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# High Level of International Risk sharing When the Productivity Growth Contains Long run Risk

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## Abstract

This theoretical paper investigates international risk sharing and its implications for equity home bias. A general equilibrium model, featuring two closed economies with nontrivial production sectors, is developed. Moreover, productivity contains a small but persistent highly correlated long run risk that becomes the major determinant of the intertemporal marginal rate of substitution (IMRS) in a model with the recursive preferences. Despite adopting the model of closed economies and autarkic asset holdings—a scenario leading to the lowest level of international risk sharing under the same conditions—our model is still able to generate international risk sharing indexes always over 96% for a broad range of parameter values, excepting two cases: where the elasticity of intertemporal substitution (EIS) is the reciprocal of the relative risk aversion (RRA); and where EIS is around 0.7. In those cases, the risk sharing index drops sharply to about 30%. This result sheds light on why the benchmark model, featuring a power utility whereby EIS is the reciprocal of RRA, generates international risk sharing as low as 30%. However, when EIS takes these values, our model's results cannot be reconciled with asset market data-model yields low volatility of the logarithms of IMRS, even lower than Hansen-Jagannathan lower bound.

The implication is that the low proportion of foreign assets in a domestic agent's portfolio, a phenomenon observed in the data, might not be a puzzle or a departure from the agent's optimality condition. After all, risk has already been well shared internationally due to the high correlations across countries of the long run productivity shocks. Hence, there is not much incentive left for an agent to hold foreign assets in her portfolio to further share the risk internationally. Therefore, equity home bias might not be a puzzle as claimed by the benchmark model, in the sense that it can be adequately reconciled with our theoretical results.

## 1. Introduction

This is a theoretical paper regarding international risk sharing and its implications for the equity home bias puzzle. Our model is a general equilibrium model featuring two closed economies with nontrivial production sectors. As a result of our closed economy setup, asset holding is autarkic in each country. Our model also adopts the Epstein-Zin-Weil recursive utility function instead of the power utility commonly used in the benchmark model. Furthermore, the productivity process contains a small but persistent long run risk and a large short run risk. The international correlation of long run risk is high -while that of the short run risk is low in order to match the correlation across countries of the overall productivity in the data.

The small but persistent long run risk becomes the major determinant of the intertemporal marginal rate of substitution (IMRS) in a model featuring the Epstein-Zin-Weil preferences. Even though the model deals with closed economies and autarkic asset holding—a scenario leading to the lowest level of international risk sharing under the same condition—our model is still able to generate international risk sharing indexes always over 96% for a broad range of parameter values, excepting two cases: where the elasticity of intertemporal substitution (EIS) is the reciprocal of the relative risk aversion (RRA); and where EIS is around 0.7. In those cases, the risk sharing index drops sharply to about 30%. This result explains why the benchmark model, with a power utility whereby EIS is the reciprocal of RRA, generates an international risk sharing index as low as 30%. Our paper shows that the

international risk sharing generated from the benchmark model is not a general result. Rather, it is a special one arising from the use of  $EIS=1/RRA$ . Generally, in cases other than that, the model generates a much higher degree of international risk sharing.

The implication of our results for the equity home bias puzzle is that the low proportion of foreign assets held in a domestic agent's portfolio, a phenomenon observed in the data, might not be a puzzle or a departure from the agent's optimality condition. After all, risk has already been well shared internationally due to the high correlations across countries of the long run productivity shocks. Hence, there is not much incentive left for the agent to hold foreign assets in order to further share her risk with foreigners. Therefore, the phenomenon of equity home bias might not be a puzzle as claimed by the benchmark model, in the sense that it can be well reconciled with our theoretical results.

The model featuring the aforementioned recursive preferences and an exogenous stochastic process with long run risk has the potential to solve other puzzles arising from applying the benchmark model in international economics and financial economics. For instance, the benchmark model generates uncovered interest parity (UIP), while data exhibit the forward premium puzzle [see Fama (1984); Backus, Foresi and Telmer (2001)]. The Backus-Smith Puzzle is another example; the volatility of the exchange rate produced in the benchmark model is much lower than that of the data [see Backus, Smith (1993)]. Third, Real Business Cycle (RBC) models have

difficulty in explaining asset prices, a phenomenon that has been dubbed the equity premium puzzle. Using reasonable parameter values, the benchmark model generates an equity premium that is much smaller and less volatile than that found in the actual data from the asset market [see Mehra, Prescott (1988)]. Despite its success in simulating quantity variables such as output, consumption, and investment, the general equilibrium RBC model is notorious for its unsatisfactory record in explaining asset prices such as equity return and exchange rates.

Since the correlations of consumption growth across countries are low, the benchmark models generate low levels of international risk sharing after applying quantity data. However, the complete asset market calls for perfect international risk sharing, which is achieved when agents across countries are holding global portfolios that are identical in composition.

We know that models will derive an intertemporal Euler equation after solving the consumer's intertemporal choice problem. In a complete market, this first-order condition implies that IMRS, also known as the stochastic discount factor (SDF), should be equalized either across agents in a closed economy or across countries in an open economy. In the benchmark model, the equality of IMRS translates into having the same consumption growth rates across countries, which in turn requires that agents across countries are holding global portfolios that are identical in composition. In reality, people hold the majority of their respective portfolios in domestic assets and only a small portion in foreign assets. The problem in reconciling the above

theoretical results with the real-world data has been dubbed the equity home bias puzzle.

Brandt, Cochrane, and Santa-Clara (2006) generated a much higher degree of international risk sharing by using asset price data instead of quantity data (such as consumption data). They maintained that as long as market prices reflect the agent's IMRS, their result of high international risk sharing holds true. Our paper is an attempt to reconcile the results of international risk sharing from both price data and quantity data; that is, we aim to generate high international risk sharing after applying quantity data.

A close look at the benchmark model reveals two features. One is the power utility, which keeps EIS equal to the reciprocal of RRA. Moreover, the benchmark model assumes that the exogenous stochastic process—be it consumption growth in an endowment economy or productivity growth in a production economy—is exposed only to *i.i.d.* shocks.

The limitation of benchmark model featuring those two properties points to a potential unified way of solving the aforementioned puzzles. The two aforementioned features might be the culprits that cause the benchmark model to generate anomalistic results. Hence, the literature modifies the benchmark model along two lines: first, replacing the power utility with the Epstein-Zin-Weil recursive preferences; and, second, by assuming an exogenous stochastic process containing long run risk instead of being exposed only to *i.i.d.* shocks. So far, the modified model has had some

success in solving the equity premium puzzle [see Bansal, Yaron (2004)], the forward premium puzzle [see Bansal, Shaliastovich (2006) Backus, Foresi and Telmer (2001)], and the Backus-Smith puzzle [see Brandt, Cochrane and Santa-Clara (2006); Colacito, Croce (2005)].

Are those two modifications justified and appropriate, or are they arbitrary changes that have been made only to fit into a theoretical exercise? First, we know that preference is unobservable. There is no clear reason for favouring a power utility against other reasonable utility forms. Second, Bansal and Yaron (2004) showed that when using consumption data only, and without relying on price data, it is hard to differentiate between the two hypotheses—first, that consumption is a random walk process, or, second, that it contains long run growth risk. To put it another way, the consumption data does not reject long run risk hypothesis in favour of a random walk hypothesis.

Based on the above justification, in this paper, we will adopt those two modifications to the benchmark model, with the goal being to shed light on international risk sharing and examine the implications for the equity home bias puzzle.

What is the intuition behind the claim that the modified model might have the potential to yield high international risk sharing after applying quantity data? Similar to the benchmark model, the modified one derives an intertemporal Euler equation as consumer's first-order condition, which

implies, in a complete market, an identical IMRS across countries. However, in the modified model the quantity implication of the first-order condition is quite different from that of the benchmark model. Thanks to the recursive preferences, equal IMRS across countries does not necessarily translate into the same consumption growth rates. Therefore, the modified model does not imply that agents across countries are holding global portfolios of identical composition—the root of the home bias puzzle.

In a model with the recursive preferences, IMRS depends on both the consumption growth rate and the return on a hypothetical asset that pays a country's aggregate consumption as its dividend. After calibrating parameters with reasonable values, IMRS in the model can be mainly determined by its second item, the return on total wealth.

Given that the productivity growth process contains long run risk—even though it is small compared with transitory risk—when long run risk is very persistent, the return on wealth is quite sensitive to it. We know that asset prices reflect not only present conditions but also the expectation of future conditions. An innovation in long run risk changes both conditions. Therefore, asset prices could be quite sensitive to long run risk. As a result, a model with small but persistent long run risk could generate a high and volatile equity premium. In this sense, long run risk literature is credited with having the potential to solve the equity premium puzzle.

After extending the long run risk literature from closed to open economies, and further assuming that long run risk has a common global origin, the model has the ability to yield high correlation between IMRS across countries. A highly correlated IMRS across countries in turn could deliver a high level of international risk sharing<sup>1</sup>.

To restore the RBC model's good record in simulating quantity variables, we can adjust the magnitude of transitory risk to make sure the model's quantity implications, such as the properties of production and consumption, are also in line with the data. The international business cycle data show that quantity variables have low volatilities and poor correlations across countries, which means the transitory risk in the model needs to be large and less correlated across countries to match the data.

In summary, in a model featuring recursive preferences and a stochastic productivity process with long run risk, if such risk comes from a common global origin, then IMRS can be highly correlated across countries, which could yield high level of international risk sharing. Thus, our model serves as medium of reconciliation between the results of international risk sharing from quantity data and from price data. Furthermore, home bias is no longer a phenomenon that cannot be reconciled with a general equilibrium model.

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1 The degree of international risk sharing is, however, not the same concept as IMRS correlation. See more detail and an index of international risk sharing in Brandt, Cochrane, and Santa-Clara (2006).

An agent can keep most of her portfolio in domestic assets and only a small portion in foreign assets, which is a scenario in line with the data, and the model can still achieve high degree of international risk sharing. It thus appears that portfolio decisions are irrelevant to the first-order condition.

Section two will review the related literature and its relevance to the problem at hand, while section three will present the specifics of our model and its log-linear approximating solution. Section four then serves to display our model's results after parameter calibration. Finally, section five will conclude our discussion.

## **2. Review of the Literature**

This paper contributes to the literature on both international risk sharing and on long run risk. Our model is an extension of long run risk model from the endowment economy to the production economy. Using a general equilibrium production model along with long run risk, our paper aims to shed light on international risk sharing. The most related previous work in this area is that of Bansal and Yaron (2004); Brandt, Cochrane, and Santa-Clara (2006); Colacito and Croce (2005); and Croce (2006).

Bansal and Yaron (2004) was the pioneering paper in the growing literature on the asset pricing field that goes under the name of "risks for the long run". Their model was based on an endowment economy featuring recursive preferences and stochastic consumption growth process with long

run risk. The model has exhibited some success in solving the equity premium puzzle.

Brandt, Cochrane, and Santa-Clara (2006) was the first paper to point out that international risk sharing is actually high after applying price data instead of applying quantity data, with the latter being a standard practice in the literature. However, since their paper lacks a theoretical model, it provides no formal explanation as to why the two approaches cannot be reconciled, nor does it explore how one might reconcile them in a single model. They concluded that a “surprisingly high level of risk sharing” holds true as long as two conditions are met. The first condition requires that asset prices reflect IMRS, and the second calls for either a complete asset market or an incomplete one with a reasonably sized uninsurable risk. Alternatively, if risks really are poorly shared, Brandt, Cochrane, and Santa-Clara (2006) maintained that exchange rates in the data are much too smooth as compared to the prediction of the model.

Colacito and Croce (2005) attempted to provide Brandt, Cochrane, and Santa-Clara (2006) with a rigorous theoretical foundation. Colacito and Croce felt that the pioneering work regarding long run risk in Bansal and Yaron (2004) might have the potential to fully achieve their goal of generating high international risk sharing from quantity data. Trying to fill a gap left by Brandt, Cochrane, and Santa-Clara (2006), Colacito and Croce (2005) set out to reconcile the results of international risk sharing from quantity data with the results from asset price data. After extending Bansal and Yaron’s (2004)

closed economy endowment model to a two-country endowment model, Colacito and Croce (2005) showed that as long as long run risks are highly corrected across two countries, the model succeeds in generating high international risk sharing from quantity data.

Croce (2006) further contributed to the long run risk literature by extending it from an endowment economy to a general equilibrium with nontrivial production sector. However, he focused on the issue of welfare cost, rather than that of international risk sharing.

Following Croce (2006), our model also features the general equilibrium production economy with long run risk. Our focus, however, is on studying the issue of international risk sharing. Similar to Colacito and Croce (2005), we attempt to provide a rigorous theoretical foundation to Brandt, Cochrane, and Santa-Clara (2006). Nevertheless, our model is a general equilibrium production model containing long run risk, whereas Colacito and Croce (2005)'s model was based on the endowment economies with long run risk.

### **3. Model**

We construct a general equilibrium model with nontrivial production sectors. There are two countries in the model, denoted respectively as home country (h) and foreign country (f). We study the degree of international risk sharing generated by the model when each country runs a closed economy and

agents' asset holdings are autarkic—a scenario leading to the lowest level of international risk sharing under the same condition.

We further assume, in each country, homogeneity among consumers and constant returns to scale in production. As a result, the model can be set up with a representative consumer and a representative firm in each country. To keep the model simple and focus on our central issue, we assume that the representative agent in each country lives infinitely and her labour supply is fixed. Moreover, we assume that there is a single good in the world economy. The good is produced in each country by its firm in a competitive environment. Since each country runs closed economy without exchanging good with another, the agent in each country derives utility solely from consuming the good produced in her country.

### 3.1. Preferences

Although the use of the power utility is standard practice in benchmark models, we abandon the use of the power utility as preference. Instead, we adopt the recursive preferences named after Epstein and Zin (1989, 1991) and Weil (1989). The main feature of those Epstein-Zin-Weil preferences is their disentangling of the elasticity of intertemporal substitution (EIS), denoted  $\psi$ , from the coefficient of the relative risk aversion (RRA), denoted  $\gamma$ . However, with a power utility, EIS always equals the reciprocal of RRA; that is,  $\psi = 1/\gamma$ . Yet, it is not clear that these two concepts should be linked so tightly. As Campbell (2003) stated: “[R]isk aversion describes the

consumer's reluctance to substitute consumption across states of the world and is meaningful even in an atemporal setting, whereas the elasticity of intertemporal substitution describes the consumer's willingness to substitute consumption over time and is meaningful even in a deterministic setting" (page 828).

The representative agents maximize the objective function, which takes the Epstein-Zin-Weil preferences as its utility form:

$$U_t^i = \left\{ (1-\beta) C_t^i \frac{1-\gamma}{\theta} + \beta \left( E_t U_{t+1}^i \right)^{\frac{1-\gamma}{\theta}} \right\}^{\frac{\theta}{1-\gamma}} \quad (1)$$

$$\forall i \in \{h, f\}$$

where  $\theta = (1-\gamma) / \left(1 - \frac{1}{\psi}\right)$  implicitly defines EIS ( $\psi$ ). When  $\gamma = 1/\psi$ , we have that  $\theta = 1$  and Equation (1) reduces to a power utility.  $U_t^i$  denotes the utility of the agent of the  $i$ th country at time  $t$ ,  $C_t^i$  is her consumption at time  $t$ , and  $\beta$  denotes the subjective discount factor, also known as the time-preference factor.

Using dynamic programming, Epstein and Zin (1989) showed that after solving the consumer's optimal consumption problem, the optimality condition—the intertemporal Euler equation—takes the form

$$E_t \left[ \beta^\theta \left( \frac{C_{t+1}^i}{C_t^i} \right)^{\frac{\theta}{\psi}} \left( R_{c,t+1}^i \right)^{\theta-1} R_{j,t+1} \right] = 1, \quad (2)$$

where  $R_{c,t+1}^i$  is the gross return on a hypothetical asset between time  $t$  and  $t+1$  which pays the  $i$ th country's aggregate consumption as its dividends in each period.  $R_{j,t+1}$  is the gross return on the  $j$ th asset between time  $t$  and  $t+1$ .

Equation (2) contains an important concept in asset pricing literature—IMRS, also known as SDF or the pricing kernel. We display it separately below:

$$M_t^i = \beta^\theta \left( \frac{C_{t+1}^i}{C_t^i} \right)^{\frac{\theta}{\psi}} \left( R_{c,t+1}^i \right)^{\theta-1} \quad (3)$$

In a benchmark model featuring the power utility, the pricing kernel  $M$  takes the form

$$M_t^i = \beta \left( \frac{C_{t+1}^i}{C_t^i} \right)^{-\gamma} \quad (4)$$

In a complete market, the pricing kernel  $M$  is unique across countries, which implies that the following equation holds in a benchmark model:

$$M_t^h = \beta \left( \frac{C_{t+1}^h}{C_t^h} \right)^{-\gamma} = \beta \left( \frac{C_{t+1}^f}{C_t^f} \right)^{-\gamma} = M_t^f \quad (5)$$

Equation (5) implicitly assumes that agents in home and foreign country have an identical RRA ( $\gamma$ ) and an identical time preference factor ( $\beta$ ). In the benchmark model, a unique pricing kernel requires that home and foreign country achieve equal consumption growth rates, which in turn necessitates that agents across countries are holding global portfolios that are identical in composition. This result from the benchmark theoretical model is in dramatic contrast to the actual data. The data shows that investors hold a majority of their portfolios in domestic assets and only a small portion in foreign assets; thus, in practice, the compositions of the global equity portfolios held by domestic and foreign investors are far from identical. As a result, the equity home bias puzzle arises.

Equation (3) is the pricing kernel  $M$  from the Epstein-Zin-Weil preferences. In a complete market, the presence of a unique pricing kernel implies that the following equation (6) holds true. Again, we assume that agents across countries share the same parameters,  $\gamma$ ,  $\beta$ , and  $\psi$ .

$$M_t^h = \beta^\theta \left( \frac{C_{t+1}^h}{C_t^h} \right)^{\frac{\theta}{\psi}} \left( R_{c,t+1}^h \right)^{\theta-1} = \beta^\theta \left( \frac{C_{t+1}^f}{C_t^f} \right)^{\frac{\theta}{\psi}} \left( R_{c,t+1}^f \right)^{\theta-1} = M_t^f \quad (6)$$

Equation (6) shows that with the recursive preferences, a unique pricing kernel does not necessarily imply equal consumption growth rates across

countries, and, in turn, does not necessarily translate to identically composed global equity portfolios held by agents in the two countries. As a result, it appears that equity home bias might not necessarily be a phenomenon that stands against the first-order condition derived from theoretical models.

### 3.2. Production

In this section, we work on the stochastic productivity growth process to introduce another line of modification to the benchmark models. In the benchmark models, the law of motion of productivity growth is often assumed being a stochastic process exposed to *i.i.d.* shocks alone. In this paper, the productivity growth process is exposed to *i.i.d.* shocks as well as long run shocks. We specify the law of motion of productivity growth containing long run risk as follows:

$$\begin{aligned}\Delta z_{t+1}^i &= \mu + x_t^i + w_{t+1}^i \\ x_t^i &= \rho x_{t-1}^i + \varepsilon_{x,t}^i \\ \forall i &\in \{h, f\},\end{aligned}\quad (7)$$

where  $z_{t+1}^i = \log Z_{t+1}^i$  and  $Z_{t+1}^i$  denote the *ith* country's total factor productivity at time  $t$ , and  $\mu$  is the logarithm of the steady state productivity level.  $w_{t+1}^i$ , which denotes the short-run component of productivity growth risk, is an *i.i.d.* random variable.  $x_t^i$  is the long-run component of productivity growth risk, with  $\rho$  measuring its persistence and  $\varepsilon_{x,t}^i$  its contemporaneous *i.i.d.* shock.

We specify the variance-covariance matrix of productivity growth shocks as follows:

$$\begin{aligned}
 & \left[ w_{t+1}^h, w_{t+1}^f, \varepsilon_{x,t+1}^h, \varepsilon_{x,t+1}^f \right] \sim N(0; \Sigma) \\
 & \Sigma = \begin{pmatrix} \sigma_w^2 \Gamma_w & 0 \\ 0 & \sigma_x^2 \Gamma_x \end{pmatrix} \\
 & \Gamma_w = \begin{pmatrix} 1 & \rho_w^{hf} \\ \rho_w^{hf} & 1 \end{pmatrix} \\
 & \Gamma_x = \begin{pmatrix} 1 & \rho_x^{hf} \\ \rho_x^{hf} & 1 \end{pmatrix},
 \end{aligned} \tag{8}$$

where  $\sigma_x^2$  denotes the variance of the long run productivity growth shocks in each country while  $\sigma_w^2$  denotes the short-run variance.  $\rho_x^{hf}$  is the correlation of the long run shocks across countries, whereas  $\rho_w^{hf}$  is the corresponding correlation of the short run risks. We assume the cross correlations between short run and long run risks, either within a country or across countries, are both zero.

A high  $\rho_x^{hf}$  and a high  $\rho$  are two critical factors that allow our model to generate high degree of international risk sharing. The reason for this is that, in a model incorporating the Epstein-Zin-Weil preferences, if long run risk is very persistent, it becomes the major determinant of IMRS. Furthermore, if long run risk is highly correlated across countries, IMRS will also be highly correlated across countries. The high correlation between IMRS across

countries could further translate to a claim that international risk sharing is high.

The rest of our model is set up as follows. In a general equilibrium production model, the firm in each country maximizes its present value to owners,<sup>2</sup> subject to the capital stock law of motion and the stochastic productivity growth process. The firm pays its worker the competitive wage rate, which is equal to the marginal product of labour. The firm then pays its shareholder dividends. We assume a Cobb-Douglas production function

$$Y_t^i = (Z_t^i N_t^i)^{1-\alpha} (K_t^i)^\alpha = (Z_t^i)^{1-\alpha} (K_t^i)^\alpha, \quad (9)$$

where  $Y_t^i$  denotes the  $i$ th country's output at time  $t$ , and  $N_t^i$  denotes its firm's labor demand for period  $t$ . At equilibrium, the firm's labour demand equals the worker's labour supply, which we normalized to unity.  $K_t^i$  is the capital stock owned by the firm of the  $i$ th country at the beginning of period  $t$ ,  $\alpha$  denotes capital's share, and  $1-\alpha$  denotes labour's share. Capital stock is chosen one period before it becomes productive, and labour can be adjusted instantaneously.

Adding production to a model makes its asset price implications even worse, since the agent can now smooth her consumption even better with

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<sup>2</sup> The firm's present value to owners is the sum total of all its current and future expected dividends discounted by a market SDF deemed valid for every owner.

production technology than in an endowment economy. To deal with this problem, a standard practice in the production-based asset pricing literature is to impose the adjustment cost in firm's investment. Following the literature, we adopt the adjustment cost in our model. Hence, the firm's capital stock evolves according to the following law of motion:

$$K_{t+1}^i = \Psi\left(\frac{I_t^i}{K_t^i}\right) K_t^i + (1-\delta) K_t^i, \quad (10)$$

where  $K_{t+1}^i$  denotes the capital stock of the firm of the  $i$ th country at the beginning of period  $t+1$ ,  $I_t^i$  is the investment made by the firm during period  $t$ ,  $\delta$  denotes the depreciation rate, and  $\Psi$  is a function form of adjustment cost, which is increasing and concave in investment  $I$  ( $\Psi > 0, \Psi' > 0, \Psi'' < 0$ ). This setting reflects the idea that changing the capital stock rapidly is more costly than changing it slowly.

The firm maximizes its value to shareholder subject to the production function (9), the law of motion of the capital stock (10), and the stochastic process of productivity growth with long run risk (7). The first-order conditions for maximizing the firm's value are:

$$\begin{aligned}
q_s &= \frac{1}{\Psi' \left( \frac{I_s}{K_s} \right)} \\
W_s &= (1-\alpha) Z_s^{1-\alpha} K_s^\alpha \\
R_{s+1} &= \left[ \alpha Z_{s+1}^{1-\alpha} K_{s+1}^{\alpha-1} + \frac{(1-\delta) + \Psi \left( \frac{I_{s+1}}{K_{s+1}} \right)}{\Psi' \left( \frac{I_{s+1}}{K_{s+1}} \right)} - \frac{I_{s+1}}{K_{s+1}} \right] \Psi' \left( \frac{I_s}{K_s} \right)
\end{aligned} \tag{11}$$

### 3.3. Log-linear Approximating Solution

After log-linearizing the system around its steady state, which contains a constant growth rate  $g$ , we get the following approximating solution, including a solution to consumption growth and IMRS (See Appendix 1 for details):

$$\begin{aligned}
c_{t+1} - c_t &= -v_{kx} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} x_{t+1} + \left[ (1-\alpha) \frac{\bar{Y}}{\bar{C}} - \frac{\bar{I}}{\bar{C}} \frac{1+g}{g+\delta} v_{kw} \right] w_{t+1} \\
&+ \left\{ (1-\alpha) \frac{\bar{Y}}{\bar{C}} + v_{kx} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{2+g-\delta}{g+\delta} - \frac{\bar{I}}{\bar{C}} \frac{1+g}{g+\delta} v_{kk} \right] \right\} x_t \\
&+ v_{kw} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{2+g-\delta}{g+\delta} - \frac{\bar{I}}{\bar{C}} \frac{1+g}{g+\delta} v_{kk} \right] w_t \\
&+ \left\{ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{1-\delta}{g+\delta} + v_{kk} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{2+g-\delta}{g+\delta} - \frac{\bar{I}}{\bar{C}} \frac{1+g}{g+\delta} v_{kk} \right] \right\} k_t \\
&= -v_{kx} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} x_{t+1} + \left[ (1-\alpha) \frac{\bar{Y}}{\bar{C}} - \frac{\bar{I}}{\bar{C}} \frac{1+g}{g+\delta} v_{kw} \right] w_{t+1} \\
&+ \left\{ (1-\alpha) \frac{\bar{Y}}{\bar{C}} + v_{kx} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{(1-\delta) + (1-\rho)(1+g)}{g+\delta} - \frac{\bar{I}}{\bar{C}} \frac{1+g}{g+\delta} v_{kk} \right] \right\} x_t \\
&+ v_{kw} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{2+g-\delta}{g+\delta} - \frac{\bar{I}}{\bar{C}} \frac{1+g}{g+\delta} v_{kk} \right] w_t \\
&+ \left\{ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{1-\delta}{g+\delta} + v_{kk} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{2+g-\delta}{g+\delta} - \frac{\bar{I}}{\bar{C}} \frac{1+g}{g+\delta} v_{kk} \right] \right\} k_t
\end{aligned} \tag{12}$$

$$\begin{aligned}
m_{t+1} &= -\frac{\theta}{\psi}(c_{t+1} - c_t) + (\theta - 1)r_{a,t+1} \\
&= \eta_{m,\varepsilon_x} \varepsilon_{x,t+1} + \eta_{m,w} w_{t+1} + E_t(m_{t+1}) \\
&= \gamma v_{kx} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \varepsilon_{x,t+1} \\
&\quad + \frac{n_1 \left( \frac{1}{\psi} - \gamma \right)}{(1-\rho n_1)} \left\{ (1-\alpha) \frac{\bar{Y}}{\bar{C}} + \frac{n_1 v_{kx}}{(1-n_1 v_{kk})} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{1-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} + v_{kk} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2+g-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right) \right] \right\} \varepsilon_{x,t+1} \\
&\quad + \gamma \left[ v_{kw} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} - (1-\alpha) \frac{\bar{Y}}{\bar{C}} \right] w_{t+1} \\
&\quad + n_1 \left( \frac{1}{\psi} - \gamma \right) v_{kw} \left\{ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2+g-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right. \\
&\quad \left. + \frac{n_1}{(1-n_1 v_{kk})} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{1-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} + v_{kk} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2+g-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right) \right] \right\} w_{t+1} \\
&\quad - \frac{1}{\psi} \left\{ (1-\alpha) \frac{\bar{Y}}{\bar{C}} + v_{kx} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{(1-\delta) + (1-\rho)(1+g)}{g+\delta} - v_{kk} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right] \right\} x_t \\
&\quad - \frac{1}{\psi} v_{kw} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2+g-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right) w_t \\
&\quad - \frac{1}{\psi} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{1-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} + v_{kk} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2+g-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right) \right] k_t
\end{aligned} \tag{13}$$

Here  $\eta_{m,\varepsilon_x}$  is the exposure of IMRS to innovation in the long run productivity shock,  $\varepsilon_{x,t+1}$ , and  $\eta_{m,w}$  is the exposure of IMRS to innovation in the short run shock,  $w_{t+1}$ .

### 3.4. Index of International Risk sharing

Brandt, Cochrane, and Santa-Clara (2006) constructed the following index to gauge the degree of international risk sharing:

$$IIRS = 1 - \frac{\sigma^2(m_{t+1}^f - m_{t+1}^d)}{\sigma^2(m_{t+1}^f) + \sigma^2(m_{t+1}^d)}, \quad (14)$$

where IIRS denotes the index of international risk sharing,  $m$  is the logarithm of IMRS, and  $\sigma^2$  denotes unconditional variances. The use of unconditional variances is justified by two considerations. First, since we are using a discrete-time model, starting with  $E_t(m_{t+1}R_{t+1})=1$ , we can condition down to  $E(mR)=1$ ;<sup>3</sup> second, after calibration, the results of our model can be compared with the unconditional moments obtained from the data.

Brandt, Cochrane, and Santa-Clara (2006) emphasized the difference between the index and the correlation between  $m$  across countries. They argued that “[r]isk sharing requires that domestic and foreign marginal utility growth are *equal*, not just perfectly correlated, and our index detects violations of scale as well as of correlation” (page 672). They also provided an example to show the difference: “[F]or example, if  $\ln m^f = 2 \times \ln m^d$ , risks are not perfectly shared despite perfect correlation. In this case, our index is 0.8” (page 672).

## 4. The Model’s Results After Parameterization

### 4.1. Parameterization

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<sup>3</sup> See Brandt, Cochrane, and Santa-Clara (2006) footnote 5 for more detail.

In the literature on calibration, parameter values are chosen “[F] or purposes of ‘calibration’ in a quarterly model” (Campbell (1994) p467). Following this practice, we choose the parameter values as follows.  $g$ , the logarithm of growth rate at steady state, is chosen to be 0.005 (2% at an annual rate).  $r$ , logarithm of steady state return on risky asset, is set at 0.015 (6% at an annual rate).  $\beta$ , subjective discount factor, is set at 0.99923 (0.997 at an annual rate); depreciate rate  $\delta$  is set at 0.025 (10% at an annual rate);<sup>4</sup> capital’s share, denoted as  $\alpha$ , is set to be 0.36, a standard value in literature.  $\zeta$ , a parameter related to investment adjustment cost, with  $\frac{1}{\zeta}$  being the elasticity of investment-capital ratio with respect to marginal  $q$ , is set at 0.8306.<sup>5</sup>  $\rho$ , the persistence of long run productivity risk, is set at 0.987.<sup>6</sup>  $n_1$ , the parameter of linearization of the return to consumption equity defined by  $n_1 = \frac{\bar{P}}{\bar{P} + \bar{C}}$ , is set at 0.9965, an average number between 0.996 from Campbell (2003) and 0.997 from Bansal and Yaron (2004).  $\frac{\bar{K}}{\bar{C}}$ , steady state level of capital-consumption ratio, is chosen to be 12.5544, an average level between 11.3755 in Campbell (1994) and 13.7333 in Uhlig (1999);

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4 These four parameter values are in line with Kydland and Prescott (1982).

5 Eberly (1997) estimated the elasticity for the U.S., denoted  $\frac{1}{\zeta}$ , at a 95% confidence interval of [1.08, 1.36].

6 Croce (2006) set this number at 0.98.

Analogously,  $\frac{\bar{Y}}{\bar{C}}$ , steady state level of output-consumption ratio, is set at 1.3387, an average between 1.3423 and 1.3350, from Campbell (1994) and Uhlig (1999) respectively. To cover the whole range of possibility, we set EIS,  $\psi \in \{20,10,5,2,1.7,1.5,1.1,1,0.9,0.71,0.5,0.2,0.1,1/15\}$  and RRA,  $\gamma \in \{15,10,5\}$ . This range covers the cases of both  $\psi \neq \frac{1}{\gamma}$  and  $\psi = \frac{1}{\gamma}$ . This range also includes  $\psi = 1$ . Table 1 summarizes the key parameter values chosen for our model.

Parameter Values			
	Parameter	Quarterly Value	Equivalent Annual Rate
1	Steady state growth rate $g$	0.005	2%
2	Steady state return on risky asset $r$	0.015	6%
3	Subjective discount factor $\beta$	0.99923	0.997
4	Depreciation $\delta$	0.025	10%
5	Capital's share $\alpha$	0.36	
6	Parameter related to Adjustment cost	0.8306	
7	Persistence of long run productivity shock	0.987	
8	$n1$	0.9965	
9	Steady state capital-consumption ratio	12.5544	
10	Steady state output-consumption ratio	1.3387	

11	EIS	$\left\{ \begin{array}{l} 20, 10, 5, 2, \\ 1.7, 1.5, 1.1, 1, \\ 0.9, 0.71, 0.5, \\ 0.2, 0.1, 1/15 \end{array} \right\}$
12	RRA	$\{15, 10, 5\}$

We calibrate the variance-covariance matrix of productivity growth shocks to match the data. Boldrin, Christiano, and Fisher (2001) parameterized the quarterly volatility of the log productivity growth,  $\sigma_{\Delta z}$ , to be 0.018. Since Equation (7) shows that a productivity process is constituted by an AR (1) long run risk and an *i.i.d.* short run risk, the overall productivity's unconditional variance is:

$$\sigma_{\Delta z}^2 = \sigma_x^2 + \sigma_w^2 = \frac{1}{1-\rho^2} \sigma_\varepsilon^2 + \sigma_w^2 = 0.018^2 \quad (15)$$

Backus, Kehoe, and Kydland (1995) summarized productivity's international correlations based on international business cycles data. For example,  $corr(z_{eu}, z_{us}) = 0.56$  and  $corr(z_{japan}, z_{us}) = 0.58$ . We take the average of the two, 0.57, as our parameter value for the international correlation of the log productivity growth. Then, we get the following equation:

$$\begin{aligned}
\text{corr}(\Delta z^h, \Delta z^f) &= \frac{\text{cov}(x^h, x^f) + \text{cov}(w^h, w^f)}{\sigma_{\Delta z}^2} \\
&= \frac{\frac{1}{1-\rho} \text{corr}(\varepsilon^h, \varepsilon^f) \sigma_{\varepsilon}^2 + \text{corr}(w^h, w^f) \sigma_w^2}{\frac{1}{1-\rho} \sigma_{\varepsilon}^2 + \sigma_w^2} \quad (16) \\
&= \frac{\frac{1}{1-\rho} \times 1 + \text{corr}(w^h, w^f) n}{\frac{1}{1-\rho} + n} = 0.57,
\end{aligned}$$

where  $n = \frac{\sigma_w^2}{\sigma_{\varepsilon}^2}$ , the ratio between the variance of the short run shock to the variance of the long run shock. Recalling that our model assumes a small but persistent long run shock, along with a large short run shock,  $n$  needs to be a large number. We set  $n = 64$ . Equations (15) and (16) imply that  $\sigma_w = 1.42\%$  and  $\text{corr}(w^h, w^f) = 0.31$ . Consequently, the variance of the long run shock,  $\sigma_{\varepsilon}^2$ , explains about 0.97% of the variance of the productivity growth. We assume that the small but persistent long run productivity growth shocks have a common global origin, and that, as a result, the correlation across the two countries of long run productivity growth shocks is set to be 1, i.e.,  $\text{corr}(\varepsilon^h, \varepsilon^f) = 1$ . Table 2 displays a summary of our choices of parameters for the productivity process.

Table 2

Parameter Values for the Productivity Process

	Parameter	Quarterly Value
1	Unconditional standard deviation of productivity growth	0.018
2	International correlation of productivity growth	0.57
3	N, the ratio between the variance of the short run shock to the variance of the long run shock	64
4	International correlation of long run productivity growth shocks	1
	Parameter	Value Generated in the Model
5	Standard deviation of short run productivity growth shocks	1.42%
6	International correlation of short run productivity growth shocks	0.31

## 4.2. Results

Table 3 reports our model's results for  $\eta_{m,\varepsilon_x}$  and  $\eta_{m,w}$  after parameterization, where  $\eta_{m,\varepsilon_x}$  captures IMRS's exposure to innovation in the long-run component,  $\varepsilon_{x,t+1}$ , and  $\eta_{m,w}$  is IMRS's exposure to innovation in the short-run component,  $w_{t+1}$ . Put it another way,  $\eta_{m,\varepsilon_x}$  is the innovation in  $m_{t+1}$  driven

by the innovation in  $\varepsilon_{x,t+1}$ , and  $\eta_{m,w}$  is the innovation in  $m_{t+1}$  driven by the innovation in  $w_{t+1}$ .

Table 3			
Model results for $\eta_{m,\varepsilon_x}$ and $\eta_{m,w}$			
EIS	RRA	$\eta_{m,\varepsilon_x}$	$\eta_{m,w}$
20	15	-612.07	-12.85
20	10	-407.37	-8.57
20	5	-202.66	-4.28
10	15	-550.07	-12.85
10	10	-365.49	-8.57
10	5	-180.90	-4.28
5	15	-466.87	-12.85
5	10	-309.15	-8.57
5	5	-151.43	-4.28
2	15	-306.84	-12.85
2	10	-201.06	-8.57
2	5	-95.28	-4.28
1.7	15	-269.74	-12.85
1.7	10	-176.19	-8.57
1.7	5	-82.64	-4.28
1.5	15	-238.79	-12.85
1.5	10	-155.54	-8.57
1.5	5	-72.28	-4.28
1.1	15	-151.77	-12.85
1.1	10	-97.99	-8.57
1.1	5	-44.22	-4.28
1	15	-121.66	-12.85
1	10	-78.30	-8.57
1	5	-34.95	-4.28
0.9	15	-86.30	-12.85

0.9	10	-55.35	-8.57
0.9	5	-24.40	-4.28
0.71	15	1.97	-12.85
0.71	10	1.05	-8.57
0.71	5	0.14	-4.28
0.5	15	157.27	-12.85
0.5	10	96.40	-8.57
0.5	5	35.53	-4.28
0.2	15	681.82	-12.85
0.2	10	338.05	-8.57
0.203	5	0.59	-4.28
0.1	15	778.05	-12.85
0.10113	10	-3.88	-8.57
0.1	5	-822.08	-4.28
0.0675	15	-1.78	-12.85
1/15	10	-1301.78	-8.57
1/15	5	-2554.82	-4.28

Table 3 shows that  $\eta_{m,\varepsilon_x}$  is much larger than  $\eta_{m,w}$ . According to our results, on average,  $\eta_{m,\varepsilon_x}$  is more than 33 times greater than  $\eta_{m,w}$ . This is pivotal for our model to be able to generate high international risk sharing. In a model featuring recursive preferences and a stochastic productivity process with long run risk, if  $\eta_{m,\varepsilon_x}$  is much larger than  $\eta_{m,w}$ , a small but persistent long run risk dominates a large short run risk to become a major determinant of IMRS  $m_{t+1}$ . Furthermore, if long run risk is highly correlated across countries, so is IMRS. As a result, the model generates a high degree

of international risk sharing, even though the two countries run closed economies and agents' asset holding are autarkic.

Finally, Table 4 presents our model's results for IIRS, index of international risk sharing.

Table 4		
Model results for Index of International Risk sharing (IIRS)		
EIS	RRA	IIRS
20	15	99.9497%
20	10	99.9496%
20	5	99.9491%
10	15	99.9378%
10	10	99.9374%
10	5	99.9361%
5	15	99.9137%
5	10	99.9125%
5	5	99.9088%
2	15	99.8005%
2	10	99.7935%
2	5	99.7701%
1.7	15	99.7420%
1.7	10	99.7313%
1.7	5	99.6948%
1.5	15	99.6711%
1.5	10	99.6556%
1.5	5	99.6016%
1.1	15	99.1915%
1.1	10	99.1388%
1.1	5	98.9457%
1	15	98.7499%

1	10	98.6607%
1	5	98.3277%
0.9	15	97.5597%
0.9	10	97.3709%
0.9	5	96.6545%
0.71	15	31.9534%
0.71	10	31.6154%
0.71	5	31.0348%
0.5	15	99.2465%
0.5	10	99.1104%
0.5	5	98.3801%
0.2	15	99.9595%
0.2	10	99.9268%
0.203	5	31.7722%
0.1	15	99.9689%
0.10113	10	38.6128%
0.1	5	99.9969%
0.0675	15	31.7849%
1/15	10	99.9951%
1/15	5	99.9997%

Our model produces international risk sharing levels above 96% for most parameter values of EIS and RRA. Only when EIS is around either 1/RRA or 0.7 does IIRS drop sharply, falling to levels as low as approximately 30%.

Using exchange rate and equity data, Brandt, Cochrane, and Santa-Clara (2006) reported international risk sharing levels above 98%: 0.986 (US vs. UK); 0.985 (US vs. Germany); 0.980 (US vs. Japan).

Adopting an endowment economy model with long run consumption risk, Colacito and Croce (2005) produced an international correlation of discount factor growth as high as 92%.

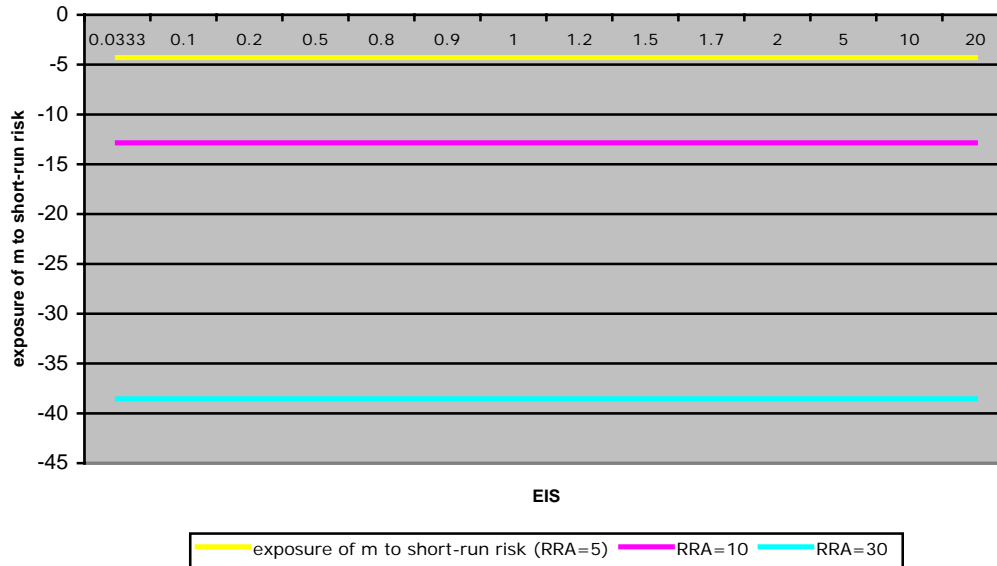
#### 4.3. The Intuition Behind $\eta_{m,w}$ (IMRS's Exposure to Short-run Shock)

Recall IMRS from Equation (3):

$$M_t^i = \beta^\theta \left( \frac{C_{t+1}^i}{C_t^i} \right)^{\frac{\theta}{\psi}} \left( R_{c,t+1}^i \right)^{\theta-1}$$

To examine the contemporaneous effect of  $w_{t+1}$  on the system while eliminating the intertemporal channel, we may equivalently consider a case in which  $\psi = 0$ , which means that a change in interest rates has no impact on future consumption, that is, there is no intertemporal effect. If  $\psi = 0$ , then it follows that  $\theta \rightarrow 0$ ,  $\theta - 1 \rightarrow -1$ , and  $-\frac{\theta}{\psi} \rightarrow 1 - \gamma < 0$ . A positive  $w_{t+1}$  causes both  $c_{t+1}$  and  $R_{c,t+1}$  to rise, and, as a result,  $m_{t+1}$  falls; it then follows that  $\eta_{m,w}$  is negative. The larger  $\gamma$  is, the more  $m_{t+1}$  decreases, and the more negative  $\eta_{m,w}$  becomes. Therefore, we can say that  $\eta_{m,w}$  is decreasing in  $\gamma$ . Chart 1 depicts the relationship between  $\eta_{m,w}$  and parameters EIS and RRA.

Chart 1 Relationship between the Exposure of m to Short-run Risk and Parameters EIS and RRA



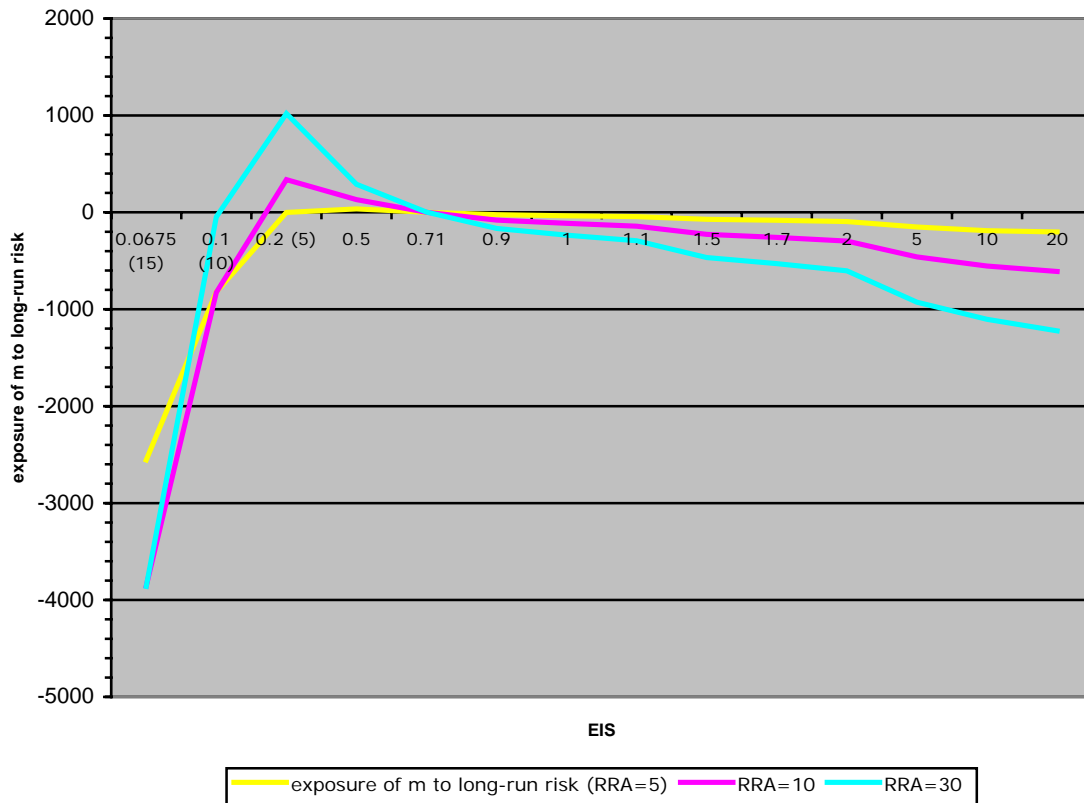
#### 4.4. The Intuition behind $\eta_{m,\varepsilon_x}$ (IMRS's Exposure to Long-run Shock)

Table 2 reveals that for  $\psi > \frac{1}{\gamma}$  (or  $\frac{1}{\psi} < \gamma$ ),  $\eta_{m,\varepsilon_x}$  is negative for large  $\psi$  and

positive for small  $\psi$ ; for  $\psi \leq \frac{1}{\gamma}$ , a case implying a very small  $\psi$ ,

$\eta_{m,\varepsilon_x}$  becomes negative again. Chart 2 depicts this result.

Chart 2 Relationship between the Exposure of m to Long-run Risk and Parameters EIS and RRA



The result is intuitive. In most situations, a positive long-run productivity shock  $\varepsilon_{x,t+1}$  drives the return on consumption equity  $R_{c,t+1}$  upwards. In regard to long-run shock's impact on consumption, on one hand, a positive  $\varepsilon_{x,t+1}$  raises consumption, because of the income effect; on the other hand, a positive  $\varepsilon_{x,t+1}$  increases the agent's desire to borrow, since future output will be higher due to the persistence of the positive productivity shock. This causes interest rates to rise, thus creating a negative substitution effect on consumption that becomes stronger as the parameter  $\psi$  becomes greater.

When  $\psi$  takes a large value, the substitution effect could be greater than the income effect, and consumption could then fall. For the values of  $\psi$  under consideration in this paper, the income effect always dominates over the substitution effect, and consumption rises after a positive long-run productivity growth shock  $\varepsilon_{x,t+1}$ .

Recalling Equation (3), with recursive preferences, IMRS depends on both  $R_{c,t+1}$  and  $c_{t+1}$ :

$$M_t^i = \beta^\theta \left( \frac{C_{t+1}^i}{C_t^i} \right)^{\frac{\theta}{\psi}} (R_{c,t+1}^i)^{\theta-1}$$

$$\eta_{m,\varepsilon_x} = -\frac{\theta}{\psi} \eta_{c,\varepsilon_x} + (\theta-1) \eta_{R_c,\varepsilon_x}$$

Considering the case in which  $\psi > \frac{1}{\gamma}$ , if  $\psi$  is larger than one ( $\psi > 1$ ), we find that  $\theta < 0$ ; if  $\psi$  is smaller than one ( $\psi < 1$ ), then  $\theta > 1$ . In the case of  $\psi > 1$ , a positive  $\varepsilon_{x,t+1}$  causes  $R_{c,t+1}$  to rise and  $c_{t+1}$  to either fall or rise.  $\theta < 0$  implies that  $\theta-1 < 0$  and  $-\frac{\theta}{\psi} > 0$ . Therefore, a rise in  $R_{c,t+1}$  leads to a fall in  $m_{t+1}$ , and a fall or rise in  $c_{t+1}$  results in a respective fall or rise in  $m_{t+1}$ . Since the impact of  $\varepsilon_{x,t+1}$  on  $R_{c,t+1}$ , denoted  $\eta_{R_c,\varepsilon_x}$ , always dominates<sup>7</sup> over its

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<sup>7</sup> When  $\psi = 1$ , it is a weak domination.

impact on  $c_{t+1}$ , denoted  $\eta_{c,\varepsilon_x}$ , the overall result is that a positive  $\varepsilon_{x,t+1}$  causes  $m_{t+1}$  to fall, which implies a negative  $\eta_{m,\varepsilon_x}$ .

If  $\psi < 1$  but is still larger than  $\frac{1}{\gamma}$ , we have  $\theta > 1$ , which implies  $\theta - 1 > 0$  and  $-\frac{\theta}{\psi} < 0$ . Hence, a rise in  $R_{c,t+1}$  due to a positive shock  $\varepsilon_{x,t+1}$  leads to a rise in  $m_{t+1}$ , while a rise in  $c_{t+1}$  causes a fall in  $m_{t+1}$ . The overall effect of  $\varepsilon_{x,t+1}$  on  $m_{t+1}$  is positive and, therefore,  $\eta_{m,\varepsilon_x}$  is positive.

If  $\psi = 1$ ,  $\varepsilon_{x,t+1}$  equally affects both  $R_{c,t+1}$  and  $c_{t+1}$ , and we have  $\eta_{R_c,\varepsilon_x} = \eta_{c,\varepsilon_x}$ . In that case,  $\eta_{m,\varepsilon_x}$  becomes

$$\begin{aligned}\eta_{m,\varepsilon_x} &= -\frac{\theta}{\psi}\eta_{c,\varepsilon_x} + (\theta - 1)\eta_{R_c,\varepsilon_x} = \left(-\frac{\theta}{\psi} + \theta - 1\right)\eta_{R_c,\varepsilon_x} \\ &= -\gamma\eta_{R_c,\varepsilon_x} < 0\end{aligned}$$

In addition, our simulation shows that  $r_{c,t+1}$  falls when  $\psi \in [*, 1)$ , where  $\frac{1}{\gamma} < * < 1$ . Our simulation reports  $*$  is around  $[0.7, 0.9]$ . When EIS falls into this range,  $A_2$ , the exposure to long-run shock of the logarithm of price-consumption ratio<sup>8</sup> becomes negative. This means that the price-consumption ratio  $p_{c,t}$  falls after a positive long-run shock  $\varepsilon_{x,t+1}$ . Recall that

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<sup>8</sup>  $p_{c,t} = A_0 + A_1 k_t + A_2 x_t + A_3 w_t$

$r_{c,t+1} = n_0 + n_1 p_{c,t+1} - p_{c,t} + c_{t+1} - c_t$  ; a lower price-consumption ratio thus contributes to a fall in  $r_{c,t+1}$ .

If  $\psi$  is very small and satisfies  $\psi < \frac{1}{\gamma}$ , we get  $0 < \theta < 1$ , which in turn implies  $\theta - 1 < 0$  and  $-\frac{\theta}{\psi} < 0$ . As a result, a rise in  $R_{c,t+1}$  causes  $m_{t+1}$  to fall, and a rise in  $c_{t+1}$  also causes  $m_{t+1}$  to fall. Therefore, a positive  $\varepsilon_{x,t+1}$  causes  $m_{t+1}$  to fall, when  $\psi$  takes very small values. In that case,  $\eta_{m,\varepsilon_x}$  is negative.

Table 5 summarizes the intuition behind  $\eta_{m,\varepsilon_x}$ .

Table 5										
Intuition behind $\eta_{m,\varepsilon_x}$										
	parameter	$\theta$	$-\frac{\theta}{\psi}$	$c_{t+1}$	$\theta - 1$	$R_{c,t+1}$	Exposure of m to $c_{t+1}$	Exposure of m to $R_{c,t+1}$	Innovation in m	$\eta_{m,\varepsilon}$
EIS > 1/ RRA	EIS > 1	< 0	> 0	Fall/ rise	< 0	rise	Fall/ rise	fall	fall	-
	EIS = 1			rise		rise			fall	-
	* < EIS < 1	> 1	< 0	rise	> 0	fall	fall	fall	fall	-
	1/RRA < EIS < *	> 1	< 0	rise	> 0	rise	fall	rise	rise	+
EIS ≤ 1/RRA		(0, 1]	< 0	rise	< 0	rise	fall	fall	fall	-

#### 4.5. Results for Consumption Properties

Business cycle data for industrialized countries show a quarterly volatility of log consumption growth of approximately 1.4%<sup>9</sup>. In addition, the quarterly correlations across countries of the log consumption growth are: 0.43 (U.S. vs. U.K.); 0.39 (U.S. vs. Germany); and 0.30 (U.S. vs. Japan).<sup>10</sup> Our model produces results that are in line with these data. Table 6 summarizes our model's results for the volatility and international correlation of consumption growth rates.

One of the striking facts shown by international business cycle data is that the international correlations of consumption growth are surprisingly smaller than the correlations of output growth or productivity growth across countries, whereas the standard models predict the opposite. Fortunately, our model has no difficulty in generating results consistent with the data. After setting the correlations across countries of productivity growths to be 0.57 to match the data, our model delivers international correlations of consumption growth around 30%–40%, which is lower than the correlation of productivity growth, as the data depicts.

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<sup>9</sup> See Croce (2006) for more detail.

<sup>10</sup> The data is from Table 2 of Backus, Kehoe, and Kydland (1992).

EIS	RRA	Standard deviation of consumption growth	International correlation of consumption growth
20	15	1.2175%	0.3101
20	10	1.2175%	0.3101
20	5	1.2175%	0.3101
10	15	1.2179%	0.3106
10	10	1.2179%	0.3106
10	5	1.2179%	0.3106
5	15	1.2195%	0.3123
5	10	1.2195%	0.3123
5	5	1.2195%	0.3123
2	15	1.2282%	0.3220
2	10	1.2282%	0.3220
2	5	1.2282%	0.3220
1.7	15	1.2317%	0.3259
1.7	10	1.2317%	0.3259
1.7	5	1.2317%	0.3259
1.5	15	1.2352%	0.3297
1.5	10	1.2352%	0.3297
1.5	5	1.2352%	0.3297
1.1	15	1.2480%	0.3434
1.1	10	1.2480%	0.3434
1.1	5	1.2480%	0.3434
1	15	1.2535%	0.3491
1	10	1.2535%	0.3491
1	5	1.2535%	0.3491
0.9	15	1.2607%	0.3565
0.9	10	1.2607%	0.3565
0.9	5	1.2607%	0.3565

0.71	15	1.2826%	0.3783
0.71	10	1.2826%	0.3783
0.71	5	1.2826%	0.3783
0.5	15	1.3363%	0.4273
0.5	10	1.3363%	0.4273
0.5	5	1.3363%	0.4273
0.2	15	1.7569%	0.6687
0.2	10	1.7569%	0.6687
0.2	5	1.7569%	0.6687
0.1	15	2.7202%	0.8618
0.1	10	2.7202%	0.8618
0.1	5	2.7202%	0.8618
1/15	15	3.7921%	0.9289
1/15	10	3.7921%	0.9289
1/15	5	3.7921%	0.9289

Table 6 shows that changing the value of RRA has virtually no impact on the second moments of aggregate variables such as consumption growth. This result is consistent with the findings in Tallarini (2000): “[R]isk aversion seems to affect asset market implications and the elasticity of intertemporal substitution affects quantity dynamics” (page 509). He thus points to a way to improve the RBC model’s performance in regard to asset prices. Since the model’s performance with respect to quantity variables mainly depends on EIS, while its performance in asset prices depends on RRA, by fixing EIS but increasing RRA, RBC models achieve better results in simulating asset prices while retaining their satisfactory records in simulating quantity variables.

Table 6 reveals that both the volatility and international correlation of consumption growth are decreasing in EIS ( $\psi$ ). This result is intuitive. EIS is a parameter controlling the magnitude of the intertemporal effect. Equation (12) shows that EIS affects the exposure to long-run risk of consumption growth, but will not affect exposure to short-run risk. Moreover, the greater the value of EIS, the more an agent has incentive to smooth her consumption over time, which results in less volatility of consumption growth.

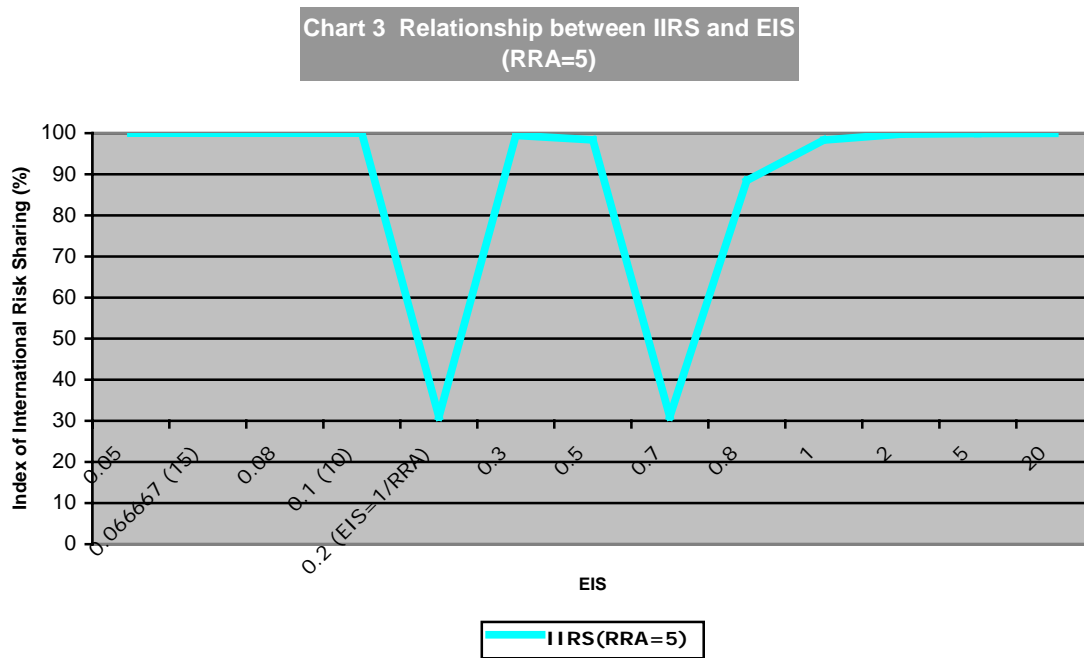
Our simulation results demonstrate a decrease in the exposure to long-run risk of consumption growth in EIS. This is again because of the incentive towards consumption smoothing. Given that the international correlation of long-run risk is assumed to be much larger than the correlation of short-run risk, the greater the value of EIS, the less long-run component will dominate the short-run in the innovation to consumption growth, and, consequently, the lower the international correlation of consumption growth will be.

#### 4.6. Properties of the Model's IIRS Results

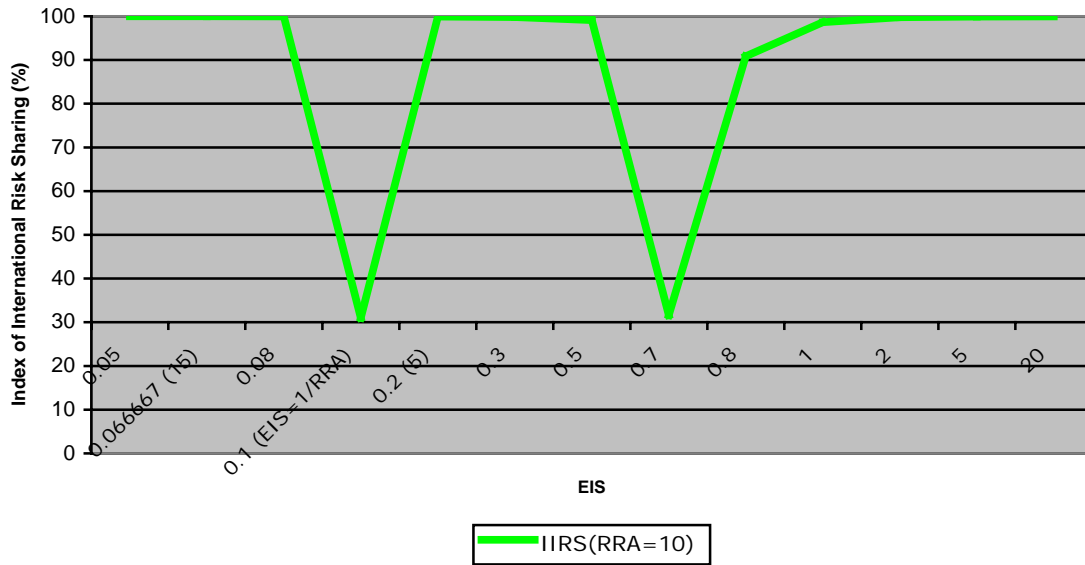
Table 4 shows the following properties of our model's IIRS results. Given fixed values for RRA, we find a W-shaped relation between IIRS and EIS, with IIRS twice reaching its local minimum at either around  $EIS=1/RRA$  or around  $EIS=0.7$ . On the other hand, given fixed values of EIS, the relationship between IIRS and RRA appears to be V-shaped, with IIRS hitting bottom at around  $EIS=1/RRA$ . Our results confirm that when  $EIS=1/RRA$ , IIRS drops sharply and international risk sharing is low, a finding consistent

with that of the benchmark model featuring a power utility whereby EIS always equals the reciprocal of RRA.

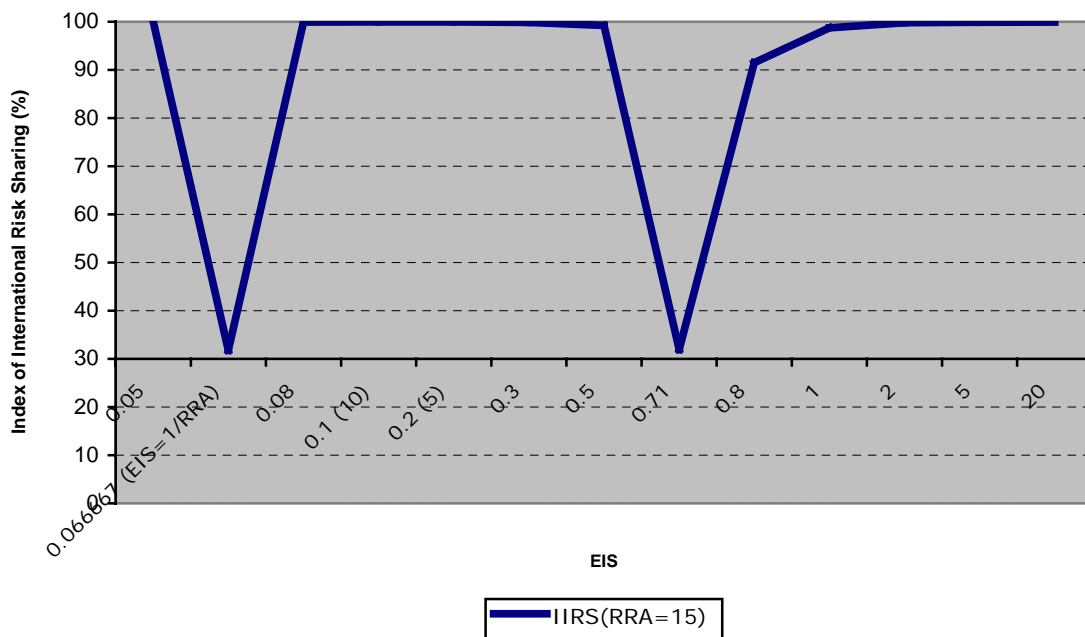
Charts 3, 4, and 5, presented below, demonstrate the W-shaped relation between IIRS and EIS, with RRA fixed at 5, 10, and 15, respectively.



**Chart 4 Relationship between IIRS and EIS (RRA=10)**



**Chart 5 Relationship between IIRS and EIS (RRA=15)**



Colacito and Croce (2005) reported a V-shaped relation between EIS and the international correlation of IMRS<sup>11</sup>, with correlation reaching its lowest level at around  $EIS=1/RRA$ .<sup>12</sup> However, we reported a W-shaped relation between EIS and IIRS, with two stationary points found around  $EIS=1/RRA$  and  $EIS=0.7$ .

From the point of view of empirical research, whether the estimator of EIS ( $\psi$ ) is larger or smaller than unity is still an open question. Hansen and Singleton (1983), Attanasio and Weber (1989), Attanasio and Vissing-Jorgensen (2003), Guvenen (2006), and Vissing-Jorgensen (2002) all estimated EIS to be greater than unity. In contrast, others, such as Hall (1988), Campbell (1999), Lustig and Van Nieuwerburgh (2005) and Favero (2005) estimated its value to be smaller than unity. However, Bansal and Yaron (2004) pointed out that estimates like those done by Hall (1988) depend on the assumption that economic volatility is homoskedastic. The presence of time-varying second moments (heteroskedasticity) leads to a serious downward bias in estimates for EIS when using a regression approach such as that used in Hall (1988). This bias might help explain the large estimates for EIS found in the long-run literature.

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11 International correlation of IMRS is related to IIRS. However they are not the same concept.

12 See Fig.2 in Colacito and Croce (2005) for more detail.

In the benchmark model, EIS - being treated as the reciprocal of RRA - is smaller than 1. Most of the studies in the long-run risk literature consider cases where EIS is greater than unity. Bansal and Yaron (2004), for example, studied a case where  $\psi=1.5, \gamma \in \{7.5, 10\}$ , and Colacito and Croce (2005) considered a case where  $\psi=2, \gamma=4.25$ . Similarly, Croce (2006) focused on a case where  $\psi \in \{0.8, 1, 1.5\}, \gamma=16$ .

Our results demonstrate that the magnitude of IIRS is sensitive to the value of EIS. When reporting the relationship between IIRS and RRA with EIS fixed, we considered the value of EIS as being confined to a broad range from 1/15 to 20, covering all cases where  $EIS < 1$ ,  $EIS = 1$ , and  $EIS > 1$ . Charts 6, 7, 8, and 9 report the relationship between IIRS and RRA, when EIS is fixed at 0.7, 1, 1.5 and 2, respectively. It appears that the relationship is V-shaped, with IIRS reaching its lowest level at around  $RRA = 1/EIS$ .

Chart 6 Relationship between IIRS and RRA  
(EIS=0.7)

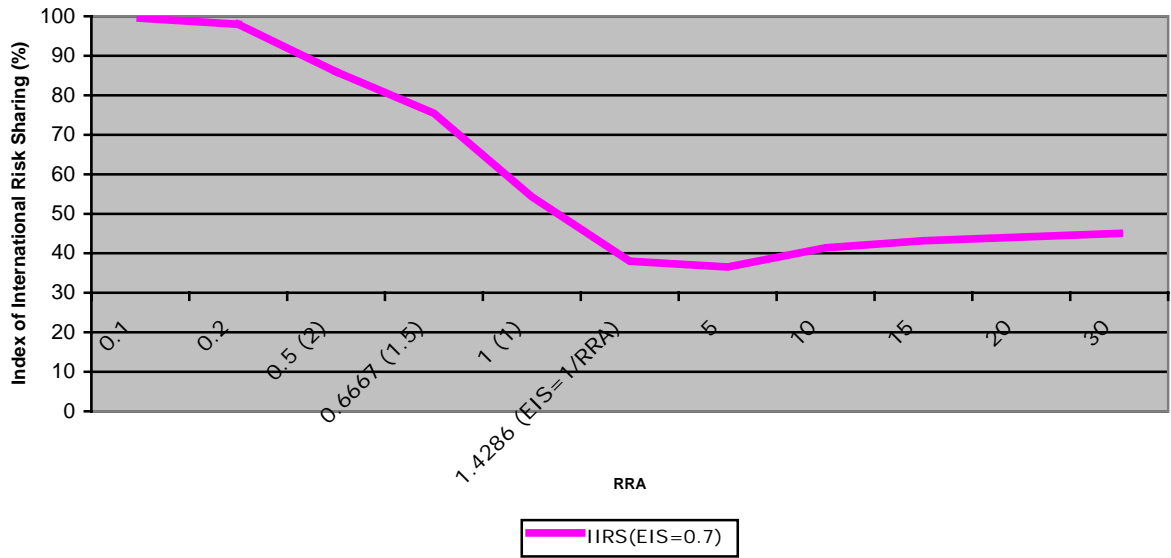
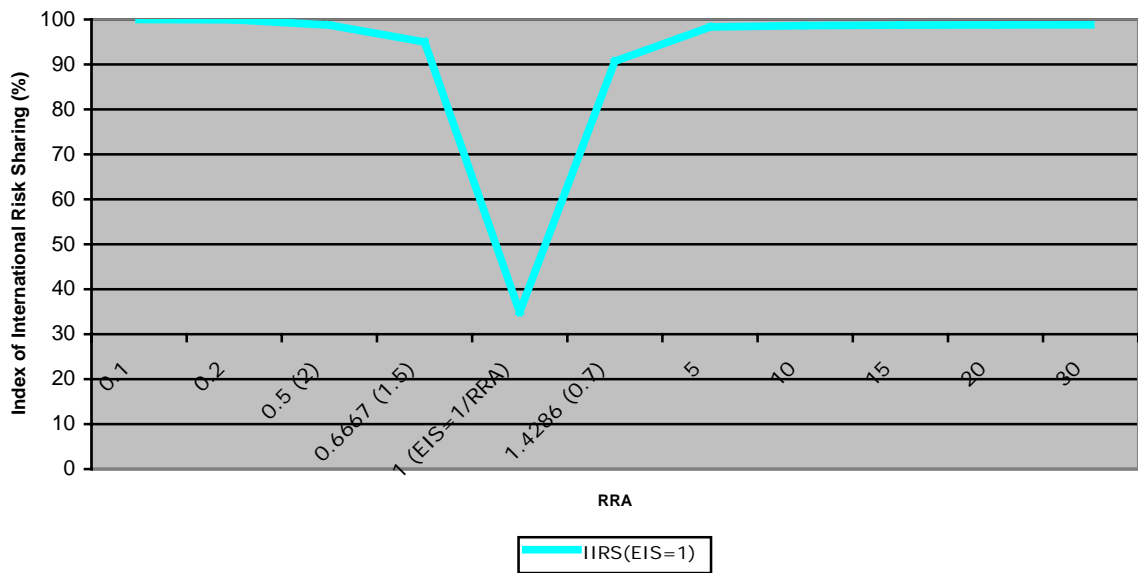
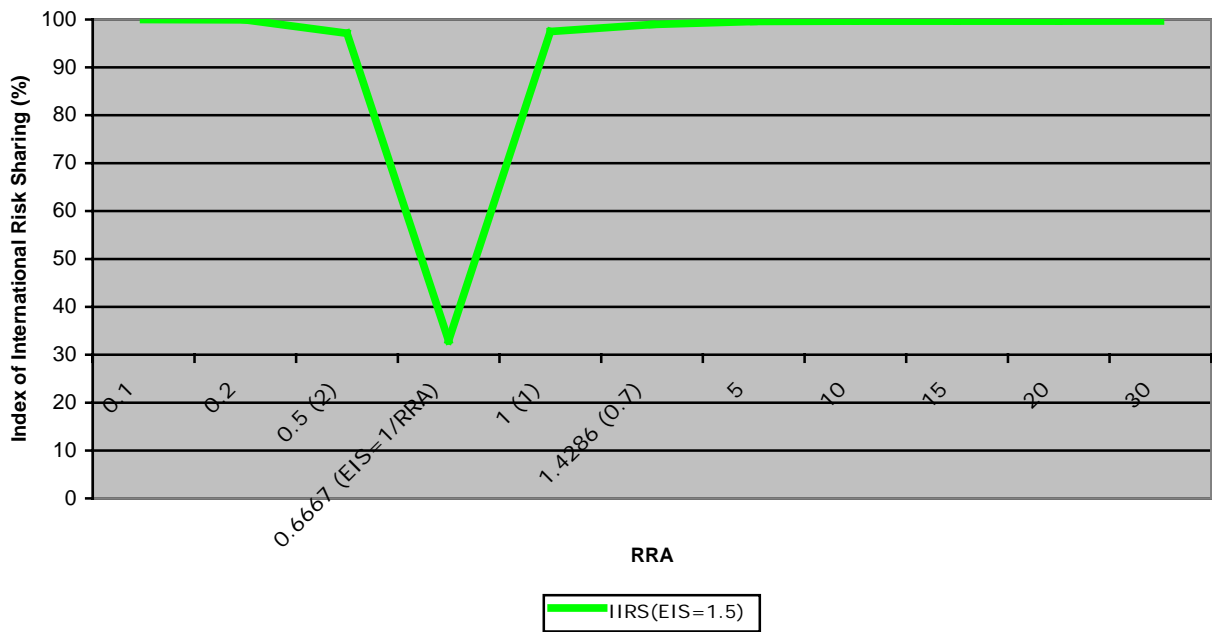


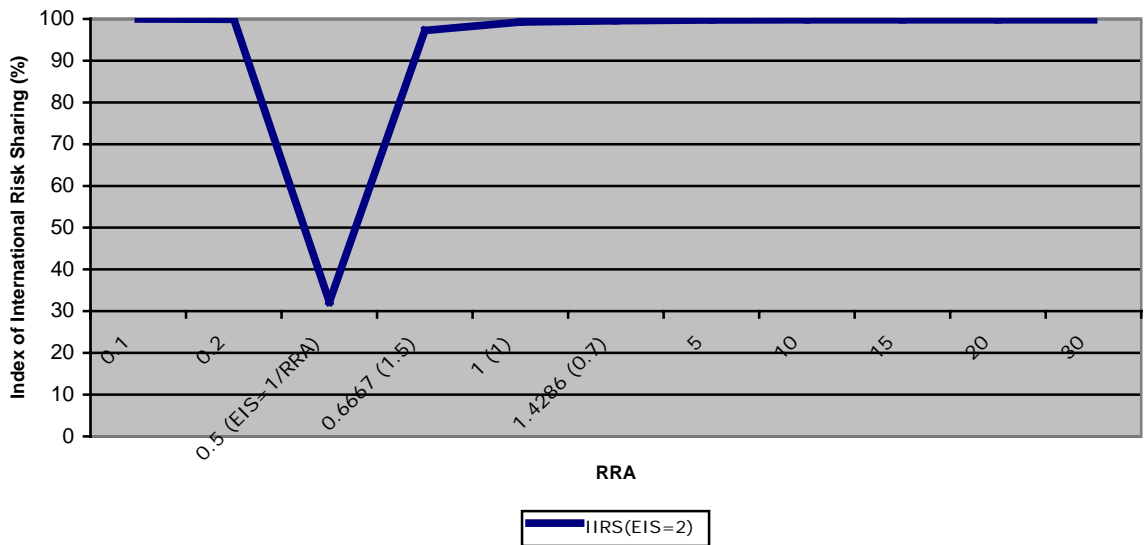
Chart 7 Relationship between IIRS and RRA  
(EIS=1)



**Chart 8 Relationship between IIRS and RRA  
(EIS=1.5)**



**Chart 10 Relationship between IIRS and RRA  
(EIS=2)**



#### 4.7. The Intuition Behind the W-shaped Relation between IIRS and EIS

The question quickly arises: Why does our model produce a W-shaped relation between IIRS and EIS, rather than a V-shaped relation such as that in Colacito and Croce (2005)? The answer is to be found in carrying out an analysis of  $\eta_{m,\varepsilon_x}$ , IMRS's exposure to long-run shock. Chart 2 shows  $\eta_{m,\varepsilon_x}$  changing sign twice, once around  $EIS=1/RRA$  and again around  $EIS=0.7$ . Around each sign change,  $\eta_{m,\varepsilon_x}$  is quite small so that  $|\eta_{m,\varepsilon_x}| < |\eta_{m,w}|$ , that is, IMRS's exposure to long run shocks is smaller than that to short-run shock. Recall that we assumed the international correlation for short-run shock is much smaller than for long-run shock. Hence, the model produces two stationary points in the graph of the relationship between IIRS and EIS. To put it another way, our model delivered low levels of international risk sharing when IMRS's exposure to short-run shock dominates that to long-run shock.

#### 4.8. Regarding the Possibility that EIS and IIRS Match the Predictions of the Benchmark Model

Is it possible that EIS is indeed around  $1/RRA$  or around 0.7 and that IIRS is as low as 30%, a prediction supported by the benchmark model? To answer this question, we looked at the model's results for the standard deviation of the logarithms of IMRS ( $\sigma(m)$ ). Even though there is no data for  $\sigma(m)$ , since IMRS is unobservable, Hensen-Jagannathan (1991), by using asset market data, developed the following lower bound for  $\sigma(m)$ :

$$\sigma(m) \geq \frac{E_t[r_{i,t+1} - r_{f,t+1}] + \sigma_i^2 / 2}{\sigma_i}$$

Recall that our model deals with closed economies and autarkic asset holdings. If the agent is allowed to hold foreign assets,  $\sigma(m)$  will become smaller due to the diversifying away of some idiosyncratic risks. This implies that  $\sigma(m)$  will probably decrease in data drawn from open economies and our results for  $\sigma(m)$  have upward bias. Using a quarterly data set, Campbell (2003) reported an estimate of the lower bound for  $\sigma(m)$  above 30% for most industrialized countries. Thus, the value of  $\sigma(m)$  produced by our model, in order to be reasonable, must be above 30%; those below 30% cannot be reconciled with asset market data.

Model's result on the volatility of the logarithms of IMRS		
EIS	RRA	Standard deviation of consumption growth
20	15	6.77
20	10	4.50
20	5	2.24
10	15	6.08
10	10	4.04
10	5	2.00
5	15	5.16
5	10	3.42
5	5	1.67

2	15	3.40
2	10	2.23
2	5	1.05
1.7	15	2.99
1.7	10	1.95
1.7	5	0.92
1.5	15	2.65
1.5	10	1.72
1.5	5	0.80
1.1	15	1.69
1.1	10	1.09
1.1	5	0.49
1	15	1.36
1	10	0.87
1	5	0.39
0.9	15	0.97
0.9	10	0.62
0.9	5	0.28
0.71	15	0.18
0.71	10	0.12
0.71	5	0.06
0.5	15	1.75
0.5	10	1.07
0.5	5	0.40
0.2	15	7.54
0.2	10	3.74
0.2	5	0.06
0.1	15	8.60
0.1	10	0.13
0.1	5	9.08
1/15	15	0.18
1/15	10	14.39
1/15	5	28.23

Table 7 shows that  $\sigma(m)$  is below 30%, particularly when EIS is around either  $1/RRA$  or 0.7. Parameterizing EIS with values around these two levels, the model produces low IIRS as well as low  $\sigma(m)$ , with the latter being even lower than the Hansen-Jagannathan lower bound. Therefore, with EIS around either  $1/RRA$  or 0.7, the model produced results that cannot be reconciled with asset prices data. In order for our model to produce satisfactory results on both quantity variables and asset prices, we instead parameterized EIS with values neither around  $1/RRA$  nor around 0.7. With these new EIS values, our model produces levels of international risk sharing that reaches levels always over 96%. So far, our results demonstrate that a theoretical model that is able to successfully reconcile both quantity data and asset price data also produces very high levels of international risk sharing, a result contrary to that of the benchmark model.

## 5. Conclusion

This is a theoretical paper on the study of international risk sharing and its implications for equity home bias. Despite adopting the model of closed economies and autarkic asset holdings—a scenario leading to the lowest level of international risk sharing under the same conditions—our model is still able to generate international risk sharing indexes always over 96% for a broad range of parameter values. However, there are two exceptions: the case in which the value of EIS is around the reciprocal of RRA and the case in which it is around 0.7. In those cases, the risk sharing index drops sharply to

about 30%. This result sheds light on why the benchmark model featuring a power utility whereby EIS is the reciprocal of RRA generates an international risk sharing value as low as 30%. Using asset market data to build the Hansen-Jagannathan lower bound for the volatility of the logarithms of IMRS, we can exclude EIS at values around either  $1/RRA$  or 0.7, since it is at those values that the model produced the volatility even smaller than the lower bound. In addition, our model's consumption results fit well with the actual consumption data.

The implication of the high international risk sharing generated in closed economy model with long run risk is that the low proportion of foreign assets in a domestic agent's portfolio, a phenomenon observed in the data, might not be a puzzle or a departure from the agent's first-order condition. After all, risk has already been well shared internationally due to the high correlations across countries of long-run productivity shocks. Hence, there is not much incentive left for an agent to hold foreign assets in her portfolio in order to further share risk internationally. Therefore, equity home bias might not be a puzzle as claimed by the benchmark model, in the sense that it can be well reconciled with the theoretical result of our model.

Following the practices found in the long-run literature, our model could be extended by introducing time-varying productivity volatility. We can then determine whether our results for international risk sharing will be robust under the introduction of stochastic volatility.

## Appendix

1. Consumer's problem:

$$\text{Consumer's FOC: } E_t \left[ \beta^\theta \left( \frac{C_{t+1}^i}{C_t^i} \right)^{\frac{\theta}{\Psi}} \left( R_{c,t+1}^i \right)^{\theta-1} R_{j,t+1} \right] = 1,$$

2. Firm's problem (with adjustment cost)

Firm's FOC:

$$q_s = \frac{1}{\Psi' \left( \frac{I_s}{K_s} \right)}$$

$$W_s = (1-\alpha) Z_s^{1-\alpha} K_s^\alpha$$

$$R_{s+1} = \left[ \alpha Z_{s+1}^{1-\alpha} K_{s+1}^{\alpha-1} + \frac{(1-\delta) + \Psi \left( \frac{I_{s+1}}{K_{s+1}} \right)}{\Psi' \left( \frac{I_{s+1}}{K_{s+1}} \right)} - \frac{I_{s+1}}{K_{s+1}} \right] \Psi' \left( \frac{I_s}{K_s} \right) \quad (\text{B1})$$

The log productivity growth evolution process:

$$\Delta z_{t+1}^i = \mu + x_t^i + w_{t+1}^i, x_t^i = \rho x_{t-1}^i + \epsilon_{x,t}^i, \forall i \in \{h, f\},$$

3. Market clearing conditions:

$$Y_t = C_t + I_t = Z_t^{1-\alpha} K_t^\alpha, K_{t+1} = \Psi \left( \frac{I_t}{K_t} \right) K_t + (1-\delta) K_t$$

## 4. Log-linear approximation of FOC around the steady state

$$\begin{aligned}
(1-\alpha)z_t + \alpha k_t &= \frac{\bar{C}}{\bar{Y}}c_t + \frac{\bar{I}}{\bar{Y}}i_t = \frac{\bar{C}}{\bar{Y}}c_t + \left(1 - \frac{\bar{C}}{\bar{Y}}\right)i_t \\
k_{t+1} &= \frac{G-(1-\delta)}{G}i_t + \frac{1-\delta}{G}k_t = \frac{g+\delta}{1+g}i_t + \frac{1-\delta}{1+g}k_t \\
r_{t+1} &= \frac{\alpha(1-\alpha)\bar{Y}}{1+r}\frac{\bar{I}}{\bar{K}}z_{t+1} + \frac{\zeta(1+g)^2}{(g+\delta)(1+r)}k_{t+2} + \left[\alpha\zeta\frac{\bar{Y}}{\bar{K}} + \zeta(1-\delta)\right]\frac{(1+g)}{(g+\delta)(1+r)}k_t \\
&\quad - \left\{\frac{\alpha(1-\alpha)\bar{Y}}{1+r}\frac{\bar{I}}{\bar{K}} + \frac{\zeta(1+g)^2}{(g+\delta)(1+r)} + \left[\alpha\zeta\frac{\bar{Y}}{\bar{K}} + \zeta(1-\delta)\right]\frac{(1+g)}{(g+\delta)(1+r)}\right\}k_{t+1} \\
\Delta z_{t+1}^i &= \mu + x_t^i + w_{t+1}^i, x_t^i = \rho x_{t-1}^i + \epsilon_{x,t}^i
\end{aligned}$$

5. The method of undetermined coefficients (state variables are  $k_t$ ,  $x_t$  and

$$w_t(\Delta z_t).)$$

$$\begin{aligned}
(1-\alpha)z_t + \alpha k_t &= \frac{\bar{C}}{\bar{Y}}c_t + \left(1 - \frac{\bar{C}}{\bar{Y}}\right)i_t = \frac{\bar{C}}{\bar{Y}}c_t + \left(1 - \frac{\bar{C}}{\bar{Y}}\right)\left(\frac{1+g}{g+\delta}k_{t+1} - \frac{1-\delta}{g+\delta}k_t\right) \\
k_{t+1} &= \frac{g+\delta}{1+g}i_t + \frac{1-\delta}{1+g}k_t \\
c_{t+1} - c_t &= -v_{kx}\frac{1+g}{g+\delta}\frac{\bar{I}}{\bar{C}}\epsilon_{x,t+1} + \left[(1-\alpha)\frac{\bar{Y}}{\bar{C}} - \frac{\bar{I}}{\bar{C}}\frac{1+g}{g+\delta}v_{kw}\right]w_{t+1} \\
&\quad + \left\{(1-\alpha)\frac{\bar{Y}}{\bar{C}} + v_{kx}\left[\alpha\frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}}\frac{(1-\delta)+(1-\rho)(1+g)}{g+\delta} - \frac{\bar{I}}{\bar{C}}\frac{1+g}{g+\delta}v_{kk}\right]\right\}x_t \\
&\quad + v_{kw}\left[\alpha\frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}}\frac{2+g-\delta}{g+\delta} - \frac{\bar{I}}{\bar{C}}\frac{1+g}{g+\delta}v_{kk}\right]w_t \\
&\quad + \left\{\alpha\frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}}\frac{1-\delta}{g+\delta} + v_{kk}\left[\alpha\frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}}\frac{2+g-\delta}{g+\delta} - \frac{\bar{I}}{\bar{C}}\frac{1+g}{g+\delta}v_{kk}\right]\right\}k_t
\end{aligned} \tag{B2}$$

6. The return on the consumption asset:  $r_{c,t+1}$

$$r_{c,t+1} = n_0 + n_1 p_{c,t+1} - p_{c,t} + c_{t+1} - c_t$$

$$p_{c,t} = \log\left(\frac{P_{c,t}}{C_t}\right),$$

where  $P_{c,t}$  is the price of the asset which pays the aggregate consumption as its dividend for each period;  $n_1$  is the parameter of linearization defined by

$$n_1 = \frac{1}{1 + \exp(\bar{c} - \bar{p})} = \frac{\bar{P}}{\bar{P} + \bar{C}}.$$

$$p_{c,t} = A_0 + A_1 k_t + A_2 x_t + A_3 w_t$$

$$\begin{aligned} r_{c,t+1} = & \left( n_1 A_2 - \frac{\bar{I}}{\bar{C}} \frac{1+g}{g+\delta} v_{kx} \right) x_{t+1} + \left[ n_1 A_3 + (1-\alpha) \frac{\bar{Y}}{\bar{C}} - \frac{\bar{I}}{\bar{C}} \frac{1+g}{g+\delta} v_{kw} \right] w_{t+1} \\ & + \left\{ n_1 A_1 v_{kx} - A_2 + (1-\alpha) \frac{\bar{Y}}{\bar{C}} + v_{kx} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{2+g-\delta}{g+\delta} - \frac{\bar{I}}{\bar{C}} \frac{1+g}{g+\delta} v_{kk} \right] \right\} x_t \\ & + \left\{ n_1 A_1 v_{kw} - A_3 + v_{kw} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{2+g-\delta}{g+\delta} - \frac{\bar{I}}{\bar{C}} \frac{1+g}{g+\delta} v_{kk} \right] \right\} w_t \\ & + \left\{ n_1 A_1 v_{kk} - A_1 + \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{1-\delta}{g+\delta} + v_{kk} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{2+g-\delta}{g+\delta} - \frac{\bar{I}}{\bar{C}} \frac{1+g}{g+\delta} v_{kk} \right] \right\} k_t \end{aligned} \quad (\text{B3})$$

## 7. Solution to $m_{t+1}$

Since consumption asset is one kind of assets, it should satisfy the following

FOC condition:  $E_t[m_{t+1} + r_{c,t+1}] = 0$ . with recursive preferences,

$$m_{t+1} = -\frac{\theta}{\Psi}(c_{t+1} - c_t) + (\theta - 1)r_{c,t+1} \quad (\text{B4}).$$

Hence:

$$E_t \left[ -\frac{\theta}{\Psi}(c_{t+1} - c_t) + (\theta - 1)r_{c,t+1} + r_{c,t+1} \right] = 0, E_t \left[ -\frac{\theta}{\Psi}(c_{t+1} - c_t) + \theta r_{c,t+1} \right] = 0$$

If we assume homoskedasity, the following equation holds true:

$$-\frac{\theta}{\psi} E_t [c_{t+1} - c_t] + \theta E_t [r_{c,t+1}] = 0 \quad (\text{B5})$$

Then we get:

$$\begin{aligned} A_1 &= \frac{\left(1 - \frac{1}{\psi}\right)}{(1 - n_1 v_{kk})} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{1 - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} + v_{kk} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2 + g - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1 + g}{g + \delta} \frac{\bar{I}}{\bar{C}} \right) \right] \\ A_2 &= \frac{\left(1 - \frac{1}{\psi}\right)}{(1 - \rho n_1)} \left\{ (1 - \alpha) \frac{\bar{Y}}{\bar{C}} + \frac{v_{kx} n_1}{(1 - n_1 v_{kk})} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{1 - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} + v_{kk} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2 + g - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1 + g}{g + \delta} \frac{\bar{I}}{\bar{C}} \right) \right] \right. \\ &\quad \left. + v_{kx} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{(1 - \delta) + (1 - \rho)(1 + g)}{g + \delta} - v_{kk} \frac{1 + g}{g + \delta} \frac{\bar{I}}{\bar{C}} \right] \right\} \\ A_3 &= v_{kw} \left(1 - \frac{1}{\psi}\right) \left\{ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2 + g - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1 + g}{g + \delta} \frac{\bar{I}}{\bar{C}} \right. \\ &\quad \left. + \frac{n_1}{(1 - n_1 v_{kk})} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{1 - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} + v_{kk} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2 + g - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1 + g}{g + \delta} \frac{\bar{I}}{\bar{C}} \right) \right] \right\} \end{aligned} \quad (\text{B6})$$

Substituting  $A_1$ ,  $A_2$  and  $A_3$  into Equation (B3), we get:

$$\begin{aligned} r_{c,t+1} &= \frac{n_1 \left(1 - \frac{1}{\psi}\right)}{(1 - \rho n_1)} \left\{ (1 - \alpha) \frac{\bar{Y}}{\bar{C}} + \frac{v_{kx} n_1}{(1 - n_1 v_{kk})} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{1 - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} + v_{kk} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2 + g - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1 + g}{g + \delta} \frac{\bar{I}}{\bar{C}} \right) \right] \right\} \varepsilon_{x,t+1} \\ &\quad + v_{kx} \frac{1 + g}{g + \delta} \frac{\bar{I}}{\bar{C}} \varepsilon_{x,t+1} + (1 - \alpha) \frac{\bar{Y}}{\bar{C}} w_{t+1} - v_{kw} \frac{1 + g}{g + \delta} \frac{\bar{I}}{\bar{C}} w_{t+1} \\ &\quad + v_{kw} n_1 \left(1 - \frac{1}{\psi}\right) \left\{ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2 + g - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1 + g}{g + \delta} \frac{\bar{I}}{\bar{C}} \right. \\ &\quad \left. + \frac{n_1}{(1 - n_1 v_{kk})} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{1 - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} + v_{kk} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2 + g - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1 + g}{g + \delta} \frac{\bar{I}}{\bar{C}} \right) \right] \right\} w_{t+1} \\ &\quad + \frac{1}{\psi} \left\{ (1 - \alpha) \frac{\bar{Y}}{\bar{C}} + v_{kx} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{(1 - \delta) + (1 - \rho)(1 + g)}{g + \delta} - v_{kk} \frac{1 + g}{g + \delta} \frac{\bar{I}}{\bar{C}} \right] \right\} x_t \\ &\quad + \frac{1}{\psi} v_{kw} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2 + g - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1 + g}{g + \delta} \frac{\bar{I}}{\bar{C}} \right) w_t + \frac{1}{\psi} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{1 - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} + v_{kk} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2 + g - \delta}{g + \delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1 + g}{g + \delta} \frac{\bar{I}}{\bar{C}} \right) \right] k_t \end{aligned} \quad (\text{B7})$$

Substituting Equations (B2) and (B7) into (B4), we get:

$$\begin{aligned}
m_{t+1} &= -\frac{\theta}{\psi}(c_{t+1} - c_t) + (\theta - 1)r_{a,t+1} \\
&= \gamma v_{kx} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \varepsilon_{x,t+1} \\
&\quad + \frac{n_1 \left( \frac{1}{\psi} - \gamma \right)}{(1-\rho n_1)} \left\{ (1-\alpha) \frac{\bar{Y}}{\bar{C}} + \frac{n_1 v_{kx}}{(1-n_1 v_{kk})} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{1-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} + v_{kk} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2+g-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right) \right] \right. \\
&\quad \left. + v_{kx} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{(1-\delta) + (1-\rho)(1+g)}{g+\delta} - v_{kk} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right] \right\} \varepsilon_{x,t+1} \\
&\quad + \gamma \left[ v_{kw} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} - (1-\alpha) \frac{\bar{Y}}{\bar{C}} \right] w_{t+1} - \frac{1}{\psi} v_{kw} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2+g-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right) w_t \\
&\quad + n_1 \left( \frac{1}{\psi} - \gamma \right) v_{kw} \left\{ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2+g-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right. \\
&\quad \left. + \frac{n_1}{(1-n_1 v_{kk})} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{1-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} + v_{kk} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2+g-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right) \right] \right\} w_{t+1} \\
&\quad - \frac{1}{\psi} \left\{ (1-\alpha) \frac{\bar{Y}}{\bar{C}} + v_{kx} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}}{\bar{C}} \frac{(1-\delta) + (1-\rho)(1+g)}{g+\delta} - v_{kk} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right] \right\} x_t \\
&\quad - \frac{1}{\psi} \left[ \alpha \frac{\bar{Y}}{\bar{C}} + \frac{1-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} + v_{kk} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2+g-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} - v_{kk} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right) \right] k_t
\end{aligned} \tag{B8}$$

### 8. Solutions to $v_{kk}$ , $v_{kx}$ , and $v_{kw}$

We are half done. Equations (B2) and (B8) express  $c_{t+1} - c_t$  and  $m_{t+1}$  as functions of both parameters and  $v_{kk}$ ,  $v_{kx}$ ,  $v_{kw}$ . Next, we solve  $v_{kk}$ ,  $v_{kx}$ ,  $v_{kw}$  as functions of parameters.

From Equation (B1), we express the investment return  $r_{t+1}$  as follows:

$$\begin{aligned}
r_{t+1} &= \frac{\alpha(1-\alpha)\bar{Y}}{1+r}\bar{K}(z_t+\mu) + v_{kx}\frac{\zeta(1+g)^2}{(g+\delta)(1+r)}\varepsilon_{x,t+1} + \frac{\alpha(1-\alpha)\bar{Y}}{1+r}\bar{K}w_{t+1} + v_{kw}\frac{\zeta(1+g)^2}{(g+\delta)(1+r)}w_{t+1} \\
&+ v_{kk}v_{kw}\frac{\zeta(1+g)^2}{(g+\delta)(1+r)}w_t + v_{kx}\frac{\zeta\rho(1+g)^2}{(g+\delta)(1+r)}x_t + \frac{\alpha(1-\alpha)\bar{Y}}{1+r}\bar{K}x_t + v_{kk}v_{kx}\frac{\zeta(1+g)^2}{(g+\delta)(1+r)}x_t \\
&- v_{kx}\left\{\frac{\alpha(1-\alpha)\bar{Y}}{1+r}\bar{K} + \frac{\zeta(1+g)^2}{(g+\delta)(1+r)} + \left[\alpha\zeta\frac{\bar{Y}}{\bar{K}} + \zeta(1-\delta)\right]\frac{(1+g)}{(g+\delta)(1+r)}\right\}x_t \\
&- v_{kw}\left\{\frac{\alpha(1-\alpha)\bar{Y}}{1+r}\bar{K} + \frac{\zeta(1+g)^2}{(g+\delta)(1+r)} + \left[\alpha\zeta\frac{\bar{Y}}{\bar{K}} + \zeta(1-\delta)\right]\frac{(1+g)}{(g+\delta)(1+r)}\right\}w_t \\
&+ v_{kk}v_{kx}\frac{\zeta(1+g)^2}{(g+\delta)(1+r)}k_t + \left[\alpha\zeta\frac{\bar{Y}}{\bar{K}} + \zeta(1-\delta)\right]\frac{(1+g)}{(g+\delta)(1+r)}k_t \\
&- v_{kk}\left\{\frac{\alpha(1-\alpha)\bar{Y}}{1+r}\bar{K} + \frac{\zeta(1+g)^2}{(g+\delta)(1+r)} + \left[\alpha\zeta\frac{\bar{Y}}{\bar{K}} + \zeta(1-\delta)\right]\frac{(1+g)}{(g+\delta)(1+r)}\right\}k_t
\end{aligned}
\tag{B9}$$

The investment return also satisfies the FOC condition:  $E_t[m_{t+1} + r_{t+1}] = 0$ . In

the case of homoskedacity, we get:  $E_t m_{t+1} + E_t r_{t+1} = 0$

The coefficient of  $x_t$  in  $E_t[m_{t+1} + r_{t+1}]$  is:

$$\begin{aligned}
v_{kx} &= \frac{\frac{1}{\Psi}(1-\alpha)\bar{Y}}{\Gamma} \\
\Gamma &= -\left[\alpha\frac{\bar{Y}}{\bar{C}} + \frac{\bar{I}(1-\delta) + (1-\rho)(1+g)}{\bar{C}}\right] - \frac{\zeta(1-\rho)(1+g)^2}{(g+\delta)(1+r)} \\
&- \left\{\left[\alpha\zeta\frac{\bar{Y}}{\bar{K}} + \zeta(1-\delta)\right]\frac{(1+g)}{(g+\delta)(1+r)}\right\} + v_{kk}\left[\frac{\zeta(1+g)^2}{(g+\delta)(1+r)} - \frac{1+g}{g+\delta}\frac{\bar{I}}{\bar{C}}\right]
\end{aligned}
\tag{B10}$$

The coefficient of  $w_t$  in  $E_t[m_{t+1} + r_{t+1}]$  is:  $v_{kw} = 0$  (B11)

The coefficient of  $k_t$  in  $E_t[m_{t+1} + r_{t+1}]$  is:

$$\begin{aligned}
0 = & v_{kk} v_{kk} \left[ \frac{\zeta(1+g)^2}{(g+\delta)(1+r)} + \frac{1}{\psi} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right] + \left[ \alpha \zeta \frac{\bar{Y}}{\bar{K}} + \zeta(1-\delta) \right] \frac{(1+g)}{(g+\delta)(1+r)} - \frac{1}{\psi} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{1-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} \right) \\
& - v_{kk} \left\{ \frac{1}{\psi} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2+g-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} \right) + \frac{\alpha(1-\alpha)}{1+r} \frac{\bar{Y}}{\bar{K}} + \frac{\zeta(1+g)^2}{(g+\delta)(1+r)} + \frac{(1+g) \left[ \alpha \zeta \frac{\bar{Y}}{\bar{K}} + \zeta(1-\delta) \right]}{(g+\delta)(1+r)} \right\}
\end{aligned}$$

(B12)

From Equation (B12), we get solution to  $v_{kk}$  as a function of model's parameters.

$$Av_{kk}^2 + Bv_{kk} + C = 0$$

$$\begin{aligned}
-B = & \frac{1}{\psi} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{2+g-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} \right) + \frac{\alpha(1-\alpha)}{1+r} \frac{\bar{Y}}{\bar{K}} + \frac{\zeta(1+g)^2}{(g+\delta)(1+r)} + \frac{(1+g) \left[ \alpha \zeta \frac{\bar{Y}}{\bar{K}} + \zeta(1-\delta) \right]}{(g+\delta)(1+r)} \\
4AC = & 4 \times \left[ \frac{\zeta(1+g)^2}{(g+\delta)(1+r)} + \frac{1}{\psi} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right] \times \left\{ \left[ \alpha \zeta \frac{\bar{Y}}{\bar{K}} + \zeta(1-\delta) \right] \frac{(1+g)}{(g+\delta)(1+r)} - \frac{1}{\psi} \left( \alpha \frac{\bar{Y}}{\bar{C}} + \frac{1-\delta}{g+\delta} \frac{\bar{I}}{\bar{C}} \right) \right\} \\
2A = & 2 \times \left[ \frac{\zeta(1+g)^2}{(g+\delta)(1+r)} + \frac{1}{\psi} \frac{1+g}{g+\delta} \frac{\bar{I}}{\bar{C}} \right]
\end{aligned}$$

Solution to  $v_{kk}$  is delivered by substituting  $v_{kk}$  into Equation (B10).

9. Solutions to  $m_{t+1}$  and  $c_{t+1} - c_t$  as functions of model's parameters:

Finally, substituting  $v_{kk}$ ,  $v_{kx}$ ,  $v_{kw}$  into Equations (B8) and (B2), we can

express  $c_{t+1} - c_t$  and  $m_{t+1}$  as functions of model's parameters.

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