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October 2006

Online at <http://mpa.ub.uni-muenchen.de/440/>
MPRA Paper No. 440, posted 07. November 2007 / 01:02

Computability of simple games: A complete investigation of the sixty-four possibilities

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October 2006

Abstract

Classify simple games into sixteen “types” in terms of the four conventional axioms: monotonicity, properness, strongness, and non-weakness. Further classify them into sixty-four classes in terms of finiteness (existence of a finite carrier) and computability. For each such class, we either show that it is empty or give an example of a game belonging to it. We observe that if a type contains an infinite game, then it contains both computable infinite games and noncomputable ones. This strongly suggests that computability is logically, as well as conceptually, unrelated to the conventional axioms.

Journal of Economic Literature Classifications: C71, C69, D71, D90.

Keywords: Voting games, infinitely many players, axiomatic method, complete independence, algorithms, Turing computability, recursion theory.

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1 Introduction

Rules can be classified into two classes according to a given axiom (also called a property or a condition): those satisfying the axiom and those violating it. If there are four axioms, rules can be classified into sixteen (2^4) classes. If some of these sixteen classes are empty, it means that the conjunction of some of these axioms implies the disjunction of the others (for example, if there does *not* exist a rule satisfying Axioms 1 and 2 but violating Axioms 3 and 4, then Axioms 1 and 2 together imply Axioms 3 or 4). If, on the other hand, each of the sixteen classes is nonempty, the four axioms are “completely independent” in the sense that there is nothing inconsistent about any combination of their truth values. This is exactly what May [14] investigated after proposing [13] four “independent” axioms characterizing simple majority rule: he showed that the four axioms are “completely independent” in the sense above.¹

We would like to do the same for (*simple*) *games* (voting games). These are the coalitional games that assign either 0 or 1 to each coalition—those assigned 1 are winning coalitions and those assigned 0 are losing coalitions. We consider *six axioms* for simple games. Four of them are conventional: *monotonicity*, *properness*, *strongness*, and *nonweakness*. Another axiom is *finiteness* (existence of a finite carrier), which distinguishes finite games (those ignoring all except fixed, finitely many players) from infinite ones. The other axiom is “*computability*,” which is the focus of this paper.

We start with the four conventional axioms. These axioms classify simple games into sixteen classes, which we call (*conventional*) *types*. Unfortunately, these axioms are not “completely independent” (though they are “independent”). For example, it is well known that there exist no weak, non-proper games. We therefore cannot hope for the “complete independence” of our six axioms.

Let us start over. Our focus is *computability*, an axiom that we introduce and characterize in a companion paper [11]. We are interested in the relation of this new axiom to the four existing ones. In other words, we want to investigate how computability restricts the games of each type (some of which are finite and the others are infinite). This suggests the following requirements for independence: If a particular type is empty (i.e., it contains no games), we require nothing—it is not computability or lack of it that is responsible for nonexistence of games of that type. If a particular type is

¹A set of consistent (compatible) axioms are “independent” [13] if for each axiom, there are rules violating the axiom but satisfying the others. As far as the axioms in a characterization are concerned, applying this notion of “independence” is the prevalent practice in the axiomatic method today, as discussed in Thomson [22]. Thomson states that an axiomatic study should offer among other things an analysis of logical relations between axioms. We have chosen to give a complete—rather than partial—analysis, in the spirit of earlier theorists like May [14] and Arrow [3, footnote 27, page 102].

nonempty, however, it should contain both computable games and noncomputable ones. Put differently, we say that “computability is *independent* of the four axioms (within a class of games)” if for each of the sixteen types, there is a computable game of that type (in that class) if and only if there is a noncomputable game of that type (in that class).

One of our main findings is (Proposition 3) that *computability is independent of the four conventional axioms within the class of infinite games*. (The analogue of Proposition 3 does not hold for the class of *finite* games. This is because all finite games are computable.) In fact, we come close to saying that computability is independent of the four conventional axioms (within the class of *all* games). The conditions for the independence are satisfied for fifteen out of the sixteen types. The only exception is the type consisting exclusively of dictatorial (hence computable) games. This strongly suggests that computability is logically, as well as conceptually, unrelated to the conventional axioms. (What is behind this terminology is Thomson’s discussion [22] of logical and conceptual independence. We do not define “conceptual independence” mathematically.)

We make these findings through complete investigation of the sixty-four (2^6) classes with respect to the six axioms. The results of the investigation is summarized in Table I in Section 3. Though the main result of the companion paper is cited here as Theorem 4, the full force of the theorem is not needed to prove the results.

One can make other interesting observations from Table I. We discuss in Section 3 observations involving several entries of Table I. Proposition 3 is an example, since it involves the last two columns of the table. Observations involving a single entry also provide useful information. For example, the last entry on the line corresponding to Type 1 indicates that *one can find a computable infinite game without sacrificing the conventional axioms*, which are voting-theoretically desirable. The companion paper [11] only exhibit *noncomputable* games satisfying the axioms and computable games *not* necessarily satisfying the axioms.

The rest of the Introduction gives the background briefly. It is fully discussed in the companion paper [11].

One can think of simple games as representing voting methods; alternatively, as representing “manuals” or “contracts.” They have been central to the study of social choice (e.g., Peleg [19]). For this reason, the paper can be viewed as a contribution to the foundations of *computability analysis of social choice*, which studies algorithmic properties of social decision-making.²

²This literature includes Kelly [10], Lewis [12], Bartholdi et al. [5, 6], and Mihara [15, 16, 17]. These are mainly concerned with the complexity of rules in themselves. Closely related are the studies of the complexity of *solutions* for rules (cooperative games), such as Deng and Papadimitriou [7] and Fang et al. [9].

The importance of computability in social choice theory would be unarguable. First, the use of the language by social choice theorists suggests the importance: For example, Arrow [3] uses words such as “*process or rule*” or “*procedure*.” Second, there is a normative reason: Computability of social choice rules formalizes the notion of “due process.” (Richter and Wong [20] give further justifications for studying computability-based economic theories.)

Like Downs [8] and Banks et al. [4], we consider an infinite set of players. Roughly speaking, a simple game is *computable* if there is a Turing program (finite algorithm) that can decide from a description (by integer) of each coalition whether it is winning or losing. Since each member of a coalition should be describable, we assume that the set N of (the names of) players is countable, say, $N = \mathbb{N} = \{0, 1, 2, \dots\}$. Also, we describe coalitions by a Turing program that can decide for the name of each player whether she is in the coalition. Since each Turing program has its code number (Gödel number), the coalitions describable in this manner are describable by an integer, as desired. (Such coalitions are called *recursive* coalitions.)

A population of countably many *players* arises not only in voting but in other contexts, depending on how we interpret a “player”:

Simulating future generations One may consider countably many *players* (people) extending into the indefinite future.

Uncertainty One may consider finitely many *persons* facing countably many states of the world [15]: each *player* can be interpreted as a particular *person* in a particular *state*.

Team management Putting the right people (and equipment) in the right places is basic to team management. To ensure “due process,” can a manager of a company write a “manual” (computable simple game³) elaborating the conditions that a team must meet?

Fix a particular task such as operating an exclusive agency of the company. A *team* consists of members (people) and equipment. The manager’s job is to organize a team that satisfactorily performs the task. Each member (or equipment) is described by attributes such as skills, position, availability at a particular time and place. Each such *attribute* can be thought of as a particular yes/no question, and there are countably many such questions.⁴

³Like Anderlini and Felli [2], who view contracts as algorithms, we view “manuals” as algorithms.

⁴A “state” in Al-Najjar et al. [1] is a countable sequence of 0’s and 1’s (answers to countably many questions). They assume that the set of states is countable. One could think of their “state” as a coalition in our paper, their “finite contract” as an algorithm composed of finitely many computable games, and their “undescrivable event” as a non-computable game satisfying certain conditions. We can find many other interpretations

Here, each *player* can be interpreted as a particular attribute of a particular member (or equipment). In other words, each *coalition* is identified with a 0-1 “matrix” of finitely many rows (each row specifying a member) and countably many columns (each column specifying a particular attribute).⁵

2 Framework

2.1 Simple games

Let $N = \mathbb{N} = \{0, 1, 2, \dots\}$ be a countable set of (the names of) players. Any **recursive** (algorithmically decidable) subset of N is called a **(recursive) coalition**.

Intuitively, a simple game describes in a crude manner the power distribution among *observable* (or describable) coalitions (subsets of players). We assume that only **recursive** coalitions are observable. According to *Church’s thesis* [21, 18], the recursive coalitions are the sets of players for which there is an algorithm that can decide for the name of each player whether she is in the set.⁶ Note that **the class REC of recursive coalitions** forms a **Boolean algebra**; that is, it includes N and is closed under union, intersection, and complementation.

Formally, a **(simple) game** is a collection $\omega \subseteq \text{REC}$ of (recursive) coalitions. We will be explicit when we require that $N \in \omega$. The coalitions in ω are said to be **winning**. The coalitions not in ω are said to be **losing**. One can regard a simple game as a function from REC to $\{0, 1\}$, assigning the value 1 or 0 to each coalition, depending on whether it is winning or losing.

We introduce from the theory of cooperative games a few basic notions of simple games [19, 23]. A simple game ω is said to be **monotonic** if for all coalitions S and T , the conditions $S \in \omega$ and $T \supseteq S$ imply $T \in \omega$. ω is **proper** if for all recursive coalitions S , $S \in \omega$ implies $S^c := N \setminus S \notin \omega$. ω is **strong** if for all coalitions S , $S \notin \omega$ implies $S^c \in \omega$. ω is **weak** if $\omega = \emptyset$ or the intersection $\bigcap \omega = \bigcap_{S \in \omega} S$ of the winning coalitions is nonempty. The members of $\bigcap \omega$ are called **veto players**; they are the players that belong

having a structure similar to this example, such as elaborating the conditions for granting tenure to junior faculty members at academic institutions [1] and deciding whether a certain act is legal or not.

⁵This example illustrates that the desirability of the (conventional) axioms depends on the context. Monotonicity makes sense here, but may be too optimistic (adding a member may turn an acceptable team into an unacceptable one). Properness may be irrelevant or even undesirable (ensuring that a given task can be performed by two non-overlapping teams may be important from the viewpoint of reliability). This observation suggests the importance of finding games that violate some of the axioms.

⁶Soare [21] and Odifreddi [18] give a more precise definition of *recursive sets* as well as detailed discussion of recursion theory. Mihara’s papers [15, 16] contain short reviews of recursion theory.

to all winning coalitions. (The set $\bigcap \omega$ of veto players may or may not be observable.) ω is **dictatorial** if there exists some i_0 (called a **dictator**) in N such that $\omega = \{S \in \text{REC} : i_0 \in S\}$. Note that a dictator is a veto player, but a veto player is not necessarily a dictator. It is immediate to prove the following well-known lemmas:

Lemma 1 *If a simple game is weak, it is proper.*

Lemma 2 *A simple game is dictatorial if and only if it is strong and weak.*

A **carrier** of a simple game ω is a coalition $S \subset N$ such that

$$T \in \omega \iff S \cap T \in \omega$$

for all coalitions T . We observe that if S is a carrier, then so is any coalition $S' \supseteq S$. Slightly abusing the word, we sometimes say a game is **finite** if it has a finite carrier; otherwise, the game is **infinite**.

2.2 An indicator for simple games

To define the notions of computability for simple games, we introduce an indicator for them. In order to do that, we first represent each recursive coalition by a characteristic index (Δ_0 -index). Here, a number e is a **characteristic index** for a coalition S if φ_e (the partial function computed by the Turing program with code number e) is the characteristic function for S . Intuitively, a characteristic index for a coalition describes the coalition by a Turing program that can decide its membership. The indicator then assign the value 0 or 1 to each number representing a coalition, depending on whether the coalition is winning or losing. When a number does not represent a recursive coalition, the value is undefined.

Given a simple game ω , its **δ -indicator** is the partial function δ_ω on \mathbb{N} defined by

$$\delta_\omega(e) = \begin{cases} 1 & \text{if } e \text{ is a characteristic index for a recursive set in } \omega, \\ 0 & \text{if } e \text{ is a characteristic index for a recursive set not in } \omega, \\ \uparrow & \text{if } e \text{ is not a characteristic index for any recursive set.} \end{cases} \quad (1)$$

Note that δ_ω is well-defined since each $e \in \mathbb{N}$ can be a characteristic index for at most one set.

2.3 The computability notion

We now introduce the notion of δ -computable simple games. We start by giving an intuition underlying the notion of δ -computability. A number (characteristic index) representing a coalition (equivalently, a Turing program that can decide the membership of the coalition) is presented by an

inquirer to the aggregator (planner), who will compute whether the coalition is winning or not. The aggregator cannot know a priori which indices will possibly be presented to her. (There are, of course, indices unlikely to be used by humans. But the aggregator cannot a priori rule out some of the indices.) So, *the aggregator should be ready to give an answer whenever a characteristic index for some recursive set is presented to her.* This intuition justifies the following condition of computability.

δ -computability δ_ω has an extension to a partial recursive function.

Among various notions of computability that one could conceive of, this notion is the only one that we find [17, 11] defensible.

3 Overview of the Results

This section gives a summary of the results in Sections 5–6.

We first classify games according to the conventional axioms (monotonicity, properness, strongness, and nonweakness) and the axioms concerning simplicity (computability and finiteness).

We identify an axiom (property) for games with the class of games satisfying the axiom. Let A_1 denote (the class of games satisfying the axiom of) monotonicity, A_2 properness, A_3 strongness, and A_4 nonweakness. We can classify games into $2^4 = 16$ classes $C_1 \cap C_2 \cap C_3 \cap C_4$ according to these four axioms, where each C_j is either A_j or A_j^c (where A_j^c is the complement of A_j ; A_j^c is identified with the negation $\neg A_j$ of A_j). Each class $C_1 \cap C_2 \cap C_3 \cap C_4$ is represented by a string (binary word) $b_1 b_2 b_3 b_4$ of length four, consisting of +’s (1’s) and –’s (0’s), where each b_j is either + or –, depending on $C_j = A_j$ or $C_j = A_j^c$. For example, the class $A_1 \cap A_2^c \cap A_3^c \cap A_4$ of monotonic, nonproper, nonstrong, nonweak games are represented by the string $+- -+$. The **(conventional) type** of a simple game is (the string $b_1 b_2 b_3 b_4$ representing) the class $C_1 \cap C_2 \cap C_3 \cap C_4$ to which the game belongs. For ease of reference, we give a label (decimal number) to each type, as shown in the first column of Table I. (The labels are obtained by first identifying the string with a binary number, next converting the binary number into a decimal number, and finally subtracting the decimal number from 16.)

Similarly, we can classify games into $2^2 = 4$ classes according to δ -computability and finiteness (existence of a finite carrier).

Altogether, we can classify games into $16 \times 4 = 64$ classes according to the six axioms. For each such class, we can ask the question whether there exists a simple game in the class. The answers to those questions are given in Sections 5–6. Table I summarizes the answers.

We are mainly interested in the relation of computability to the four conventional axioms. What can we observe from Table I? For example, we can see from row (2), *there is a computable game of type (+ + +–), but*

Table I: Existence of Games in Different Classes

Types	With Finite Carrier		Without Finite Carrier	
	Non	Computable	Non	Computable
1(++++)	no	yes	yes	yes
2(+++-)	no	<i>yes</i>	<i>no</i>	<i>no</i>
3(++-+)	no	yes	yes	yes
4(++--)	no	yes	yes	yes
5(+ - ++)	no	yes	yes	yes
6(+ - +-)	no	no	no	no
7(+ - -+)	no	yes	yes	yes
8(+ - --)	no	no	no	no
9(- + ++)	no	yes	yes	yes
10(- + +-)	no	no	no	no
11(- + -+)	no	yes	yes	yes
12(- + --)	no	yes	yes	yes
13(- - ++)	no	yes	yes	yes
14(- - +-)	no	no	no	no
15(- - -+)	no	yes	yes	yes
16(- - --)	no	no	no	no

The types are defined by the four conventional axioms: monotonicity, properness, strongness, and nonweakness. For example, row (2) indicates that among the monotonic (+), proper (+), strong (+), weak (-, because not nonweak) games, there exist no finite noncomputable ones, there exist finite computable ones, there exist no infinite noncomputable ones, and there exist no infinite computable ones. Note that except for row (2), the last three columns are identical.

not a noncomputable game of the same type. (In fact, the type consists of dictatorial games.) This means that computability is not “independent of” the four axioms in the sense to be made precise below. *For each of the other fifteen types, however, there is a computable game of that type if and only if there is a noncomputable game of that type.* Hence, we could almost say that computability is “unrelated to” the four axioms. In fact, if we restrict our attention to the infinite games (games without a finite carrier), we can say this:

Proposition 3 *Axiom C (δ -computability) is independent of the axioms A_1 (monotonicity), A_2 (properness), A_3 (strongness), and A_4 (nonweakness) within the class I of infinite games in the sense that for each of the $2^4 = 16$ types $C_1 \cap C_2 \cap C_3 \cap C_4$, the following condition is satisfied:*

$$C \cap I \cap (C_1 \cap C_2 \cap C_3 \cap C_4) \neq \emptyset \iff C^c \cap I \cap (C_1 \cap C_2 \cap C_3 \cap C_4) \neq \emptyset.$$

The exact sense of the statement “computability is independent of the four axioms” mentioned above can be defined by replacing the class I in the statement of the proposition with the class of all simple games. Therefore, computability is not independent of the four axioms if and only if it is *restrictive* in the sense that some nonempty type consists only of computable games or only of noncomputable games.

We leave this section with two interesting observations involving the last three (instead of two as in Proposition 3) columns of the table: From rows (6), (8), (10), (14), (16), we conclude that *if there does not exist a finite computable game of a particular type, then there does not exist a game of that type.* From the other rows except row (2), we conclude that *if there exists an infinite (non)computable game of a particular type, then there exists a finite computable game of that type.*

4 Preliminary Results

This section summarizes the results in the companion paper [11] and notation needed in Sections 5–6.

To give a background, we first mention Theorem 4, which characterizes δ -computable simple games in terms of sets of “determining strings.” Roughly speaking, in a computable game, finitely many players determine whether a coalition is winning or losing. Though we cannot tell in advance which finite set of players determines that, we can list such sets in an effective manner.

Notation. We identify a natural number k with the finite set $\{0, 1, 2, \dots, k-1\}$, which is an initial segment of \mathbb{N} . Given a coalition $S \subseteq N$, we write $S \cap k$ to represent the coalition $\{i \in S : i < k\}$ consisting of the members of S whose name is less than k . We call $S \cap k$ the **k -initial segment of S** , and view it either as a subset of \mathbb{N} or as the string $S[k]$ of length k

of 0's and 1's (representing the restriction of its characteristic function to $\{0, 1, 2, \dots, k-1\}$). \parallel

Definition 1. Consider a simple game. A string τ (of 0's and 1's) of length $k \geq 0$ is **winning determining** if any coalition $G \in \text{REC}$ extending τ (in the sense that τ is an initial segment of G , i.e., $G \cap k = \tau$) is winning; τ is **losing determining** if any coalition $G \in \text{REC}$ extending τ is losing. A string is **determining** if it is either winning determining or losing determining. A string is **nondetermining** if it is not determining.

Theorem 4 (Kumabe and Mihara [11]) *A simple game ω is δ -computable if and only if there are an r.e. set T_0 of losing determining strings and an r.e. set T_1 of winning determining strings such that (the characteristic function for) any coalition has an initial segment in T_0 or in T_1 .*

Note that $T_0 \cup T_1$ in the theorem does not necessarily contain all determining strings.

It is easy [11] to prove from Theorem 4 that *games having a finite carrier are computable* and that *computable games have both finite winning coalitions and cofinite losing coalitions*. (Mihara [17] proves these results without resorting to the theorem.) As already seen in Section 3, the property of having a finite carrier is an important criterion for classifying games in this paper.

Though the proof of Theorem 4 is rather involved, the full force of this theorem is not needed in this paper.

First, *to construct computable games*, we only need the “if” direction, whose proof is straightforward. The following proposition restates the “if” direction of the theorem in a more convenient form:

Proposition 5 *Let T_0 and T_1 be recursively enumerable sets of (nonempty) strings such that any coalition has an initial segment in T_0 or in T_1 but not both. Let ω be the simple game defined by $S \in \omega$ if and only if S has an initial segment in T_1 . Then T_1 consists only of winning determining strings, T_0 consists only of losing determining strings, and ω is δ -computable.*

Proof. The assertion that T_1 consists of winning determining strings is obvious from the definition of ω . To see that T_0 consists of losing determining strings, let $\alpha \in T_0$. Suppose a coalition S extends α but S is winning. Thus there is a string $\beta \in T_1$ that S extends. This implies that S has an initial segment in both T_0 and T_1 , contrary to the assumption. Computability of ω is immediate from the “if” direction of Theorem 4 \blacksquare

Second, *to construct noncomputable games*, we only need the following proposition, whose proof is much easier than that of Theorem 4 (it can

also be derived as a corollary of Theorem 4). It states that for δ -computable simple games, (the characteristic function for) every coalition S has an initial segment $S \cap k$ that is determining. (The number $k - 1$ may be greater than the greatest element, if any, of S .) Recall that $S[k]$ is $S \cap k$ viewed as a string:

Proposition 6 (Kumabe and Mihara [11]) *Suppose that a δ -computable simple game is given. (i) If a coalition S is winning, then it has an initial segment $S[k]$ (for some $k \in \mathbb{N}$) that is winning determining. (ii) If S is losing, then it has an initial segment $S[k]$ that is losing determining.*

We conclude the section with a summary of notation that the reader should take notice of.

Notation. For a partial function f , $f(\alpha) \downarrow$ means $f(\alpha)$ is defined; $f(\alpha) \uparrow$ means $f(\alpha)$ is undefined. $\varphi_k(\cdot)$ denotes the k th partial recursive function of one variable—it is computed by the Turing program with code (Gödel) number k .

Let α and β be strings (of 0's and 1's).

Then α^c denotes the string of the length $|\alpha|$ such that $\alpha^c(i) = 1 - \alpha(i)$ for each $i < |\alpha|$; for example, $0110100100^c = 1001011011$. Occasionally, a string α is identified with the set $\{i : \alpha(i) = 1\}$. (Note however that α^c is occasionally identified with the set $\{i : \alpha(i) = 0\}$, but never with the set $\{i : \alpha(i) = 1\}^c$.)

$\alpha\beta$ (or $\alpha * \beta$) denotes the concatenation of α followed by β .

$\alpha \subseteq \beta$ means that α is an initial segment of β (β extends α); $\alpha \subseteq A$ means that α is an initial segment of a set A .

Strings α and β are **incompatible** if neither $\alpha \subseteq \beta$ nor $\beta \subseteq \alpha$ (i.e., there is $k < \min\{|\alpha|, |\beta|\}$ such that $\alpha(k) \neq \beta(k)$). \parallel

5 Games with Finite Carriers

We start with the class of finite games (games having a finite carrier). Any game in this class is δ -computable.

In the following, for each of the sixteen conventional types (with respect to monotonicity, properness, strongness, and nonweakness), we either give an example of a finite game of that type or give a proof that there exists no such game. We give each example by exhibiting finite sets T_0 and T_1 satisfying the condition of Proposition 5.

1. (+ + + +) A monotonic, proper, strong, nonweak game. Let $T_0 = \{00, 010, 100\}$ and $T_1 = \{11, 011, 101\}$.
2. (+ + + -) A monotonic, proper, strong, weak game. Let $T_0 = \{0\}$ and $T_1 = \{1\}$. Player 0 is a dictator.

3. (+ + -+) A monotonic, proper, nonstrong, nonweak game. Let $T_0 = \{00, 010, 0110, 100, 1010\}$ and $T_1 = \{11, 1011, 0111\}$.
4. (+ + --) A monotonic, proper, nonstrong, weak game. Let $T_0 = \{0, 10\}$ and $T_1 = \{11\}$.
5. (+ - ++) A monotonic, nonproper, strong, nonweak game. Let $T_0 = \{00\}$ and $T_1 = \{1, 01\}$.
6. (+ - +-) A monotonic, nonproper, strong, weak game. By Lemma 1, there is no such game.
7. (+ - -+) A monotonic, nonproper, nonstrong, nonweak game. Let $T_0 = \{00, 1000, 1001, 0110, 0100\}$ and $T_1 = \{11, 1011, 1010, 0101, 0111\}$.
8. (+ ---) A monotonic, nonproper, nonstrong, weak game. By Lemma 1, there is no such game.
9. (- + ++) A nonmonotonic, proper, strong, nonweak game. Let $T_0 = \{1\}$ and $T_1 = \{0\}$.
10. (- + +-) A nonmonotonic, proper, strong, weak game. By Lemma 2, any such game is dictatorial. But any dictatorial game is monotonic. So there is no such game.
11. (- + -+) A nonmonotonic, proper, nonstrong, nonweak game. Let $T_0 = \{1, 01\}$ and $T_1 = \{00\}$.
12. (- + --) A nonmonotonic, proper, nonstrong, weak game. Let $T_0 = \{1, 00\}$ and $T_1 = \{01\}$.
13. (- - ++) A nonmonotonic, nonproper, strong, nonweak game. Let $T_0 = \{10\}$ and $T_1 = \{0, 11\}$.
14. (- - +-) A nonmonotonic, nonproper, strong, weak game. By Lemma 1, there is no such game.
15. (- - -+) A nonmonotonic, nonproper, nonstrong, nonweak game. Let $T_0 = \{01, 10\}$ and $T_1 = \{00, 11\}$.
16. (- - --) A nonmonotonic, nonproper, nonstrong, weak game. By Lemma 1, there is no such game.

6 Games without Finite Carriers

We consider infinite games (games without finite carriers) in this section.

6.1 An assortment of noncomputable games

We first give examples of infinite *noncomputable* simple games. We have noted in Section 4 (immediately after Theorem 4) that all *computable* games belong to the class of games that have both finite winning coalitions and cofinite losing coalitions. To show that variety is not lost even if we restrict our games to this class, all the examples are chosen from the class.

Three examples (namely, types 1, 3, and 9) in this section are based on the following lemma. The construction of other examples is simpler.

Lemma 7 *Let A be a recursive set. Let T_0 and T_1 be recursively enumerable, nonempty sets of (nonempty) strings such that any coalition has an initial segment in T_0 or in T_1 but not both. Let ω be the simple game defined by $S \in \omega$ if and only if either $S = A$ or $[S \neq A^c$ and S has an initial segment in $T_1]$. Then we have the following:*

(i) *$S \notin \omega$ if and only if either $S = A^c$ or $[S \neq A$ and S has an initial segment in $T_0]$.*

(ii) *ω has a finite winning coalition and a cofinite losing coalition.*

(iii) *Suppose further that either A is infinite and has an initial segment in T_0 or A^c is infinite and has an initial segment in T_1 . Then ω is δ -noncomputable.*

Proof. (i) From the definition of ω and the assumption that any coalition S has a initial segment in T_0 or T_1 but not both, we have

$$\begin{aligned} S \notin \omega &\iff S \neq A \text{ and } [S = A^c \text{ or } S \text{ has no initial segment in } T_1] \\ &\iff [S \neq A \text{ and } S = A^c] \text{ or} \\ &\quad [S \neq A \text{ and } S \text{ has no initial segment in } T_1] \\ &\iff [S = A^c] \text{ or } [S \neq A \text{ and } S \text{ has an initial segment in } T_0]. \end{aligned}$$

(ii) Choose a string α from the nonempty set T_1 . Let $\beta = \alpha * A(|\alpha|)$. Then $\beta \neq A^c$ since $\beta(|\alpha|) = A(|\alpha|) \neq A^c(|\alpha|)$. Since β has the initial segment $\alpha \in T_1$, $\beta \in \omega$ by the definition of ω . We have obtained a finite winning coalition, namely β . To obtain a cofinite losing coalition, choose $\alpha \in T_0$ and let $\beta = \alpha * A^c(|\alpha|)$. Then by (i), $B := \{i : \beta(i) = 1 \text{ or } \beta(i) \uparrow\}$ is a cofinite losing set.

(iii) Suppose A is infinite and has an initial segment $A[k]$ in T_0 . Suppose ω is δ -computable. Then, by Proposition 6, the winning coalition A has an initial segment $A[k']$ that is a winning determining string. Let $\hat{k} = \max\{k, k'\}$. Then on the one hand, $A[\hat{k}]$, which is different from A and has an initial segment in T_0 , is losing by (i). On the other hand, $A[\hat{k}]$ is winning since it extends the winning determining string $A[k']$. We have obtained a contradiction. The case where A^c is infinite and has an initial segment in T_1 is similar. ■

In the rest of this section, we give, for each of the sixteen conventional types of simple games, an example of an infinite noncomputable game of that type, if such a game exists.

1. (+ + + +) A monotonic, proper, strong, nonweak game. Let $A = \{0\}^c = \{1, 2, 3, \dots\} \supseteq 0$, $T_1 = \{1\}$, and $T_0 = \{0\}$. The game ω defined by Lemma 7 satisfies the properties [11, Section 6.1].
2. (+ + + -) A monotonic, proper, strong, weak game. By Lemma 2, such a game has a dictator, violating the property that it has no finite carrier.
3. (+ + - +) A monotonic, proper, nonstrong, nonweak game. Let $A = \{0, 1\}^c = \{2, 3, 4, \dots\} \supseteq 00$, $T_1 = \{11, 1011, 0111\}$, and $T_0 = \{00, 010, 0110, 100, 1010\}$. Define ω by $S \in \omega$ if and only if either $S = A$ or $[S \neq A^c$ and S has an initial segment in $T_1]$. Then Lemma 7 applies. Note that all the elements of $T_1 \cup T_0$ except 11 and 00 are determining and 111 is winning determining. To show that ω is monotonic, suppose $S \in \omega$ and $T \supseteq S$. If $S = A$, then $T \supseteq 1011$ or $T \supseteq 0111$ or $T = N \supseteq 111$, implying that $T \in \omega$. Otherwise, $S \supseteq \alpha$ for some $\alpha \in T_1$. Then, $T \supseteq 1011$ or $T \supseteq 0111$ or $T = N \supseteq 1111$, implying that $T \in \omega$.

To show that it is proper, suppose $S, S^c \in \omega$. If $S = A$, then contradiction is obtained since $S^c = A^c \neq A$. Otherwise, $S \neq A^c$ and $S[2] = 11$ or $S[4] = 1011$ or 0111 . This contradicts the condition that S^c has an initial segment in T_0 .

It is nonstrong since $\{1, 3\} \supseteq 010$ is losing and $\{1, 3\}^c \supseteq 1010$ is losing. It is nonweak since the winning coalitions $\{0, 1, 4\} \supseteq 11001$, $\{0, 2, 3\} \supseteq 1011$, $\{1, 2, 3\} \supseteq 0111$ have an empty intersection.

4. (+ + - -) A monotonic, proper, nonstrong, weak game. Let $A = \{0, 1\}$, $T_1 = \{11\}$, and $T_0 = \{0, 10\}$. Define ω by $S \in \omega$ if and only if $S \neq A$ and S has an initial segment in T_1 . Then, we have $S \notin \omega$ if and only if $S = A$ or S has an initial segment in T_0 . To show that ω is noncomputable, suppose it is not. Then, by Proposition 6, the losing coalition A has an initial segment $A[k]$ that is losing determining, where $k \geq 2$ without loss of generality. But $A[k] * 1$ is winning, contradiction. The proofs of the remaining properties are easy.
5. (+ - + +) A monotonic, nonproper, strong, nonweak game. Let $A = \{0, 1\}^c = \{2, 3, 4, \dots\}$, $T_1 = \{1, 01\}$ and $T_0 = \{00\}$. Define ω by $S \in \omega$ if and only if $S = A$ or S has an initial segment in T_1 . Then, we have $S \notin \omega$ if and only if $S \neq A$ and S has an initial segment in T_0 . The proofs of all the properties are easy.

6. (+ - +-) A monotonic, nonproper, strong, weak game. There is no such game.
7. (+ - -+) A monotonic, nonproper, nonstrong, nonweak game. Let $A = \{0, 1\}^c = \{2, 3, 4, \dots\}$, $T_1 = \{11, 1011, 1010, 0101, 0111\}$, and $T_0 = \{00, 1000, 1001, 0110, 0100\}$. Define ω by $S \in \omega$ if and only if $S = A$ or S has an initial segment in T_1 . The proofs of all the properties are easy.
8. (+ - --) A monotonic, nonproper, nonstrong, weak game. There is no such game.
9. (- + ++) A nonmonotonic, proper, strong, nonweak game. Let $A = N \supseteq 1$, $T_1 = \{0\}$, and $T_0 = \{1\}$. Define ω by $S \in \omega$ if and only if either $S = A$ or $[S \neq A^c$ and S has an initial segment in $T_1]$. Then Lemma 7 applies. It is nonmonotonic since $\{1\} \supseteq 0 \in \omega$ but $\{0, 1\} \supseteq 1 \notin \omega$. To show that it is proper and strong, note the condition equivalent to $S \notin \omega$ given in Lemma 7 (i): $S = A^c$ or $[S \neq A$ and S has an initial segment in $T_0]$. This condition is in turn equivalent to $S^c \in \omega$ since S has an initial segment in T_0 if and only if S^c has an initial segment in T_1 in this example. It is nonweak since the disjoint sets $\{1\} \supseteq 01$ and $\{2\} \supseteq 001$ are winning.
10. (- + +-) A nonmonotonic, proper, strong, weak game. There is no such game.
11. (- + -+) A nonmonotonic, proper, nonstrong, nonweak game. Let $A = \{0, 1\}^c = \{2, 3, 4, \dots\}$, $T_1 = \{00\}$, and $T_0 = \{1, 01\}$. Define ω by $S \in \omega$ if and only if $S \neq A$ and S has an initial segment in T_1 . The proofs of all the properties are easy.
12. (- + --) A nonmonotonic, proper, nonstrong, weak game. Let $A = \{1\}$, $T_1 = \{01\}$, $T_0 = \{1, 00\}$. Define ω by $S \in \omega$ if and only if $S \neq A$ and S has an initial segment in T_1 . The proofs of all the properties are easy.
13. (- - ++) A nonmonotonic, nonproper, strong, nonweak game. Let $A = \{1\}^c = \{0, 2, 3, 4, \dots\}$, $T_1 = \{0, 11\}$, and $T_0 = \{10\}$. Define ω by $S \in \omega$ if and only if $S = A$ or S has an initial segment in T_1 . The proofs of all the properties are easy.
14. (- - +-) A nonmonotonic, nonproper, strong, weak game. There is no such game.
15. (- - -+) A nonmonotonic, nonproper, nonstrong, nonweak game. Let $A = N$, $T_1 = \{00, 11\}$, and $T_0 = \{01, 10\}$. Define ω by $S \in \omega$ if and only if $S \neq A$ and S has an initial segment in T_1 . The proofs of all the properties are easy.

16. (---) A nonmonotonic, nonproper, nonstrong, weak game. There is no such game.

6.2 A class of computable, monotonic, proper, strong, non-weak games without finite carriers

In this section, *we construct for each recursive set A , an infinite, computable, monotonic, proper, strong, nonweak simple game $\omega[A]$* . We do so with a view to constructing examples of various types in Section 6.3. For this reason, the construction is long and elaborate. The reader who finds this construction too complicated might want to consult our paper [11, Section 6.2] first, in which a less elaborate construction of an infinite computable game appears. The construction there is shorter, though not very useful for constructing examples in Section 6.3.

Our approach is to construct recursively enumerable sets T_0 and T_1 of strings (of 0's and 1's) satisfying the conditions of Proposition 5. We first construct certain sets F_s of strings for $s \in \{0, 1, 2, \dots\}$. We then specify an algorithm for enumerating the elements of T_0 and T_1 using the sets F_s , and construct a simple game $\omega[A]$ according to Proposition 5. We conclude that the game is computable by checking (Lemma 14) that T_0 and T_1 satisfy the conditions of Proposition 5. Finally, we show (Lemmas 16, 17, and 18) that the game satisfies the desired properties.

Before constructing sets T_0 and T_1 of determining strings, we introduce the notions of p-strings and d-strings. Roughly speaking, a p-string consists of 10's or 01's; A d-string is a concatenation of a p-string followed by 00 or 11. More formally, a string α is a **p-string** if $|\alpha|$ is even and for each $2k < |\alpha|$, we have $\alpha(2k)\alpha(2k+1) \in \{10, 01\}$ (i.e., $\alpha(2k+1) = 1 - \alpha(2k)$). Examples of a p-string include the empty string, 01, 0101, 0110, and 1001011010. Note that any substring of even length of a p-string is a p-string. Denote by α^- the substring $\alpha[|\alpha| - 1]$ of α with length $|\alpha| - 1$. In other words, $\alpha = \alpha^- * \alpha(|\alpha| - 1)$. A string α (of even length) is a **d-string** if α^{--} is a p-string and $\alpha(|\alpha| - 2)\alpha(|\alpha| - 1) \in \{00, 11\}$ (i.e., $\alpha(|\alpha| - 2) = \alpha(|\alpha| - 1)$). In other words, a d-string α is of the form $\alpha^{--} * 00$ or $\alpha^{--} * 11$ for some p-string α^{--} .

Lemma 8 (i) *Any string of even length either is a p-string or extends a d-string.* (ii) *Any two distinct d-strings α and β are incompatible. That is, we have neither $\alpha \subseteq \beta$ nor $\beta \subseteq \alpha$.*

Proof. (i) Let α be a string of even length. Find the least k with $2k + 1 < |\alpha|$ such that $\alpha(2k)\alpha(2k + 1) \in \{00, 11\}$. If there is not such a k , then α is a p-string. If there is such a k , then the substring $\alpha[2k + 2]$ is a d-string.

(ii) Assume α and β are different d-strings. If $|\alpha| = |\beta|$, then clearly they are incompatible (since they are different). Otherwise, assume $|\alpha| > |\beta|$ without loss of generality. Since these length are even, we have $|\alpha| - 2 \geq |\beta|$. Suppose α and β are compatible. Then $\alpha \supset \beta$ in this case. In fact, the inequality above implies that $\alpha^{--} \supseteq \beta$. But this is impossible since $\alpha^{--}(|\beta| - 2)\alpha^{--}(|\beta| - 1) \in \{10, 01\}$ (a pair in a p-string) on the one hand, while $\beta(|\beta| - 2)\beta(|\beta| - 1) \in \{00, 11\}$ (the tail of a d-string) on the other hand. ■

Let $\{k_s\}_{s=0}^\infty$ be an effective listing (recursive enumeration) of the members of the recursively enumerable set $\{k : \varphi_k(2k) \in \{0, 1\}\}$, where $\varphi_k(\cdot)$ is the k th partial recursive function of one variable. We can assume without loss of generality that $k_0 \geq 1$ and all the elements k_s are distinct. Thus,

$$\text{CRec} \subset \{k : \varphi_k(2k) \in \{0, 1\}\} = \{k_0, k_1, k_2, \dots\},$$

where CRec is the set of characteristic indices for recursive sets.

Let $l_0 = 2k_0 + 2 \geq 4$ and for $s > 0$, let $l_s = \max\{l_{s-1}, 2k_s + 2\}$. Then $\{l_s\}$ is an nondecreasing sequence of even numbers and $l_s > 2k_s + 1$ for each s . Note also that $l_s \geq l_{s-1} > 2k_{s-1} + 1$, $l_s \geq l_{s-2} > 2k_{s-2} + 1$, etc. imply that $l_s > 2k_s + 1, 2k_{s-1} + 1, 2k_{s-2} + 1, \dots, 2k_0 + 1$.

For each s , let F_s be the finite set of p-strings $\alpha = \alpha(0)\alpha(1) \cdots \alpha(l_s - 1) \supseteq 10$ of length $l_s \geq 4$ such that

$$(1) \alpha(2k_s) = \varphi_{k_s}(2k_s) \text{ and for each } s' < s, \alpha(2k_{s'}) = 1 - \varphi_{k_{s'}}(2k_{s'}).$$

Note that (1) imposes no constraints on $\alpha(2k)$ for $k \notin \{k_0, k_1, k_2, \dots, k_s\}$, while it actually imposes constraints for all k in the set, since $|\alpha| = l_s > 2k_s, 2k_{s-1}, 2k_{s-2}, \dots, 2k_0$. We observe that if $\alpha \in F_s \cap F_{s'}$, then $s = s'$. Let $F = \bigcup_s F_s$. Then F is recursive and we have the following:

Lemma 9 *Any two distinct elements in F are incompatible.*

Proof. Let $\alpha, \beta \in F$ such that $|\alpha| \leq |\beta|$, without loss of generality. If α and β have the same length, then the conclusion follows since otherwise they become identical strings. If $l_s = |\alpha| < |\beta| = l_{s'}$, then $s < s'$ and by (1), $\alpha(2k_s) = \varphi_{k_s}(2k_s)$ on the one hand, but $\beta(2k_s) = 1 - \varphi_{k_s}(2k_s)$ on the other hand. So $\alpha(2k_s) \neq \beta(2k_s)$. ■

Let f be a recursive bijection from F onto \mathbb{N} (f can be obtained by enumerating the elements of F one by one, assigning 0 to the first element enumerated, 1 to the second element enumerated, and so on). Regarding f as a partial function on the set of strings, we have $f(\alpha) \downarrow$ (i.e., $f(\alpha)$ is defined) if and only if $\alpha \in F$.

Lemma 10 *Let $\alpha \supseteq 10$ be a p-string of length l_s . Then the following statements are equivalent: (i) no substring of α is in F ; (ii) for each $s' \leq s$, $\alpha[l_{s'}] \notin F$; (iii) for each $s' \leq s$, $f(\alpha[l_{s'}]) \uparrow$; (iv) for each $s' \leq s$, $\alpha(2k_{s'}) = 1 - \varphi_{k_{s'}}(2k_{s'})$.*

Proof. The definition of F implies that $\alpha \in F$ only if $|\alpha| = l_s$ for some s . Hence the equivalence of (i), (ii), and (iii) is immediate. We next show that (ii) and (iv) are equivalent. The direction from (iv) to (ii) is clear from (1). To see the other direction, suppose that (iv) is not the case; we derive the negation of (ii). For some $s' \leq s$, we have $\alpha(2k_{s'}) = \varphi_{k_{s'}}(2k_{s'})$. Choose the least such s' . Then ($s' = 0$ or) for any $s'' < s'$, $\alpha(2k_{s''}) = 1 - \varphi_{k_{s''}}(2k_{s''})$. So $\alpha[l_{s'}] \in F_{s'}$ by (1), since $\alpha[l_{s'}] \supseteq 10$ is a p-string of length $l_{s'}$. Thus (ii) is violated. ■

Let A be a recursive set. The game $\omega[A]$ will be defined via the sets $T_0 := T_0^A$ and $T_1 := T_1^A$ of strings, constructed by enumerating the elements as follows: For each s and $\alpha \in F_s$ (having a length l_s and extending 10),

- (2.i) for each p-string α' that is a proper substring of α , if $s = 0$ or $|\alpha'| \geq l_{s-1}$, then enumerate $\alpha' * 11$ in T_1 and $\alpha' * 00$ in T_0 ;
- (2.ii) if $f(\alpha) \in A$, enumerate α in T_1 ; if $f(\alpha) \notin A$, enumerate α in T_0 (note that $f(\alpha) \downarrow$ since $\alpha \in F$);
- (3) if a string β is enumerated in T_1 (or in T_0) above, then enumerate β^c in T_0 (or in T_1 , respectively).

Clearly, T_0 and T_1 are recursively enumerable because of this generating algorithm. We observe that the sets T_0 and T_1 consist of

- d-strings (11, 00, and those extending 10 enumerated at (2.i) and those extending 01 enumerated at (3) via (2.i)) and
- p-strings (those extending 10 enumerated at (2.ii) and those extending 01 enumerated at (3) via (2.ii)).

We also observe that $11 \in T_1$, $00 \in T_0$, $T_0 \cap T_1 = \emptyset$, and $\alpha \in T_0 \Leftrightarrow \alpha^c \in T_1$.

Define a game $\omega[A]$ by $S \in \omega[A]$ if and only if S has an initial segment in T_1 . Lemma 14 establishes computability of $\omega[A]$ (as well as the assertion that T_0 consists of losing determining strings and T_1 consists of winning determining strings) by way of Proposition 5.

Lemma 11 *Let α, β be distinct strings in $T_0 \cup T_1$. Then α and β are incompatible. In particular, if $\alpha \in T_0$ and $\beta \in T_1$, then α and β are incompatible.*

Proof. Obviously, neither α nor β is an empty string. Since T_0 and T_1 consist of p-strings and d-strings, there are three cases to consider:

Case (pp): Both α and β are p-strings. Then either α or α^c is enumerated at (2.ii) of the generating algorithm and so $\alpha \in F$ or $\alpha^c \in F$. Similarly, $\beta \in F$ or $\beta^c \in F$. If $\alpha \in F$ and $\beta \in F$, then α and β are incompatible, since any two distinct elements of F are incompatible by Lemma 9. If $\alpha \in F$ and $\beta^c \in F$, then $\alpha \supset 10$ and $\beta \supset 01$, so they are incompatible. The other two subcases are similar.

Case (pd): one of α or β is a p-string and the other is a d-string. Without a loss of generality, α is a p-string and β is a d-string. Suppose α and β are compatible. Then, $\beta \supset \alpha$. In fact, $\beta^{--} \supseteq \alpha$. As in (pp) above, either $\alpha \in F$ or $\alpha^c \in F$. Also, since either β or β^c is enumerated at (2.i) of the algorithm, we have either (pd.i) $\beta^{--} \subset \tilde{\beta}$ for some $\tilde{\beta} \in F$ or (pd.ii) $(\beta^c)^{--} \subset \hat{\beta}$ for some $\hat{\beta} \in F$. *Subcase: $\alpha \in F$ and (pd.i).* α and $\tilde{\beta}$ and both in F . So they are incompatible by Lemma 9, contradicting the fact that $\alpha \subseteq \beta^{--} \subset \tilde{\beta}$. *Subcase: $\alpha \in F$ and (pd.ii).* Then $\alpha \supseteq 10$ but $\beta \supset 01$, a contradiction. *Subcase: $\alpha^c \in F$ and (pd.i).* Similar to the second subcase. *Subcase: $\alpha^c \in F$ and (pd.ii).* Similar to the first subcase.

Case (dd): Both α and β are d-strings. Immediate from Lemma 8. ■

Notation. We write $f(\beta) \downarrow \in A$ if $f(\beta) \in A$ (which requires $f(\beta) \downarrow$); we write $f(\beta) \downarrow \notin A$ if $f(\beta) \downarrow$ but $f(\beta) \notin A$.

Lemma 12 *Let $\alpha \supset 1$ be a string of length l_s .*

- (i) α extends a string in T_1 if and only if (i.a) for some $s' \leq s$, $f(\alpha[l_{s'}]) \downarrow \in A$ (in this case, $\alpha[l_{s'}] \in T_1$) or (i.b) α extends a d-string $\alpha' = (\alpha')^{--} * 11$ such that no substring of $(\alpha')^{--}$ is in F (in this case, $\alpha' \in T_1$).
- (ii) α extends a string in T_0 if and only if (ii.a) for some $s' \leq s$, $f(\alpha[l_{s'}]) \downarrow \notin A$ (in this case, $\alpha[l_{s'}] \in T_0$) or (ii.b) α extends a d-string $\alpha' = (\alpha')^{--} * 00$ such that no substring of $(\alpha')^{--}$ is in F (in this case, $\alpha' \in T_0$).
- (iii) α does not extend a string in $T_0 \cup T_1$ if and only if α is a p-string and no substring of α is in F .

Proof. (i) (\implies). Assume $\alpha \supseteq 11$. Then (i.b) is satisfied by letting $\alpha' = 11$.

Assume $\alpha \supseteq 10$ extends a string $\alpha' \in T_1$. Suppose first that α' is enumerated in T_1 by applying (2.i) of the generating algorithm. (We show (i.b) holds.) Then $\alpha' = (\alpha')^{--} * 11$ and $(\alpha')^{--}$ is properly extended by some element in F_s . Since any two different elements in F are incompatible by Lemma 9, no substring of $(\alpha')^{--}$ is in F . So (i.b) holds. Suppose next that α' is enumerated in T_1 by applying (2.ii). Then $f(\alpha') \in A$. Since $\alpha' = \alpha[l_{s'}]$ for some $s' \leq s$, we obtain (i.a). Finally, the case where $\alpha' \supseteq 10$ is enumerated

in T_1 by applying (3) is impossible, since every string enumerated at (3) extends 0.

(\Leftarrow). Assume $\alpha \supseteq 11$. Since $11 \in T_1$, the left hand side of (i) holds.

Assume $\alpha \supseteq 10$ and either (i.a) or (i.b) holds.

Suppose (i.a) first. By the definition of f , $\alpha[l_{s'}] \in F_{s'}$. Since $f(\alpha[l_{s'}]) \in A$, we have $\alpha[l_{s'}] \in T_1$ by (2.ii). So α extends a string in T_1 .

Suppose (i.b) next: α extends a d-string $\alpha' = (\alpha')^{--} * 11$ such that no substring of $(\alpha')^{--}$ is in F . We show that α' is in T_1 .

Suppose $(\alpha')^{--} \subset \alpha[l_0]$ first. Since l_0 is even and $(\alpha')^{--}$ is a p-string of even length $< l_0$, we have $|(\alpha')^{--}| \leq l_0 - 2$. Since $l_0 := 2k_0 + 2$, we can find a p-string β of length l_0 that is an extension of $(\alpha')^{--}$ such that $\beta(2k_0) = \varphi_{k_0}(2k_0)$. Then $\beta \in F_0$ and by (2.i) (for β and $(\alpha')^{--}$ instead of α and α' , respectively), $\alpha' = (\alpha')^{--} * 11 \in T_1$.

Otherwise, there is s'' such that $0 < s'' \leq s$ and $\alpha[l_{s''-1}] \subseteq (\alpha')^{--} \subset \alpha[l_{s''}]$. Since α' is a d-string, $(\alpha')^{--}$ is a p-string. As $\alpha[l_{s''-1}] \subseteq (\alpha')^{--}$ and no substring of $(\alpha')^{--}$ is in F , $\alpha[l_{s''-1}]$ is a p-string of which no substring is in F . By Lemma 10, for each $t \leq s'' - 1$, we have $\alpha[l_{s''-1}](2k_t) = 1 - \varphi_{k_t}(2k_t)$.

Since $\alpha[l_{s''-1}] \subseteq (\alpha')^{--} \subset \alpha[l_{s''}]$, we have $l_{s''-1} < l_{s''}$. Hence $l_{s''} := \max\{l_{s''-1}, 2k_{s''} + 2\} = 2k_{s''} + 2$. Since $|(\alpha')^{--}|$ and $l_{s''}$ are even, $|(\alpha')^{--}| \leq 2k_{s''}$. We can find a p-string β of length $l_{s''}$ that is an extension of $(\alpha')^{--}$ such that $\beta(2k_{s''}) = \varphi_{k_{s''}}(2k_{s''})$. Therefore, for each $t \leq s'' - 1$, we have $\beta[l_{s''-1}](2k_t) = (\alpha')^{--}[l_{s''-1}](2k_t) = 1 - \varphi_{k_t}(2k_t)$. So $\beta \in F_{s''}$ by (1). Then since $|(\alpha')^{--}| \geq l_{s''-1}$, we have by (2.i) (for β and $(\alpha')^{--}$ instead of α and α' , respectively), $\alpha' = (\alpha')^{--} * 11 \in T_1$.

(ii) Similar to (i).

(iii) (\Rightarrow). Suppose that α does not extend a string in $T_0 \cup T_1$. Then the negations of (i.a) and of (ii.a) imply for each $t \leq s$, $f(\alpha[l_t]) \uparrow$, which implies by Lemma 10 that no substring of α is in F . Furthermore, (since no substring of α is in F) the negations of (i.b) and of (ii.b) imply that α does not extend a d-string. By Lemma 8 (i), α is a p-string.

(\Leftarrow). Suppose that α is a p-string and no substring of α is in F . Since α is a p-string, no substring of α is a d-string. So α does not satisfy (i.b) or (ii.b). Since no substring α' of α is in F , we have for such α' , $f(\alpha') \uparrow$. So α does not satisfy (i.a) or (ii.a). Therefore, α does not extend a string in $T_0 \cup T_1$. ■

Lemma 13 *Let $\alpha \supset 1$ be a string of length l_s such that $\alpha(2k_s) = \varphi_{k_s}(2k_s)$. Then α extends a string in $T_0 \cup T_1$.*

Proof. If $\alpha \supseteq 11$, the conclusion follows immediately, since $11 \in T_1$.

Suppose $\alpha \supseteq 10$. We prove the lemma by induction on s . Assume $s = 0$. If α is a p-string, then $\alpha \in F_0$. By (2.ii) of the generating algorithm for T_0 and T_1 , we obtain $\alpha \in T_0 \cup T_1$. Otherwise, by Lemma 8 (i), α extends

a d-string β . Since $|\beta^{--}| < l_0 \leq l_s$ for all s , no substring of β^{--} is in F (because F consists of certain strings of length l_s for some s). By Lemma 12 (i.b) or (ii.b), α extends a string (namely β) in $T_0 \cup T_1$.

Assume the lemma holds for $s-1$. If for some $s' < s$, $\alpha(2k_{s'}) = \varphi_{k_{s'}}(2k_{s'})$ then by the induction hypothesis, $\alpha[l_{s'}]$ extends a string in $T_0 \cup T_1$. So α extends a string in $T_0 \cup T_1$. Otherwise, for each $s' < s$, $\alpha(2k_{s'}) = 1 - \varphi_{k_{s'}}(2k_{s'})$. If α is a p-string then $\alpha \in F$ by (1), hence it is in $T_0 \cup T_1$ by (2.ii) of the construction. If α is not a p-string then by Lemma 8 (i), α extends a d-string β . Then $|\beta^{--}| < l_s$. Since $\beta \subseteq \alpha$ and for each $s' < s$, $\alpha(2k_{s'}) = 1 - \varphi_{k_{s'}}(2k_{s'})$, no substring of β^{--} is in F by (1). By Lemma 12 (i.b) or (ii.b), α extends a string (namely β) in $T_0 \cup T_1$. ■

Lemma 14 *Any coalition $S \in \text{REC}$ has an initial segment in T_0 or in T_1 , but not both.*

Proof. We show that S has an initial segment in $T_0 \cup T_1$. Lemma 11 implies that S does not have initial segments in both T_0 and T_1 . (The assertion following “In particular” in Lemma 11 is sufficient for this, but we can actually show the stronger statement that S has exactly one initial segment in $T_0 \cup T_1$.)

If $S \supseteq 1$, suppose φ_k is the characteristic function for S . Then $k \in \{k_0, k_1, k_2, \dots\}$ since this set contains the set CRec of characteristic indices. So $k = k_s$ for some s . By Lemma 13, the initial segment $S[l_s]$ (i.e., $\varphi_{k_s}[l_s]$) extends a string in $T_0 \cup T_1$. So, S has an initial segment in $T_0 \cup T_1$.

If $S \supseteq 0$, then $S^c \supseteq 1$ has an initial segment in $T_0 \cup T_1$ by the argument above. So, S has an initial segment in $T_1 \cup T_0$. ■

Next, we show that the game $\omega[A]$ has the desired properties. Before showing monotonicity, we need the following lemma. For strings α and β with $|\alpha| \leq |\beta|$, we say β *properly contains* α if for each $k < |\alpha|$, $\alpha(k) \leq \beta(k)$ and for some $k' < |\alpha|$, $\alpha(k') < \beta(k')$; we say β *is properly contained by* α if for each $k < |\alpha|$, $\beta(k) \leq \alpha(k)$ and for some $k' < |\alpha|$, $\beta(k') < \alpha(k')$.

Lemma 15 *Let α and β be strings such that $l_s = |\alpha| \leq |\beta|$ for some s . (i) If α extends a string in T_1 and β properly contains α , then β extends a string in T_1 . (ii) If α extends a string in T_0 and β is properly contained by α , then β extends a string in T_0 .*

Proof. We only prove (i). The proof for (ii) is similar. Suppose that α extends a string in T_1 and that β properly contains α .

Case 1: $\alpha \supseteq 1$. In this case, (i.a) or (i.b) of Lemma 12 holds.

First assume (i.a) is the case: we can choose an $s' \leq s$ such that $f(\alpha[l_{s'}]) \downarrow \in A$ (in this case, $\alpha[l_{s'}] \in T_1$). If β extends $\alpha[l_{s'}]$, clearly the

conclusion holds. Otherwise, since $|\beta| \geq l_s \geq l_{s'}$, $\alpha[l_{s'}]$ and β are incompatible; that is, there exists $k < l_{s'}$ such that $\alpha[l_{s'}](k) \neq \beta(k)$. Choose the least such k ; since β properly contains α , we have $\alpha[l_{s'}](k) = 0$ and $\beta(k) = 1$. Let $\beta' = \beta[k](= \alpha[k])$. Note that $f(\alpha[l_{s'}]) \downarrow$ implies $\alpha[l_{s'}] \in F$, which in turn implies $\alpha[l_{s'}]$ is a p-string.

Suppose k is even. We will show that β extends $\beta' * 11 \in T_1$. Since $k < l_{s'}$ and $l_{s'}$ is also even, we have $k + 1 < l_{s'}$, so that $\alpha[l_{s'}](k + 1) \downarrow$. Since $\alpha[l_{s'}]$ is a p-string, $\beta(k + 1) \geq \alpha[l_{s'}](k + 1) = 1 - \alpha[l_{s'}](k) = 1$. So $\beta(k)\beta(k + 1) = 11$. Hence $\beta' * 11 \subseteq \beta[l_s]$. Since $\alpha[l_{s'}] \in F$, no proper substring of $\alpha[l_{s'}]$ is in F . As $\beta' \subset \alpha[l_{s'}]$, no substring of β' is in F . So by Lemma 12 (i.b), $\beta[l_s]$ extends a string (namely, $\beta' * 11$) in T_1 .

Suppose k is odd. We will show that β extends $(\beta')^- * 11 \in T_1$. Since $\alpha[l_{s'}]$ is a p-string, $\beta(k - 1) = \alpha[l_{s'}](k - 1) = 1 - \alpha[l_{s'}](k) = 1$. So $\beta(k - 1)\beta(k) = 11$. Hence $(\beta')^- * 11 \subseteq \beta[l_s]$. Since no proper substring of $\alpha[l_{s'}]$ is in F and $(\beta')^- \subset \alpha[l_{s'}]$, no substring of $(\beta')^-$ is in F . So by Lemma 12 (i.b), $\beta[l_s]$ extends a string (namely, $(\beta')^- * 11$) in T_1 .

Next assume (i.b) is the case: α extends a d-string $\alpha' = (\alpha')^{--} * 11$ such that no substring of $(\alpha')^{--}$ is in F (in this case, $\alpha' \in T_1$). Choose the least $k \leq |\alpha|$ such that $\alpha(k) \neq \beta(k)$; we have $\alpha(k) = 0$ and $\beta(k) = 1$. Let $\beta' = \beta[k](= \alpha[k])$. Since $\alpha'(|\alpha| - 2) = \alpha'(|\alpha| - 1) = 1$, either $k > |\alpha| - 1$ or $k < |\alpha| - 2 = |(\alpha')^{--}|$. If $k > |\alpha| - 1$, we get $\beta' \supseteq \alpha'$. This implies $\beta \supseteq \beta' \supseteq \alpha' \in T_1$; hence β extends a string in T_1 . Otherwise, we have $k < l := |(\alpha')^{--}|$ and $\beta' \subset (\alpha')^{--}$.

Suppose k is even. Since $k < l$ and l is also even, we have $k + 1 < l$, so that $(\alpha')^{--}(k + 1) \downarrow$. Since α is a p-string, $\beta(k + 1) \geq (\alpha')^{--}(k + 1) = 1 - (\alpha')^{--}(k) = 1$. So $\beta(k)\beta(k + 1) = 11$. Hence $\beta' * 11 \subseteq \beta[l_s]$. Since no substring of $(\alpha')^{--}$ is in F and $\beta' \subset (\alpha')^{--}$, no substring of β' is in F . So by Lemma 12 (i.b), $\beta[l_s]$ extends a string (namely, $\beta' * 11$) in T_1 .

Suppose k is odd. Since $(\alpha')^{--}$ is a p-string, $\beta(k - 1) = (\alpha')^{--}(k - 1) = 1 - (\alpha')^{--}(k) = 1$. So $\beta(k - 1)\beta(k) = 11$. Hence $(\beta')^- * 11 \subseteq \beta[l_s]$. Since no substring of $(\alpha')^{--}$ is in F and $(\beta')^- \subset (\alpha')^{--}$, no substring of $(\beta')^-$ is in F . So by Lemma 12 (i.b), $\beta[l_s]$ extends a string (namely, $(\beta')^- * 11$) in T_1 .

Case 2: $\alpha \supseteq 0$. First note that assertion (ii) for Case 1 can be proved by an argument similar to the proof of assertion (i) for Case 1 above (use Lemma 12 (ii) instead of Lemma 12 (i)). By the construction of T_1 and T_0 , $\alpha^c \supseteq 1$ extends a string in T_0 and β^c is properly contained by α^c . Applying assertion (ii) for Case 1, we obtain that β^c extends a string in T_0 . Hence β extends a string in T_1 . ■

Note that the preceding proof shows that β actually extends a *d-string* unless it extends $\alpha[l_{s'}]$.

Lemma 16 *The game $\omega[A]$ is monotonic.*

Proof. Suppose $B \in \omega[A]$ and $B' \supseteq B$. By the definition of $\omega[A]$, B has an initial segment $\alpha \in T_1$. Choose the least s such that $l_s \geq |\alpha|$. Then the initial segment $B[l_s]$ extends $\alpha \in T_1$. Let $\beta = B'[l_s]$. Then either $\beta = B[l_s]$ or β properly contains $B[l_s]$.

If $\beta = B[l_s]$, then clearly β extends $\alpha \in T_1$ and so does B' . Therefore, $B' \in \omega[A]$. Otherwise, β properly contains $B[l_s]$, which extends $\alpha \in T_1$. By Lemma 15 (i), β extends a string in T_1 and so does B' . Therefore, $B' \in \omega[A]$. ■

Lemma 17 *The game $\omega[A]$ is proper and strong.*

Proof. It suffices to show that $S^c \in \omega \Leftrightarrow S \notin \omega$. From the observations that T_0 and T_1 consist of determining strings and that $\alpha^c \in T_0 \Leftrightarrow \alpha \in T_1$, we have: $S^c \in \omega$ iff S^c has an initial segment in T_1 iff S has an initial segment in T_0 iff $S \notin \omega$. ■

Lemma 18 *The game $\omega[A]$ is nonweak and does not have a finite carrier.*

Proof. We construct a set B such that for infinitely many l , the l -initial segment $B[l]$ has an extension that is winning and an extension that is losing. Let $B \supseteq 10$ be a set such that for each k_s , $B(2k_s) = 1 - \varphi_{k_s}(2k_s)$ and any initial segment of B of even length is a p-string. Let s be such that $l_{s+1} > l_s$.

Then $l_{s+1} := \max\{l_s, 2k_{s+1} + 2\} = 2k_{s+1} + 2$ and $2k_{s+1} + 2 > l_s$ implies (since both sides are even numbers) that $2k_{s+1} \geq l_s$. By the definition of B , for each $t \leq s$, we have $B(2k_t) = 1 - \varphi_{k_t}(2k_t)$ and $2k_t < l_s$ (the last inequality from the observation that $l_s > 2k_s + 1, 2k_{s-1} + 1, 2k_{s-2} + 1, \dots, 2k_0 - 1$). Then since $2k_{s+1} \geq l_s$, there is a p-string $\alpha \supseteq B[l_s]$ of length l_{s+1} such that $\alpha(2k_{s+1}) = \varphi_{k_{s+1}}(2k_{s+1})$ and for each $t \leq s$, $\alpha(2k_t) = 1 - \varphi_{k_t}(2k_t)$. Then by (1), $\alpha \in F_{s+1}$ and $|\alpha^{--}| = |\alpha| - 2 = l_{s+1} - 2 = 2k_{s+1} \geq l_s$. So by (2.i) of the generating algorithm, $\alpha^{--} * 11 \in T_1$ and $\alpha^{--} * 00 \in T_0$.

There are infinitely many such s . It follows that any initial segment of B has an extension in T_1 and an extension in T_0 . This means that the game has no finite carrier.

To show nonweakness, we give three (winning) coalitions in T_1 whose intersection is empty. First, 10 (in fact any initial segment of the coalition $B \supseteq 10$) has extensions α in T_1 and β in T_0 by the argument above. So 01 has the extension β^c in T_1 . Clearly, the intersection of the winning coalitions $11 \in T_1$, $\alpha \supseteq 10$, and $\beta^c \supseteq 01$ is empty. ■

Note that the proof that $\omega[A]$ has no finite carrier depends on (2.i), but not (2.ii) or (3), of the generating algorithm.

6.3 An assortment of computable games without finite carriers

In this section, we give, for each of the sixteen conventional types of simple games, an example of an infinite computable game of that type, if such a game exists. Most of the examples are based on the game $\omega[A]$ in Section 6.2.

1. (+ + + +) A monotonic, proper, strong, nonweak game. $\omega[A]$ is such a game.
2. (+ + + -) A monotonic, proper, strong, weak game. By Lemma 2, such a game has a dictator, violating the property that it has no finite carrier.
3. (+ + - +) A monotonic, proper, nonstrong, nonweak game. Let $\omega = \omega[\emptyset] \cap \omega[\mathbb{N}]$; that is, $S \in \omega$ if and only if $S \in \omega[\emptyset]$ and $S \in \omega[\mathbb{N}]$.

To show ω is proper, suppose $S \in \omega$ and $S^c \in \omega$. Then $S \in \omega[\mathbb{N}]$ and $S^c \in \omega[\mathbb{N}]$, contradicting the properness of $\omega[\mathbb{N}]$.

To show ω is nonstrong, let $\alpha \in F$. We show that both α and α^c are losing. On the one hand, we have $\alpha \in T_0^\emptyset$ by (2.ii) of the generating algorithm. Since T_0^\emptyset consists of losing determining strings, $\alpha \notin \omega[\emptyset]$. Hence $\alpha \notin \omega$. On the other hand, we have $\alpha \in T_1^\mathbb{N}$ by (2.ii). Hence $\alpha^c \in T_0^\mathbb{N}$. Since $T_0^\mathbb{N}$ consists of losing determining strings, $\alpha^c \notin \omega[\mathbb{N}]$. Hence $\alpha^c \notin \omega$, as desired.

Computability, monotonicity, and nonweakness of ω are immediate from the corresponding properties of $\omega[A]$. The proof that ω does not have a finite carrier is similar to the proof for $\omega[A]$.

4. (+ + - -) A monotonic, proper, nonstrong, weak game. In the construction of (the sets T_0 and T_1 for) $\omega[A]$ in Section 6.2, replace (2.i), (2.ii), and (3) by

(2*.i) for each p-string α' that is a proper substring of α , if $s = 0$ or $|\alpha'| \geq l_{s-1}$, then enumerate $1 * \alpha' * 11$ in T_1 and $1 * \alpha' * 00$ in T_0 ; furthermore, enumerate 0 in T_0 ;

(2*.ii) if $f(\alpha) \in A$, enumerate $1 * \alpha$ in T_1 ; if $f(\alpha) \notin A$, enumerate $1 * \alpha$ in T_0 ;

(3*) if a string $\beta = 1 * \beta'$ is enumerated in T_1 (or in T_0) above, then enumerate $1 * (\beta')^c$ in T_0 (or in T_1 , respectively).

Let T'_0 and T'_1 be the sets T_0 and T_1 in the original (Section 6.2) construction of $\omega[A]$ renamed. We observe that $\beta = 1 * \beta' \in T_i$ if and only if $\beta' \in T'_i$.

We first show that any coalition S has exactly one initial segment in $T_0 \cup T_1$. This is immediate if $S \supseteq 0$. So, suppose $S \supseteq 1$. Define S'

by $S'(k) = S(k + 1)$ for all k . Then, by the proof of Lemma 14 for $\omega[A]$, S' has exactly one initial segment $S'[k]$ in $T'_0 \cup T'_1$. From the observation above, $S[k + 1] = 1 * S'[k] \in T_0 \cup T_1$ for a unique k , which is what we wanted.

To show the game is monotonic, it suffices to show Lemma 15 (i) holds for the newly defined game. Suppose that α, β satisfy the assumption of the lemma and that α extends a string $\hat{\alpha}$ in T_1 and β properly contains α . Then, $\hat{\alpha} \supseteq 1$; write $\hat{\alpha} = 1 * \hat{\alpha}'$. Then $\hat{\alpha}' \in T'_1$ from the observation above. We can write $\beta = 1 * \beta'$. Then β' either extends or properly contains $\hat{\alpha}' \in T'_1$. If β' extends $\hat{\alpha}' \in T'_1$, then β extends $1 * \hat{\alpha}' \in T_1$, as desired. Otherwise, β' properly contains $\hat{\alpha}' \in T'_1$. By Lemma 15 for the original game $\omega[A]$ (the condition that $l_s = |\alpha|$ can be ignored for our purpose), β' extends a string $\hat{\beta} \in T'_1$. So, $\beta = 1 * \hat{\beta}$ extends $1 * \hat{\beta} \in T_1$, as desired.

The game is weak (hence proper by Lemma 1) since every winning coalition extends 1; in other words, 0 is a veto player. It is nonstrong since $\{0\} \supseteq 100 \in T_0$ implies $\{0\} \notin \omega$, while $\{0\}^c \supseteq 0 \in T_0$ implies $\{0\}^c \notin \omega$. The proof that the game is computable and has no finite carrier is similar to the proofs for $\omega[A]$.

5. (+ - ++) A monotonic, nonproper, strong, nonweak game. Let $\omega = \omega[\emptyset] \cup \omega[\mathbb{N}]$; that is, $S \in \omega$ if and only if $S \in \omega[\emptyset]$ or $S \in \omega[\mathbb{N}]$.

To show ω is nonproper, let $\alpha \in F$. We show that both α and α^c are winning. On the one hand, we have $\alpha \in T_1^{\mathbb{N}}$ by (2.ii). So $\alpha \in \omega[\mathbb{N}]$, implying $\alpha \in \omega$. On the other hand, we have $\alpha \in T_0^{\emptyset}$ by (2.ii). Hence $\alpha^c \in T_1^{\emptyset}$. So $\alpha^c \in \omega[\emptyset]$. Hence $\alpha^c \in \omega$, as desired.

To show ω is strong, suppose $S \notin \omega$ and $S^c \notin \omega$. Then $S \notin \omega[\mathbb{N}]$ and $S^c \notin \omega[\mathbb{N}]$, contradicting the strongness of $\omega[\mathbb{N}]$.

Computability and monotonicity of ω are immediate from the corresponding properties of $\omega[A]$. Nonweakness is immediate from nonproperness by Lemma 1. The proof that ω does not have a finite carrier is similar to the proof for $\omega[A]$.

6. (+ - +-) A monotonic, nonproper, strong, weak game. There is no such game.

7. (+ - -+) A monotonic, nonproper, nonstrong, nonweak game. Let A be the set of even numbers. In the construction of $\omega[A]$, replace (2.ii) and (3) by

(2*.ii) if $f(\alpha) \in A$, enumerate α and α^c in T_1 ; if $f(\alpha) \notin A$, enumerate α and α^c in T_0 ;

(3*) if a string β is enumerated in T_1 (or in T_0) by applying (2.i), then enumerate β^c in T_0 (or in T_1 , respectively).

To show the game is monotonic, it suffices to show Lemma 15 (i) holds. Suppose that α, β satisfy the assumption of the lemma and that α extends a string α' in T_1 and β properly contains α . Let T'_0 and T'_1 be the sets T_0 and T_1 in the original construction of $\omega[A]$ renamed. Note that the replacement of (2.ii) and (3) by (2*.ii) and (3*) only affects p-strings, but not d-strings; hence the set of d-strings in T_1 is the same as the set of d-strings in T'_1 , the set of d-strings in T_0 is the same as the set of d-strings in T'_0 , and the set of p-strings in $T_0 \cup T_1$ is the same as the set of p-strings in $T'_0 \cup T'_1$. If α' is a d-string in T_1 , it is in T'_1 . Lemma 15 (i) implies that β extends a string in T'_1 . In fact, an inspection of the proof of Lemma 15 reveals that β extends a d-string in T'_1 , unless $\beta \supseteq \alpha'$, in which case the conclusion is obvious. So assume $\beta \not\supseteq \alpha'$. Then β extends a d-string in T'_1 ; hence it extends a d-string in T_1 , as desired. If α' is a p-string in T_1 , it is in $T'_1 \cup T'_0$. If $\alpha' \in T'_1$, then Lemma 15 (i) implies that β extends a string in T'_1 . So the rest of the proof is similar. If $\alpha' \in T'_0$, then Lemma 15 (ii) implies that β^c extends a string in T'_0 . Assume $\beta \not\supseteq \alpha'$ as before. Then β^c extends a d-string in T'_0 ; hence it extends a d-string in T_0 . By (3*), β extends a d-string in T_1 , as desired.

The game is nonproper since (2*.ii) implies that there is a string $\alpha \in F$ such that the coalitions $\{i : \alpha(i) = 1\}$ and $\{i : \alpha(i) = 1\}^c$ (which extends α^c) are winning. Similarly, it is nonstrong since there is a string $\alpha \in F$ such that the coalitions above are losing. It is nonweak by Lemma 1 since it is nonproper. The proof that the game is computable and has no finite carrier is similar to the proofs for $\omega[A]$.

8. (+ - - -) A monotonic, nonproper, nonstrong, weak game. There is no such game.
9. (- + + +) A nonmonotonic, proper, strong, nonweak game. In the construction of $\omega[A]$, replace (2.i) by

(2*.i) for each p-string $\alpha' \neq \emptyset$ that is a proper substring of α , if $s = 0$ or $|\alpha'| \geq l_{s-1}$, then enumerate $\alpha' * 11$ in T_1 and $\alpha' * 00$ in T_0 ; furthermore, enumerate 00 in T_1 .

By (3) of the construction, $11 \in T_0$. (In other words, the game is constructed from the sets $T_0 := T'_0 \cup \{11\} \setminus \{00\}$ and $T_1 := T'_1 \cup \{00\} \setminus \{11\}$, where T'_0 and T'_1 are T_0 and T_1 in the original construction of $\omega[A]$ renamed.) Since 00 is winning and 11 is losing, the game is nonmonotonic. It is also nonweak since 00 (or an empty coalition) is winning. For the remaining properties, the proofs are similar to the proofs for $\omega[A]$.

10. (- + + -) A nonmonotonic, proper, strong, weak game. There is no such game.

11. $(- + - +)$ A nonmonotonic, proper, nonstrong, nonweak game. In the construction of $\omega[A]$, replace (2.i) and (3) by

(2*.i) for each p-string $\alpha' \neq \emptyset$ that is a proper substring of α , if $s = 0$ or $|\alpha'| \geq l_{s-1}$, then enumerate $\alpha' * 11$ in T_1 and $\alpha' * 00$ in T_0 ; furthermore, enumerate 00 and 11 in T_0 ;

(3*) if a string $\beta \notin \{00, 11\}$ is enumerated in T_1 (or in T_0) above, then enumerate β^c in T_0 (or in T_1 , respectively).

(In other words, the game is constructed from the sets $T_0 := T'_0 \cup \{11\}$ and $T_1 := T'_1 \setminus \{11\}$, where T'_0 and T'_1 are T_0 and T_1 in the original construction of $\omega[A]$ renamed.)

The game is nonmonotonic since N is losing but there are winning coalitions. It is proper since it is a subset of $\omega[A]$, which is proper. It is nonstrong since $11, 00 \in T_0$ implies that the coalitions $\{0, 1\}$, $\{0, 1\}^c$ are losing.

To show nonweakness, find a $\beta \in T_1$ such that $|\beta| = l_{t+1}$ for some t (e.g., let $\beta = \alpha^{--} * 11$ in the proof of Lemma ??, with s replaced by t). Choose an s such that $l_{t+1} < l_s < l_{s+1}$. Following the proof of Lemma ??, we can find $\alpha \in F_{s+1}$ such that $|\alpha^{--}| \geq l_s$, $\alpha^{--} * 11 \in T_1$, and $\alpha^{--} * 00 \in T_0$. Then $(\alpha^c)^{--} * 11 \in T_1$. Nonweakness follows since the intersection of winning coalitions β (regarded as the coalition $\{i : \beta(i) = 1\}$), $\alpha^{--} * 11 \in T_1$, and $(\alpha^c)^{--} * 11$ is empty.

The proofs of computability and nonexistence of a finite carrier are similar to the proofs for $\omega[A]$.

12. $(- + - -)$ A nonmonotonic, proper, nonstrong, weak game. Let $A = \mathbb{N}$. In the construction of $\omega[A] = \omega[\mathbb{N}]$, replace (2.i) by

(2*.i) for each p-string α' that extends 1010 or 1001 and is a proper substring of α , if $s = 0$ or $|\alpha'| \geq l_{s-1}$, then enumerate $\alpha' * 11$ in T_1 and $\alpha' * 00$ in T_0 ; furthermore, enumerate d-strings 11 and 1000 in T_1 and strings 1011 and 0 in T_0 .

and remove (3). To show that any coalition S has an initial segment in $T_0 \cup T_1$, suppose that S extends 1010 or 1001. (The other cases are immediate.) Let T'_0 and T'_1 be T_0 and T_1 in the original construction of $\omega[\mathbb{N}]$ renamed. Then, by Proposition 14, S has an initial segment $S[k]$ in $T'_0 \cup T'_1$, where $k \geq 4$ without loss of generality. If $S[k]$ is enumerated in $T'_0 \cup T'_1$ by applying (2.ii), then it is enumerated in $T_0 \cup T_1$ by applying (2.ii). So, the conclusion follows. If $S[k]$ is enumerated in $T'_0 \cup T'_1$ by applying (2.i), then $S[k]$ is equal to $\alpha' * 11$ or $\alpha' * 00$ for some p-string α' satisfying the requirements in (2.i). Clearly, α' extends 1010 or 1001. So, $S[k]$ is enumerated in $T_0 \cup T_1$ by applying (2*.i). So the conclusion follows.

To show that no coalition S has initial segments in both T_0 and T_1 , it suffices to show that a string α enumerated in T_0 by (2*.i) and a p-string β enumerated in T_1 by (2.ii) are incompatible. (Note that all $\alpha \in F$ are enumerated in T_1 and none in T_0 by (2.ii).) Since $\beta \supset 10$, it is incompatible with $0 \in T_1$. All the other strings enumerated by (2*.i) are d-strings, so α and β are compatible only if α extends β , which in turn extends (since $\beta \in F$ is of length ≥ 4) 1001 or 1010. Then, $\alpha = \alpha' * 00$ for some α' , so as above, $\alpha \in T'_0$; similarly, $\beta \in T'_1$. This implies that α and β are incompatible.

The game ω defined above is nonmonotonic since 1000 is winning but 1011 is not. To see ω is weak (hence proper by Lemma 1), note that any winning coalition extends 1; so the intersection contains a veto player 0. The game is nonstrong because $0, 1011 \in T_0$ imply that the coalitions $\{1\}$ and $\{1\}^c$ are losing. The proofs of computability and nonexistence of a finite carrier are similar to the proofs for $\omega[A]$.

13. $(- - ++)$ A nonmonotonic, nonproper, strong, nonweak game. In the construction of $\omega[A]$, replace (2.i) and (3) by

(2*.i) for each p-string $\alpha' \neq \emptyset$ that is a proper substring of α , if $s = 0$ or $|\alpha'| \geq l_{s-1}$, then enumerate $\alpha' * 11$ in T_1 and $\alpha' * 00$ in T_0 ; furthermore, enumerate 00 and 11 in T_1 ;

(3*) if a string $\beta \notin \{00, 11\}$ is enumerated in T_1 (or in T_0) above, then enumerate β^c in T_0 (or in T_1 , respectively).

(In other words, the game is constructed from the sets $T_0 := T'_0 \setminus \{00\}$ and $T_1 := T'_1 \cup \{00\}$, where T'_0 and T'_1 are T_0 and T_1 in the original construction of $\omega[A]$ renamed.)

The game is nonmonotonic since \emptyset is winning but there are losing coalitions. It is nonproper since the coalitions $\{0, 1\}$, $\{0, 1\}^c$ are winning. It is strong since its subset $\omega[A]$ is strong. It is nonweak by Lemma 1 since it is nonproper. The proofs of computability and nonexistence of a finite carrier are similar to the proofs for $\omega[A]$.

14. $(- - +-)$ A nonmonotonic, nonproper, strong, weak game. There is no such game.

15. $(- - - +)$ A nonmonotonic, nonproper, nonstrong, nonweak game. In the construction of $\omega[A]$, replace (2.i) and (3) by

(2*.i) for each p-string α' that extends 1010 or 1001 and is a proper substring of α , if $s = 0$ or $|\alpha'| \geq l_{s-1}$, then enumerate $\alpha' * 11$ in T_1 and $\alpha' * 00$ in T_0 ; furthermore, enumerate d-strings 00, 11, 1000, and 0111 in T_0 and d-strings 1011 and 0100 in T_1 ;

(3*) if a string $\beta \notin \{00, 11, 1000, 0111, 1011, 0100\}$ is enumerated in T_1 (or in T_0) above, then enumerate β^c in T_0 (or in T_1 , respectively).

The game is nonmonotonic since 0100 is winning but 0111 is not. The game is nonproper since 1011, 0100 $\in T_1$ imply that the coalitions $\{1\}$ and $\{1\}^c$ are winning. It is nonstrong since 1000, 0111 $\in T_0$ imply $\{0\}$ and $\{0\}^c$ are losing. It is nonweak by Lemma 1 since it is nonproper. The proofs of computability and nonexistence of a finite carrier are similar to the proofs for $\omega[A]$.

16. (---) A nonmonotonic, nonproper, nonstrong, weak game. There is no such game.

References

- [1] N. I. Al-Najjar, L. Anderlini, and L. Felli. Undescribable events. *Review of Economic Studies*, 73:849–868, 2006.
- [2] L. Anderlini and L. Felli. Incomplete written contracts: Undescribable states of nature. *Quarterly Journal of Economics*, 109:1085–1124, 1994.
- [3] K. J. Arrow. *Social Choice and Individual Values*. Yale University Press, New Haven, 2nd edition, 1963.
- [4] J. S. Banks, J. Duggan, and M. L. Breton. Social choice and electoral competition in the general spatial model. *Journal of Economic Theory*, 126:194–234, 2006.
- [5] J. Bartholdi III, C. A. Tovey, and M. A. Trick. Voting schemes for which it can be difficult to tell who won the election. *Social Choice and Welfare*, 6:157–165, 1989.
- [6] J. J. Bartholdi III, C. A. Tovey, and M. A. Trick. The computational difficulty of manipulating an election. *Social Choice and Welfare*, 6:227–241, 1989.
- [7] X. Deng and C. H. Papadimitriou. On the complexity of cooperative solution concepts. *Mathematics of Operations Research*, 19:257–266, 1994.
- [8] A. Downs. *An Economic Theory of Democracy*. Addison-Wesley, Boston, 1957.
- [9] Q. Fang, S. Zhu, M. Cai, and X. Deng. On computational complexity of membership test in flow games and linear production games. *International Journal of Game Theory*, 31:39–45, 2002.

- [10] J. S. Kelly. Social choice and computational complexity. *Journal of Mathematical Economics*, 17:1–8, 1988.
- [11] M. Kumabe and H. R. Mihara. Computability of simple games: A characterization and application to the core. MPRA Paper, July 2006. <http://mpra.ub.uni-muenchen.de/437/>
- [12] A. A. Lewis. An infinite version of Arrow’s Theorem in the effective setting. *Mathematical Social Sciences*, 16:41–48, 1988.
- [13] K. O. May. A set of independent, necessary and sufficient conditions for simple majority decision. *Econometrica*, 20:680–84, 1952.
- [14] K. O. May. A note on the complete independence of the conditions for simple majority decision. *Econometrica*, 21:172–173, 1953.
- [15] H. R. Mihara. Arrow’s Theorem and Turing computability. *Economic Theory*, 10(2):257–76, Aug. 1997.
- [16] H. R. Mihara. Arrow’s theorem, countably many agents, and more visible invisible dictators. *Journal of Mathematical Economics*, 32(3):267–287, 1999.
- [17] H. R. Mihara. Nonanonymity and sensitivity of computable simple games. *Mathematical Social Sciences*, 48(3):329–341, 2004.
- [18] P. Odifreddi. *Classical Recursion Theory: The Theory of Functions and Sets of Natural Numbers*. Elsevier, Amsterdam, 1992.
- [19] B. Peleg. Game-theoretic analysis of voting in committees. In K. J. Arrow, A. K. Sen, and K. Suzumura, editors, *Handbook of Social Choice and Welfare*, volume 1, chapter 8, pages 395–423. Elsevier, Amsterdam, 2002.
- [20] M. K. Richter and K.-C. Wong. Computable preference and utility. *Journal of Mathematical Economics*, 32:339–354, 1999.
- [21] R. I. Soare. *Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets*. Springer-Verlag, Berlin, 1987.
- [22] W. Thomson. On the axiomatic method and its recent applications to game theory and resource allocation. *Social Choice and Welfare*, 18:327–386, 2001.
- [23] R. J. Weber. Games in coalitional form. In R. J. Aumann and S. Hart, editors, *Handbook of Game Theory*, volume 2, chapter 36, pages 1285–1303. Elsevier, Amsterdam, 1994.