ASD and DSD cannot be replaced by the lower order ASD and DSD, provide examples to illustrate that TASD could be 'the opposite of' or 'the same as' their counterpart TASD, and provide example to illustrate existence of TASD (TDSD) does not imply the existence of its counterpart TDSD (TASD).

The paper is organized as follows. We begin by introducing definitions and notations in the next section. Section 3 develops several theorems and properties for the ASD and DSD. Section 4 provides examples for ASD and DSD to illustrate all the properties developed in this paper. Section 5 concludes our findings.

2   Definitions and Notations

Let $\overline{\overline{\mathbb{R}}}$ be the set of extended real numbers and $\Omega = [a, b]$ be a subset of $\overline{\overline{\mathbb{R}}}$ in which $a < b$. Let $\mathbb{B}$ be the Borel σ-field of Ω and μ be a *measure* on $(\Omega, \mathbb{B})$. We first define the functions F^A and F^D of the measure μ on the support Ω as

$$F_1^A(x) \equiv F(x) \equiv \mu[a, x] \quad \text{and} \quad F_1^D(x) \equiv \mu[x, b] \qquad \text{for all} \quad x \in \Omega\,. \qquad (2.1)$$

Function F is a *cumulative distribution function (CDF)*[1] or simply *distribution function* and μ is a *probability measure* if $\mu(\Omega) = 1$. All functions are assumed to be measurable and all random variables are assumed to satisfy $F_1^A(a) = 0$ and $F_1^D(b) = 0$. It is well known in probability theory that for any random variable X with an associated probability measure P, there exists a unique induced probability measure μ on $(\Omega, \mathbb{B})$ and a distribution function F such that F satisfies (2.1) and $\mu(B) = P(X^{-1}(B)) = P(X \in B)$ for any $B \in \mathbb{B}$.

An integral written in the form of $\int_A f(t)\, d\,\mu(t)$ or $\int_A f(t)\, d\,F(t)$ is a Lebesgue-Stieltjes integral for integrable function $f(t)$. If the integrals have the same value for all A among $(c, d]$, $[c, d)$, or $[c, d]$, then we use the notation $\int_c^d f(t)\, d\,\mu(t)$ instead. In addition, if μ is a Borel measure with $\mu(c, d] = d - c$, then we write the integral as $\int_c^d f(t)\, dt$.

Random variables, denoted by X and Y, defined on Ω are considered together with their corresponding distribution functions F and G, and their corresponding probability density functions f and g, respectively. The following notations will be used throughout this paper:

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[1]In this paper, the definition of F is slightly different from the "traditional" definition of a distribution function.

$$\mu_F = \mu_X = E(X) = \int_a^b t \, d\, F(t) \quad, \quad \mu_G = \mu_Y = E(Y) = \int_a^b t \, d\, G(t) \quad,$$

$$H_j^A(x) = \int_a^x H_{j-1}^A(y)\, dy \quad, \quad H_j^D(x) = \int_x^b H_{j-1}^D(y)\, dy \quad j = 2,3; \tag{2.2}$$

$$h(x) = H_0^A(x) = H_0^D(x) \quad,$$

where $h = f$ or g and $H = F$ or G.[2] In (2.2), $\mu_F = \mu_X$ is the mean of X, whereas $\mu_G = \mu_Y$ is the mean of Y.

We note that the definition of H_i^A can be used to develop the stochastic dominance theory for risk averters (see, for example, Quirk and Saposnik 1962) and thus we could call this type of SD ascending stochastic dominance (ASD) and call H_i^A the ith order ASD integral or the ith order cumulative probability as H_i^A is integrated in ascending order from the leftmost point of downside risk. On the other hand, H_i^D can be used to develop the stochastic dominance theory for risk seekers (see, for example, Hammond, 1974) and thus we could call this type of SD descending stochastic dominance (DSD) and call H_i^D the ith order DSD integral or the ith order reversed cumulative probability as H_i^D is integrated in descending order from the rightmost point of upside profit. Typically, risk averters prefer assets that have a smaller probability of losing, especially in downside risk while risk seekers prefer assets that have a higher probability of gaining, especially in upside profit. To make a choice between two assets F or G, risker averters will compare their corresponding i^{th} order ASD integrals F_i^A and G_i^A and choose F if F_i^A is smaller since it has a smaller probability of losing. On the other hand, risk seekers will compare their corresponding i^{th} order DSD integrals F_i^D and G_i^D and choose F if F_i^D is bigger since it has a higher probability of gaining. In this paper we will study the properties of ASD and DSD in detail, especially for the third order SD. We next define the first-, second-, and third-order ASDs that are applied to risk averters; and then define the first-, second-, and third-order DSDs that are applied to risk seekers. The following definitions of stochastic dominance are widely used; see, for example, Li and Wong (1999):

Definition 2.1 *Given two random variables X and Y with F and G as their respective distribution functions, X is at least as large as Y and F is at least as large as G in the sense of:*

1. *FASD, denoted by $X \succeq_1 Y$ or $F \succeq_1 G$, if and only if $F_1^A(x) \leq G_1^A(x)$ for each x in $[a, b]$,*

2. *SASD, denoted by $X \succeq_2 Y$ or $F \succeq_2 G$, if and only if $F_2^A(x) \leq G_2^A(x)$ for each x in $[a, b]$,*

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[2]The above definitions are commonly used in the literature; see for example, Wong and Li (1999) and Anderson (2004).

3. TASD, denoted by $X \succeq_3 Y$ or $F \succeq_3 G$, if and only if $F_3^A(x) \leq G_3^A(x)$ for each x in $[a, b]$ and $\mu_X \geq \mu_Y$,

where FASD, SASD, and TASD stand for first-, second-, and third-order ascending stochastic dominance, respectively.

If, in addition, there is a subinterval $I \subset [a, b]$ such that for any $x \in I$ such that $F_i^A(x) < G_i^A(x)$ for $i = 1, 2$ and 3, we say that X is larger than Y and F is larger than G in the sense of SFASD, SSASD, and STASD, denoted by $X \succ_1 Y$ or $F \succ_1 G$, $X \succ_2 Y$ or $F \succ_2 G$, and $X \succ_3 Y$ or $F \succ_3 G$, respectively, where SFASD, SSASD, and STASD stand for strictly first-, second-, and third-order ascending stochastic dominance, respectively.

Definition 2.2 *Given two random variables X and Y with F and G as their respective distribution functions, X is at least as large as Y and F is at least as large as G in the sense of:*

1. FDSD, denoted by $X \succeq^1 Y$ or $F \succeq^1 G$, if and only if $F_1^D(x) \geq G_1^D(x)$ for each x in $[a, b]$,

2. SDSD, denoted by $X \succeq^2 Y$ or $F \succeq^2 G$, if and only if $F_2^D(x) \geq G_2^D(x)$ for each x in $[a, b]$,

3. TDSD, denoted by $X \succeq^3 Y$ or $F \succeq^3 G$, if and only if $F_3^D(x) \geq G_3^D(x)$ for each x in $[a, b]$ and $\mu_X \geq \mu_Y$,

where FDSD, SDSD, and TDSD stand for first-, second-, and third-order descending stochastic dominance, respectively.

If, in addition, there is a subinterval $I \subset [a, b]$ such that for any $x \in I$ such that $F_i^D(x) > G_i^D(x)$ for $i = 1, 2$ and 3, we say that X is larger than Y and F is larger than G in the sense of SFDSD, SSDSD, and STDSD, denoted by $X \succ^1 Y$ or $F \succ^1 G$, $X \succ^2 Y$ or $F \succ^2 G$, and $X \succ^3 Y$ or $F \succ^3 G$, respectively, where SFDSD, SSDSD, and STDSD stand for strictly first-, second-, and third-order descending stochastic dominance, respectively.

We note that if $F \succeq_i G$ or $F \succ_i G$, then $-H_j^A$ is a distribution function for any $j > i$, and there exists a unique measure μ such that $\mu[a, x] = -H_j^A(x)$ for any $x \in [a, b]$. Similarly, if $F \succeq^i G$ or $F \succ^i G$, then H_j^D is a distribution function for any $j > i$. H_j^A and H_j^D are defined in (2.2).

The stochastic dominance approach is regarded as one of the most useful tools for ranking investment prospects when there is uncertainty, since ranking assets has been proven to be equivalent to expected-utility maximization for the preferences of investors/decision makers with different types of utility functions. Before we carry on our discussion, we first state different types of utility functions as shown in the following definition:

Definition 2.3 *For $n = 1, 2, 3, U_n^A, U_n^{SA}, U_n^D$ and U_n^{SD} are sets of utility functions u such that:*

$$U_n^A(U_n^{SA}) = \{u : (-1)^i u^{(i)} \leq (<) 0 , i = 1, \cdots, n\},$$
$$U_n^D(U_n^{SD}) = \{u : u^{(i)} \geq (>) 0 , i = 1, \cdots, n\}.$$

where $u^{(i)}$ is the ith derivative of the utility function u.

Note that in Definition 2.3 'increasing' means 'nondecreasing' and 'decreasing' means 'nonincreasing'. We also note that in Definition 2.3, $U_1^A = U_1^D$ and $U_1^{SA} = U_1^{SD}$. We note that the theory can be easily extended to satisfy utilities defined in Definition 2.3 to be non-differentiable.[3] It is noted that investors in U_n^A are risk averse while investors in U_n^D are risk seeking. Refer to Figure 1 for the shape of utility functions in U_2^A and U_2^D and refer to Figure 2 for the shape of the first derivatives of the utility functions in U_3^A and U_3^D respectively.

It is well known that a positive third derivative for the utility function is a necessary, but not sufficient condition for decreasing absolute risk aversion (DARA). Menezes, Geiss and Tressler (1980) show that one cumulative distribution function is an increase in downside risk from another if and only if the latter is preferred to the former by all decision makers whose utility function has a positive third derivative. Utility functions in U_3^A have a non-negative third derivative. This implies the empirically attractive feature of DARA. On the other hand, if we find DARA of any utility u is increasing, we could conclude that $u''' > 0$ and $u \in U_3^D$. Post and Levy (2005) suggest that a third-order polynomial utility function implies that investors care only about the first three central moments of the return distribution (mean, variance, and skewness). On the other hand, Post and Versijp (2007) suggest that third-order stochastic dominance (TSD) efficiency applies if and only if a portfolio is optimal for some nonsatiable, risk-averse, and skewness-loving investor. Fong, et al. (2008) comment that third order stochastic dominance adds to risk aversion with the assumption of skewness preference.

An individual chooses between F and G in accordance with a consistent set of preferences satisfying the von Neumann-Morgenstern (1944) consistency properties. Accordingly, F is (strictly) preferred to G, or equivalently, X is (strictly) preferred to Y if

$$\Delta Eu \equiv E[u(X)] - E[u(Y)] \geq 0(> 0), \tag{2.3}$$

where $E[u(X)] \equiv \int_a^b u(x)dF(x)$ and $E[u(Y)] \equiv \int_a^b u(x)dG(x)$.

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[3]Readers may refer to Wong and Ma (2008) and the references there for more information. In this paper, we will skip the discussion of non-differentiable utilities.

3   The Theory

We first state the following basic result linking the ASD and DSD of the first three orders to expected-utility maximization for risk-averse and risk-seeking investors :

Theorem 3.1   *Let X and Y be random variables with distribution functions F and G, respectively. Suppose u is a utility function. For $j = 1, 2$ and 3, we have* [4]

1. $X \succeq_j Y$ if and only if $E\big[u(X)\big] \geq E\big[u(Y)\big]$ for any u in U_j^A, and

2. $F \succeq^j G$ if and only if $E\big[u(X)\big] \geq E\big[u(Y)\big]$ for any u in U_j^D.

There are many papers that obtain findings similar to the results in the above proposition for orders 1 and 2. For example, Hadar and Russell (1971) and Bawa (1975) prove the ascending stochastic dominance results for continuous density functions and continuously differentiable utility functions. Hanoch and Levy (1969) and Tesfatsion (1976) prove the first and second order ascending stochastic dominance for general distribution functions. Rothschild and Stiglitz (1970, 1971) study the special case of distributions with equal means and have proposed a condition that is equivalent to the second order ascending stochastic dominance results. Meyer (1977) discusses second order stochastic dominance for risk averters and risk seekers. Stoyan (1983) proves the first and second order stochastic dominance results for risk averters as well as risk seekers.

The result in Theorem 3.1 that is still controversial is the result of order 3 because for order 3 of ASD, some suggest that both conditions (i) $F_3^A(x) \leq G_3^A(x)$ for each x in $[a, b]$ and (ii) $\mu_X \geq \mu_Y$ as stated in Definition 2.1 are necessary while some suggest that condition (ii) is redundant. For example, Schmid (2005) proves that (i) implies (ii) and thus he suggests that condition (ii) is not necessary. One could draw similar arguments for DSD. In this paper, we confirm that the condition $\mu_X \geq \mu_Y$ in Definitions 2.1 and 2.2 is necessary in order to obtain the result of order 3 in Theorem 3.1. Without this condition, the assertions of Theorem 3.1 do not hold for the case $j = 3$. We will construct examples in our illustration section to show that $\mu_f \geq \mu_g$ is not related to $F_3^A(x) \leq G_3^A(x)$. One could easily modify our example to construct another example to show that $\mu_f \geq \mu_g$ is not related to $F_3^D(x) \geq G_3^D(x)$.

We are now ready to discuss some other relationship between the third orders of ASD and DSD. Before we do so, we first discuss the proposition that hierarchy exists in SD.

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[4]Since most of the established properties of SD require the "strict" form but not the "weak" form of SD, from now on, we will discuss only the "strict" form of SD in our paper. Thus, for $j = 1, 2$ and 3, we will use "$\succ_j$" to represent both "$\succ_j$" and "$\succeq_j$", "$\succ^j$" to represent both "$\succ^j$" and "$\succeq^j$", and U_j^J represent for both U_j^J and U_j^{SJ} for $J = A$ and D if no confusion occurs.

Theorem 3.2 *For any pair of random variables X and Y, for $i = 1$ and 2, we have:*

1. if $X \succeq_i^A Y$, then $X \succeq_{i+1}^A Y$; and

2. if $X \succeq_i^D Y$, then $X \succeq_{i+1}^D Y$.

The proof of Theorem 3.2 is straightforward. The results of this theorem suggest practitioners to report the ASD and DSD results to the lowest order in empirical analyses. Levy and Levy (2002) show that it is possible for ASD to be 'the opposite' of DSD in their second orders and that F dominates G in SDSD, but G dominates F in SASD. We extend their result to include ASD and DSD to the third order SD as stated in the following theorem:

Theorem 3.3 *For any pair of random variables X and Y, if F and G have the same mean which is finite, and if either $X \succeq_2^A Y$ or $Y \succeq_2^D X$, then we have*

$$X \succeq_3^A Y \quad and \quad Y \succeq_3^D X . \tag{3.1}$$

The proof of Theorem 3.3 is straightforward. Levy and Levy (2002) show that if X and

Y have the same mean which is finite, then $X \succeq_2^A Y$ if and only if $X \succeq_2^D Y$. The result of Theorem 3.3 could then be obtained by applying Theorem 3.2.

From Theorem 3.3, we find that the dominance relationships of X and Y are reversed for ASD and DSD. One may wonder whether the relationships of ASD and DSD are always of different directions? The answer is NO. We construct a theorem to show this possibility as follows:

Theorem 3.4 *For any random variables X and Y, if either $X \succeq_1^A Y$ or $X \succeq_1^D Y$, then we have*

$$X \succeq_3^A Y \quad and \quad X \succeq_3^D Y . \tag{3.2}$$

The proof of Theorem 3.4 could be obtained by applying Lemma 3 in Li and Wong (1999) and Theorem 3.2 in this paper. One might argue that the third orders ASD and DSD in both Theorems 3.3 and 3.4 are trivial. We get the third orders ASD and DSD because the second orders or the first order ASD and DSD relationships exist. One might wonder whether there is any non-trivial third order ASD and DSD relationship. Or, more specifically, one might ask: it is possible that there are X and Y such that they do not possess first- and second order ASD and DSD but there exist third order ASD and DSD and there is a relationship between their third order ASD and DSD. Our answer is YES and we derive one as follows:

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Theorem 3.5 *If F and G satisfy $\mu_F = \mu_G$ and either*

$$F_3^A(b) = G_3^A(b) \ \text{ or } \ F_3^D(a) = G_3^D(a), \tag{3.3}$$

then

$$F \succeq_3^A G \quad \text{if and only if} \quad G \succeq_3^D F.$$

4 Illustration

Some papers suggest that the condition $\mu_X \geq \mu_Y$ stated in Definition 2.1 is not necessary to obtain the result in Theorem 3.1. For example, Schmid (2005) proves that $F_3^A(x) \leq G_3^A(x)$ implies $\mu_X \geq \mu_Y$ and thus he suggests condition $\mu_X \geq \mu_Y$ is not necessary. In this paper, we confirm that the condition $\mu_X \geq \mu_Y$ in both Definitions 2.1 and 2.2 is necessary in order to obtain the result of order 3 in Theorem 3.1. Without this condition, the assertions of Theorem 3.1 do not hold for the case $j = 3$. In this section, we construct the following example to illustrate that $\mu_f \geq \mu_g$ is not related to $F_3^A(x) \leq G_3^A(x)$. One could easily modify our example to construct another example to show that $\mu_f \geq \mu_g$ is not related to $F_3^D(x) \geq G_3^D(x)$.

Example 4.1 $\underline{\mu_f \geq \mu_g \text{ is not related to } F_3^A(x) \leq G_3^A(x)}$

a. *We first construct an example in which $G_3^A(x) > F_3^A(x)$ for all x but yet $\mu_g > \mu_f$. Let $F(x) = x$, the uniform distribution on [0, 1]. Let $G(x)$ be such that*

$$G(x) = \begin{cases} \frac{3x}{2} & 0 \leq x \leq 0.24, \\ 0.24 + \frac{x}{2} & 0.24 \leq x \leq 0.74, \\ \frac{3x}{2} - 0.5 & 0.74 \leq x \leq 1. \end{cases}$$

The left panel of Figure 4.1 shows the plot of $F(x)$ and $G(x)$ while the right panel of Figure 4.1. shows the plot of $F_3^A(x)$, $G_3^A(x)$ and $G_3^A(x) - F_3^A(x)$. We can see that $\mu_g = 0.505 > 0.5 = \mu_f$ and $G_3^A(x) - F_3^A(x) \geq 0$ for all $0 \leq x \leq 1$.

b. *Next, we construct an example where $G_3^A(x) > F_3^A(x)$ for all x yet $\mu_g < \mu_f$. Again, let $F(x) = x$, the uniform distribution on [0, 1]. Let $G(x)$ be such that*

$$G(x) = \begin{cases} \frac{3x}{2} & 0 \leq x \leq 0.26, \\ 0.26 + \frac{x}{2} & 0.26 \leq x \leq 0.76, \\ \frac{3x}{2} - 0.5 & 0.76 \leq x \leq 1. \end{cases}$$

Figure 4.2 (left) shows the plot of $F(x)$ and $G(x)$ while Figure 4.2 (right) shows the plot of $F_3^A(x)$, $G_3^A(x)$ and $G_3^A(x) - F_3^A(x)$. We can see that $\mu_g = 0.495 < 0.5 = \mu_f$ while $G_3^A(x) - F_3^A(x) \geq 0$ for all $0 \leq x \leq 1$.

Figure 4.1: Plots of $F(x)$, $G(x)$, $F_3^A(x)$, $G_3^A(x)$, and their differences
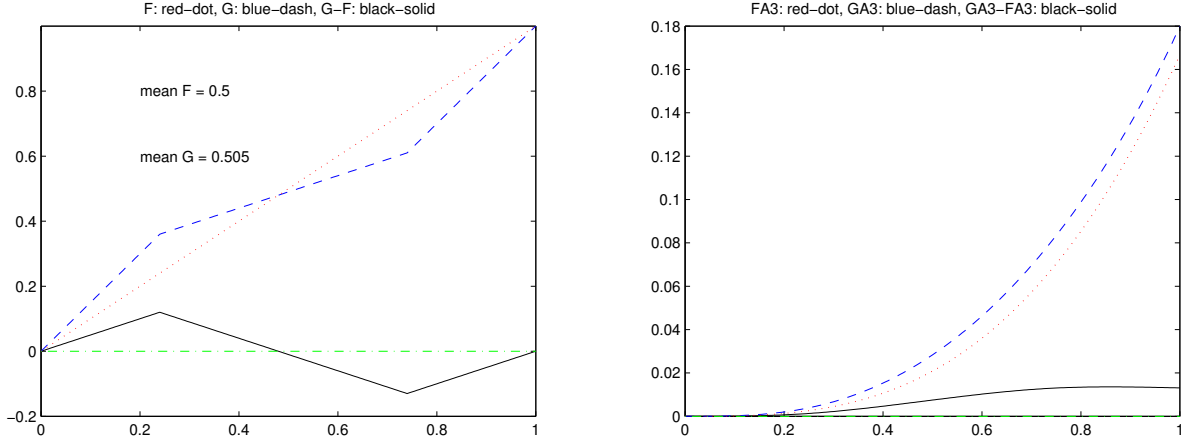
Dotted red line — $F(x)$ or $F_3^A(x)$;
Dashed blue line— $G(x)$ or $G_3^A(x)$
Solid blue line — $G_1^A(x) - F_1^A(x)$ or $G_3^A(x) - F_3^A(x)$

Note : The left panel shows $F(x)$ and $G(x)$ and their difference $G_1^A(x) - F_1^A(x)$ while the right panel shows $F_3^A(x)$, $G_3^A(x)$, and their difference $G_3^A(x) - F_3^A(x)$

From this example, we can conclude that (i) $G_3^A(x) \geq F_3^A(x)$ for all x and (ii) $\mu_g < \mu_f$ have no relationship at all.

To construct an example to illustrate Theorem 3.3, we use the Production/Operations Management example demonstrated by Wong (2007) who modifies the example from Weeks (1985) and Dillinger et al. (1992). The example is shown as follows:

Example 4.2 *A production/operations system needs extra capacity to satisfy the expected increased demand. Two mutually exclusive alternative sites have been identified and the profit (x) with their associated probabilities f and g have been estimated as shown in Table 4.1.*

We use the ASD and DSD integrals H_j^A and H_j^D for $H = F$ and G and $j = 1, 2$ and 3 as defined in (2.2). To make the comparison easier, we define their differentials

$$GF_j^A = G_j^A - F_j^A \quad and \quad GF_j^D = G_j^D - F_j^D \tag{4.1}$$

for $j = 1, 2$ and 3 and present the results of the ASD and DSD integrals with their differentials for the first three orders in Tables 4.1 and 4.2.

In this example, our results show that there are no first order ASD or DSD between F and G but we have $F \succeq_j^A G$ and $G \succeq_j^D F$ for $j = 2$ and 3. Thus, this example illustrates Theorem 3.3.

To illustrates Theorem 3.4, we use Experiments 1 in Levy and Levy (2002) as follows:

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Table 4.1: The Profits of two Locations and their ASD Integrals and Integral Differentials

Profit (in million)	Probability		FASD Integrals		SASD Integrals		TASD Integrals		ASD Integral Differentials		
x	f	g	F_1^A	G_1^A	F_2^A	G_2^A	F_3^A	G_3^A	GF_1^A	GF_2^A	GF_3^A
1	0.5	0.5	0.5	0.5	0	0	0	0	0	0	0
2	0	0	0.5	0.5	0.5	0.5	0.25	0.25	0	0	0
3	0	0.1	0.5	0.6	1	1	1	1	0.1	0	0
4	0.25	0.05	0.75	0.65	1.5	1.6	2.25	2.3	-0.1	0.1	0.05
5	0.25	0.35	1	1	2.25	2.25	4.125	4.225	0	0	0.1

Note: The ASD integrals H_j^A is defined in (2.2) for $H = F$ or G and the integral differential GF_j^A is defined in (4.1) for $j = 1, 2$ and 3.

Table 4.2: The Profits of two Locations and their DSD Integrals and Integral Differentials

| Profit (in million)	Probability		FDSD Integrals		SDSD Integrals		TDSD Integrals		DSD Integral Differentials		
x	f	g	F_1^D	G_1^D	F_2^D	G_2^D	F_3^D	G_3^D	GF_1^D	GF_2^D	GF_3^D
1	0.5	0.5	1	1	1.75	1.75	3.125	3.225	0	0	0.1
2	0	0	0.5	0.5	1.25	1.25	1.625	1.725	0	0	0.1
3	0	0.1	0.5	0.5	0.75	0.75	0.625	0.725	0	0	0.1
4	0.25	0.05	0.5	0.4	0.25	0.35	0.125	0.175	-0.1	0.1	0.05
5	0.25	0.35	0.25	0.35	0	0	0	0	0.1	0	0

Note: The DSD integral H_j^D is defined in (2.2) for $H = F$ or G and the integral differential GF_j^D is defined in (4.1) for $j = 1, 2$ and 3.

Figure 4.2: Plots of $F(x)$, $G(x)$, $F_3^A(x)$, $G_3^A(x)$, and their differences
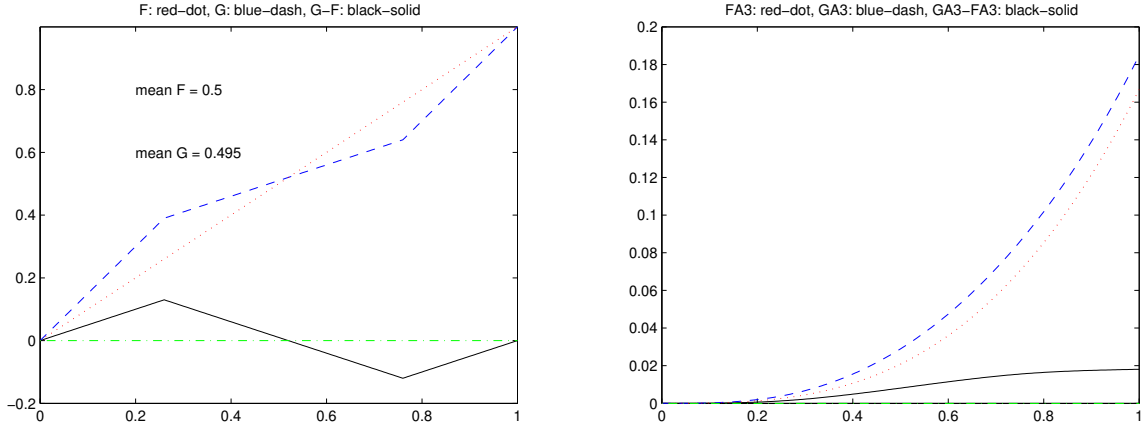
Dotted red line — $F(x)$ or $F_3^A(x)$;
Dashed blue line— $G(x)$ or $G_3^A(x)$
Solid blue line — $G_1^A(x) - F_1^A(x)$ or $G_3^A(x) - F_3^A(x)$

Note : The left panel shows $F(x)$ and $G(x)$ and their difference $G_1^A(x) - F_1^A(x)$ while the right panel shows $F_3^A(x)$, $G_3^A(x)$, and their difference $G_3^A(x) - F_3^A(x)$

Example 4.3 *The gains one month later and their probabilities for an investor who invests \$10,000 either in stock A or in stock B is shown in the following experiment:*

Experiments 1			
Stock A		Stock B	
Gain (in thousand)	Probability	Gain (in thousand)	Probability
0.5	0.3	-0.5	0.1
2	0.3	0	0.1
5	0.4	0.5	0.1
		1	0.2
		2	0.1
		5	0.4

We let X and Y be the gain or profit for investing in Stocks A and B with the corresponding probability functions f and g and the corresponding cumulative probability functions F and G, respectively. Thereafter, we depict the ASD and DSD integral differentials GF_j^A and GF_j^D for the gain of investing in Stocks A and B in Table 4.3 in which GF_j^A and GF_j^D are defined in (4.1) for $j = 1, 2$ and 3.

From Table 4.3, we obtain $X \succeq_1^A Y$ and $X \succeq_1^D Y$, $X \succeq_2^A Y$ and $X \succeq_2^D Y$, as well as $X \succeq_3^A Y$ and $X \succeq_3^D Y$. This example illustrates Theorem 3.4.

In the above examples, we find that we have both SASD, SDSD, TASD, and TDSD for a pair of random variables. Is it possible to have TASD and TDSD but no SASD or

Table 4.3: The ASD and DSD integral differentials for the gain of investing in Stocks A and B.

Profit (in million)	Probability		ASD Integral Differentials			DSD Integral Differentials		
x	f	g	GF_1^A	GF_2^A	GF_3^A	GF_1^D	GF_2^D	GF_3^D
-1	0	0	0	0	0	0	-0.35	-0.6625
-0.5	0	0.1	0.1	0	0	0	-0.35	-0.4875
0	0	0.1	0.2	0.05	0.0125	-0.1	-0.3	-0.325
0.5	0.3	0.1	0	0.15	0.0625	-0.2	-0.2	-0.2
1	0	0.2	0.2	0.15	0.1375	0	-0.2	-0.1
2	0.3	0.1	0	0.35	0.3875	-0.2	0	0
5	0.4	0.4	0	0.35	1.4375	0	0	0

Note: The integral differentials GF_j^A and GF_j^D are defined in (4.1) for $j = 1, 2$ and 3.

SDSD? The answer is YES and this is exactly what Theorem 3.5 tells us. Thus, herewith we construct an example to illustrate Theorem 3.5 as follows:

Example 4.4 *Consider*

$$F(x) = \frac{x+1}{2}, \quad -1 \leq x \leq 1$$

and

$$G(t) = \begin{cases} 0 & -1 \leq x \leq -3/4, \\ x + \frac{3}{4} & -3/4 \leq x \leq -1/4, \\ \frac{1}{2} & -1/4 \leq x \leq 0, \\ x + \frac{1}{2} & 0 \leq x \leq 1/4, \\ \frac{3}{4} & 1/4 \leq x \leq 3/4, \\ x & 3/4 \leq x \leq 1. \end{cases}$$

Figure 4.3: Plots of $F_i^A(x)$, $G_i^A(x)$, $F_i^D(x)$, $G_i^D(x)$, $i = 1, 2, 3$, and their differences

In Figure 4.3 (first-row, left panel), we draw $F(x)$, $G(x)$, and their difference. Notice that both distributions have the same zero mean. In Figure 4.3 (first-row, right panel), we draw F_2^A and G_2^A and their difference. Notice that the difference has both positive and negative values. This means that we do not have $F \succeq_2^A G$ or $G \succeq_2^A F$. In Figure 4.3 (second row, left panel), we draw F_3^A and G_3^A and their difference. We see that the difference is non-positive. This means that $F \succeq_3^A G$. From the figure or by simple calculation, we can find that $F_3^A(b) = G_3^A(b) = 2/3$, so the conditions of Theorem 3.5 hold and we expect $G \succeq_3^D F$. In Figure 4.3 (second row, left and right panel and in third row, right panel), we draw F_i^D and G_i^D, $i = 1, 2, 3$ and their differences. We see from Figure 4.3 (third row,

right panel) that the difference $G_3^A - F_3^A$ is nonnegative. This means that indeed $G \succeq_3^D F$ as predicted by Theorem 3.5.

We now ask: is it possible that we have TASD and TDSD but no SASD or SDSD, and the conditions of Theorem 3.5 do not hold? The answer is YES and we construct an example to illustrate this possibility.

Example 4.5   *Consider*

$$F(x) = x \quad \text{and} \quad G(x) = \begin{cases} 2x & 0 \leq x \leq 0.2, \\ 2/5 & 0.2 \leq x \leq 0.4, \\ x/3 + 4/15 & 0.4 \leq x \leq 0.7, \\ (5x - 2)/3 & 0.7 \leq x \leq 1. \end{cases}$$

In Figure 4.4 (first-row, left panel), we draw $F(x)$, $G(x)$, and their difference. Notice that $\mu_F = 0.5 \neq 0.52 = \mu_G$. Thus the conditions of Theorem 3.5 do not hold. In Figure 4.4 (first-row, right panel), we draw F_2^A and G_2^A and their difference. Notice that the difference has both positive and negative values. This means that we do not have $F \succeq_2^A G$ or $G \succeq_2^A F$. In Figure 4.4 (second row, left panel), we draw F_3^A and G_3^A and their difference. We see that the difference is non-negative. This means that $G \succeq_3^A F$. From the figure we can see that $F_3^A(b) \neq G_3^A(b)$, so the conditions of Theorem 3.5 do not hold. In Figure 4.4 (second row, left and right panel and in third row, right panel), we draw F_i^D and G_i^D, $i = 1, 2, 3$ and their differences. We see from Figure 4.4 (third row, right panel) that the difference $G_3^A - F_3^A$ is nonnegative. This means that indeed $G \succeq_3^D F$.

In the above examples, we find that we have both TASD and TDSD for a pair of random variables. Is it possible to have TASD but no TDSD or vice versa? The answer is YES and we construct an example in which there exists F and G such that $G \succeq_3^D F$ but neither $F \succeq_3^A G$ nor $G \succeq_3^A F$ holds. We also construct an example in which there exists F and G such that $G \succeq_3^A F$ but neither $F \succeq_3^D G$ nor $G \succeq_3^D F$ holds.

Example 4.6   <u>$F \succeq_3^A G$ and $F \succeq_3^D G$ are not related</u>

a. *We construct an example in which there exists F and G such that $G \succeq_3^D F$ but neither $F \succeq_3^A G$ nor $G \succeq_3^A F$ holds.*

   Consider

$$F(t) = \begin{cases} 4(t + 1)/5 & -1 \leq t \leq -3/4, \\ 2t/5 + 1/2 & -3/4 \leq t \leq -1/4, \\ (4t + 3)/5 & -1/4 \leq t \leq 0, \\ 1 - G(-t) & 0 \leq t \leq 1, \end{cases} \quad \text{and} \quad G(t) = t$$

   In Figure 4.5 (first-row, left panel), we draw $F(x)$, $G(x)$, and their difference. In Figure 4.5 (first-row, right panel), we draw F_3^A and G_3^A and their difference. Notice

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that the difference has both positive and negative values showing that we do not have
$F \succeq_3^A G$ *or* $G \succeq_3^A F$. *In Figure 4.5 (second row, left panel), we draw* F_3^D *and* G_3^D
and their difference. We see that the difference is non-negative showing that we have
$G \succeq_3^D F$.

b. *Next we construct an example in which there exists* F *and* G *such that* $G \succeq_3^A F$ *but*
neither $F \succeq_3^D G$ *nor* $G \succeq_3^D F$ *holds.*

Consider

$$F(t) = t \quad and \quad G(t) = \begin{cases} 4t/5 & 0 \le t \le 0.4, \\ 3t/4 + 1/5 & 0.4 \le t \le 0.8, \\ 0.8 & 0.8 \le t \le 0.9, \\ 2t - 1 & 0.9 \le t \le 1, \end{cases}$$

In Figure 4.6 (first-row, left panel), we draw $F(x)$, $G(x)$, *and their difference. In*
Figure 4.6 (first-row, right panel), we draw F_3^A *and* G_3^A *and their difference. Notice*
that the difference is non-negative showing that $F \succeq_3^A G$. *In Figure 4.6 (second row,*
left panel), we draw F_3^D *and* G_3^D *and their difference. We see that the difference has*
both positive and negative values showing that we do not have $F \succeq_3^D G$ *or* $G \succeq_3^D F$.

In this example, one can easily show that we do not have $F \succeq_3^A G$ or $G \succeq_3^A F$ but we
have $G \succeq_3^D F$.[5] The above corollary and example show that under some regularities, F is
'the same' as G in the sense of TASD and TDSD. One may wonder whether this 'same
direction property' could appear in FASD vs FDSD and SASD vs SDSD. In the following
corollary, we show that this is possible.

In this example, one can easily show that there is no SASD and no SDSD dominance
but $F \succeq_3^A G$ and $G \succeq_3^D F$.[6] The above corollary provides the conditions in which F is
'the opposite' of G and the above example shows that there exist pairs of distributions
which are 'opposites' in the third order but not in the second order. On the other hand,
we find that under some regularities, F becomes 'the same' as G in the sense of TASD
and TDSD as shown in the corollary below:

In this example, one can easily show that there is no SASD and no SDSD dominance
but $F \succeq_3^A G$ and $G \succeq_3^D F$.[7] The above corollary provides the conditions in which F is
'the opposite' of G and the above example shows that there exist pairs of distributions
which are 'opposites' in the third order but not in the second order. On the other hand,
we find that under some regularities, F becomes 'the same' as G in the sense of TASD
and TDSD as shown in the corollary below:

In fact, if some of the assumptions are not satisfied, there exists F and G such that (a)
$G \succeq_3^A F$ but neither $F \succeq_3^D G$ nor $G \succeq_3^D F$ holds, and (b) $G \succeq_3^D F$ but neither $F \succeq_3^A G$ nor

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[5]The working is available on request.
[6]The working is available on request.
[7]The working is available on request.

$G \succeq_3^A F$ holds. To illustrates the case in (a) we use Experiments 2 in Levy and Levy (2002) as shown in Example 4.7. To illustrates the case in (b), one could simply let $Y = -X$ where Y is the gain defined in Example 4.7, then one could obtain SD relationship as shown in (b) by applying Lemma 3a in Li and Wong (1999) and the results from Example 4.7.

Example 4.7   *The gains one month later and their probabilities for an investor who invests $10,000 either in stock A or in stock B is shown in the following experiment:*

Stock A		Stock B	
Gain (in thousand)	Probability	Gain (in thousand)	Probability
-1.6	0.25	-1	0.25
-0.2	0.25	-0.8	0.25
1.2	0.25	0.8	0.25
1.6	0.25	2	0.25

We let X and Y be the gain or profit for investing in Stocks A and B with the corresponding probability functions f and g and the corresponding cumulative probability functions F and G, respectively. Thereafter, we depict the ASD and DSD integral differentials GF_j^A and GF_j^D for the gain of investing in Stocks A and B in Table 4.3 in which GF_j^A and GF_j^D are defined in (4.1) for $j = 1, 2$ and 3.

Table 4.4: The ASD and DSD integral differentials for the gain of investing in Stocks A and B.

Profit (in million)	Probability		ASD Integral Differentials			DSD Integral Differentials		
x	f	g	GF_1^A	GF_2^A	GF_3^A	GF_1^D	GF_2^D	GF_3^D
-1.6	0.25	0	-0.25	0	0	0	0	-0.04
-1	0	0.25	0	-0.15	-0.045	0.25	-0.15	0.005
-0.8	0	0.25	0.25	-0.15	-0.075	0	-0.15	0.035
-0.2	0.25	0	0	0	-0.12	-0.25	0	0.08
0.8	0	0.25	0.25	0	-0.12	0	0	0.08
1.2	0.25	0	0	0.1	-0.1	-0.25	0.1	0.06
1.6	0.25	0	-0.25	0.1	-0.06	0	0.1	0.02
2	0	0.25	0	0	-0.04	0.25	0	0

Note: The integral differentials GF_j^A and GF_j^D are defined in (4.1) for $j = 1, 2$ and 3.

From Table 4.4, we obtain if some of the assumptions are not satisfied, there exists (b) $G \succeq_3^A F$ but neither $F \succeq_3^D G$ nor $G \succeq_3^D F$ holds. For (a), $G \succeq_3^D F$ but neither $F \succeq_3^A G$ nor $G \succeq_3^A F$ holds, a similar example can be obtained.

5 Concluding Remarks

In this paper, we develop some properties for the ASD and DSD theory We first discuss the basic property of ASD and DSD linking the ASD and DSD of the first three orders to expected-utility maximization for risk-averse and risk-seeking investors. Thereafter, we prove that a hierarchy exists in both ASD and DSD relationships and that the higher orders of ASD and DSD cannot be replaced by the lower orders of ASD and DSD. Furthermore, we study conditions in which third order ASD preferences will be 'the opposite of' or 'the same as' their counterpart third order DSD preferences. In addition, we construct examples to illustrate all the properties developed in this paper. The theory developed in this paper provides investors with tools to identify first, second, and third order ASD and DSD prospects and thus they could make wiser choices on their investment decision.

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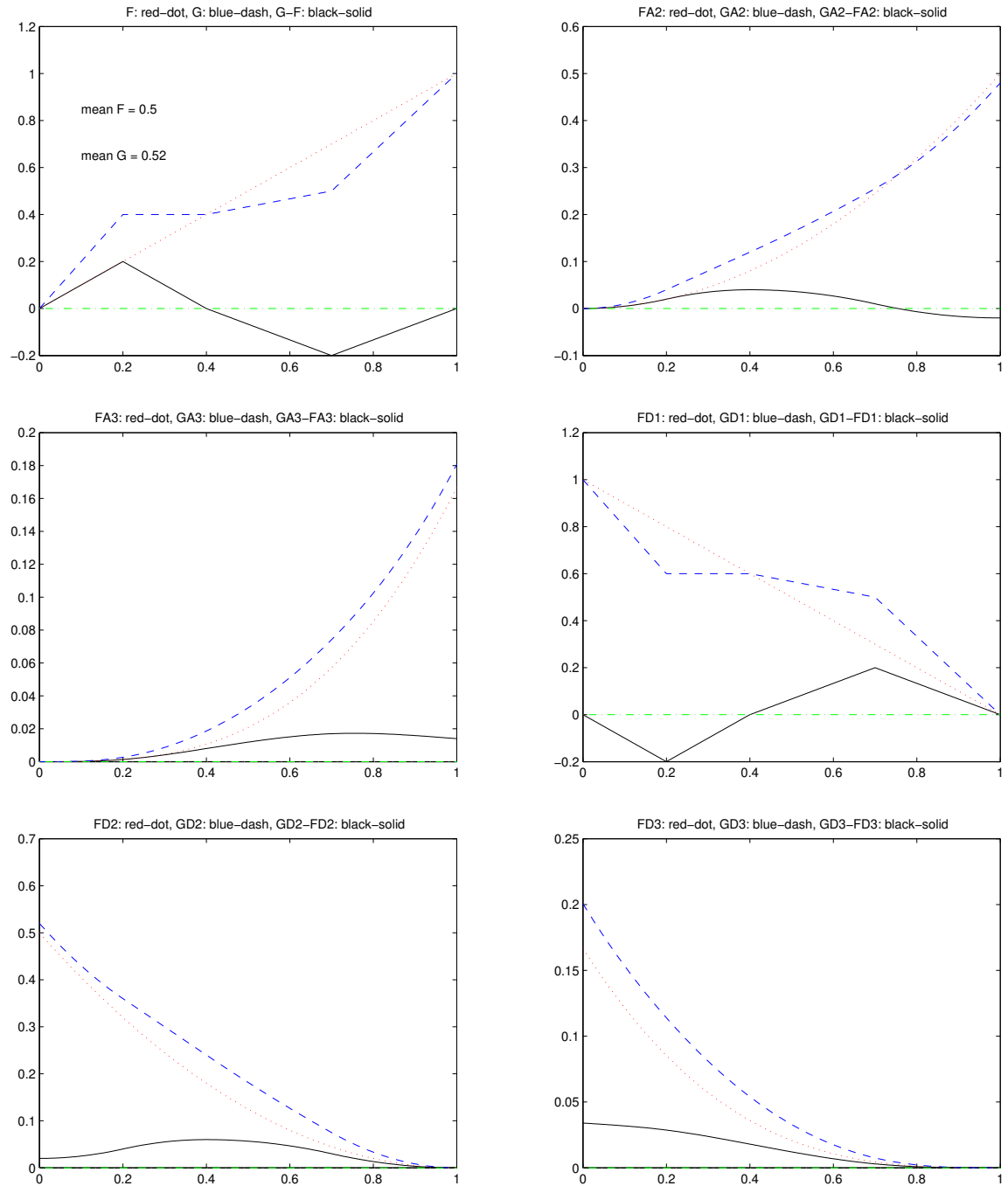Figure 4.4: Plots of $F_i^A(x)$, $G_i^A(x)$, $F_i^D(x)$, $G_i^D(x)$, $i = 1, 2, 3$, and their differences

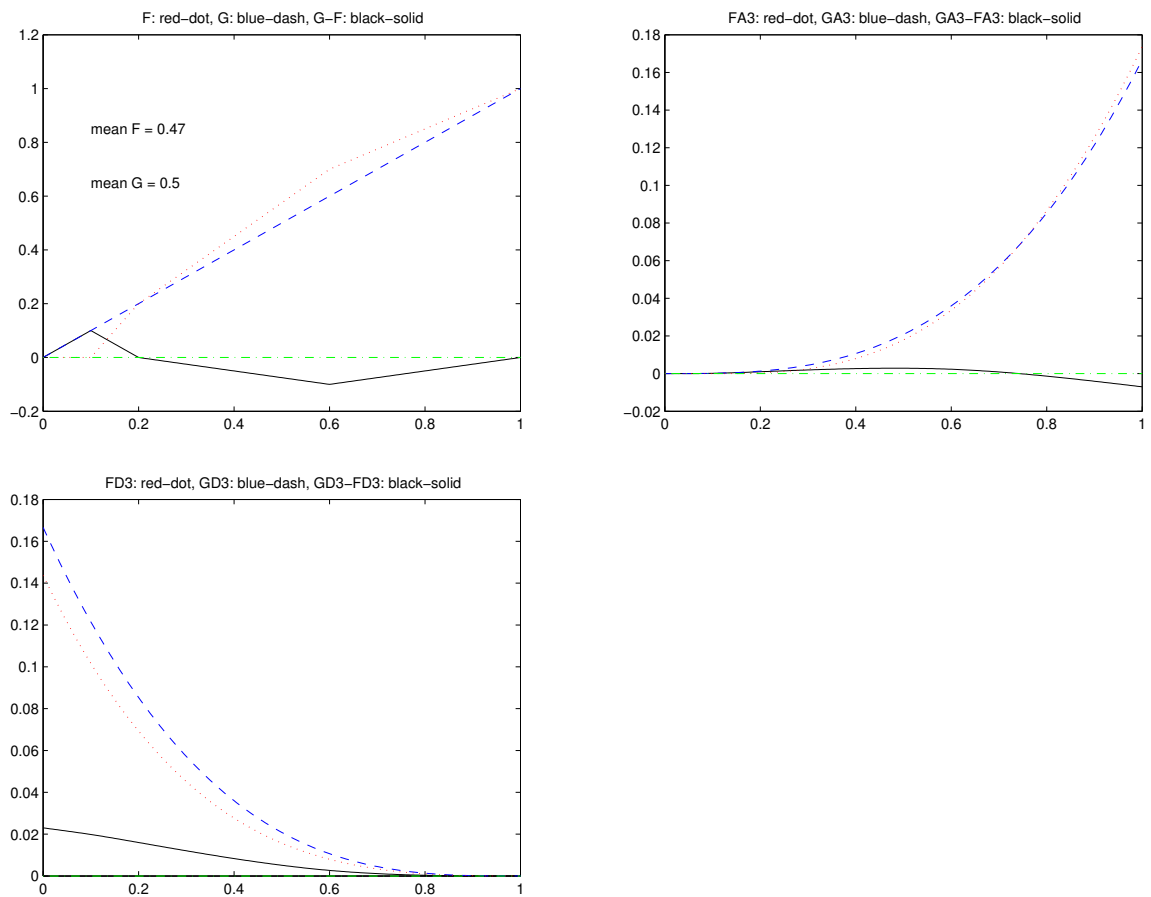Figure 4.5: Plots of $F(x)$, $G(x)$, $F_3^A(x)$, $G_3^A(x)$, $F_3^D(x)$, $G_3^D(x)$, and their differences

Figure 4.6: Plots of $F(x)$, $G(x)$, $F_3^A(x)$, $G_3^A(x)$, $F_3^D(x)$, $G_3^D(x)$, and their differences