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# Exchange Options

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**Abstract.** The contract is described and market examples given. Essential theoretical developments are introduced and cited chronologically. The principles and techniques of hedging and unique pricing are illustrated for the two simplest nontrivial examples: the classical Black-Scholes/Merton/Margrabe exchange option model brought somewhat up-to-date from its form three decades ago, and a lesser exponential Poisson analogue to illustrate jumps. Beyond these, a simplified Markovian SDE/PDE line is sketched in an arbitrage-free semimartingale setting. Focus is maintained on construction of a hedge using Itô's formula and on unique pricing, now for general homogenous payoff functions. Clarity is primed as the multivariate log-Gaussian and exponential Poisson cases are worked out.

Numeraire invariance is emphasized as the primary means to reduce dimensionality by one to the projective space where the SDE dynamics are specified and the PDEs solved (or expectations explicitly calculated). Predictable representation of a homogenous payoff with deltas (hedge ratios) as partial derivatives or partial differences of the option price function is highlighted. Equivalent martingale measures are utilized to show unique pricing with bounded deltas (and in the nondegenerate case unique hedging) and to exhibit the PDE or closed-form solutions as numeraire-deflated conditional expectations in the usual way. Homogeneity, change of numeraire, and extension to dividends are discussed.

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## 1. INTRODUCTION

**1.1. Definition and examples.** A (European) exchange option is a contract that gives the buyer the right to exchange two (possibly dividend-paying) assets  $A$  and  $B$  at a fixed expiration time  $T$ , say to receive  $A$  and deliver (pay)  $B$ ; so, the option payoff is

$$(A_T - B_T)^+ := \max(A_T - B_T, 0).$$

(American and Bermudan exchange options are complicated by early optimal exercise and not discussed here.) An ordinary (European) call or put on an asset struck at  $K$  can be viewed as in [9] as an option to exchange the asset with the  $T$ -maturity zero-coupon bond of principal  $K$ . More generally, a call or put on an  $s$ -maturity forward contract ( $s \geq T$ ) on a (say) zero-dividend asset is equivalent to an option to exchange the asset at time  $T$  with an  $s$ -maturity zero-coupon bond. Options to exchange two stocks or commodities provide good hypothetical examples but are not prevalent in the market place.

Exchange options are related to spread options with time- $T$  payoffs of the form  $(X - Y)^+$ , given two prescribed time- $T$  observables  $X$  and  $Y$ . A common structure is a CMS spread option, with  $X$  and  $Y$  say the 20-year and 2-year spot swap rates at time  $T$ . A spread option can be viewed as an exchange option when there exist (or can be replicated) two zero-dividend assets  $A$  and  $B$  such that  $A_T = X$  and  $B_T = Y$ . In the CMS case,  $A$  and  $B$  can be taken as the time- $T$  coupons of two CMS bonds or swaps. Exchange options on dividend-paying assets are in practice reduced to the zero-dividend case in a similar way.

Interest-rate swaptions, including caplets and floorlets as one-period special cases, can be viewed both as ordinary call or put options struck at par on coupon bonds, or more directly as options to exchange the fixed and floating cashflow legs of a swap. The latter is the standard as it imposes the classical assumption of a lognormal ratio  $A_T/B_T$  on the forward swap rate (a swap-curve concept) rather than on the forward coupon bond price.

An exchange option is related to its reverse by parity:  $(Y - X)^+ = (X - Y)^+ + Y - X$ . (So an American option to exchange two fixed zero-dividend assets is not exercised early.)

**1.2. Pricing and hedging approaches.** The exchange option is an special case of a path-independent contingent claim with payoff a homogenous function of the underlying asset prices at expiration. It is governed by the same general theory. One makes sure that the underlying assets are arbitrage free which implies there are no free lunches in a strong sense. If the payoff can be attained by a sufficiently regular self-financing trading strategy (SFTS) (e.g. bounded number of shares or “deltas”) then the law of one price holds and the option price at each time is defined as the value of the self-financing portfolio. Otherwise arbitrage-free pricing is not unique. We will not discuss this case, but only mention that one approach then chooses a linear pricing kernel (e.g., the minimal measure) among the many then available, and another is nonlinear based on expected utility maximization.

Payoff replication by a SFTS is a question of predictable representation. As the payoff in this case is a path-independent function of the underliers, it natural that the option price and deltas too be functions of time and the underliers at that time. This has been the traditional Markovian approach, beginning with Black and Scholes [1] and immediate extension by Merton [9]. Their simple choice of a geometric Brownian motion for the

underlying asset in [1] and more generally of a deterministic-volatility forward price process in [9] meant that the underlying stochastic differential equation (SDE) and the associated partial differential equation (PDE) had constant coefficients (in log-state). Itô's formula was applied to construct a riskless hedge, with the deltas (hedge ratios) simply given by partial derivatives of the *option price function*, the unique solution to the PDE.

Black and Scholes constructed a SFTS for a call option struck at  $K$  by dynamically rebalancing long positions on the underlying asset  $A$  financed by shorting the riskless money market asset  $B^* = (e^{rt})$ , post an initial investment equal the option price. Merton's extension to stochastic interest rate  $r$  treated the call option as an option to exchange the asset  $A$  with the  $T$ -maturity zero-coupon bond  $B$  of principal  $K$ . The Black-Scholes model corresponded to a deterministic bond price  $B_t = e^{-r(T-t)}K$ , but now in general  $B$  had infinite variation. The former's simplicity was nonetheless recaptured by exploiting the homogenous symmetry of the option payoff to reduce dimensionality by one, in effect a projective transformation that hedged the forward option contract with trades in the forward asset  $A/B$ . The relevant volatility was accordingly the forward price volatility.

Margrabe [8] extended [9] to an option to exchange any two correlated assets assuming constant volatilities. The option price was derived as a two-dimensional integral of the payoff against the bivariate normal distribution and reduced to the univariate normal distribution. The result was a formula similar to Black-Scholes/Merton's, highlighting again the volatility of the assets' ratio  $A/B$ . The deltas showed that the option is replicated as in [9] by dynamically going long in  $A$  and short in  $B$ , with no trades in any other asset.

Martingale theory leads to a conceptual as well as computationally practical representation of solutions to the PDEs that describe option prices as a conditional expectation of terminal payoff. Harrison and Kreps [5] and Harrison and Pliska [6] developed in related papers an equivalent martingale measure framework that not only made this fruitful representation of the option price available, but laid a more general and probabilistic formulation of the notion of a dynamic hedge, or its mirror image, a replicating SFTS. Such arbitrage-free semimartingale approach does permit path dependency, yet accommodates Markovian SDE/PDE models even nicer. They took the money market asset  $B^*$  as a tradable entering any hedge, giving it a general stochastic form  $B_t^* = e^{\int_0^t r_s ds}$  for discounting payoffs before conditional expectation. In concert with Black and Scholes but contradistinction to Merton, the finite variation asset  $B^*$  was their exclusive choice of numeraire.

With the advent of the forward measure sometime later it was evident that Merton's choice of an infinite variation zero-coupon bond  $B$  as the financing hedge instrument fitted equivalent martingale measure theory perfectly well, leading no less to quicker derivations of concrete pricing formulae than  $B^*$ , as discounting is conveniently performed outside the expectation (see, e.g., [7] and [4]). Another useful numeraire was one by Neuberger [10] to price interest-rate swaptions. Viewed as an option to exchange the fixed and floating swap cashflows, the asset's ratio  $A/B$  represents the forward swap rate here. Assumed in [10] to have deterministic volatility yielded a model that has since served as industry standard to quote swaption implied volatilities. It is noteworthy that here the ratio  $A/B$  has deterministic volatility but  $A$  and  $B$  themselves decidedly do not. In time, El-Karoui

et. al. [4] showed that one can basically change numeraire to any asset  $B$  and associate to it an equivalent measure under which  $A/B$  is a martingale for every other asset  $A$ .

Today option pricing and hedging theory has advanced farther and in many directions. Especially relevant to our discussion of exchange options is the principle of numeraire invariance and arbitrage-free modelling. For in-depth study of these and related topics we refer to Duffie [3] and Delbaen and Schachermayer [2] among other excellent books. Our approach is to concentrate the modelling in “projective coordinate”  $X := A/B$ , while minding that imposed conditions should be invariant under the transformation  $X \mapsto 1/X$ .

## 2. THE DETERMINISTIC-VOLATILITY AND EXPONENTIAL-POISSON MODELS

The option to exchange two assets with a *deterministic volatility*  $\sigma(t)$  of the asset price ratio  $X = A/B$  is celebrated as the simplest nontrivial example in option pricing theory. Its classical Black-Scholes/Merton option price function and explicit representation of the “deltas” (“hedge ratios”) illustrate the principles that underline options in many assets with arbitrary homogenous payoffs and more general dynamics. There is another concrete albeit little known example with simple jumps in  $X$  involving the Poisson rather than the normal distribution. The pattern is similar, the main difference being that the deltas are the partial differences rather than the partial derivatives of the option price function.

We fix throughout a stochastic basis  $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$  with time horizon  $t \in [0, T]$ ,  $T > 0$ . In this section we fix two *zero-dividend* assets with price processes  $A = (A_t)$  and  $B = (B_t)$ .

**2.1. The exchange option price process.** When  $A$  and  $B$  are semimartingales, we call a pair  $(\delta^A, \delta^B)$  of (locally) bounded predictable processes a (locally) bounded self-financing trading strategy (SFTS) (see more generally Sec. 3.1) if  $C = C_0 + \int \delta^A dA + \int \delta^B dB$ , where

$$(2.1) \quad C = \delta^A A + \delta^B B.$$

Clearly  $C$  is then a semimartingale,  $\Delta C = \delta^A \Delta A + \delta^B \Delta B$ , hence  $C_- = \delta^A A_- + \delta^B B_-$ .

The differential form of the self-financing equation is often handy:

$$(2.2) \quad dC = \delta^A dA + \delta^B dB.$$

SFTSs form a linear space. If there exists a unique *bounded* SFTS  $(\delta^A, \delta^B)$  such that

$$(2.3) \quad C_T = (A_T - B_T)^+,$$

then one is justified to call  $C$  *the exchange option price process* and  $\delta^A$  and  $\delta^B$  the deltas.

Assume now the semimartingales  $A$  and  $B$  are positive and have positive left limits. The numeraire invariance principle (see Sec 2.3 and more comprehensively Sec 3.2) states that if  $(\delta^A, \delta^B)$  is a locally bounded SFTS then  $C = \delta^A A + \delta^B B$  satisfies

$$d\left(\frac{C}{B}\right) = \delta^A d\left(\frac{A}{B}\right).$$

(Ditto by symmetry with  $A$  as numeraire.) This is useful for uniqueness. Numeraire invariance also states the converse: if  $C$  is a semimartingale and  $\delta^A$  a locally bounded

predictable process such that  $d(\frac{C}{B}) = \delta^A d(\frac{A}{B})$ , then  $(\delta^A, \delta^B)$  is a SFTS and (2.1) and (2.2) hold, where  $\delta^B = \frac{C_-}{B_-} - \delta^A \frac{A_-}{B_-}$ . This reduces existence to finding an  $F_0$  and  $\delta^A$  such that

$$\left(\frac{A_T}{B_T} - 1\right)^+ = F_0 + \int_0^T \delta_t^A d\left(\frac{A_t}{B_t}\right).$$

The exchange option price process is then the semimartingale  $C = B(F_0 + \int \delta^A d(\frac{A}{B}))$ .

Numeraire invariance in effect reduces general option pricing and hedging to a market where one of the asset price processes equals 1 identically. The remaining task is to find the above “projective” predictable representation of the ratio payoff against the ratio process.

**2.2. Deterministic-volatility exchange option model.** Let  $\sigma(t) > 0$  be a continuous positive function. Define the *Black-Scholes/Merton projective option price function*

$$(2.4) \quad f(t, x) := x\delta_A(t, x) + \delta_B(t, x)$$

on  $t \leq T$ ,  $x > 0$ , where  $\delta_A(T, x) := 1_{x>1}$ ,  $\delta_B(T, x) := -1_{x>1}$ , and for  $t < T$ ,

$$(2.5) \quad \delta_A(t, x) := N\left(\frac{\log x}{\sqrt{\nu_t}} + \frac{\sqrt{\nu_t}}{2}\right), \quad \delta_B(t, x) := -N\left(\frac{\log x}{\sqrt{\nu_t}} - \frac{\sqrt{\nu_t}}{2}\right),$$

where  $\nu_t := \int_t^T \sigma_s^2 ds$  and  $N(\cdot)$  is the normal distribution function. The function  $f(t, x)$  is continuous, and on  $t < T$  is  $C^1$  in  $t$  and analytic in  $x$ . Also,  $-1 \leq \delta_B \leq 0 \leq \delta_A \leq 1$ , and

$$f(T, x) = (x - 1)^+, \quad \frac{\partial f}{\partial x}(t, x) = \delta_A(t, x).$$

As is well known and seen in Sec. 2.9 or 3.6, the function  $f(t, x)$  is the unique  $C^{1,2}$  (on  $t < T$ ) solution with bounded partial  $\frac{\partial f}{\partial x}(t, x)$  subject to  $f(T, x) = (x - 1)^+$  of the PDE

$$(2.6) \quad \frac{\partial f}{\partial t}(t, x) + \frac{1}{2}\sigma^2(t)x^2 \frac{\partial^2 f}{\partial x^2}(t, x) = 0.$$

Assume now  $A = BX$  for some positive continuous semimartingale  $X > 0$  satisfying

$$(2.7) \quad d[\log X]_t = \sigma^2(t)dt. \quad (A = BX)$$

Under this assumption, one traditionally defines *the exchange option price process*  $C$  by

$$(2.8) \quad C := BF, \quad F = (F_t), \quad F_t := f(t, X_t).$$

Clearly,  $C_T = (A_T - B_T)^+$ . The definition is justified using the continuous semimartingales

$$(2.9) \quad \delta_t^A := \delta_A(t, X_t) = \frac{\partial f}{\partial x}(t, X_t), \quad \delta_t^B := \delta_B(t, X_t) = F_t - \delta_t^A X_t.$$

Clearly,  $C = \delta^A A + \delta^B B$ , and the deltas are bounded:  $0 \leq \delta^A \leq 1$  and  $-1 \leq \delta^B \leq 0$ . Since  $f(t, x)$  satisfies the PDE (2.6) (as directly verified) and  $\frac{\partial f}{\partial x}(t, X_t) = \delta_t^A$ , by Itô's formula the continuous semimartingale  $F := (f(t, X_t))$  satisfies the predictable representation

$$(2.10) \quad dF = \delta^A dX.$$

If at this stage we assume  $B$  is a semimartingale, then  $A$  and  $C$  are semimartingales too, and by the invariance principle next,  $dC = \delta^A dA + \delta^B dB$  and  $(\delta^A, \delta^B)$  is a bounded SFTS.

**2.3. Numeraire invariance.** Let  $X$  and  $F$  be two semimartingales and  $\delta^A$  be a locally bounded predictable process such that  $dF = \delta^A dX$ . Set  $\delta^B = F - \delta^A X$ . Clearly  $\delta^B = F_- - \delta^A X_-$  since  $\Delta F = \delta^A \Delta X$ . Let  $B$  be any semimartingale. Set  $A = BX$ ,  $C = BF$ . Clearly  $C = \delta^A A + \delta^B B$ . We claim  $dC = \delta^A dA + \delta^B dB$ , so  $(\delta^A, \delta^B)$  is a SFTS.

Indeed, this follows by applying Itô's product rule to  $BF$ , then substituting  $dF = \delta^A dX$  and  $F_- = \delta^B + \delta^A X_-$ , followed by Itô's product rule on  $BX$ :

$$\begin{aligned} dC &= d(BF) = B_- dF + F_- dB + d[B, F] \\ &= B_- \delta^A dX + (\delta^B + \delta^A X_-) dB + \delta^A d[B, X] \\ &= \delta^A d(BX) + \delta^B dB = \delta^A dA + \delta^B dB. \end{aligned}$$

Conversely and similarly, if  $A$  and  $B$  are semimartingales with  $B, B_- > 0$  and  $(\delta^A, \delta^B)$  is a SFTS, then  $dF = \delta^A dX$ , where  $X = A/B$ ,  $F = C/B$ , and  $C = \delta^A A + \delta^B B$ .

**2.4. Exponential Poisson exchange option model.** Assume that the two zero-dividend asset price processes  $A$  and  $B$  satisfy  $A = BX$ , where  $X$  is a semimartingale satisfying

$$(2.11) \quad X_t = X_0 e^{\beta P_t - (e^\beta - 1)\lambda t}$$

for some constants  $\beta \neq 0$ ,  $\lambda > 0$  and semimartingale  $P$  such that  $[P] = P$  and  $P_0 = 0$  (so,  $P_t = \sum_{s \leq t} 1_{\Delta P_s \neq 0}$ ). Define the projective option price function  $f(t, x)$ ,  $x > 0$  by

$$(2.12) \quad f(t, x) := \sum_{n=0}^{\infty} (x e^{\beta n - (e^\beta - 1)\lambda(T-t)} - 1)^+ \frac{\lambda^n}{n!} (T-t)^n e^{-\lambda(T-t)},$$

and exchange option price process by

$$(2.13) \quad C := BF, \quad F = (F_t), \quad F_t := f(t, X_t).$$

Clearly  $f(T, x) = (x - 1)^+$  and  $C_T = (A_T - B_T)^+$ . One has the predictable representation

$$(2.14) \quad dF = \delta^A dX$$

as shown shortly, where

$$(2.15) \quad \delta_t^A := \delta_A(t, X_{t-}), \quad \delta_A(t, x) := \frac{f(t, e^\beta x) - f(t, x)}{(e^\beta - 1)x}.$$

Thus by numeraire invariance  $(\delta^A, \delta^B)$  is a SFTS if  $A$  and  $B$  are semimartingales, where

$$(2.16) \quad \delta^B := F - \delta^A X = F_- - \delta^A X_-.$$

Moreover, it is bounded. Indeed, since  $|(e^\beta y - 1)^+ - (y - 1)^+| \leq |e^\beta - 1|y$  for any  $y > 0$ ,

$$0 \leq \delta_A(t, x) \leq \sum_{n=0}^{\infty} e^{\beta n - (e^\beta - 1)\lambda(T-t)} \frac{\lambda^n}{n!} (T-t)^n e^{-\lambda(T-t)} = 1.$$

Hence,  $0 \leq \delta^A \leq 1$ . Similarly,  $-1 \leq \delta^B \leq 0$ .

We caution that this model is arbitrage free only when  $\mathbb{P}\{P_t = n\} > 0$  for all  $t > 0$  and  $n \in \mathbb{N}$ , e.g., when  $P$  is Poisson process under an equivalent measure, as in Sec. 2.10 below.

**2.5. Derivation of the predictable representation (2.14).** To show  $dF = \delta^A dX$ , we first note that  $[P]^c = 0$  since  $[P] = P$ ; hence  $(\Delta P)^2 = \Delta P$  and  $P_t = [P]_t = \sum_{s \leq t} \Delta P_s$ . If  $v(p)$ ,  $p \in \mathbb{R}$ , is any function, then clearly  $V = (v(P_t))$  is a semimartingale and we have

$$\Delta V_t = v(P_t) - v(P_{t-}) = (v(P_t) - v(P_{t-}))\Delta P_t = (v(P_{t-} + 1) - v(P_{t-}))\Delta P_t.$$

Hence, as  $V$  is clearly the sum of its jumps,

$$V_t - v(0) = \sum_{s \leq t} \Delta V_s = \sum_{s \leq t} (v(P_{s-} + 1) - v(P_{s-}))\Delta P_s = \int_0^t (v(P_{s-} + 1) - v(P_{s-}))dP_s.$$

Likewise,  $(u(t, P_t))$  is a semimartingale for any  $C^1$  in  $t$  function  $u(t, p)$ ,  $p \in \mathbb{R}$ , and one has

$$(2.17) \quad du(t, P_t) = \frac{\partial u}{\partial t}(t, P_{t-})dt + (u(t, P_{t-} + 1) - u(t, P_{t-}))dP_t.$$

Now, define the function

$$(2.18) \quad x(t, p) := X_0 e^{\beta p - (e^\beta - 1)\lambda t}. \quad (p \in \mathbb{R})$$

Clearly  $X_t = x(t, P_t)$ . Applying (2.17) to the function  $x(t, p)$  and using that

$$\frac{\partial x}{\partial t}(t, p) = -x(t, p)(e^\beta - 1)\lambda, \quad x(t, p + 1) - x(t, p) = x(t, p)(e^\beta - 1),$$

(or alternatively applying Itô's formula to  $x(t, P_t)$  and simplifying) yields

$$(2.19) \quad dX_t = X_{t-}(e^\beta - 1)d(P_t - \lambda t).$$

Next, define the function of  $t \leq T$  and  $p \in \mathbb{R}$ ,

$$(2.20) \quad u(t, p) := f(t, x(t, p)) = \sum_{n=0}^{\infty} (X_0 e^{\beta(p+n) - (e^\beta - 1)\lambda T} - 1)^+ \frac{\lambda^n}{n!} (T - t)^n e^{-\lambda(T-t)}.$$

Clearly,  $u(t, P_t) = F_t$ . One readily verifies that  $u(t, p)$  satisfies the partial difference equation

$$(2.21) \quad \frac{\partial u}{\partial t}(t, p) + \lambda(u(t, p + 1) - u(t, p)) = 0.$$

Hence by (2.17) we have,

$$(2.22) \quad dF_t = (u(t, P_{t-} + 1) - u(t, P_{t-}))d(P_t - \lambda t).$$

Combining this with (2.19) and the fact that clearly

$$u(t, p + 1) - u(t, p) = f(t, e^\beta x(t, p)) - f(t, x(t, p)),$$

we conclude that, as desired,

$$(2.23) \quad dF_t = \frac{f(t, e^\beta X_{t-}) - f(t, X_{t-})}{(e^\beta - 1)X_{t-}} dX_t.$$

**2.6. The homogenous option price function.** There is an alternative derivation of the self-financing equation  $dC = \delta^A dA + \delta^B dB$  much along the original lines in [9] that does not employ numeraire invariance. It is related to a curious family of two-dimensional PDEs satisfied by the *homogenized* option price function generally in continuous SDE models.

Let  $f(t, x)$ ,  $x > 0$ , be any  $C^{1,2}$  function, e.g., the projective option price function (2.4). The Poisson model is excluded because its projective function  $f(t, x)$  is *not*  $C^1$  in  $x$  (though convex, absolutely continuous and piecewise analytic in  $x$ ). Define the function

$$(2.24) \quad c(t, a, b) := bf(t, \frac{a}{b}). \quad (a, b > 0)$$

Since  $c(t, a, b)$  is homogenous of degree 1 in  $(a, b)$ , we have by Euler's formula,

$$(2.25) \quad c(t, a, b) = \frac{\partial c}{\partial a}(t, a, b)a + \frac{\partial c}{\partial b}(t, a, b)b.$$

A laborious repeated application of the chain rule on (2.24) gives

$$(2.26) \quad a^2 \frac{\partial^2 c}{\partial a^2}(t, a, b) = b^2 \frac{\partial^2 c}{\partial b^2}(t, a, b) = -ab \frac{\partial^2 c}{\partial a \partial b}(t, a, b) = bx^2 \frac{\partial^2 f}{\partial x^2}(t, x), \quad x := \frac{a}{b}.$$

Let  $\sigma(t)$ ,  $\sigma_A(t, a, b)$ ,  $\sigma_B(t, a, b)$ ,  $\sigma_{AB}(t, a, b)$  be any functions ( $x, a, b > 0$ ) such that

$$(2.27) \quad \sigma^2(t) = \sigma_A^2(t, a, b) + \sigma_B^2(t, a, b) - 2\sigma_{AB}(t, a, b).$$

Using (2.26), (2.27), and  $\frac{\partial c}{\partial t}(t, a, b) = b \frac{\partial f}{\partial t}(t, \frac{a}{b})$ , we see that  $c(t, a, b)$  satisfies the PDE

$$(2.28) \quad \frac{\partial c}{\partial t} + \frac{1}{2}\sigma_A^2(t, a, b)a^2 \frac{\partial^2 c}{\partial a^2} + \frac{1}{2}\sigma_B^2(t, a, b)b^2 \frac{\partial^2 c}{\partial b^2} + \sigma_{AB}(t, a, b)ab \frac{\partial^2 c}{\partial a \partial b} = 0$$

if and only if  $f(t, x)$  satisfies the PDE (2.6):  $\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(t)x^2 \frac{\partial^2 f}{\partial x^2} = 0$ .

Let now  $f(t, x)$  stand for the Black-Scholes/Merton option price function (2.4). Clearly,

$$\frac{\partial c}{\partial a}(t, a, b) = \frac{\partial f}{\partial x}(t, \frac{a}{b}) = \delta_A(t, \frac{a}{b}).$$

This combined with the Euler's formula (2.25) and the definition (2.4)  $f := \delta_A x + \delta_B$  give

$$\frac{\partial c}{\partial b}(t, a, b) = \delta_B(t, \frac{a}{b}).$$

Assume  $A$  and  $B$  are positive semimartingales with positive left limits and  $X := A/B$  has deterministic volatility:  $d[X]_t = X_t^2 \sigma^2(t) dt$  for some continuous function  $\sigma(t) > 0$ . Whence, the deltas are conveniently the sensitivities of the homogenous option price function:

$$(2.29) \quad \delta_t^A = \frac{\partial c}{\partial a}(t, A_t, B_t), \quad \delta_t^B = \frac{\partial c}{\partial b}(t, A_t, B_t).$$

Since  $X$  is continuous, we also have  $\delta_t^A = \frac{\partial c}{\partial a}(t, A_{t-}, B_{t-})$ , ditto  $\delta_t^B$ . Sec. 2.2 yields  $dC = \delta^A dA + \delta^B dB$  with  $C_t = B_t f(t, X_t) = c(t, A_t, B_t)$ . Therefore, by (2.29) and Itô's formula,

$$(2.30) \quad \frac{\partial c}{\partial t} dt + \frac{1}{2} \frac{\partial^2 c}{\partial a^2} d[A]_t^c + \frac{1}{2} \frac{\partial^2 c}{\partial b^2} d[B]_t^c + \frac{\partial^2 c}{\partial a \partial b} d[A, B]_t^c = 0,$$

where the partials are evaluated at  $(t, A_{t-}, B_{t-})$  and  $[\cdot]^c$  is the bracket continuous part. (The jump term in Itô's formula vanishes as it equals  $\sum_{s \leq t} (\Delta C_s - \delta_s^A \Delta A_s - \delta_s^B \Delta B_s) = 0$ .)

Coming to the main point, assume now  $d[\log A]_t = \sigma_A^2(t, A_t, B_t)dt$  for some function  $\sigma_A$  and similarly  $d[\log B] = \sigma_B^2 dt$  and  $d[\log A, \log B] = \sigma_{AB} dt$ . Then Eq. (2.27) holds using  $\log X = \log A - \log B$ . Since  $f(t, x)$  satisfies the PDE (2.6), the PDE (2.28) follows as before by the chain rule. But, (2.28) implies (2.30), which by Itô's formula in turn implies the self-financing equation  $dC = \delta^A dA + \delta^B dB$  with  $\delta^A$  and  $\delta^B$  given by (2.29).

**2.7. Change of numeraire.** The solution  $c(t, a, b)$  to the PDE (2.28) subject to  $c(T, a, b) = (a - b)^+$  can be expressed in a form  $\mathbb{E}(X - Y)^+$  for some random variables  $X$  and  $Y > 0$  with means  $a$  and  $b$ . Expectations of this form often become more tractable by a change of measure as in [4]. Define the equivalent probability measure  $\mathbb{Q}$  by  $\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{Y}{\mathbb{E}Y}$ . Clearly,

$$(2.31) \quad \mathbb{E}^{\mathbb{Q}}\left(\frac{X}{Y}\right) = \frac{\mathbb{E}(X)}{\mathbb{E}(Y)}. \quad \left(\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{Y}{\mathbb{E}(Y)}\right)$$

Replacing  $X$  by  $(X - Y)^+$  in (2.31) and using the homogeneity to factor out  $Y$ , we get

$$(2.32) \quad \mathbb{E}(X - Y)^+ = \mathbb{E}(Y)\mathbb{E}^{\mathbb{Q}}\left(\frac{X}{Y} - 1\right)^+.$$

If  $X/Y$  is  $\mathbb{Q}$ -lognormally distributed then (2.32) with the aid of (2.31) readily yields,

$$(2.33) \quad \mathbb{E}(X - Y)^+ = \mathbb{E}(X)N\left(\frac{\log(\mathbb{E}X/\mathbb{E}Y)}{\sqrt{\nu^{\mathbb{Q}}}} + \frac{\sqrt{\nu^{\mathbb{Q}}}}{2}\right) - \mathbb{E}(Y)N\left(\frac{\log(\mathbb{E}X/\mathbb{E}Y)}{\sqrt{\nu^{\mathbb{Q}}}} - \frac{\sqrt{\nu^{\mathbb{Q}}}}{2}\right),$$

where  $\nu^{\mathbb{Q}} := \text{var}^{\mathbb{Q}}[\log(X/Y)]$ . When  $X$  and  $Y$  are bivariate lognormally distributed, it is not difficult to show that  $X/Y$  is lognormally distributed in both  $\mathbb{P}$  and  $\mathbb{Q}$  with the same log-variance  $\nu^{\mathbb{Q}} = \nu := \text{var}[\log(X/Y)]$ . Then  $\nu^{\mathbb{Q}}$  can be replaced with  $\nu$  in (2.33). This occurs when the functions  $\sigma_A$ ,  $\sigma_B$  and  $\sigma_{AB}$  in (2.28) are independent of  $a$  and  $b$ , as in [8].

**2.8. Uniqueness.** Assume  $A$  and  $B$  are positive semimartingales with positive left limits such that  $X := A/B$  is square-integrable martingale under an equivalent probability measure  $\mathbb{Q}$  and  $d\langle X \rangle_t^{\mathbb{Q}} = X_{t-}^2 \sigma_t^2 dt$  for some nowhere zero continuous process  $\sigma$ , where  $\langle X \rangle^{\mathbb{Q}}$  is the  $\mathbb{Q}$ -compensator of  $[X]$ . (Of course,  $\langle X \rangle^{\mathbb{Q}} = [X]$  if  $X$  is continuous.) Let  $(\delta^A, \delta^B)$  be a SFTS and set  $C := \delta^A A + \delta^B B$ . We claim that  $\delta^A = \delta^B = 0$  if  $C_T = 0$  and  $\delta^A$  is bounded.

Indeed, set  $F := C/B$ . By numeraire invariance  $dF = \delta^A dX$ . Hence  $F$  is a  $\mathbb{Q}$ -square-integrable martingale since  $X$  is and  $\delta^A$  is bounded. Thus,  $F = 0$  since  $F_T = C_T/B_T = 0$ . Hence,  $0 = d\langle F \rangle^{\mathbb{Q}} = (\delta^A)^2 X_-^2 \sigma^2 dt$ . But,  $X_- \sigma \neq 0$ . Thus,  $\delta^A = 0$  and  $\delta^B = F - \delta^A X = 0$ .

**2.9. Deterministic-volatility model uniqueness.** Assume that  $A$  and  $B$  are positive semimartingales with positive left limits and  $X := A/B$  is an Itô process following

$$(2.34) \quad \frac{dX_t}{X_t} = \mu_t dt + \sigma_t dZ_t, \quad (X := \frac{A}{B})$$

where  $Z$  is a Brownian motion and  $\mu$  and  $\sigma > 0$  are continuous adapted processes with  $\sigma$  bounded and  $\mathbb{E}[e^{\frac{1}{2} \int_0^T (\frac{\mu_t}{\sigma_t})^2 dt}] < \infty$ . Let  $(\delta^A, \delta^B)$  be a SFTS with  $\delta^A$  bounded. Set  $C := \delta^A A + \delta^B B$ . We claim that  $\delta^A = \delta^B = 0$  if  $C_T = 0$ . Indeed, the process

$$M := \mathcal{E}\left(-\int \frac{\mu}{\sigma} dZ\right) = e^{-\int \frac{\mu}{\sigma} dZ - \frac{1}{2} \int (\frac{\mu}{\sigma})^2 dt},$$

is then a positive martingale with  $M_0 = 1$ . Define the equivalent probability measure  $\mathbb{Q}$  by  $d\mathbb{Q} = M_T d\mathbb{P}$ . The process  $W := Z + \int \frac{\mu}{\sigma} dt$  is a  $\mathbb{Q}$ -Brownian motion because  $[W]_t = t$  and  $W$  is  $\mathbb{Q}$ -local martingale as  $MW$  is a local martingale using Itô's product rule:

$$d(MW) - WdM = MdW + d[W, M] = M(dZ + \frac{\mu}{\sigma} dt) - M\frac{\mu}{\sigma} d[Z] = MdZ.$$

Moreover,  $dX = X\sigma dW$  by (2.34). Therefore  $X$  is a  $\mathbb{Q}$ -square integrable martingale (in fact in  $\mathcal{H}^p(\mathbb{Q})$  for all  $p > 0$ ) since  $\sigma$  is bounded. The claim thus follows by Sec. 2.8.

As a corollary of the proof,  $F := C/B$  is  $\mathbb{Q}$ -square-integrable martingale because  $\delta^A$  is bounded and by numeraire invariance  $dF = \delta^A dX$ . In particular,  $C_t = B_t \mathbb{E}^{\mathbb{Q}}[C_T/B_T | \mathcal{F}_t]$ .

Assume now  $\sigma_t$  is *deterministic*. The results of Sec. 2.2 hold since  $d[\log X] = \sigma_t^2 dt$ . But we can now derive them more conceptually. Indeed, both conditioned on  $\mathcal{F}_t$  and unconditionally,  $X_T/X_t$  is  $\mathbb{Q}$ -lognormally distributed with mean 1 and log-variance  $\int_t^T \sigma_s^2 ds$  since  $X_T = X_t e^{\int_t^T \sigma_s dW_s - \frac{1}{2} \int_t^T \sigma_s^2 ds}$ . Hence

$$(2.35) \quad f(t, X_t) = \mathbb{E}^{\mathbb{Q}}[(X_T - 1)^+ | \mathcal{F}_t], \quad \text{where } f(t, x) := \mathbb{E}^{\mathbb{Q}}(x \frac{X_T}{X_t} - 1)^+,$$

which function readily equals the Black-Scholes/Merton option price function (2.4). Thus,  $F := (f(t, X_t))$  is a  $\mathbb{Q}$ -martingale. Therefore Itô's formula implies that  $f(t, x)$  satisfies the PDE (2.6) and  $dF = \delta^A dX$  where  $\delta^A := \frac{\partial f}{\partial x}(t, X_t)$ . Numeraire invariance now yields  $(\delta^A, \delta^B := F - \delta^A X)$  is a SFTS. Clearly  $C_T = (A_T - B_T)^+$  where  $C := \delta^A A + \delta^B B = BF$ .

**2.10. Exponential Poisson model uniqueness.** Let  $\beta \neq 0$  be a constant and  $\kappa$  and  $\lambda$  be positive continuous adapted processes such that  $\lambda$  is bounded and  $\mathbb{E} e^{\int_0^T (\frac{\lambda_t}{\kappa_t} - 1)^2 \kappa_t dt} < \infty$ . Let  $P$  be semimartingale satisfying  $[P] = P$  with  $P_0 = 0$  and compensator  $\int \kappa dt$ . Assume that  $A$  and  $B$  are positive semimartingales with positive left limits and  $X := \frac{A}{B}$  satisfies

$$(2.36) \quad dX_t = X_{t-}(e^\beta - 1)(dP_t - \lambda_t dt).$$

Using  $de^{\beta P} = (e^\beta - 1)e^{\beta P} dP$  or as in Sec. 2.5, this is equivalent to the integrated form

$$(2.37) \quad X_t = X_0 e^{\beta P_t - (e^\beta - 1) \int_0^t \lambda_s ds}.$$

Let  $(\delta^A, \delta^B)$  be a SFTS with  $\delta^A$  bounded. Set  $C := \delta^A A + \delta^B B$ . We claim  $\delta^A = \delta^B = 0$  if  $C_T = 0$ . Indeed,  $\mathbb{E} e^{(\int (\frac{\lambda}{\kappa} - 1)(dP - \kappa dt))_T} = \mathbb{E} e^{\int_0^T (\frac{\lambda_t}{\kappa_t} - 1)^2 \kappa_t dt} < \infty$ , so the positive local martingale

$$M := \mathcal{E}\left(\int \left(\frac{\lambda}{\kappa} - 1\right)(dP - \kappa dt)\right) = e^{-\int (\lambda - \kappa) dt} \prod_{s \leq \cdot} \left(1 + \left(\frac{\lambda_s}{\kappa_s} - 1\right) \Delta P_s\right)$$

is a martingale. Define the equivalent probability measure  $\mathbb{Q}$  by  $d\mathbb{Q} = M_T d\mathbb{P}$ . Then  $N := P - \int \lambda dt$  is a  $\mathbb{Q}$ -local martingale as  $MN$  is a local martingale by Itô's product rule:

$$\begin{aligned} d(MN) - N_- dM &= M_- dN + d[M, N] \\ &= M_- (dP - \lambda dt) + M_- \left(\frac{\lambda}{\kappa} - 1\right) dP = M_- \frac{\lambda}{\kappa} (dP - \kappa dt). \end{aligned}$$

Therefore by (2.36)  $X$  is a  $\mathbb{Q}$ -square-integrable martingale (in fact in  $\mathcal{H}^p(\mathbb{Q})$  all  $p > 0$ ) since  $\lambda$  is bounded. Thus, by Sec. (2.8),  $\delta^A = \delta^B = 0$  if  $C_T = 0$ , as claimed.

As a corollary to the proof,  $F := C/B$  is  $\mathbb{Q}$ -square-integrable martingale because  $\delta^A$  is bounded and by numeraire invariance  $dF = \delta^A dX$ . In particular,  $C_t = B_t \mathbb{E}^{\mathbb{Q}}[C_T/B_T | \mathcal{F}_t]$ .

Assume now  $\lambda$  is a positive constant. By (2.37) we are in a special case of the exponential Poisson model. Further,  $P$  is a  $\mathbb{Q}$ -Poisson process with intensity  $\lambda$  since  $[P] = P$ . We now have uniqueness, but additionally, the previous results follow more conceptually as follows.

Conditioned on  $\mathcal{F}_t$ ,  $P_T - P_t$  is  $\mathbb{Q}$ -Poisson distributed with mean  $\lambda(T-t)$ . Its unconditional  $\mathbb{Q}$ -distribution is identical. Thus the  $\mathcal{F}_t$ -conditional and the unconditional  $\mathbb{Q}$ -distribution of  $X_T/X_t$  are identical and are exponentially Poisson distributed with mean 1. Hence

$$(2.38) \quad f(t, X_t) = \mathbb{E}^{\mathbb{Q}}[(X_T - 1)^+ | \mathcal{F}_t], \quad \text{where } f(t, x) := \mathbb{E}^{\mathbb{Q}}(x \frac{X_T}{X_t} - 1)^+,$$

which function readily equals that defined in (2.12). Thus,  $F := (f(t, X_t))$  is a  $\mathbb{Q}$ -martingale. Using this and (2.17) one shows that  $F$  satisfies (2.23) and with it that the pair  $(\delta^A, \delta^B)$  as defined in (2.15), (2.16) is a bounded SFTS for the exchange option.

**2.11. Extension to dividends.** Consider two assets with positive price processes  $\hat{A}$  and  $\hat{B}$  and continuous dividend yields  $y_t^A$  and  $y_t^B$ . When there exist traded or replicable zero-dividend assets  $A$  and  $B$  such that  $A_T = \hat{A}_T$  and  $B_T = \hat{B}_T$  (if not there is little hope of replication), it is natural to define the price process of the option to exchange  $\hat{A}$  and  $\hat{B}$  to be that of the option to exchange  $A$  and  $B$ . If  $y^A$  and  $y^B$  are deterministic, then consistently with the treatment of dividends in [9],  $A$  and similarly  $B$  is simply given by

$$A_t := a \tilde{A}_t = e^{-\int_t^T y_s^A ds} \hat{A}_t, \quad \tilde{A}_t := e^{\int_0^t y_s^A ds} \hat{A}_t, \quad a := e^{-\int_0^T y_t^A dt}.$$

Note  $A/B$  is a semimartingale if and only if  $\hat{A}/\hat{B}$  is, in which case  $[\log A/B] = [\log \hat{A}/\hat{B}]$ .

In general,  $\tilde{A}_t$  is the price of the zero-dividend asset that initially buys one share of  $\hat{A}$  and thereon continually reinvests all dividends in  $\hat{A}$  itself. What is required is that the four zero-dividend assets  $A, \tilde{A}, B$  and  $\tilde{B}$  be arbitrage-free among each other (see Sec. 3.3).

For instance, say  $\hat{A}$  and  $\hat{B}$  are the yen/dollar and yen/Euro exchange rates viewed as yen-denominated dividend assets. Then  $A$  is the yen-value of the U.S.  $T$ -maturity zero-coupon bond and  $\tilde{A}$  is the yen-value of the U.S. money market asset. This exchange option is equivalent to a Euro-denominated call struck at 1 on the Euro/dollar exchange rate  $\hat{A}/\hat{B}$ . The ratio  $A/B$  is the *forward* Euro/dollar exchange rate. If it has deterministic volatility, we are as in a setting of [7] which yields the same pricing formula as that from Sec 2.2.

### 3. PRICING AND HEDGING OPTIONS WITH HOMOGENOUS PAYOFFS

We took some shortcuts above to quickly presents the main results for two of the simplest and among the most interesting examples. But, a better understanding of the principles at work requires generalization to contingent claims  $C$  on many assets with price processes  $A = (A^1, \dots, A^m) > 0$  and a path-independent payoff  $C_T = h(A_T)$  given as a homogenous function  $h(a)$ ,  $a \in \mathbb{R}_+^m$ , of the asset prices  $A_T$  at expiration time  $T$ . Combined with an underlying SDE and the resulting PDE, such Markovian setting utilizes the invariance principle and equivalent martingale measures to derive unique pricing and construct a

SFTS that replicates the given payoff  $h(A_T)$  in general. The construction is explicit in the multivariate extensions of the deterministic-volatility and exponential Poisson models.

The homogeneity of the payoff function  $h(a)$  implies  $h(A_T) = A_T^m g(X_T)$  where  $g(x) := h(x, 1)$ ,  $x \in \mathbb{R}_+^n$ ,  $n := m - 1$ , and  $X := (\frac{A^1}{A^m}, \dots, \frac{A^n}{A^m})$ . Once a predictable representation  $F = F_0 + \delta' \cdot X$ ,  $F_T = g(X_T)$  is found, then by numeraire invariance  $\delta := (\delta', \delta^m)$  will be a SFTS with payoff  $h(A_T)$ , where  $\delta^m := F_- - \sum_{i=1}^n \delta^i X_- = F - \sum_{i=1}^n \delta^i X$ . Uniqueness of pricing requires boundedness of partial derivatives (or differences) of  $h(a)$  (or  $g(x)$ ) and that  $A$  be arbitrage free, meaning  $X$  is a martingale under an equivalent measure. Arbitrage freedom holds “generically” when the matrix  $(\langle X^i, X^j \rangle)$  is nonsingular, basically a “no redundant asset” condition. Then the SFTS itself is unique, namely the constructed one.

Libor and swap derivatives are among of contingent claims with homogenous payoffs.

**3.1. Self-financing trading strategies.** By a SFTS we mean a pair  $(\delta, A)$  of an  $m$ -dimensional semimartingale  $A = (A^1, \dots, A^m)$  and an  $A$ -integrable predictable vector process  $\delta = (\delta^1, \dots, \delta^m)$  such that (with  $\delta \cdot A$  denoting the  $m$ -dimensional stochastic integral)

$$(3.1) \quad \sum_{i=1}^m \delta^i A^i = \sum_{i=1}^m \delta_0^i A_0^i + \delta \cdot A.$$

We then say  $\delta$  is a *SFTS for A*. This is equivalent to saying that the SFTS *price process*

$$(3.2) \quad C := \sum_{i=1}^m \delta^i A^i$$

satisfies  $C = C_0 + \delta \cdot A$ . Clearly  $C$  is then a semimartingale,  $\Delta C = \sum_i \delta^i \Delta A_i$ , and hence

$$C_- = \sum_{i=1}^m \delta^i A_-^i.$$

If  $\delta^i$  are bounded (say by  $b$ ) and  $A^i$  are martingales then the SFTS price process  $C$  is a *martingale* because  $C$  is then a local martingale that is dominated by a martingale  $M$ :

$$|C_t| \leq b \sum_i |A_t^i| = b \sum_i |\mathbb{E}[A_T^i | \mathcal{F}_t]| \leq b \sum_i \mathbb{E}[|A_T^i| | \mathcal{F}_t] =: M_t.$$

As suggested by when  $\delta$  is locally bounded, we often use the differential form

$$(3.3) \quad dC = \sum_{i=1}^m \delta^i dA^i$$

of the equation  $C = C_0 + \delta \cdot A$  as a convenient symbolic equivalent in calculations. One interprets the  $A^i$  as prices of  $m$  zero-dividend assets and the  $\delta_t^i$  as the number of shares invested in them at time  $t$ . Then  $C_t$  indicates the resultant self-financing portfolio price by (3.2), and (3.3) is the self-financing equation, saying that the change  $dC$  in the portfolio price is due only to the changes  $dA^i$  in the asset prices with no financing from outside.

Assume for the remainder of this subsection that  $A$  is continuous and  $C_t = c(t, A_t)$  for some  $C^{1,2}$  function  $c(t, a)$ .<sup>1</sup> Then by (3.3) and Itô's formula we have,

$$\frac{\partial c}{\partial t}(t, A_t)dt + \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t) d[A^i, A^j]_t = \sum_{i=1}^m (\delta_t^i - \frac{\partial c}{\partial a_i}(t, A_t)) dA_t^i.$$

In particular the right hand side has finite variation. Hence  $\delta_t^i = \frac{\partial c}{\partial a_i}(t, A_t)$  when  $d[A^i, A^j]$  are absolutely continuous and the  $m \times m$  matrix  $(\frac{d}{dt}[A^i, A^j])$  is nonsingular. Then we get

$$\frac{\partial c}{\partial t}(t, A_t)dt + \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t) d[A^i, A^j]_t = 0,$$

and, by (3.2),  $c(t, A_t) = \sum_i \frac{\partial c}{\partial a_i}(t, A_t) A_t^i$ . If further the support of  $A_t$  is a cone, it follows by Euler's formula that  $c(t, a)$  must be *homogenous of degree 1* in  $a$  on that cone.

Instead of the nonsingularity assumption above, suppose there exists a locally bounded predictable process  $r = (r_t)$  such that the processes  $M^i := e^{-\int r dt} A^i$  are local martingales under an equivalent measure. Then,  $dA^i = r A^i dt + e^{\int r dt} dM^i$ , so by Itô's formula and (2.2),

$$\frac{\partial c}{\partial t}(t, A_t)dt + \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t) d[A^i, A^j]_t = r_t (C_t - \sum_{i=1}^m \frac{\partial c}{\partial a_i}(t, A_t) A_t^i) dt.$$

Hence if one assumes as in [9] that  $c(t, a)$  is homogenous of degree 1 in  $a$ , then the right hand side vanishes again. Given a homogenous payoff function  $h(a)$ , Sec. 3.7 below constructs under suitable assumptions such a function  $c(t, a)$  with  $c(T, a) = h(a)$  and  $\frac{\partial c}{\partial a_i}(t, A_t) = \delta_t^i$ .

**3.2. The invariance principle.** *Let  $(\delta, A)$  be a SFTS and  $S$  a (scalar) semimartingale such that  $\delta$  is  $SA := (SA^1, \dots, SA^m)$ -integrable. Then  $(\delta, SA)$  is a SFTS. Consequently,*

$$(3.4) \quad d(SC) = \sum_{i=1}^m \delta^i d(SA^i),$$

where  $C := \sum_i \delta^i A^i = C_0 + \delta \cdot A$ , that is,  $SC = S_0 C_0 + \delta \cdot (SA)$ . Indeed, by Itô's product rule, then substituting for  $dC$  and  $C_-$  and regrouping, followed by Itô's product rule again,

$$\begin{aligned} d(SC) &= S_- dC + C_- dS + d[S, C] \\ &= S_- \sum_{i=1}^m \delta^i dA^i + \sum_{i=1}^m \delta^i A_-^i dS + \sum_{i=1}^m \delta^i d[S, A^i] \\ &= \sum_{i=1}^m \delta^i (S_- dA^i + A_-^i dS + d[S, A^i]) = \sum_{i=1}^m \delta^i d(SA^i). \end{aligned}$$

<sup>1</sup>Clearly then the restriction of (any such)  $c(t, a)$  to the support of  $A$  is unique, and if  $\hat{c}(t, a)$  is any function that equals  $c(t, a)$  on the support of  $A$ , then  $C_t = \hat{c}(t, A_t)$  too. If the support of  $A_t$  is a proper surface, e.g., if  $A^m$  is deterministic as in the Black-Scholes model, or more generally  $A_t^m = a_m(t, A_t^1, \dots, A_t^{m-1})$  as in Markovian short-rate models, then obviously there exist infinitely many nonhomogeneous functions  $\hat{c}(t, a)$  such that  $C_t = \hat{c}(t, A_t)$ . (A homogenous such function also exists under assumptions as in Sec. 3.7.)

Interpreting  $S$  as an exchange rate, this result, referred to as *numeraire invariance*, means that the self-financing property is independent of the choice of base currency.

If  $S, S_- > 0$ , then applied to the semimartingale  $1/S$  we see that  $\delta$  is a SFTS for  $A$  if and only if it is one for  $SA$ . Thus, if (3.2) holds then (3.3) and (3.4) are equivalent.

Assume now  $A^m, A_-^m > 0$  and  $m \geq 2$ . Define the  $n := m - 1$  dimensional semimartingale

$$(3.5) \quad X := \left( \frac{A^1}{A^m}, \dots, \frac{A^n}{A^m} \right), \quad n := m - 1.$$

Taking  $S = 1/A^m$ , it follows that  $\delta$  is a SFTS for  $A$  if and only if it is a SFTS for  $A/A^m = (X, 1)$ , i.e., if and only if  $F := C/B$  satisfies  $F = F_0 + \delta' \cdot X$  where  $\delta' := (\delta^1, \dots, \delta^n)$ . In this case clearly  $F = \sum_{i=1}^n \delta^i X^i + \delta^m$  and  $F_- = \sum_{i=1}^n \delta^i X_-^i + \delta^m$  as  $\Delta F = \delta' \cdot \Delta X$ . Thus,

$$(3.6) \quad \delta^m = F - \sum_{i=1}^n \delta^i X^i = F_- - \sum_{i=1}^n \delta^i X_-^i. \quad (F := \frac{C}{A^m})$$

When  $m = 1$  a similar argument shows that  $\delta$  must be a constant, as intuitively obvious.

Conversely, suppose  $\delta'$  is an  $X$ -integrable process and  $F$  is a process such that  $F = F_0 + \delta' \cdot X$ . Define  $\delta^m$  by either of the above formula - the other then holds as before. Obviously then  $\delta = (\delta', \delta^m)$  is a SFTS for  $(X, 1)$  with price process  $F$ . Hence by numeraire invariance  $\delta$  is a SFTS for  $A$  with price process  $C = BF$ , provided  $\delta$  is  $A$ -integrable.

Numeraire invariance thus shows that *in order to find a SFTS with a given time- $T$  payoff  $C_T$  it is sufficient to find processes  $\delta'$  and  $F$  such that  $F = F_0 + \delta' \cdot X$  and  $F_T = C_T/A_T^m$ .*

We often use the differential form  $dF = \sum_{i=1}^n \delta^i dX^i$  of the equation  $F = F_0 + \delta' \cdot X$ . Substituting the latter in (3.6) reveals that  $\delta^m$  is determined by  $\delta'$  and  $F_0$ . As such, one interprets the  $m$ -th asset as the “numeraire asset” chosen to finance an otherwise arbitrary trading strategy  $\delta'$  in the other assets, post an initial investment of  $C_0 = A_0^m F_0$ .

**3.3. Arbitrage-free semimartingales and uniqueness.** We call a semimartingale  $A = (A^1, \dots, A^m)$ ,  $m \geq 2$ , arbitrage free if there exists a positive (scalar) semimartingale  $S$  with  $S_- > 0$  such that  $SA^i$  are martingales for all  $i$ . Such a process  $S$  is called a state price density or deflator for  $A$ . The law of one price justifies the terminology:

*If  $A$  is arbitrage free and  $\delta$  is a bounded SFTS for  $A$  then  $SC$  is a martingale where  $C := \sum_{i=1}^m \delta^i A^i$ ; consequently  $C = 0$  if  $C_T = 0$ .*

Indeed, by numeraire invariance  $\delta$  is then a SFTS for  $SA$  with price process  $SC$ . Hence by Sec. 3.1,  $SC$  is a martingale, implying  $SC = 0$  if  $C_T = 0$ , and with it  $C = 0$ , as claimed.

A simple and well-known argument yields that *if  $A^m, A_-^m > 0$  then  $A$  is arbitrage free if and only if there exists an equivalent probability measure  $\mathbb{Q}$  such that  $X$  is a  $\mathbb{Q}$ -martingale, where  $X := (\frac{A^1}{A^m}, \dots, \frac{A^n}{A^m})$ ,  $n := m - 1$ .<sup>2</sup> A corollary is that  $C/A^m$  is then a  $\mathbb{Q}$ -martingale for the price process  $C := \sum_i \delta^i A^i$  of any bounded SFTS  $\delta$ , and hence  $C_t = A_t^m \mathbb{E}^{\mathbb{Q}}[C_T/A_T^m | \mathcal{F}_t]$ .*

<sup>2</sup>Indeed, first assume  $A$  is arbitrage-free and let  $S$  be a state price density. The martingale  $M := \frac{SA^m}{\mathbb{E}[S_0 A_0^m]}$  clearly satisfies  $\mathbb{E} M_T = 1$ . Hence the equivalent measure  $\mathbb{Q}$  defined by  $d\mathbb{Q} = M_T d\mathbb{P}$  is a probability measure. Since  $MX^i = \frac{SA^i}{\mathbb{E}[S_0 A_0^m]}$  is a martingale,  $X^i$  is a  $\mathbb{Q}$ -martingale by the Bayes' rule. Conversely, assume  $X^i$  are  $\mathbb{Q}$ -martingales for some  $\mathbb{Q}$ . Define  $M_t := \mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}} | \mathcal{F}_t] > 0$ . Then (the right continuous version

Indeed, by numeraire invariance,  $\delta$  is then a SFTS for  $A/A^m$  with price process  $F := C/A^m$ . Hence,  $F$  is a  $\mathbb{Q}$ -martingale by Sec 3.1 applied in measure  $\mathbb{Q}$  since  $A/A^m$  is.

Assume now that  $X$  is a  $\mathbb{Q}$ -square-integrable martingale. Then  $F := C/A^m$  is a  $\mathbb{Q}$ -square-integrable provided only that  $\delta^i$  are bounded for  $i \leq n$  as  $dF = \sum_{i=1}^n \delta^i dX^i$ . Further,  $d\langle F \rangle^\mathbb{Q} = \sum_{i,j=1}^n \delta^i \delta^j d\langle X^i, X^j \rangle^\mathbb{Q}$ . Therefore if  $d\langle X^i \rangle^\mathbb{Q}$  are absolutely continuous and the  $n \times n$  matrix  $(\frac{d}{dt} \langle X^i, X^j \rangle^\mathbb{Q})$  is nonsingular, then *given any random variable  $R$ , there exists at most one SFTS  $\delta$  for  $A$  such that  $\sum_{i=1}^m \delta_T^i A_T^i = R$  and  $\delta^i$  are bounded for  $i \leq n$ .*

**3.4. Projective continuous Markovian SFTS.** Let  $X = (X^1, \dots, X^n)$  be a continuous vector martingale. In what follows  $x \in \mathbb{R}_+^n$  if  $X > 0$  (the main case of interest), otherwise  $x \in \mathbb{R}^n$ . Let  $g(x)$  be a Borel function of linear growth (so  $\mathbb{E}|g(X_T)| < \infty$ ), and  $f(t, x)$  be a function,  $C^{1,2}$  on  $t < T$ . Set  $m := n + 1$  and define the  $C^1$  functions

$$(3.7) \quad \delta_i(t, x) := \frac{\partial f}{\partial x_i}(t, x), \quad i \leq n, \quad \delta_m(t, x) := f(t, x) - \sum_{i=1}^n \delta_i(t, x)x_i,$$

and the continuous vector process

$$(3.8) \quad \delta = (\delta^1, \dots, \delta^m), \quad \delta_t^i := \delta_i(t, X_t).$$

First suppose that

$$(3.9) \quad f(t, X_t) = \mathbb{E}[g(X_T) | \mathcal{F}_t].$$

Then the process  $F := (f(t, X_t))$  is a martingale, and since  $X^i$  are too, Itô's formula yields,

$$(3.10) \quad dF_t = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t, X_t) dX_t^i,$$

and

$$(3.11) \quad \frac{\partial f}{\partial t}(t, X_t) dt + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t) d[X^i, X^j]_t = 0.$$

Clearly  $F_T = g(X_T)$  and (3.10) implies  $\delta$  is a SFTS for  $(X, 1)$  with price process  $F$ .

Conversely, suppose that  $f(t, x)$  satisfies (3.11) or by Itô's formula equivalently (3.10). By (3.10)  $\delta$  is a SFTS for  $(X, 1)$  with price process  $F := f(t, X_t)$ . Thus by Sec. 3.1, if  $\delta_i(t, x)$  are bounded then  $F$  is a martingale and if further  $f(T, x) = g(x)$  then (3.9) holds. Moreover, as in Sec 3.3,  $\delta$  given by (3.8) is then the *unique* bounded SFTS for  $(X, 1)$  with payoff  $g(X_T)$ , provided  $d[X^i, X^j] = X^i X^j \sigma^{ij} dt$  for some nonsingular matrix process  $(\sigma_t^{ij})$ .

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of  $M = (M_t)$  is a martingale (so  $M_- > 0$ ). By the Bayes' rule  $MX^i$  are martingales since  $X^i$  are  $\mathbb{Q}$ -martingales. Set  $S := M/A^m$ . Then  $S, S_- > 0$  and  $SA^i = MX^i$ . Thus  $S$  is deflator, as desired. Further, since  $SC$  is a martingale for any bounded SFTS  $\delta$ , by the Bayes' rule  $SC/M = C/A^m$  is a  $\mathbb{Q}$ -martingale.

**3.5. Example: projective deterministic volatility.** Let  $X = (X^1, \dots, X^n) > 0$  be a continuous  $n$ -dimensional martingale such that

$$(3.12) \quad d[X^i, X^j]_t = X_t^i X_t^j \sigma_{ij}(t) dt$$

for some  $n^2$  deterministic continuous functions  $\sigma_{ij}(t)$ . So,  $d[\log X^i, \log X^j]_t = \sigma_{ij}(t) dt$ . Conditioned on  $\mathcal{F}_t$  and unconditionally,  $X_T/X_t$  is then *multivariately lognormally distributed*, with mean  $(1, \dots, 1)$  and log-covariance matrix  $(\int_t^T \sigma_{ij}(s) ds)$ . Let  $P(t, T, z)$ , denote its distribution function. Let  $g(x)$  be a Borel function of linear growth. Define the function

$$(3.13) \quad f(t, x) := \mathbb{E}[g(x_1 \frac{X_T^1}{X_t^1}, \dots, x_n \frac{X_T^n}{X_t^n})].$$

Obviously  $f(T, x) = g(x)$ . Clearly  $f(t, x)$  can also be represented in two other ways as

$$f(t, x) = \int_{\mathbb{R}_+^n} g(x_1 z_1, \dots, x_n z_n) P(t, T, dz) = \mathbb{E}[g(x_1 \frac{X_T^1}{X_t^1}, \dots, x_n \frac{X_T^n}{X_t^n}) | \mathcal{F}_t].$$

Eq. (3.9) holds by the second equality, and  $f(t, x)$  is  $C^1$  in  $t$  and smooth (even analytic) in  $x$  on  $t < T$  as seen by changing variable in the integral to  $y^i = x^i z^i$  and differentiating under the integral sign in the first equality. Therefore by (3.11),  $f(t, x)$  satisfies the PDE

$$(3.14) \quad \frac{\partial f}{\partial t} + \frac{1}{2} \sum_{i,j=1}^n \sigma_{ij}(t) x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} = 0$$

on the support of  $X$ , (3.10) holds, and  $\delta$  is a SFTS for  $(X, 1)$  with price process  $F := (f(t, X_t))$ , a martingale by (3.9). If  $g(x)$  is  $dx$ -absolutely continuous with bounded partials  $\frac{\partial g}{\partial x_i}$  (as  $L_{loc}^1$  functions) then  $g(x)$  has linear growth,  $\mathbb{E}|g(X_T)|^p < \infty$  for  $p > 0$ , and

$$\frac{\partial f}{\partial x_i}(t, x) = \mathbb{E}[\frac{X_T^i}{X_t^i} \frac{\partial g}{\partial x_i}(x_1 \frac{X_T^1}{X_t^1}, \dots, x_n \frac{X_T^n}{X_t^n})].$$

Thus  $\delta_i(t, x) = \frac{\partial f}{\partial x_i}(t, x)$  are bounded. If  $g(x) - \sum \frac{\partial g}{\partial x_i} x_i$  is bounded then so is  $\delta_m(t, x)$  as

$$\delta_m(t, x) = \mathbb{E}[g(x \frac{X_T}{X_t}) - \sum_{i=1}^n \frac{\partial g}{\partial x_i}(x \frac{X_T}{X_t}) \frac{X_T^i}{X_t^i}].$$

It further follows that if  $f(t, x)$  is any  $C^{1,2}$  function with bounded partials  $\frac{\partial f}{\partial x_i}(t, x)$  satisfying  $f(T, x) = 0$  for all  $x$  and the PDE (3.14), then  $F := (f(t, X_t)) = 0$ . Indeed, (3.10) then holds by PDE (3.14) and Itô's formula, implying  $F$  is a square-integrable martingale since  $X$  is; thus  $F = 0$  since  $F_T = 0$ . As such,  $f(t, x) = 0$  identically if the support of  $X_t$  equals  $\mathbb{R}_+^n$  for every  $t$ . This is so if the matrix  $(\sigma_{ij}(t))$  is nonsingular at least near 0, and it is "generically" so even when the matrix has rank 1 but is time dependent.

**3.6. Projective continuous SDE SFTS.** Continuous Markovian positive martingales  $X = (X^1, \dots, X^n)$  often arise as solutions to an SDE system of the form

$$(3.15) \quad dX_t^i = X_t^i \sum_{j=1}^k \varphi_{ij}(t, X_t) dW_t^j,$$

where  $W^1, \dots, W^k$  are independent Brownian motions and  $\varphi_{ij}(t, x)$ ,  $x \in \mathbb{R}_+^n$ , are continuous bounded functions. As is well known, for each  $s \leq T$  and  $x \in \mathbb{R}_+^n$ , there is a unique continuous semimartingale  $X^{s,x} = (X_t^{s,x})$  on  $[s, T]$  with  $X_s^{s,x} = x$  satisfying this SDE; moreover  $X^{s,x}$  is a positive square-integrable martingale (in fact in all  $\mathcal{H}^p$ ) since  $\varphi_{ij}(t, x)$  are bounded. Fixing an  $X_0 \in \mathbb{R}_+^n$ , the solution on  $[0, T]$  starting at  $X_0$  at time 0 is denoted  $X = X^{0, X_0}$ . The Markov property holds: for any Borel function  $g(x)$  of linear growth,

$$(3.16) \quad \mathbb{E}[g(X_T) | \mathcal{F}_t] = f(t, X_t), \quad \text{where } f(t, x) := \mathbb{E}g(X_T^{t,x}).$$

Clearly  $f(T, x) = g(x)$ . (Intuitively,  $f(t, x) = \mathbb{E}[g(X_T) | X_t = x]$ .)

Thus if we assume  $\varphi_{ij}(t, x)$  are such that  $f(t, x)$  is  $C^{1,2}$  on  $t < T$  for every bounded (hence of linear growth) Borel function  $g(x)$ , then the assumptions of Sec. 2.3 are satisfied and the conclusions hold. In particular (3.10) then holds, and since

$$d[X^i, X^j] = X^i X^j \sigma_{ij}(t, X) dt, \quad \text{where } \sigma_{ij}(t, x) := \sum_{l=1}^k \varphi_{il}(t, x) \varphi_{jl}(t, x),$$

it follows from (3.11) that, at least on the support of  $X$ ,  $f(t, x)$  satisfies the PDE

$$(3.17) \quad \frac{\partial f}{\partial t}(t, x) + \frac{1}{2} \sum_{i,j=1}^n x_i x_j \sigma_{ij}(t, x) \frac{\partial^2 f}{\partial x_i \partial x_j}(t, x) = 0.$$

In the deterministic volatility case the functions  $\varphi_{ij}$  and hence  $\sigma_{ij}$  are independent of  $x$  and simply  $X_T^{t,x} = x X_T / X_t$ , explaining why in this special case  $f(t, x)$  is also given by (3.13).

In general, if  $g(x)$  is absolutely continuous with bounded derivatives and the probability transition function of  $X$  is sufficiently regular, one shows as in the deterministic volatility case that the  $x$ -partials of  $f$  (the deltas) are bounded and thereby concludes uniqueness.

If  $\sigma_{ij}(t, x)$  are homogenous of degree 0 in  $x$ , then the uniqueness and symmetry of PDE (3.17) under dilation in  $x$  imply that  $f(t, x)$  is homogenous of degree 1 in  $x$  if  $g(x)$  is so. By Euler's formula then  $\delta_m(t, x) = 0$  in (3.7), implying  $(\delta^1, \dots, \delta^n)$  is a SFTS for  $X$ .

**3.7. Homogenous continuous Markovian SFTS.** Let  $A = (A^1, \dots, A^m)$  be a semimartingale with  $A, A_- > 0$  such that  $X^i := A^i / A^m$  are Itô processes following

$$(3.18) \quad dX_t^i = X_t^i \sum_{j=1}^k \varphi_t^{ij} (dZ_t^j + \phi^j dt), \quad (i = 1, \dots, n := m - 1)$$

where  $Z^j$  are independent Brownian motions and  $\phi^j, \varphi^{ij}$  are locally bounded predictable processes with  $\varphi^{ij}$  bounded and  $\mathbb{E} e^{\frac{1}{2} \sum_j \int_0^T (\phi_t^j)^2 dt} < \infty$ . Define the martingale

$$(3.19) \quad M := \mathcal{E} \left( - \sum_{j=1}^k \int \phi^j dZ^j \right) = e^{-\sum_{j=1}^k (\int \phi^j dZ^j + \frac{1}{2} \int (\phi^j)^2 dt)},$$

and the measure  $\mathbb{Q}$  by  $d\mathbb{Q} = M_T d\mathbb{P}$ . Then  $W^i := Z^i + \int \phi^i dt$  are  $\mathbb{Q}$ -Brownian motions and are  $\mathbb{Q}$ -independent since  $[W^k, W^l] = 0$  for  $k \neq l$ . Hence  $X^i$  are  $\mathbb{Q}$ -square-integrable martingales as  $dX^i = X^i \sum_{j=1}^k \varphi^{ij} dW^j$  and  $\varphi^{ij}$  are bounded. Thus  $A$  is arbitrage-free.

Now let  $h(a)$ ,  $a \in \mathbb{R}_+^n > 0$ , be a homogenous function of linear growth. Define  $g(x) := h(x, 1)$ ,  $x \in \mathbb{R}_+^n$ . Assume further that  $\varphi_t^{ij} = \varphi_{ij}(t, X_t)$  for some continuous bounded functions  $\varphi_{ij}(t, x)$ . Then (3.15) holds, hence Sec. 3.6 applied under measure  $\mathbb{Q}$  shows that  $X$  is  $\mathbb{Q}$ -Markovian in that  $\mathbb{E}^{\mathbb{Q}}[g(X_T) | \mathcal{F}_t] = f(t, X_t)$  as in Eq. (3.16) where  $f(t, x) = \mathbb{E}^{\mathbb{Q}} g(X_T^{t,x})$ . Thus Sec. 3.4 yields as before equations (3.10) and (3.11), and that  $\delta$  as defined in (3.8) is a SFTS for  $(X, 1)$ . Therefore by numeraire invariance  $\delta$  is a SFTS for  $A$  with price process  $C = A^m F$ . The homogeneity of  $h(a)$  further implies  $C_T = A_T^m g(X_T) = h(A_T)$ .

We have thus constructed a SFTS with the given payoff  $h(A_T)$ . As in Sec. 3.5 or 3.6 we ensure its boundedness by requiring the  $x$ -partials of  $g(x)$  or equivalently  $a$ -partials of  $h(a)$  (as  $L_{\text{loc}}^1$  functions) be bounded, and thereby get unique pricing. For (very) low dimensions  $n$ , the PDE (3.17) is suitable for numerical valuation in absence of a closed-form solution.

We can further define the homogenous option price function

$$c(t, a) := a_m f \left( t, \frac{a^1}{a^m}, \dots, \frac{a^n}{a^m} \right).$$

Then  $C_t = c(t, A_t)$ . Agreeably,  $\delta_t^i = \frac{\partial c}{\partial a_i}(t, A_t)$  by (3.7). (For  $i = m$  use Euler's formula for  $c(t, a)$ ). By the continuity of  $X$  and (3.7),  $\delta_t^i = \frac{\partial c}{\partial a_i}(t, A_{t-})$  too. Therefore by Itô's formula,

$$(3.20) \quad \frac{\partial c}{\partial t}(t, A_{t-}) dt + \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_{t-}) d[A^i, A^j]_t^c = 0.$$

(The sum of jumps term in Itô's formula drops out since  $\Delta C = \sum \delta^i \Delta A^i$ .) Using this formula or the chain rule, one can derive as in Sec. 2.6 an  $m$ -dimensional PDE for  $c(t, a)$  similar to PDE (2.28) - but the quotient-space PDE (3.17) is of one lower dimension.

**3.8. Multivariate Poisson predictable representation.** Let  $P = (P^1, \dots, P^k)$  be a vector of independent Poisson processes  $P^i$  with intensities  $\lambda_i > 0$ . For any  $C^1$  in  $t$  function  $u(t, p)$ ,  $p \in \mathbb{R}^k$ , the process  $u(t, P) = (u(t, P_t))$  is a finite activity semimartingale, and using  $[P^i, P^j] = 0$ , one has  $\Delta u(t, P) = \sum_i \Delta_i u(t, P_-) \Delta P^i$ , where

$$(3.21) \quad \Delta_i u(t, p) := u(t, p_1, \dots, p_i + 1, \dots, p_n) - u(t, p)$$

denotes the  $i$ -th forward partial difference of  $u(t, p)$  in  $p$ . This in turn readily implies

$$(3.22) \quad du(t, P) = \frac{\partial u}{\partial t}(t, P_-) dt + \sum_{i=1}^k \Delta_i u(t, P_-) dP^i.$$

Let  $v(p)$ ,  $p \in \mathbb{R}^k$ , be a function of exponential linear growth. Define the function

$$(3.23) \quad u(t, p) := \sum_{q_1, \dots, q_k=0}^{\infty} v(p + q) \prod_{i=1}^k \frac{\lambda_i^{q_i}}{q_i!} (T - t)^{q_i} e^{-\lambda_i(T-t)}. \quad (p \in \mathbb{R}^k)$$

Clearly,  $u(T, p) = v(p)$ . Since the unconditional distribution of  $P_{T-t}$  is Poisson and is the same as the distribution of  $P_T - P_t$  conditioned on  $\mathcal{F}_t$ , we have

$$u(t, p) = \mathbb{E}[v(p + P_T - P_t)] = \mathbb{E}[v(p + P_T - P_t) | \mathcal{F}_t].$$

Hence,  $u(t, P_t) = \mathbb{E}[v(P_T) | \mathcal{F}_t]$ . (Intuitively,  $u(t, p) = \mathbb{E}[v(P_T) | P_t = p]$ .) Thus the process

$$(3.24) \quad F = (F_t), \quad F_t := u(t, P_t) = \mathbb{E}[v(P_T) | \mathcal{F}_t]$$

is a martingale. But so are  $P^j - \lambda_j t$ . Therefore in view of (3.22) it follows that

$$(3.25) \quad dF = \sum_{i=1}^k \Delta_i u(t, P_-) d(P^i - \lambda_i t).$$

and  $u(t, p)$  satisfies the partial difference equation

$$(3.26) \quad \frac{\partial u}{\partial t}(t, P_{t-}) + \sum_{i=1}^k \lambda_i \Delta_i u(t, P_{t-}) = 0.$$

Since  $F_T = v(P_T)$  and  $F_0 = u(0, 0)$ , combining (3.23) and (3.25) yields the representation

$$(3.27) \quad v(P_T) = \sum_{q_1, \dots, q_k=0}^{\infty} v(q_1, \dots, q_k) \prod_{i=1}^k \frac{\lambda_i^{q_i}}{q_i!} T^{q_i} e^{-\lambda_i T} + \sum_{i=1}^k \int_0^T \Delta_i u(t, P_{t-}) d(P_t^i - \lambda_i t).$$

**3.9. Projective exponential-Poisson SFTS.** Let  $P = (P^1, \dots, P^k)$  be a vector of independent Poisson processes  $P^j$  with intensities  $\lambda_j > 0$ . Let  $X_0 \in \mathbb{R}_+^n$ ,  $n \geq k$ , and  $\beta = (\beta_{ij})$  be an  $n \times k$  matrix such that the  $n \times k$  matrix  $(e^{\beta_{ij}} - 1)$  has full rank. Then the processes  $X^i := (x_i(t, P_t))$ ,  $i = 1, \dots, n$ , are square-integrable martingales (in fact in all  $\mathcal{H}^p$ ), where

$$(3.28) \quad x_i(t, p) := X_0^i \exp\left(\sum_{j=1}^k (\beta_{ij} p_j - (e^{\beta_{ij}} - 1) \lambda_j t)\right). \quad (p \in \mathbb{R}^k)$$

Since

$$\frac{\partial x_i}{\partial t}(t, p) = -x_i(t, p) \sum_{j=1}^k (e^{\beta_{ij}} - 1) \lambda_j, \quad \Delta_j x_i(t, p) = x_i(t, p) (e^{\beta_{ij}} - 1),$$

it follows from (3.22) (or easily also from Itô's formula) that

$$(3.29) \quad dX^i = X_-^i \sum_{j=1}^k (e^{\beta_{ij}} - 1) d(P^j - \lambda_j t). \quad (X_t^i := x_i(t, P_t))$$

Let  $\alpha = (\alpha_{ij})$  be any  $n \times k$  matrix such that  $\sum_i (e^{\beta_{il}} - 1)\alpha_{ij} = \delta_{jl}$ , all  $1 \leq j, l \leq k$ . Then

$$(3.30) \quad d(P^j - \lambda_j t) = \sum_{i=1}^n \alpha_{ij} \frac{dX^i}{X^i}.$$

Now let  $g(x)$ ,  $x \in \mathbb{R}_+^n$ , be a function of linear growth, define the function

$$v(p) := g(x_1(T, p), \dots, x_n(T, p)), \quad (p \in \mathbb{R}^n)$$

and the function  $u(t, p)$  by (3.23). By Sec. 3.8,  $F := (u(t, P_t))$  is a martingale with  $F_T = v(P_T) = g(X_T)$  and is represented as (3.25). Substituting (3.30) into (3.25) yields

$$(3.31) \quad dF = \sum_{i=1}^n \delta^i dX^i,$$

where

$$(3.32) \quad \delta_t^i := \frac{1}{X_{t-}^i} \sum_{j=1}^k \alpha_{ij} \Delta_j u(t, P_{t-}).$$

Thus,  $\delta = (\delta^1, \dots, \delta^m)$  is a SFTS for  $(X, 1)$  where  $m := n + 1$  and  $\delta^m := F - \sum_{i=1}^n \delta^i X^i$ .

It is more desirable to express  $\delta$  in term of  $X$ . One has  $u(t, p) = f(t, x(t, p))$ , where

$$(3.33) \quad f(t, x) := \mathbb{E}[g(x \frac{X_T}{X_t})] = \mathbb{E}[g(x \frac{X_T}{X_t}) | \mathcal{F}_t] = \sum_{q_1, \dots, q_n=0}^{\infty} g(x_1 e^{\sum_{j=1}^n (\beta_{1j} q_j - (e^{\beta_{1j}} - 1)\lambda_j(T-t))}, \dots, x_n e^{\sum_{j=1}^n (\beta_{nj} q_j - (e^{\beta_{nj}} - 1)\lambda_j(T-t))}) \prod_{i=1}^n \frac{\lambda_i^{q_i}}{q_i!} (T-t)^{q_i} e^{-\lambda_i(T-t)}.$$

The equalities follows from the definition of  $v(p)$  above and of  $u(t, p)$  in (3.23) together with the two formulae following it. We clearly have  $f(T, x) = g(x)$  and

$$(3.34) \quad F_t := u(t, P_t) = f(t, X_t) = \mathbb{E}[g(X_T) | \mathcal{F}_t].$$

Since  $u(t, p) = f(t, x(t, p))$ , the deltas in (3.32) are given by partial differences of  $f(t, x)$  as

$$(3.35) \quad \delta_t^i = \delta_i(t, X_{t-}), \quad \text{where } \delta_i(t, x) := \frac{1}{x_i} \sum_{j=1}^k \alpha_{ij} (f(t, e^{\beta_{1j}} x_1, \dots, e^{\beta_{nj}} x_n) - f(t, x)).$$

We have unique pricing since  $(X, 1)$  is arbitrage-free (as  $X^i$  are martingales). Specifically, if  $\hat{\delta}$  is another SFTS for  $(X, 1)$  with payoff  $\hat{F}_T = g(X_T)$ , then  $\hat{F} := \sum_{i=1}^n \hat{\delta}^i X^i + \hat{\delta}^m = F$  provided that either all  $\hat{\delta}^i$ ,  $i \leq n$  are bounded or all  $\hat{\delta}^i - \delta^i$ ,  $i \leq n$  are bounded.

Indeed, then  $\hat{F} = \hat{F}_0 + \hat{\delta}' \cdot X$  is a martingale since  $X$  is a square-integrable integrable (in the second case, use also that  $F$  is a martingale). Hence  $\hat{F} = F$  as  $\hat{F}_T = F_T$ .

Moreover, if  $k = n$  we have unique hedging, i.e.,  $\hat{\delta} = \delta$  for any bounded SFTS  $\hat{\delta}$  for  $(X, 1)$  with payoff  $\hat{F}_T = g(X_T)$ . Indeed,  $\hat{F} = F$ , as before; thus, setting  $\theta^i := \hat{\delta}^i - \delta^i$ , we

have

$$0 = d\langle \hat{F} - F \rangle = \sum_{i,j=1}^n \theta^i \theta^j d\langle X^i, X^j \rangle = \sum_{i,j=1}^n \theta^i \theta^j X_-^i X_-^j \sum_{l=1}^n (e^{\beta_{il}} - 1)(e^{\beta_{jl}} - 1) \lambda_l dt,$$

the last equality following by (3.29). But the  $n \times n$  matrix  $(\sum_{l=1}^n (e^{\beta_{il}} - 1)(e^{\beta_{jl}} - 1) \lambda_l)_{i,j=1}^n$  is nonsingular. Therefore  $\theta^i = 0$ , i.e.,  $\hat{\delta}^i = \delta^i$  for  $i \leq n$ , implying  $\hat{\delta}^m = \delta^m$  too as  $\hat{F} = F$ .

One shows as in Sec. 2.4 that the processes  $\delta^i$  are bounded if  $\gamma_i(x)$  are bounded, where

$$(3.36) \quad \gamma_i(x) := \frac{1}{x_i} \sum_{j=1}^k \alpha_{ij} (g(e^{\beta_{1j}} x_1, \dots, e^{\beta_{nj}} x_n) - g(x)), \quad \gamma_m(x) := g(x) - \sum_{i=1}^n \gamma_i(x) x_i.$$

**3.10. Homogenous exponential Poisson SFTS.** Let  $A > 0$  be an  $m$ -dimensional semimartingale with  $A_- > 0$  and set  $X := (A^i/A^m)_{i=1}^n$ ,  $n := m - 1$  as before. Assume that

$$(3.37) \quad dX_t^i = X_{t-}^i \sum_{j=1}^k (e^{\beta_{ij}} - 1) (dP_t^j - \lambda_t^j dt),$$

where  $1 \leq k \leq n$ ,  $\beta_{ij}$  are constants with the  $n \times k$  matrix  $(e^{\beta_{ij}} - 1)$  has full rank,  $\lambda^j > 0$  are bounded predictable processes, and  $P^j$  are semimartingales with  $[P^j, P^l] = 0$  for  $j \neq l$  such that  $[P^j] = P^j$ ,  $P_0^j = 0$ , and  $P^j - \int \kappa^j dt$  are local martingales for some locally bounded predictable processes  $\kappa^j > 0$ . Assume further that  $\mathbb{E} \exp(\sum_{j=1}^k \int_0^T (\frac{\lambda_t^j}{\kappa_t^j} - 1)^2 \kappa_t^j dt) < \infty$ .

Due to the above growth condition, the positive local martingale

$$M := \mathcal{E} \left( \sum_{j=1}^k \int (\frac{\lambda^j}{\kappa^j} - 1) (dP^j - \kappa^j dt) \right) = e^{-\sum_{j=1}^k \int (\lambda^j - \kappa^j) dt} \prod_{s \leq \cdot} \left( 1 + \sum_{j=1}^k (\frac{\lambda_s^j}{\kappa_s^j} - 1) \Delta P_s^j \right)$$

is a martingale. Define the measure  $\mathbb{Q}$  by  $d\mathbb{Q} = M_T d\mathbb{P}$ . As in Sec. 2.10,  $\int \lambda^j dt$  are the  $\mathbb{Q}$ -compensator of  $P^j$ . This, (3.37), and boundedness of  $\lambda^j$  imply that  $X^i$  are  $\mathbb{Q}$ -square integrable martingales. Thus  $A$  is arbitrage free. As before, the SDE (3.37) integrates to

$$(3.38) \quad X_t^i = X_0^i e^{\sum_{j=1}^k \beta_{ij} P_t^j - (e^{\beta_{ij}} - 1) \int_0^t \lambda_s^j ds}.$$

Now assume  $\lambda^j$  are constant. Then  $P^j$  are  $\mathbb{Q}$ -Poisson processes with intensities  $\lambda^j$  and are independent since  $[P^j, P^l] = 0$ ,  $j \neq l$ . Sec. 3.9 applied under  $\mathbb{Q}$  implies that  $\delta$  given by (3.35) (with  $\delta^m = F - \sum_{i=1}^n \delta^i X^i$ ) is a SFTS for  $(X, 1)$  with price process  $F = (f(t, X_t))$  satisfying  $F_T = g(X_T)$ , where  $f(t, x)$  is defined explicitly by the long equation in (3.33), or equivalently,  $f(t, x) = \mathbb{E}^{\mathbb{Q}} g(x X_T / X_t)$ . Therefore, by numeraire invariance  $\delta$  is a SFTS for  $A$  with price process  $C := A^m F$  satisfying  $C_T = A^m g(X_T) = h(A_T)$  by homogeneity.

Assume finally that the payoff function  $h(a)$  is such that the functions  $\gamma_i(x)$  defined in (3.36) are bounded (e.g.,  $h(a) = \max(a^1, \dots, a^m)$ ). By Sec. 3.9, if  $k = n$  then  $\delta$  is the unique bounded SFTS for  $A$  with payoff  $C_T = h(A_T)$ . In general, since  $A$  is arbitrage free,  $\hat{C} = C$  for any other bounded SFTS  $\hat{\delta}$  for  $A$  with payoff  $\hat{C}_T = h(A_T)$ , where  $\hat{C} := \sum_i \hat{\delta}^i A^i$ .

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