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Abstract. The contract is described and market examples given. Essential theoretical developments are introduced and cited chronologically. The principles and techniques of hedging and unique pricing are illustrated for the two simplest nontrivial examples: the classical Black-Scholes/Merton/Margrabe exchange option model brought somewhat up-to-date from its form three decades ago, and a lesser exponential Poisson analogue to illustrate jumps. Beyond these, a simplified Markovian SDE/PDE line is sketched in an arbitrage-free semimartingale setting. Focus is maintained on construction of a hedge using Itô's formula and on unique pricing, now for general homogenous payoff functions. Clarity is primed as the multivariate log-Gaussian and exponential Poisson cases are worked out.

Numeraire invariance is emphasized as the primary means to reduce dimensionality by one to the projective space where the SDE dynamics are specified and the PDEs solved (or expectations explicitly calculated). Predictable representation of a homogenous payoff with deltas (hedge ratios) as partial derivatives or partial differences of the option price function is highlighted. Equivalent martingale measures are utilized to show unique pricing with bounded deltas (and in the nondegenerate case unique hedging) and to exhibit the PDE or closed-form solutions as numeraire-deflated conditional expectations in the usual way. Homogeneity, change of numeraire, and extension to dividends are discussed.

Keywords and phrases that may afford a hyperlink to other EQF articles are underlined.

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1. Introduction

1.1. **Definition and examples.** A (European) exchange option is a contract that gives the buyer the right to exchange two (possibly dividend-paying) assets A and B at a fixed expiration time T, say to receive A and deliver (pay) B; so, the option payoff is

$$(A_T - B_T)^+ := \max(A_T - B_T, 0).$$

(American and Bermudan exchange options are complicated by early optimal exercise and not discussed here.) An ordinary (European) call or put on an asset struck at K can be viewed as in [9] as an option to exchange the asset with the T-maturity zero-coupon bond of principal K. More generally, a call or put on an s-maturity forward contract ($s \geq T$) on a (say) zero-dividend asset is equivalent to an option to exchange the asset at time T with an s-maturity zero-coupon bond. Options to exchange two stocks or commodities provide good hypothetical examples but are not prevalent in the market place.

Exchange options are related to spread options with time-T payoffs of the form $(X-Y)^+$, given two prescribed time-T observables X and Y. A common structure is a CMS spread option, with X and Y say the 20-year and 2-year spot swap rates at time T. A spread option can be viewed as an exchange option when there exist (or can be replicated) two zero-dividend assets A and B such that $A_T = X$ and $B_T = Y$. In the CMS case, A and B can be taken as the time-T coupons of two CMS bonds or swaps. Exchange options on dividend-paying assets are in practice reduced to the zero-dividend case in a similar way.

Interest-rate swaptions, including caplets and floorlets as one-period special cases, can be viewed both as ordinary call or put options struck at par on coupon bonds, or more directly as options to exchange the fixed and floating cashflow legs of a swap. The latter is the standard as it imposes the classical assumption of a lognormal ratio A_T/B_T on the forward swap rate (a swap-curve concept) rather than on the forward coupon bond price.

An exchange option is related to its reverse by parity: $(Y - X)^+ = (X - Y)^+ + Y - X$. (So an American option to exchange two fixed zero-dividend assets is not exercised early.)

1.2. **Pricing and hedging approaches.** The exchange option is a special case of a *path-independent* contingent claim with payoff a *homogenous function* of the underlying asset prices at expiration. It is governed by the same general theory. One makes sure that the underlying assets are arbitrage free which implies there are no free lunches in a strong sense. If the payoff can be attained by a sufficiently regular self-financing trading strategy (SFTS) (e.g. bounded number of shares or "deltas") then the law of one price holds and the option price at each time is defined as the value of the self-financing portfolio. Otherwise arbitrage-free pricing is not unique. We will not discuss this case, but only mention that one approach then chooses a linear pricing kernel (e.g., the minimal measure) among the many then available, and another is nonlinear based on expected utility maximization.

Payoff replication by a SFTS is a question of predictable representation. As the payoff in this case is a path-independent function of the underliers, it natural that the option price and deltas too be functions of time and the underliers at that time. This has been the traditional Markovian approach, beginning with Black and Scholes [1] and immediate extension by Merton [9]. Their simple choice of a geometric Brownian motion for the underlying asset in [1] and more generally of a deterministic-volatility forward price process in [9] meant that the underlying stochastic differential equation (SDE) and the associated partial differential equation (PDE) had constant coefficients (in log-state). Itô's formula was applied to construct a riskless hedge, with the deltas (hedge ratios) simply given by partial derivatives of the *option price function*, the unique solution to the PDE.

Black and Scholes constructed a SFTS for a call option struck at K by dynamically rebalancing long positions on the underlying asset A financed by shorting the riskless money market asset $B^* = (e^{rt})$, post an initial investment equal the option price. Merton's extension to stochastic interest rate r treated the call option as an option to exchange the asset A with the T-maturity zero-coupon bond B of principal K. The Black-Scholes model corresponded to a deterministic bond price $B_t = e^{-r(T-t)}K$, but now in general B had infinite variation. The former's simplicity was nonetheless recaptured by exploiting the homogenous symmetry of the option payoff to reduce dimensionality by one, in effect a projective transformation that hedged the forward option contract with trades in the forward asset A/B. The relevant volatility was accordingly the forward price volatility.

Margrabe [8] extended [9] to an option to exchange any two correlated assets assuming constant volatilities. Taking the option price to be a function C of (t, A, B), he observed in concert with [9] that by Itô's formula the self-financing equation with $\partial C/\partial A$ and $\partial C/\partial B$ as deltas is equivalent to $C(t, A, B)$ satisfying a PDE with no drift (first order terms in A, B). With C chosen as the homogenized Black-Scholes function, it followed by Euler's formula for homogenous functions that $\partial C/\partial A$ and $\partial C/\partial B$ formed in fact a SFTS replicating the exchange option payoff. Utilizing the ratio A/B, Margrabe "*let asset two be the numeraire*" ([8], p. 179), as [9] had. The result demonstrated that the exchange option is replicated by dynamically going long in A and short in B, with no trades in any other asset.

Martingale theory leads to a conceptual as well as computationally practical representation of solutions to the PDEs that describe option prices as a conditional expectation of terminal payoff. Harrison and Kreps [5] and Harrison and Pliska [6] developed in related papers an equivalent martingale measure framework that not only made this fruitful representation of the option price available, but laid a more general and probabilistic formulation of the notion of a dynamic hedge, or its mirror image, a replicating SFTS. Their arbitrage-free semimartingale approach does permit path dependency, yet accommodates Markovian SDE/PDE models even nicer. They took the money market asset B^* as a tradable entering any hedge, giving it a general stochastic form $B_t^* = e^{\int_0^t r_s ds}$ for discounting payoffs before expectation. In concert with Black-Scholes but contradistinction to Merton and Margrabe, the finite variation asset B^* was their exclusive choice of numeraire.

With the advent of the forward measure sometime later it was evident that Merton's choice of an infinite variation zero-coupon bond B as the financing hedge instrument fitted equivalent martingale measure theory perfectly well, leading no less to quicker derivations of concrete pricing formulae than B^*, as discounting is conveniently performed outside the expectation (see, e.g., [7] and [4]). Another useful numeraire was one by Neuberger [10] to price interest-rate swaptions. Viewed as an option to exchange the fixed and floating swap cashflows, the asset's ratio A/B represents the forward swap rate here. Assumed in [10] to have deterministic volatility yielded a model that has since served as industry standard to quote swaption implied volatilities. It is noteworthy that here the ratio A/B has deterministic volatility but A and B themselves decidedly do not. In time, El-Karoui et. al. [4] showed that one can basically change numeraire to any asset B and associate to it an equivalent measure under which A/B is a martingale for every other asset A.

Today option pricing and hedging theory has advanced farther and in many directions. Especially relevant to our discussion of exchange options is the principle of numeraire invariance and arbitrage-free modelling. For in-depth study of these and related topics we refer to Duffie [3] and Delbaen and Schachermayer [2] among other excellent books. Our approach is to concentrate the modelling in "projective coordinate" $X := A/B$, while minding that imposed conditions should be invariant under the transformation $X \mapsto 1/X$.

2. The deterministic-volatility and exponential-Poisson models

The option to exchange two assets with a *deterministic volatility* $\sigma(t)$ of the asset price ratio $X = A/B$ is celebrated as the simplest nontrivial example in option pricing theory. Its classical Black-Scholes/Merton option price function and explicit representation of the "deltas" ("hedge ratios") illustrate the principles that underline options in many assets with arbitrary homogenous payoffs and more general dynamics. There is another concrete albeit little known example with simple jumps in X involving the Poisson rather than the normal distribution. The pattern is similar, the main difference being that the deltas are the partial differences rather than the partial derivatives of the option price function.

We fix throughout a stochastic basis $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$ with time horizon $t \in [0, T]$, $T > 0$. In this section we fix two *zero-dividend* assets with price processes $A = (A_t)$ and $B = (B_t)$.

2.1. **The exchange option price process.** When A and B are semimartingales, we call a pair (δ^A, δ^B) of (locally) bounded predictable processes a (locally) bounded self-financing trading strategy (SFTS) (see more generally Sec. 3.1) if $C = C_0 + \int \delta^A dA + \int \delta^B dB$, where

$$C = \delta^A A + \delta^B B. \tag{2.1}$$

Clearly C is then a semimartingale, $\Delta C = \delta^A \Delta A + \delta^B \Delta B$, hence $C_- = \delta^A A_- + \delta^B B_-$.

The differential form of the self-financing equation is often handy:

$$dC = \delta^A dA + \delta^B dB. \tag{2.2}$$

SFTSs form a linear space. If there exists a unique *bounded* SFTS (δ^A, δ^B) such that

$$C_T = (A_T - B_T)^+, \tag{2.3}$$

then one is justified to call C *the exchange option price process* and δ^A and δ^B the deltas.

Assume now the semimartingales A and B are positive and have positive left limits. The numeraire invariance principle (see Sec 2.3 and more comprehensively Sec 3.2) states that if (δ^A, δ^B) is a locally bounded SFTS then $C = \delta^A A + \delta^B B$ satisfies

$$d(\frac{C}{B}) = \delta^A d(\frac{A}{B}).$$

(Ditto by symmetry with A as numeraire.) This is useful for uniqueness. Numeraire invariance also states the converse: if C is a semimartingale and δ^A a locally bounded predictable process such that $d(\frac{C}{B}) = \delta^A d(\frac{A}{B})$, then (δ^A, δ^B) is a SFTS and (2.1) and (2.2) hold, where $\delta^B = \frac{C_-}{B_-} - \delta^A \frac{A_-}{B_-}$. This reduces existence to finding an F_0 and δ^A such that

$$(\frac{A_T}{B_T} - 1)^+ = F_0 + \int_0^T \delta_t^A d(\frac{A_t}{B_t}).$$

The exchange option price process is then the semimartingale $C = B(F_0 + \int \delta^A d(\frac{A}{B}))$.

Numeraire invariance in effect reduces general option pricing and hedging to a market where one of the asset price processes equals 1 identically. The remaining task is to find the above "projective" predictable representation of the ratio payoff against the ratio process.

2.2. **Deterministic-volatility exchange option model.** Let $\sigma(t) > 0$ be a continuous positive function. Define the *Black-Scholes/Merton projective option price function*

$$f(t, x) := x\delta_A(t, x) + \delta_B(t, x) \tag{2.4}$$

on $t \le T$, $x > 0$, where $\delta_A(T, x) := 1_{x>1}$, $\delta_B(T, x) := -1_{x>1}$, and for $t < T$,

$$\delta_A(t, x) := N(\frac{\log x}{\sqrt{\nu_t}} + \frac{\sqrt{\nu_t}}{2}), \quad \delta_B(t, x) := -N(\frac{\log x}{\sqrt{\nu_t}} - \frac{\sqrt{\nu_t}}{2}), \tag{2.5}$$

where $\nu_t := \int_t^T \sigma_s^2 ds$ and $N(\cdot)$ is the normal distribution function. The function $f(t, x)$ is continuous, and on $t < T$ is C^1 in t and analytic in x. Also, $-1 \le \delta_B \le 0 \le \delta_A \le 1$, and

$$f(T, x) = (x - 1)^+, \quad \frac{\partial f}{\partial x}(t, x) = \delta_A(t, x).$$

As is well known and seen in Sec. 2.9 or 3.6, the function $f(t, x)$ is the unique $C^{1,2}$ (on $t < T$) solution with bounded partial $\frac{\partial f}{\partial x}(t, x)$ subject to $f(T, x) = (x - 1)^+$ of the PDE

$$\frac{\partial f}{\partial t}(t, x) + \frac{1}{2}\sigma^2(t)x^2 \frac{\partial^2 f}{\partial x^2}(t, x) = 0. \tag{2.6}$$

Assume now $A = BX$ for some positive continuous semimartingale $X > 0$ satisfying

$$d[\log X]_t = \sigma^2(t)dt. \qquad (A = BX) \tag{2.7}$$

Under this assumption, one traditionally defines *the exchange option price process* C by

$$C := BF, \quad F = (F_t), \quad F_t := f(t, X_t). \tag{2.8}$$

Clearly, $C_T = (A_T - B_T)^+$. The definition is justified using the continuous semimartingales

$$\delta_t^A := \delta_A(t, X_t) = \frac{\partial f}{\partial x}(t, X_t), \qquad \delta_t^B := \delta_B(t, X_t) = F_t - \delta_t^A X_t. \tag{2.9}$$

Clearly, $C = \delta^A A + \delta^B B$, and the deltas are bounded: $0 \le \delta^A \le 1$ and $-1 \le \delta^B \le 0$. Since $f(t, x)$ satisfies the PDE (2.6) (as directly verified) and $\frac{\partial f}{\partial x}(t, X_t) = \delta_t^A$, by Itô's formula the continuous semimartingale $F := (f(t, X_t))$ satisfies the predictable representation

$$dF = \delta^A dX. \tag{2.10}$$

If at this stage we assume B is a semimartingale, then A and C are semimartingales too, and by the invariance principle next, $dC = \delta^A dA + \delta^B dB$ and (δ^A, δ^B) is a bounded SFTS.

2.3. **Numeraire invariance.** Let X and F and be two semimartingales and δ^A be a locally bounded predicable process such that $dF = \delta^A dX$. Set $\delta^B = F - \delta^A X$. Clearly $\delta^B = F_- - \delta^A X_-$ since $\Delta F = \delta^A \Delta X$. Let B be any semimartingale. Set $A = BX$, $C = BF$. Clearly $C = \delta^A A + \delta^B B$. We claim $dC = \delta^A dA + \delta^B dB$, so (δ^A, δ^B) is a SFTS.

Indeed, this follows by applying Itô's product rule to BF, then substituting $dF = \delta^A dX$ and $F_- = \delta^B + \delta^A X_-$, followed by Itô's product rule on BX:

$$\begin{aligned} dC = d(BF) &= B_- dF + F_- dB + d[B, F] \\ &= B_- \delta^A dX + (\delta^B + \delta^A X_-) dB + \delta^A d[B, X] \\ &= \delta^A d(BX) + \delta^B dB = \delta^A dA + \delta^B dB. \end{aligned}$$

Conversely and similarly, if A and B are semimartingales with $B, B_- > 0$ and (δ^A, δ^B) is a SFTS, then $dF = \delta^A dX$, where $X = A/B$, $F = C/B$, and $C = \delta^A A + \delta^B B$.

2.4. **Exponential Poisson exchange option model.** Assume that the two zero-dividend asset price processes A and B satisfy $A = BX$, where X is a semimartingale satisfying

$$X_t = X_0 e^{\beta P_t - (e^\beta - 1)\lambda t} \tag{2.11}$$

for some constants $\beta \neq 0$, $\lambda > 0$ and semimartingale P such that $[P] = P$ and $P_0 = 0$ (so, $P_t = \sum_{s \le t} 1_{\Delta P_s \neq 0}$). Define the projective option price function $f(t, x)$, $x > 0$ by

$$f(t, x) := \sum_{n=0}^{\infty} (x e^{\beta n - (e^\beta - 1)\lambda(T-t)} - 1)^+ \frac{\lambda^n}{n!} (T-t)^n e^{-\lambda(T-t)}, \tag{2.12}$$

and exchange option price process by

$$C := BF, \quad F = (F_t), \quad F_t := f(t, X_t). \tag{2.13}$$

Clearly $f(T, x) = (x-1)^+$ and $C_T = (A_T - B_T)^+$. One has the predictable representation

$$dF = \delta^A dX \tag{2.14}$$

as shown shortly, where

$$\delta^A_t := \delta_A(t, X_{t-}), \quad \delta_A(t, x) := \frac{f(t, e^\beta x) - f(t, x)}{(e^\beta - 1)x}. \tag{2.15}$$

Thus by numeraire invariance (δ^A, δ^B) is a SFTS if A and B are semimartingales, where

$$\delta^B := F - \delta^A X = F_- - \delta^A X_-. \tag{2.16}$$

Moreover, it is bounded. Indeed, since $|(e^\beta y - 1)^+ - (y-1)^+| \le |e^\beta - 1| y$ for any $y > 0$,

$$0 \le \delta_A(t, x) \le \sum_{n=0}^{\infty} e^{\beta n - (e^\beta - 1)\lambda(T-t)} \frac{\lambda^n}{n!} (T-t)^n e^{-\lambda(T-t)} = 1.$$

Hence, $0 \le \delta^A \le 1$. Similarly, $-1 \le \delta^B \le 0$.

We note that $f(t,x)$ is *not* C^1 in x (though convex, absolutely continuous and piecewise analytic in x). We also caution that this model is arbitrage free only when $\mathbb{P}\{P_t = n\} > 0$ for all $t > 0$ and $n \in \mathbb{N}$, e.g., when P is a Poisson process under an equivalent measure.

2.5. **Derivation of the predictable representation (2.14).** To show $dF = \delta^A dX$, we first note that $[P]^c = 0$ since $[P] = P$; hence $(\Delta P)^2 = \Delta P$ and $P_t = [P]_t = \sum_{s\le t} \Delta P_s$. If $v(p)$, $p \in \mathbb{R}$, is any function, then clearly $V = (v(P_t))$ is a semimartingale and we have

$$\Delta V_t = v(P_t) - v(P_{t-}) = (v(P_t) - v(P_{t-}))\Delta P_t = (v(P_{t-}+1) - v(P_{t-}))\Delta P_t.$$

Hence, as V is clearly the sum of its jumps,

$$V_t - v(0) = \sum_{s\le t} \Delta V_s = \sum_{s\le t} (v(P_{s-}+1) - v(P_{s-}))\Delta P_s = \int_0^t (v(P_{s-}+1) - v(P_{s-}))dP_s.$$

Likewise, $(u(t,P_t))$ is a semimartingale for any C^1 in t function $u(t,p)$, $p \in \mathbb{R}$, and one has

$$du(t,P_t) = \frac{\partial u}{\partial t}(t,P_{t\_})dt + (u(t,P_{t\_}+1) - u(t,P_{t\_}))dP_t. \tag{2.17}$$

Now, define the function

$$x(t,p) := X_0 e^{\beta p - (e^\beta - 1)\lambda t}. \qquad (p \in \mathbb{R}) \tag{2.18}$$

Clearly $X_t = x(t,P_t)$. Applying (2.17) to the function $x(t,p)$ and using that

$$\frac{\partial x}{\partial t}(t,p) = -x(t,p)(e^\beta - 1)\lambda, \quad x(t,p+1) - x(t,p) = x(t,p)(e^\beta - 1),$$

(or alternatively applying Itô's formula to $x(t,P_t)$ and simplifying) yields

$$dX_t = X_{t-}(e^\beta - 1)d(P_t - \lambda t). \tag{2.19}$$

Next, define the function of $t \le T$ and $p \in \mathbb{R}$,

$$u(t,p) := f(t,x(t,p)) = \sum_{n=0}^{\infty} (X_0 e^{\beta(p+n) - (e^\beta - 1)\lambda T} - 1)^+ \frac{\lambda^n}{n!}(T-t)^n e^{-\lambda(T-t)}. \tag{2.20}$$

Clearly, $u(t,P_t) = F_t$. One readily verifies that $u(t,p)$ satisfies the partial difference equation

$$\frac{\partial u}{\partial t}(t,p) + \lambda(u(t,p+1) - u(t,p)) = 0. \tag{2.21}$$

Hence by (2.17) we have,

$$dF_t = (u(t,P_{t\_}+1) - u(t,P_{t\_}))d(P_t - \lambda t). \tag{2.22}$$

Combining this with (2.19) and the fact that clearly

$$u(t,p+1) - u(t,p) = f(t,e^\beta x(t,p)) - f(t,x(t,p)),$$

we conclude that, as desired,

$$dF_t = \frac{f(t,e^\beta X_{t-}) - f(t,X_{t-})}{(e^\beta - 1)X_{t\_}} dX_t. \tag{2.23}$$

2.6. **The homogenous option price function.** There is an alternative derivation of the self-financing equation $dC = \delta^A dA + \delta^B dB$ much along the original lines in [9] and [8] that does not employ numeraire invariance. It is related to a curious family of two-dimensional PDEs satisfied by the *homogeneous Merton/Margrabe option price function* $c(t, a, b)$ below.

Let $f(t, x)$, $x > 0$, be any $C^{1,2}$ function, e.g., as in (2.4). Define the homogenized function

$$c(t, a, b) := bf(t, \frac{a}{b}). \qquad (a, b > 0) \tag{2.24}$$

Then $c(t, a, b)$ is homogenous of degree 1 in (a, b), and hence by Euler's formula

$$c(t, a, b) = \frac{\partial c}{\partial a}(t, a, b)a + \frac{\partial c}{\partial b}(t, a, b)b. \tag{2.25}$$

A laborious repeated application of the chain rule on (2.24) gives

$$a^2 \frac{\partial^2 c}{\partial a^2}(t, a, b) = b^2 \frac{\partial^2 c}{\partial b^2}(t, a, b) = -ab \frac{\partial^2 c}{\partial a \partial b}(t, a, b) = b\, x^2 \frac{\partial^2 f}{\partial x^2}(t, x), \qquad x := \frac{a}{b}. \tag{2.26}$$

Let $\sigma(t)$, $\sigma_A(t, a, b)$, $\sigma_B(t, a, b)$, $\sigma_{AB}(t, a, b)$ be any functions $(x, a, b > 0)$ such that

$$\sigma^2(t) = \sigma_A^2(t, a, b) + \sigma_B^2(t, a, b) - 2\sigma_{AB}(t, a, b). \tag{2.27}$$

Using (2.26), (2.27), and $\frac{\partial c}{\partial t}(t, a, b) = b \frac{\partial f}{\partial t}(t, \frac{a}{b})$, we see that $c(t, a, b)$ satisfies the PDE

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma_A^2(t, a, b)a^2 \frac{\partial^2 c}{\partial a^2} + \frac{1}{2}\sigma_B^2(t, a, b)b^2 \frac{\partial^2 c}{\partial b^2} + \sigma_{AB}(t, a, b)ab \frac{\partial^2 c}{\partial a \partial b} = 0 \tag{2.28}$$

if and only if $f(t, x)$ satisfies the PDE (2.6): $\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(t)x^2 \frac{\partial^2 f}{\partial x^2} = 0$.

The PDE (2.28) appeared in [8] and [9] with the σ coefficients independent of (a, b). It was further observed that if $d[\log A]_t = \sigma_A^2(t)dt$, $d[\log B]_t = \sigma_B^2(t)dt$ and $d[\log A, \log B]_t = \sigma_{AB}(t)dt$, then Itô' formula and the PDE (2.28) imply at once $dc(t, A_t, B_t) = \delta_t^A dA_t + \delta_t^B dB_t$, with δ^A and δ^B as in (2.29) below, and therefore (δ^A, δ^B) is a SFTS with price process $c(t, A, B)$ by Euler's formula (2.25). We now expand on this in a more general setting.

Let $f(t, x)$ henceforth denote the Black-Scholes/Merton projective option function (2.4), and $c(t, a, b) := bf(t, a/b)$ be the homogeneous Merton/Margrabe option function. Clearly,

$$\frac{\partial c}{\partial a}(t, a, b) = \frac{\partial f}{\partial x}(t, \frac{a}{b}) = \delta_A(t, \frac{a}{b}).$$

This combined with the Euler's formula (2.25) and the definition (2.4) $f := \delta_A x + \delta_B$ give

$$\frac{\partial c}{\partial b}(t, a, b) = \delta_B(t, \frac{a}{b}).$$

Assume A and B are positive semimartingales with positive left limits and $X := A/B$ has deterministic volatility: $d[X]_t = X_t^2 \sigma^2(t)dt$ for some continuous function $\sigma(t) > 0$. Whence, the deltas are conveniently the sensitivities of the homogenous option price function:

$$\delta_t^A = \frac{\partial c}{\partial a}(t, A_t, B_t), \qquad \delta_t^B = \frac{\partial c}{\partial b}(t, A_t, B_t). \tag{2.29}$$

Since X is continuous, we also have $\delta_t^A = \frac{\partial c}{\partial a}(t, A_{t-}, B_{t-})$, ditto δ_t^B. Sec. 2.2 yields $dC = \delta^A dA + \delta^B dB$ with $C_t = B_t f(t, X_t) = c(t, A_t, B_t)$. Therefore, by (2.29) and Itô's formula,

$$\frac{\partial c}{\partial t}dt + \frac{1}{2}\frac{\partial^2 c}{\partial a^2}d[A]_t^c + \frac{1}{2}\frac{\partial^2 c}{\partial b^2}d[B]_t^c + \frac{\partial^2 c}{\partial a \partial b}d[A,B]_t^c = 0, \tag{2.30}$$

where the partials are evaluated at (t, A_{t-}, B_{t-}) and $[\cdot]^c$ is the bracket continuous part. (The jump term in Itô's formula vanishes as it equals $\sum_{s \le t}(\Delta C_s - \delta_s^A \Delta A_s - \delta_s^B \Delta B_s) = 0$.)

Coming to the main point, assume now $d[\log A]_t = \sigma_A^2(t, A_t, B_t)dt$ for some function σ_A and similarly $d[\log B] = \sigma_B^2 dt$ and $d[\log A, \log B] = \sigma_{AB} dt$. Then Eq. (2.27) holds using $\log X = \log A - \log B$. Since $f(t,x)$ satisfies the PDE (2.6), the PDE (2.28) follows as before by the chain rule. But, (2.28) implies (2.30), which by Itô's formula in turn implies the self-financing equation $dC = \delta^A dA + \delta^B dB$ with δ^A and δ^B given by (2.29).

2.7. **Change of numeraire.** The solution $c(t,a,b)$ to the PDE (2.28) subject to $c(T,a,b) = (a-b)^+$ can be expressed in a form $\mathbb{E}\,(X-Y)^+$ for some random variables X and $Y > 0$ with means a and b. Expectations of this form often become more tractable by a change of measure as in [4]. Define the equivalent probability measure $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{Y}{\mathbb{E}\,Y}$. Clearly,

$$\mathbb{E}^{\mathbb{Q}}(\frac{X}{Y}) = \frac{\mathbb{E}(X)}{\mathbb{E}(Y)}. \qquad (\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{Y}{\mathbb{E}(Y)}) \tag{2.31}$$

Replacing X by $(X-Y)^+$ in (2.31) and using the homogeneity to factor out Y, we get

$$\mathbb{E}\,(X-Y)^+ = \mathbb{E}(Y)\mathbb{E}^{\mathbb{Q}}(\frac{X}{Y} - 1)^+. \tag{2.32}$$

If X/Y is $\mathbb{Q}$-lognormally distributed then (2.32) with the aid of (2.31) readily yields,

$$\mathbb{E}\,(X-Y)^+ = \mathbb{E}(X)N(\frac{\log(\mathbb{E}X/\mathbb{E}Y)}{\sqrt{\nu^{\mathbb{Q}}}} + \frac{\sqrt{\nu^{\mathbb{Q}}}}{2}) - \mathbb{E}(Y)N(\frac{\log(\mathbb{E}X/\mathbb{E}Y)}{\sqrt{\nu^{\mathbb{Q}}}} - \frac{\sqrt{\nu^{\mathbb{Q}}}}{2}), \tag{2.33}$$

where $\nu^{\mathbb{Q}} := \mathrm{var}^{\mathbb{Q}}[\log(X/Y)]$. When X and Y are bivariately lognormally distributed, it is not difficult to show that X/Y is lognormally distributed in both $\mathbb{P}$ and $\mathbb{Q}$ with the same log-variance $\nu^{\mathbb{Q}} = \nu := \mathrm{var}[\log(X/Y)]$. Then $\nu^{\mathbb{Q}}$ can be replaced with ν in (2.33). This occurs when the functions σ_A, σ_B and σ_{AB} in (2.28) are independent of a and b, as in [8,9].

2.8. **Uniqueness.** Assume A and B are positive semimartingales with positive left limits such that $X := A/B$ is square-integrable martingale under an equivalent probability measure $\mathbb{Q}$ and $d\langle X\rangle_t^{\mathbb{Q}} = X_{t\_}^2 \sigma_t^2 dt$ for some nowhere zero continuous process σ, where $\langle X\rangle^{\mathbb{Q}}$ is the $\mathbb{Q}$-compensator of $[X]$. (Of course, $\langle X\rangle^{\mathbb{Q}} = [X]$ if X is continuous.) Let (δ^A, δ^B) be a SFTS and set $C := \delta^A A + \delta^B B$. We claim that $\delta^A = \delta^B = 0$ if $C_T = 0$ and δ^A is bounded.

Indeed, set $F := C/B$. By numeraire invariance $dF = \delta^A dX$. Hence F is a $\mathbb{Q}$-square-integrable martingale since X is and δ^A is bounded. Thus, $F = 0$ since $F_T = C_T/B_T = 0$. Hence, $0 = d\langle F\rangle^{\mathbb{Q}} = (\delta^A)^2 X_-^2 \sigma^2 dt$. But, $X_- \sigma \neq 0$. Thus, $\delta^A = 0$ and $\delta^B = F - \delta^A X = 0$.

2.9. **Deterministic-volatility model uniqueness.** Assume that A and B are positive semimartingales with positive left limits and $X := A/B$ is an Itô process following

$$\frac{dX_t}{X_t} = \mu_t dt + \sigma_t dZ_t, \qquad (X := \frac{A}{B}) \tag{2.34}$$

where Z is a Brownian motion and μ and $\sigma > 0$ are continuous adapted processes with σ bounded and $\mathbb{E}[e^{\frac{1}{2}\int_0^T (\frac{\mu_t}{\sigma_t})^2 dt}] < \infty$. Let (δ^A, δ^B) be a SFTS with δ^A bounded. Set $C := \delta^A A + \delta^B B$. We claim that $\delta^A = \delta^B = 0$ if $C_T = 0$. Indeed, the process

$$M := \mathcal{E}(-\int \frac{\mu}{\sigma} dZ) = e^{-\int \frac{\mu}{\sigma} dZ - \frac{1}{2}\int (\frac{\mu}{\sigma})^2 dt},$$

is then a positive martingale with $M_0 = 1$. Define the equivalent probability measure $\mathbb{Q}$ by $d\mathbb{Q} = M_T d\mathbb{P}$. The process $W := Z + \int \frac{\mu}{\sigma} dt$ is a $\mathbb{Q}$-Brownian motion because $[W]_t = t$ and W is $\mathbb{Q}$-local martingale as MW is a local martingale using Itô's product rule:

$$d(MW) - WdM = MdW + d[W, M] = M(dZ + \frac{\mu}{\sigma} dt) - M\frac{\mu}{\sigma} d[Z] = MdZ.$$

Moreover, $dX = X\sigma dW$ by (2.34). Therefore X is a $\mathbb{Q}$-square integrable martingale (in fact in $\mathcal{H}^p(\mathbb{Q})$ for all $p > 0$) since σ is bounded. The claim thus follows by Sec. 2.8.

As a corollary of the proof, $F := C/B$ is $\mathbb{Q}$-square-integrable martingale because δ^A is bounded and by numeraire invariance $dF = \delta^A dX$. In particular, $C_t = B_t \mathbb{E}^{\mathbb{Q}}[C_T/B_T \,|\, \mathcal{F}_t]$.

Assume now σ_t is *deterministic*. The results of Sec. 2.2 hold since $d[\log X] = \sigma_t^2 dt$. But we can now derive them more conceptually. Indeed, both conditioned on $\mathcal{F}_t$ and unconditionally, X_T/X_t is $\mathbb{Q}$-lognormally distributed with mean 1 and log-variance $\int_t^T \sigma_s^2 ds$ since $X_T = X_t e^{\int_t^T \sigma_s dW_s - \frac{1}{2}\int_t^T \sigma_s^2 ds}$. Hence

$$f(t, X_t) = \mathbb{E}^{\mathbb{Q}}[(X_T - 1)^+ \,|\, \mathcal{F}_t], \quad \text{where } f(t, x) := \mathbb{E}^{\mathbb{Q}}(x\frac{X_T}{X_t} - 1)^+, \tag{2.35}$$

which function readily equals the Black-Scholes/Merton option price function (2.4). Thus, $F := (f(t, X_t))$ is a $\mathbb{Q}$-martingale. Therefore Itô's formula implies that $f(t, x)$ satisfies the PDE (2.6) and $dF = \delta^A dX$ where $\delta^A := \frac{\partial f}{\partial x}(t, X_t)$. Numeraire invariance now yields $(\delta^A, \delta^B := F - \delta^A X)$ is a SFTS. Clearly $C_T = (A_T - B_T)^+$ where $C := \delta^A A + \delta^B B = BF$.

2.10. **Exponential Poisson model uniqueness.** Let $\beta \neq 0$ be a constant and κ and λ be positive continuous adapted processes such that λ is bounded and $\mathbb{E}\, e^{\int_0^T (\frac{\lambda_t}{\kappa_t} - 1)^2 \kappa_t dt} < \infty$. Let P be semimartingale satisfying $[P] = P$ with $P_0 = 0$ and compensator $\int \kappa dt$. Assume that A and B are positive semimartingales with positive left limits and $X := \frac{A}{B}$ satisfies

$$dX_t = X_{t-}(e^\beta - 1)(dP_t - \lambda_t dt). \tag{2.36}$$

Using $de^{\beta P} = (e^\beta - 1)e^{\beta P_-} dP$ or as in Sec. 2.5, this is equivalent to the integrated form

$$X_t = X_0 e^{\beta P_t - (e^\beta - 1)\int_0^t \lambda_s ds}. \tag{2.37}$$

Let (δ^A, δ^B) be a SFTS with δ^A bounded. Set $C := \delta^A A + \delta^B B$. We claim $\delta^A = \delta^B = 0$ if $C_T = 0$. Indeed, $\mathbb{E}\, e^{\langle \int (\frac{\lambda}{\kappa}-1)(dP-\kappa dt)\rangle_T} = \mathbb{E}\, e^{\int_0^T (\frac{\lambda_t}{\kappa_t}-1)^2 \kappa_t dt} < \infty$, so the positive local martingale

$$M := \mathcal{E}(\int (\frac{\lambda}{\kappa} - 1)(dP - \kappa dt) = e^{-\int (\lambda-\kappa)dt} \prod_{s\leq \cdot} (1 + (\frac{\lambda_s}{\kappa_s} - 1)\Delta P_s)$$

is a martingale. Define the equivalent probability measure $\mathbb{Q}$ by $d\mathbb{Q} = M_T d\mathbb{P}$. Then $N := P - \int \lambda dt$ is a $\mathbb{Q}$-local martingale as MN is a local martingale by Itô's product rule:

$$\begin{aligned} d(MN) - N_- dM &= M_- dN + d[M, N] \\ &= M_-(dP - \lambda dt) + M_-(\frac{\lambda}{\kappa} - 1)dP = M_- \frac{\lambda}{\kappa}(dP - \kappa dt). \end{aligned}$$

Therefore by (2.36) X is a $\mathbb{Q}$-square-integrable martingale (in fact in $\mathcal{H}^p(\mathbb{Q})$ all $p > 0$) since λ is bounded. Thus, by Sec. (2.8), $\delta^A = \delta^B = 0$ if $C_T = 0$, as claimed.

As a corollary to the proof, $F := C/B$ is $\mathbb{Q}$-square-integrable martingale because δ^A is bounded and by numeraire invariance $dF = \delta^A dX$. In particular, $C_t = B_t \mathbb{E}^{\mathbb{Q}}[C_T/B_T \,|\, \mathcal{F}_t]$.

Assume now λ is a positive constant. By (2.37) we are in a special case of the exponential Poisson model. Further, P is a $\mathbb{Q}$-*Poisson process* with intensity λ since $[P] = P$. We now have uniqueness, but additionally, the previous results follow more conceptually as follows.

Conditioned on $\mathcal{F}_t$, $P_T - P_t$ is $\mathbb{Q}$-Poisson distributed with mean $\lambda(T-t)$. Its unconditional $\mathbb{Q}$-distribution is identical. Thus the $\mathcal{F}_t$- conditional and the unconditional $\mathbb{Q}$-distribution of X_T/X_t are identical and are exponentially Poisson distributed with mean 1. Hence

$$f(t, X_t) = \mathbb{E}^{\mathbb{Q}}[(X_T - 1)^+ \,|\, \mathcal{F}_t], \quad \text{where } f(t, x) := \mathbb{E}^{\mathbb{Q}}(x\frac{X_T}{X_t} - 1)^+, \tag{2.38}$$

which function readily equals that defined in (2.12). Thus, $F := (f(t, X_t))$ is a $\mathbb{Q}$-martingale. Using this and (2.17) one shows that F satisfies (2.23) and with it that the pair (δ^A, δ^B) as defined in (2.15), (2.16) is a bounded SFTS for the exchange option.

2.11. **Extension to dividends.** Consider two assets with positive price processes $\hat{A}$ and $\hat{B}$ and continuous dividend yields y_t^A and y_t^B. When there exist traded or replicable zero-dividend assets A and B such that $A_T = \hat{A}_T$ and $B_T = \hat{B}_T$ (if not there is little hope of replication), it is natural to define the price process of the option to exchange $\hat{A}$ and $\hat{B}$ to be that of the option to exchange A and B. If y^A and y^B are deterministic, then consistently with the treatment of dividends in [9], A (and similarly B) is simply given by

$$A_t := a\tilde{A}_t = e^{-\int_t^T y_s^A ds} \hat{A}_t, \qquad \tilde{A}_t := e^{\int_0^t y_s^A ds} \hat{A}_t, \qquad a := e^{-\int_0^T y_t^A dt}.$$

Note A/B is a semimartingale if and only if $\hat{A}/\hat{B}$ is, in which case $[\log A/B] = [\log \hat{A}/\hat{B}]$.

In general, $\tilde{A}_t$ is the price of the zero-dividend asset that initially buys one share of $\hat{A}$ and thereon continually reinvests all dividends in $\hat{A}$ itself. What is required is that the four zero-dividend assets A, $\tilde{A}$, B and $\tilde{B}$ be arbitrage-free among each other (see Sec. 3.3).

For instance, say $\hat{A}$ and $\hat{B}$ are the yen/dollar and yen/Euro exchange rates viewed as yen-denominated dividend assets. Then A is the yen-value of the U.S. T-maturity zero-coupon bond and $\tilde{A}$ is the yen-value of the U.S. money market asset. This exchange option is equivalent to a Euro-denominated call struck at 1 on the Euro/dollar exchange rate $\hat{A}/\hat{B}$. The ratio A/B is the *forward* Euro/dollar exchange rate. If it has deterministic volatility, we are as in a setting of [7] which yields the same pricing formula as that from Sec 2.2.

3. Pricing and hedging options with homogenous payoffs

We took some shortcuts above to quickly presents the main results for two of the simplest and among the most interesting examples. But, a better understanding of the principles at work requires generalization to contingent claims C on many assets with price processes $A = (A^1, \cdots, A^m) > 0$ and a path-independent payoff $C_T = h(A_T)$ given as a homogenous function $h(a)$, $a \in \mathbb{R}^m_+$, of the asset prices A_T at expiration time T. Combined with an underlying SDE and the resulting PDE, such Markovian setting utilizes the invariance principle and equivalent martingale measures to derive unique pricing and construct a SFTS that replicates the given payoff $h(A_T)$ in general. The construction is explicit in the multivariate extensions of the deterministic-volatility and exponential Poisson models.

The homogeneity of the payoff function $h(a)$ implies $h(A_T) = A^m_T g(X_T)$ where $g(x) := h(x,1)$, $x \in \mathbb{R}^n_+$, $n := m-1$, and $X := (\frac{A^1}{A^m}, \cdots, \frac{A^n}{A^m})$. Once a predictable representation $F = F_0 + \delta' \cdot X$, $F_T = g(X_T)$ is found, then by numeraire invariance $\delta := (\delta', \delta^m)$ will be a SFTS with payoff $h(A_T)$, where $\delta^m := F_- - \sum_{i=1}^n \delta^i X_- = F - \sum_{i=1}^n \delta^i X$. Uniqueness of pricing requires boundedness of partial derivatives (or differences) of $h(a)$ (or $g(x)$) and that A be arbitrage free, meaning X is a martingale under an equivalent measure. Arbitrage freedom holds "generically" when the matrix $(\langle X^i, X^j \rangle)$ is nonsingular, basically a "no redundant asset" condition. Then the SFTS itself is unique, namely the constructed one.

Libor and swap derivatives are among contingent claims with homogenous payoffs.

3.1. **Self-financing trading strategies.** By a SFTS we mean a pair (δ, A) of an m-dimensional semimartingale $A = (A^1, ..., A^m)$ and an A-integrable predictable vector process $\delta = (\delta^1, \cdots, \delta^m)$ such that (with $\delta \cdot A$ denoting the m-dimensional stochastic integral)

$$\sum_{i=1}^m \delta^i A^i = \sum_{i=1}^m \delta^i_0 A^i_0 + \delta \cdot A. \tag{3.1}$$

We then say δ is *a SFTS for A*. This is equivalent to saying that the SFTS *price process*

$$C := \sum_{i=1}^m \delta^i A^i \tag{3.2}$$

satisfies $C = C_0 + \delta \cdot A$. Clearly C is then a semimartingale, $\Delta C = \sum_i \delta^i \Delta A_i$, and hence

$$C_- = \sum_{i=1}^m \delta^i A^i_-.$$

If δ^i are bounded (say by b) and A^i are martingales then the SFTS price process C is a martingale because C is then a local martingale that is dominated by a martingale M:

$$|C_t| \le b \sum_i |A^i_t| = b \sum_i |\mathbb{E}[A^i_T \,|\, \mathcal{F}_t]| \le b \sum_i \mathbb{E}[|A^i_T| \,|\, \mathcal{F}_t] =: M_t.$$

As suggested by when δ is locally bounded, we often use the differential form

$$dC = \sum_{i=1}^{m} \delta^i dA^i \tag{3.3}$$

of the equation $C = C_0 + \delta \cdot A$ as a convenient symbolic equivalent in calculations. One interprets the A^i as prices of m zero-dividend assets and the δ^i_t as the number of shares invested in them at time t. Then C_t indicates the resultant self-financing portfolio price by (3.2), and (3.3) is the self-financing equation, saying that the change dC in the portfolio price is due only to the changes dA^i in the asset prices with no financing from outside.

Assume for the remainder of this subsection as a way of motivation that A is continuous and $C_t = c(t, A_t)$ for some $C^{1,2}$ function $c(t,a)$.[1] Then by (3.3) and Itô's formula we have,

$$\frac{\partial c}{\partial t}(t, A_t)dt + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t)d[A^i, A^j]_t = \sum_{i=1}^{m}(\delta^i_t - \frac{\partial c}{\partial a_i}(t, A_t))dA^i_t. \tag{3.4}$$

In particular if $\delta^i_t = \frac{\partial c}{\partial a_i}(t, A_t)$ for all i then $c(t, A_t) = \sum_i \frac{\partial c}{\partial a_i}(t, A_t)A^i_t$ by (3.2) and

$$\frac{\partial c}{\partial t}(t, A_t)dt + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t)d[A^i, A^j]_t = 0. \tag{3.5}$$

In general, $\sum_{i,j}(\delta^i - \frac{\partial c}{\partial a_i})(\delta^j - \frac{\partial c}{\partial a_j})d[A^i, A^j] = 0$ since the (left so) right hand side of (3.4) has finite variation. Thus if $d[A^i]$ are absolutely continuous and the $m \times m$ matrix $(\frac{d}{dt}[A^i, A^j])$ is nonsingular then $\delta^i_t = \frac{\partial c}{\partial a_i}(t, A_t)$, so (3.5) holds and $c(t, A_t) = \sum_i \frac{\partial c}{\partial a_i}(t, A_t)A^i_t$. If further the support of A_t is a cone, it follows $c(t,a)$ is *homogenous of degree 1* in a on that cone.

Assume $M^i := e^{-\int r dt}A^i$ are local martingales under an equivalent measure for some locally bounded predictable process r. Then $dA^i = rA^i dt + e^{\int r dt}dM^i$; so by (3.4) and (3.3)

$$\frac{\partial c}{\partial t}(t, A_t)dt + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t)d[A^i, A^j]_t = r_t(C_t - \sum_{i=1}^{m} \frac{\partial c}{\partial a_i}(t, A_t)A^i_t)dt.$$

Hence, if $c(t,a)$ is taken homogenous of degree 1 in a as in [8] and [9], then by Euler's formula (3.5) holds. Given a homogenous payoff function $h(a)$, Sec. 3.7 below constructs under suitable assumptions such a function $c(t,a)$ with $c(T,a) = h(a)$ and $\delta^i_t = \frac{\partial c}{\partial a_i}(t, A_t)$.

3.2. **The invariance principle.** *Let (δ, A) be a SFTS and S a (scalar) semimartingale such that δ is $SA := (SA^1, \cdots, SA^m)$-integrable. Then (δ, SA) is a SFTS. Consequently,*

$$d(SC) = \sum_{i=1}^{m} \delta^i d(SA^i), \tag{3.6}$$

[1]Clearly then the restriction of (any such) $c(t,a)$ to the support of A is unique, and if $\hat{c}(t,a)$ is any function that equals $c(t,a)$ on the support of A, then $C_t = \hat{c}(t, A_t)$ too. If the support of A_t is a proper surface, e.g., if $m = 2$ and A^2 is deterministic as in the Black-Scholes model or $A^2_t = a_2(t, A^1_t)$ as in Markovian short-rate models, then obviously there exist infinitely many nonhomogeneous functions $\hat{c}(t,a)$ such that $C_t = \hat{c}(t, A_t)$. (A homogenous such function also exists under some assumptions as in Sec. 3.7.)

where $C := \sum_i \delta^i A^i = C_0 + \delta \cdot A$, *that is,* $SC = S_0 C_0 + \delta \cdot (SA)$. Indeed, by Itô's product rule, then substituting for dC and C_- and regrouping, followed by Itô's product rule again,

$$d(SC) = S_- dC + C_- dS + d[S, C]$$
$$= S_- \sum_{i=1}^m \delta^i dA^i + \sum_{i=1}^m \delta^i A^i_- dS + \sum_{i=1}^m \delta^i d[S, A^i]$$
$$= \sum_{i=1}^m \delta^i (S_- dA^i + A^i_- dS + d[S, A^i]) = \sum_{i=1}^m \delta^i d(SA^i).$$

Interpreting S as an exchange rate, this result, referred to as *numeraire invariance*, means that the self-financing property is independent of the choice of base currency.

If $S, S_- > 0$, then applied to the semimartingale $1/S$ we see that δ *is a SFTS for* A *if and only if it is one for* SA. Thus, *if (3.2) holds then (3.3) and (3.6) are equivalent.*

Assume now $A^m, A^m_- > 0$ and $m \geq 2$. Define the $n := m-1$ dimensional semimartingale

$$X := (\frac{A^1}{A^m}, \cdots, \frac{A^n}{A^m}), \qquad n := m - 1.$$

Taking $S = 1/A^m$, it follows that δ *is a SFTS for* A *if and only if it is a SFTS for* $A/A^m = (X, 1)$*, i.e., if and only if* $F := C/B$ *satisfies* $F = F_0 + \delta' \cdot X$ *where* $\delta' := (\delta^1, \cdots, \delta^n)$. In this case clearly $F = \sum_{i=1}^n \delta^i X^i + \delta^m$ and $F_- = \sum_{i=1}^n \delta^i X^i_- + \delta^m$ as $\Delta F = \delta' \cdot \Delta X$. Thus,

$$\delta^m = F - \sum_{i=1}^n \delta^i X^i = F_- - \sum_{i=1}^n \delta^i X^i_-. \qquad (F := \frac{C}{A^m})$$

When $m = 1$ a similar argument shows that δ must be a constant, as intuitively obvious.

Conversely, suppose δ' is an X-integrable process and F is a process such that $F = F_0 + \delta' \cdot X$. Define δ^m by either of the above formula - the other then holds as before. Obviously then $\delta = (\delta', \delta^m)$ is a SFTS for $(X, 1)$ with price process F. Hence by numeraire invariance δ is a SFTS for A with price process $C = BF$, provided δ is A-integrable.

Numeraire invariance thus shows that *in order to find a SFTS with a given time-T payoff* C_T *it is sufficient to find processes* δ' *and* F *such that* $F = F_0 + \delta' \cdot X$ *and* $F_T = C_T / A^m_T$.

We often use the differential form $dF = \sum_{i=1}^n \delta^i dX^i$ of the equation $F = F_0 + \delta' \cdot X$. Since $\delta^m = F - \sum_{i=1}^n \delta^i X^i$, δ^m is determined by δ' and F_0. As such, one interprets the m-th asset as the "numeraire asset" chosen to finance an otherwise arbitrary trading strategy δ' in the other assets, post an initial investment of $C_0 = A^m_0 F_0$.

3.3. **Arbitrage-free semimartingales and uniqueness.** We call a semimartingale $A = (A^1, \cdots, A^m)$, $m \geq 2$, <u>arbitrage free</u> if there exists a positive (scalar) semimartingale S with $S_- > 0$ such that SA^i are martingales for all i. Such a process S is called a <u>state price density</u> or <u>deflator</u> for A. <u>The law of one price</u> justifies the terminology:

If A *is arbitrage free and* δ *is a bounded SFTS for* A *then* SC *is a martingale where* $C := \sum_{i=1}^m \delta^i A^i$*; consequently* $C = 0$ *if* $C_T = 0$.

Indeed, by numeraire invariance δ is then a SFTS for SA with price process SC. Hence by Sec. 3.1, SC is a martingale, implying $SC = 0$ if $C_T = 0$, and with it $C = 0$, as claimed.

A simple and well-known argument yields that *if* $A^m, A^m_- > 0$ *then* A *is arbitrage free if and only if there exists an equivalent probability measure* $\mathbb{Q}$ *such that* X *is a* $\mathbb{Q}$*-martingale, where* $X := (\frac{A^1}{A^m}, \cdots \frac{A^n}{A^m})$, $n := m-1$.[2] A corollary is that C/A^m *is then a* $\mathbb{Q}$*-martingale for the price process* $C := \sum_i \delta^i A^i$ *of any bounded SFTS* δ, *and hence* $C_t = A^m_t \mathbb{E}^{\mathbb{Q}}[C_T/A^m_T \mid \mathcal{F}_t]$.

Indeed, by numeraire invariance, δ is then a SFTS for A/A^m with price process $F := C/A^m$. Hence, F is a $\mathbb{Q}$-martingale by Sec 3.1 applied in measure $\mathbb{Q}$ since A/A^m is.

Assume now that X is a $\mathbb{Q}$-*square-integrable* martingale. Then $F := C/A^m$ is a $\mathbb{Q}$-*square-integrable* provided only that δ^i are bounded for $i \le n$ as $dF = \sum_{i=1}^n \delta^i dX^i$. Further, $d\langle F\rangle^{\mathbb{Q}} = \sum_{ij=1}^n \delta^i\delta^j d\langle X^i, X^j\rangle^{\mathbb{Q}}$. Therefore if $d\langle X^i\rangle^{\mathbb{Q}}$ are absolutely continuous and the $n \times n$ matrix $(\frac{d}{dt}\langle X^i, X^j\rangle^{\mathbb{Q}})$ is nonsingular, then *given any random variable* R, *there exists at most one SFTS* δ *for* A *such that* $\sum_{i=1}^m \delta^i_T A^i_T = R$ *and* δ^i *are bounded for* $i \le n$.

3.4. **Projective continuous Markovian SFTS.** Let $X = (X^1, \cdots, X^n)$ be a continuous vector martingale. In what follows $x \in \mathbb{R}^n_+$ if $X > 0$ (the main case of interest), otherwise $x \in \mathbb{R}^n$. Let $g(x)$ be a Borel function of linear growth (so $\mathbb{E}|g(X_T)| < \infty$), and $f(t,x)$ be a function, $C^{1,2}$ on $t < T$. Set $m := n+1$ and define the C^1 functions

$$\delta_i(t,x) := \frac{\partial f}{\partial x_i}(t,x),\ i \le n, \qquad \delta_m(t,x) := f(t,x) - \sum_{i=1}^n \delta_i(t,x)x_i, \tag{3.7}$$

and the continuous vector process

$$\delta = (\delta^1, \cdots, \delta^m), \qquad \delta^i_t := \delta_i(t, X_t). \tag{3.8}$$

First suppose that

$$f(t, X_t) = \mathbb{E}[g(X_T) \mid \mathcal{F}_t]. \tag{3.9}$$

Then the process $F := (f(t, X_t))$ is a martingale, and since X^i are too, Itô's formula yields,

$$dF_t = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t, X_t)dX^i_t, \tag{3.10}$$

and

$$\frac{\partial f}{\partial t}(t, X_t)dt + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)d[X^i, X^j]_t = 0. \tag{3.11}$$

Clearly $F_T = g(X_T)$ and (3.10) implies δ is a SFTS for $(X, 1)$ with price process F.

Conversely, suppose that $f(t,x)$ satisfies (3.11) or by Itô's formula equivalently (3.10). By (3.10) δ is a SFTS for $(X, 1)$ with price process $F := f(t, X_t)$. Thus by Sec. 3.1, if

[2] Indeed, first assume A is arbitrage-free and let S be a state price density. The martingale $M := \frac{SA^m}{\mathbb{E}[S_0 A^m_0]}$ clearly satisfies $\mathbb{E}\, M_T = 1$. Hence the equivalent measure $\mathbb{Q}$ defined by $d\mathbb{Q} = M_T d\mathbb{P}$ is a probability measure. Since $MX^i = \frac{SA^i}{\mathbb{E}[S_0 A^m_0]}$ is a martingale, X^i is a $\mathbb{Q}$-martingale by the Bayes' rule. Conversely, assume X^i are $\mathbb{Q}$-martingales for some $\mathbb{Q}$. Define $M_t := \mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t] > 0$. Then (the right continuous version of) $M = (M_t)$ is a martingale (so $M_- > 0$). By the Bayes' rule MX^i are martingales since X^i are $\mathbb{Q}$-martingales . Set $S := M/A^m$. Then $S, S_- > 0$ and $SA^i = MX^i$. Thus S is deflator, as desired. Further, since SC is a martingale for any bounded SFTS δ, by the Bayes' rule $SC/M = C/A^m$ is a $\mathbb{Q}$-martingale.

$\delta_i(t,x)$ are bounded then F is a martingale and if further $f(T,x) = g(x)$ then (3.9) holds. Moreover, as in Sec 3.3, δ given by (3.8) is then the *unique* bounded SFTS for $(X,1)$ with payoff $g(X_T)$, provided $d[X^i, X^j] = X^i X^j \sigma^{ij} dt$ for some nonsingular matrix process (σ_t^{ij}).

3.5. **Example: projective deterministic volatility.** Let $X = (X^1, \cdots, X^n) > 0$ be a continuous n-dimensional martingale such that

$$d[X^i, X^j]_t = X_t^i X_t^j \sigma_{ij}(t) dt \tag{3.12}$$

for some n^2 *deterministic* continuous functions $\sigma_{ij}(t)$. So, $d[\log X^i, \log X^j]_t = \sigma_{ij}(t)dt$. Conditioned on $\mathcal{F}_t$ and unconditionally, X_T/X_t is then *multivariately lognormally distributed*, with mean $(1, \cdots, 1)$ and log-covariance matrix $(\int_t^T \sigma_{ij}(s)ds)$. Let $P(t,T,z)$, denote its distribution function. Let $g(x)$ be a Borel function of linear growth. Define the function

$$f(t,x) := \mathbb{E}[g(x_1 \frac{X_T^1}{X_t^1}, \cdots, x_n \frac{X_T^n}{X_t^n})]. \tag{3.13}$$

Obviously $f(T,x) = g(x)$. Clearly $f(t,x)$ can also be represented in two other ways as

$$f(t,x) = \int_{\mathbb{R}_+^n} g(x_1 z_1, \cdots, x_n z_n) P(t,T,dz) = \mathbb{E}[g(x_1 \frac{X_T^1}{X_t^1}, \cdots, x_n \frac{X_T^n}{X_t^n}) \,|\, \mathcal{F}_t].$$

Eq. (3.9) holds by the second equality, and $f(t,x)$ is C^1 in t and smooth (even analytic) in x on $t < T$ as seen by changing variable in the integral to $y^i = x^i z^i$ and differentiating under the integral sign in the first equality. Therefore by (3.11), $f(t,x)$ satisfies the PDE

$$\frac{\partial f}{\partial t} + \frac{1}{2} \sum_{i,j=1}^n \sigma_{ij}(t) x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} = 0 \tag{3.14}$$

on the support of X, (3.10) holds, and δ is a SFTS for $(X,1)$ with price process $F := (f(t,X_t))$, a martingale by (3.9). If $g(x)$ is dx-absolutely continuous with bounded partials $\frac{\partial g}{\partial x_i}$ (as L^1_{loc} functions) then $g(x)$ has linear growth, $\mathbb{E}|g(X_T)|^p < \infty$ for $p > 0$, and

$$\frac{\partial f}{\partial x_i}(t,x) = \mathbb{E}[\frac{X_T^i}{X_t^i} \frac{\partial g}{\partial x_i}(x_1 \frac{X_T^1}{X_t^1}, \cdots, x_n \frac{X_T^n}{X_t^n})].$$

Thus $\delta_i(t,x) = \frac{\partial f}{\partial x_i}(t,x)$ are bounded. If $g(x) - \sum \frac{\partial g}{\partial x_i} x_i$ is bounded then so is $\delta_m(t,x)$ as

$$\delta_m(t,x) = \mathbb{E}[g(x\frac{X_T}{X_t}) - \sum_{i=1}^n \frac{\partial g}{\partial x_i}(x\frac{X_T}{X_t})\frac{X_T^i}{X_t^i})].$$

It further follows that if $f(t,x)$ is any $C^{1,2}$ function with bounded paritals $\frac{\partial f}{\partial x_i}(t,x)$ satisfying $f(T,x) = 0$ for all x and the PDE (3.14), then $F := (f(t,X_t)) = 0$. Indeed, (3.10) then holds by PDE (3.14) and Itô's formula, implying F is a square-integrable martingale since X is; thus $F = 0$ since $F_T = 0$. As such, $f(t,x) = 0$ identically if the support of X_t equals $\mathbb{R}_+^n$ for every t. This is so if the matrix $(\sigma_{ij}(t))$ is nonsingular at least near 0, and it is "generically" so even when the matrix has rank 1 but is time dependent.

3.6. **Projective continuous SDE SFTS.** Continuous Markovian positive martingales $X = (X^1, \cdots, X^n)$ often arise as solutions to an SDE system of the form

$$dX_t^i = X_t^i \sum_{j=1}^{k} \varphi_{ij}(t, X_t) dW_t^j, \tag{3.15}$$

where $W^1, \cdots, W^k$ are independent Brownian motions and $\varphi_{ij}(t,x)$, $x \in \mathbb{R}_+^n$, are continuous bounded functions. As is well known, for each $s \le T$ and $x \in \mathbb{R}_+^n$, there is a unique continuous semimartingale $X^{s,x} = (X_t^{s,x})$ on $[s,T]$ with $X_s^{s,x} = x$ satisfying this SDE; moreover $X^{s,x}$ is a positive square-integrable martingale (in fact in all $\mathcal{H}^p$) since $\varphi_{ij}(t,x)$ are bounded. Fixing an $X_0 \in \mathbb{R}_+^n$, the solution on $[0,T]$ starting at X_0 at time 0 is denoted $X = X^{0,X_0}$. The Markov property holds: for any Borel function $g(x)$ of linear growth,

$$\mathbb{E}[g(X_T) \,|\, \mathcal{F}_t] = f(t, X_t), \quad \text{where } \ f(t,x) := \mathbb{E}\, g(X_T^{t,x}). \tag{3.16}$$

Clearly $f(T,x) = g(x)$. (Intuitively, $f(t,x) = \mathbb{E}[g(X_T) \,|\, X_t = x]$.)

Thus if we assume $\varphi_{ij}(t,x)$ are such that $f(t,x)$ is $C^{1,2}$ on $t < T$ for every bounded (hence of linear growth) Borel function $g(x)$, then the assumptions of Sec. 2.3 are satisfied and the conclusions hold. In particular (3.10) then holds, and since

$$d[X^i, X^j] = X^i X^j \sigma_{ij}(t, X) dt, \quad \text{where } \ \sigma_{ij}(t,x) := \sum_{l=1}^{k} \varphi_{il}(t,x)\varphi_{jl}(t,x),$$

it follows from (3.11) that, at least on the support of X, $f(t,x)$ satisfies the PDE

$$\frac{\partial f}{\partial t}(t,x) + \frac{1}{2} \sum_{i,j=1}^{n} x_i x_j \sigma_{ij}(t,x) \frac{\partial^2 f}{\partial x_i \partial x_j}(t,x) = 0. \tag{3.17}$$

In the deterministic volatility case the functions φ_{ij} and hence σ_{ij} are independent of x and simply $X_T^{t,x} = x X_T / X_t$, explaining why in this special case $f(t,x)$ is also given by (3.13).

In general, if $g(x)$ is absolutely continuous with bounded derivatives and the probability transition function of X is sufficiently regular, one shows as in the deterministic volatility case that the x-partials of f (the deltas) are bounded and thereby concludes uniqueness.

If $\sigma_{ij}(t,x)$ are homogenous of degree 0 in x, then assumed uniqueness and symmetry of PDE (3.17) under dilation in x imply that $f(t,x)$ is homogenous of degree 1 in x if $g(x)$ is so. By Euler's formula then $\delta_m(t,x) = 0$ in (3.7), implying $(\delta^1, \cdots, \delta^n)$ is a SFTS for X.

3.7. **Homogenous continuous Markovian SFTS.** Let $A = (A^1, \cdots, A^m)$ be a semimartingale with $A, A_- > 0$ such that $X^i := A^i / A^m$ are Itô processes following

$$dX_t^i = X_t^i \sum_{j=1}^{k} \varphi_t^{ij} (dZ_t^j + \phi^j dt), \qquad (i = 1, \cdots, n := m-1) \tag{3.18}$$

where Z^j are independent Brownian motions and ϕ^j, φ^{ij} are locally bounded predictable processes with φ^{ij} bounded and $\mathbb{E}\, e^{\frac{1}{2}\sum_j \int_0^T (\phi_t^j)^2 dt} < \infty$. Define the martingale

$$M := \mathcal{E}(-\sum_{j=1}^{k} \int \phi^j dZ^j) = e^{-\sum_{j=1}^{k}(\int \phi^j dZ^j + \frac{1}{2}\int (\phi^i)^2 dt)}, \tag{3.19}$$

and the measure $\mathbb{Q}$ by $d\mathbb{Q} = M_T d\mathbb{P}$. Then $W^i := Z^i + \int \phi^i dt$ are $\mathbb{Q}$-Brownian motions and are $\mathbb{Q}$-independent since $[W^k, W^l] = 0$ for $k \neq l$. Hence X^i are $\mathbb{Q}$-square-integrable martingales as $dX^i = X^i \sum_{j=1}^k \varphi^{ij} dW^j$ and φ^{ij} are bounded. *Thus A is arbitrage-free.*

Now let $h(a)$, $a \in \mathbb{R}_+^n > 0$, be a homogenous function of linear growth. Define $g(x) := h(x, 1)$, $x \in \mathbb{R}_+^n$. Assume further that $\varphi_t^{ij} = \varphi_{ij}(t, X_t)$ for some continuous bounded functions $\varphi_{ij}(t, x)$. Then (3.15) holds, hence Sec. 3.6 applied under measure $\mathbb{Q}$ shows that X is $\mathbb{Q}$-Markovian in that $\mathbb{E}^{\mathbb{Q}}[g(X_T) \,|\, \mathcal{F}_t] = f(t, X_t)$ where $f(t, x) = \mathbb{E}^{\mathbb{Q}} g(X_T^{t,x})$, as in Eq. (3.16). Thus by Sec 3.6, equations (3.10) and (3.11) of Sec. 3.4 hold and δ as defined in (3.8) is a SFTS for $(X, 1)$. Therefore by numeraire invariance δ is a SFTS for A with price process $C = A^m F$. The homogeneity of $h(a)$ further implies $C_T = A_T^m g(X_T) = h(A_T)$.

We have thus constructed a SFTS with the given payoff $h(A_T)$. As in Sec. 3.5 or 3.6 we ensure its boundedness by requiring the x-partials of $g(x)$ or equivalently a-partials of $h(a)$ (as L_{loc}^1 functions) be bounded, and thereby get unique pricing. For (very) low dimensions n, the PDE (3.17) is suitable for numerical valuation in absence of a closed-form solution.

We can further define the homogenous option price function

$$c(t, a) := a_m f(t, \frac{a^1}{a^m}, \cdots, \frac{a^n}{a^m}).$$

Then $C_t = c(t, A_t)$. Agreeably, $\delta_t^i = \frac{\partial c}{\partial a_i}(t, A_t)$ by (3.7). (For $i = m$ use Euler's formula for $c(t, a)$). By the continuity of X and (3.7), $\delta_t^i = \frac{\partial c}{\partial a_i}(t, A_{t_-})$ too. Therefore by Itô's formula,

$$\frac{\partial c}{\partial t}(t, A_{t_-})dt + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_{t_-}) d[A^i, A^j]_t^c = 0. \tag{3.20}$$

(The sum of jumps term in Itô's formula drops out since $\Delta C = \sum \delta^i \Delta A^i$.) This yields the PDE $\frac{\partial c}{\partial t} + \frac{1}{2}\sum_{i,j} a_i a_j \sigma_{ij}^A(t, a) \frac{\partial^2 c}{\partial a_i \partial a_j} = 0$ for the special case $d[A^i, A^j]_t = A_t^i A_t^j \sigma_{ij}^A(t, A_t)dt$ for some functions $\sigma_{ij}^A(t, a)$. The quotient-space PDE (3.17) is more fundamental as it holds in general (even when A is discontinuous), has one lower dimension, and under the change of variable $L^i = \frac{X^i}{X^{i+1}} - 1$ ($L^n = X^n - 1$) transforms to the Libor market model PDE.

3.8. **Multivariate Poisson predictable representation.** Let $P = (P^1, \cdots, P^k)$ be a vector of independent Poisson processes P^i with intensities $\lambda_i > 0$. For any C^1 in t function $u(t, p)$, $p \in \mathbb{R}^k$, the process $u(t, P) = (u(t, P_t))$ is a finite activity semimartingale, and using $[P^i, P^j] = 0$, one has $\Delta u(t, P) = \sum_i \Delta_i u(t, P_-)\Delta P^i$, where

$$\Delta_i u(t, p) := u(t, p_1, \cdots, p_i + 1, \cdots p_n) - u(t, p) \tag{3.21}$$

denotes the i-th forward partial difference of $u(t,p)$ in p. This in turn readily implies

$$du(t,P) = \frac{\partial u}{\partial t}(t,P_-)dt + \sum_{i=1}^{k} \Delta_i u(t,P_-)dP^i. \tag{3.22}$$

Let $v(p)$, $p \in \mathbb{R}^k$, be a function of exponential linear growth. Define the function

$$u(t,p) := \sum_{q_1,\cdots,q_k=0}^{\infty} v(p+q) \prod_{i=1}^{k} \frac{\lambda_i^{q_i}}{q_i!}(T-t)^{q_i} e^{-\lambda_i(T-t)}. \quad (p \in \mathbb{R}^k) \tag{3.23}$$

Clearly, $u(T,p) = v(p)$. Since the unconditional distribution of P_{T-t} is Poisson and is the same as the distribution of $P_T - P_t$ conditioned on $\mathcal{F}_t$, we have

$$u(t,p) = \mathbb{E}[v(p + P_T - P_t)] = \mathbb{E}[v(p + P_T - P_t) \,|\, \mathcal{F}_t].$$

Hence, $u(t,P_t) = \mathbb{E}[v(P_T) \,|\, \mathcal{F}_t]$. (Intuitively, $u(t,p) = \mathbb{E}[v(P_T) \,|\, P_t = p]$.) Thus the process

$$F = (F_t), \quad F_t := u(t,P_t) = \mathbb{E}[v(P_T) \,|\, \mathcal{F}_t] \tag{3.24}$$

is a martingale. But so are $P^j - \lambda_j t$. Therefore in view of (3.22) it follows that

$$dF = \sum_{i=1}^{k} \Delta_i u(t,P_-)d(P^i - \lambda_i t). \tag{3.25}$$

and $u(t,p)$ satisfies the partial difference equation

$$\frac{\partial u}{\partial t}(t,P_{t_-}) + \sum_{i=1}^{k} \lambda_i \Delta_i u(t,P_{t_-}) = 0. \tag{3.26}$$

Since $F_T = v(P_T)$ and $F_0 = u(0,0)$, combining (3.23) and (3.25) yields the representation

$$v(P_T) = \sum_{q_1,\cdots,q_k=0}^{\infty} v(q_1,\cdots,q_k) \prod_{i=1}^{k} \frac{\lambda_i^{q_i}}{q_i!} T^{q_i} e^{-\lambda_i T} + \sum_{i=1}^{k} \int_0^T \Delta_i u(t,P_{t-})d(P_t^i - \lambda_i t). \tag{3.27}$$

3.9. **Projective exponential-Poisson SFTS.** Let $P = (P^1,\cdots,P^k)$ be a vector of independent Poisson processes P^j with intensities $\lambda_j > 0$. Let $X_0 \in \mathbb{R}_+^n$, $n \geq k$, and $\beta = (\beta_{ij})$ be an $n \times k$ matrix such that the $n \times k$ matrix $(e^{\beta_{ij}} - 1)$ has full rank. Then the processes $X^i := (x_i(t,P_t))$, $i = 1,\cdots,n$, are square-integrable martingales (in fact in all $\mathcal{H}^p$), where

$$x_i(t,p) := X_0^i \exp\left(\sum_{j=1}^{k} (\beta_{ij} p_j - (e^{\beta_{ij}} - 1)\lambda_j t)\right). \quad (p \in \mathbb{R}^k) \tag{3.28}$$

Since

$$\frac{\partial x_i}{\partial t}(t,p) = -x_i(t,p) \sum_{j=1}^{k} (e^{\beta_{ij}} - 1)\lambda_j, \quad \Delta_j x_i(t,p) = x_i(t,p)(e^{\beta_{ij}} - 1),$$

it follows from (3.22) (or easily also from Itô's formula) that

$$dX^i = X^i_- \sum_{j=1}^k (e^{\beta_{ij}} - 1) d(P^j - \lambda_j t). \qquad (X^i_t := x_i(t, P_t)) \tag{3.29}$$

Let $\alpha = (\alpha_{ij})$ be any $n \times k$ matrix such that $\sum_i (e^{\beta_{il}} - 1)\alpha_{ij} = \delta_{jl}$, all $1 \leq j, l \leq k$. Then

$$d(P^j - \lambda_j t) = \sum_{i=1}^n \alpha_{ij} \frac{dX^i}{X^i_-}. \tag{3.30}$$

Now let $g(x)$, $x \in \mathbb{R}^n_+$, be a function of linear growth, define the function

$$v(p) := g(x_1(T, p), \cdots, x_n(T, p)), \qquad (p \in \mathbb{R}^n)$$

and the function $u(t, p)$ by (3.23). By Sec. 3.8, $F := (u(t, P_t))$ is a martingale with $F_T = v(P_T) = g(X_T)$ and is represented as (3.25). Substituting (3.30) into (3.25) yields

$$dF = \sum_{i=1}^n \delta^i dX^i, \tag{3.31}$$

where

$$\delta^i_t := \frac{1}{X^i_{t-}} \sum_{j=1}^k \alpha_{ij} \Delta_j u(t, P_{t-}). \tag{3.32}$$

Thus, $\delta = (\delta^1, \cdots, \delta^m)$ is a SFTS for $(X, 1)$ where $m := n + 1$ and $\delta^m := F - \sum_{i=1}^n \delta^i X^i$.

It is more desirable to express δ in term of X. One has $u(t, p) = f(t, x(t, p))$, where

$$f(t, x) := \mathbb{E}[g(x \frac{X_T}{X_t})] = \mathbb{E}[g(x \frac{X_T}{X_t}) \,|\, \mathcal{F}_t] = \tag{3.33}$$

$$\sum_{q_1, \cdots, q_n = 0}^{\infty} g(x_1 e^{\sum_{j=1}^n (\beta_{1j} q_j - (e^{\beta_{1j}} - 1)\lambda_j (T-t))}, \cdots, x_n e^{\sum_{j=1}^n (\beta_{nj} q_j - (e^{\beta_{nj}} - 1)\lambda_j (T-t))}) \prod_{i=1}^n \frac{\lambda_i^{q_i}}{q_i!} (T-t)^{q_i} e^{-\lambda_i (T-t)}.$$

The equalities follows from the definition of $v(p)$ above and of $u(t, p)$ in (3.23) together with the two formulae following it. We clearly have $f(T, x) = g(x)$ and

$$F_t := u(t, P_t) = f(t, X_t) = \mathbb{E}[g(X_T) \,|\, \mathcal{F}_t]. \tag{3.34}$$

Since $u(t, p) = f(t, x(t, p))$, the deltas in (3.32) are given by partial differences of $f(t, x)$ as

$$\delta^i_t = \delta_i(t, X_{t-}), \;\text{ where }\; \delta_i(t, x) := \frac{1}{x_i} \sum_{j=1}^k \alpha_{ij} (f(t, e^{\beta_{1j}} x_1, \cdots, e^{\beta_{nj}} x_n) - f(t, x)). \tag{3.35}$$

We have unique pricing since $(X, 1)$ is arbitrage-free (as X^i are martingales). Specifically, *if $\hat{\delta}$ is another SFTS for $(X, 1)$ with payoff $\hat{F}_T = g(X_T)$, then $\hat{F} := \sum_{i=1}^n \hat{\delta}^i X^i + \hat{\delta}^m = F$ provided that either all $\hat{\delta}^i$, $i \leq n$ are bounded or all $\hat{\delta}^i - \delta^i$, $i \leq n$ are bounded.*

Indeed, then $\hat{F} = \hat{F}_0 + \hat{\delta}' \cdot X$ is a martingale since X is a square-integrable integrable (in the second case, use also that F is a martingale). Hence $\hat{F} = F$ as $\hat{F}_T = F_T$.

Moreover, *if $k = n$ we have unique hedging, i.e., $\hat{\delta} = \delta$ for any bounded SFTS $\hat{\delta}$ for $(X, 1)$ with payoff $\hat{F}_T = g(X_T)$.* Indeed, $\hat{F} = F$, as before; thus, setting $\theta^i := \hat{\delta}^i - \delta^i$, we have

$$0 = d\langle \hat{F} - F\rangle = \sum_{i,j=1}^{n} \theta^i\theta^j d\langle X^i, X^j\rangle = \sum_{i,j=1}^{n} \theta^i\theta^j X^i_- X^j_- \sum_{l=1}^{n}(e^{\beta_{il}} - 1)(e^{\beta_{jl}} - 1)\lambda_l dt,$$

the last equality following by (3.29). But the $n \times n$ matrix $(\sum_{l=1}^n (e^{\beta_{il}} - 1)(e^{\beta_{jl}} - 1)\lambda_l)_{i,j=1}^n$ is nonsingular. Therefore $\theta^i = 0$, i.e., $\hat{\delta}^i = \delta^i$ for $i \le n$, implying $\hat{\delta}^m = \delta^m$ too as $\hat{F} = F$.

One shows as in Sec. 2.4 that the processes δ^i are bounded if $\gamma_i(x)$ are bounded, where

$$\gamma_i(x) := \frac{1}{x_i}\sum_{j=1}^{k} \alpha_{ij}(g(e^{\beta_{1j}}x_1, \cdots, e^{\beta_{nj}}x_n) - g(x)), \quad \gamma_m(x) := g(x) - \sum_{i=1}^{n}\gamma_i(x)x_i. \tag{3.36}$$

3.10. **Homogenous exponential Poisson SFTS.** Let $A > 0$ be an m-dimensional semimartingale with $A_- > 0$ and set $X := (A^i/A^m)_{i=1}^n$, $n := m - 1$ as before. Assume that

$$dX^i_t = X^i_{t-}\sum_{j=1}^{k}(e^{\beta_{ij}} - 1)(dP^j_t - \lambda^j_t dt), \tag{3.37}$$

where $1 \le k \le n$, β_{ij} are constants with the $n \times k$ matrix $(e^{\beta_{ij}} - 1)$ of full rank, $\lambda^j > 0$ are bounded predictable processes, and P^j are semimartingales with $[P^j, P^l] = 0$ for $j \ne l$ such that $[P^j] = P^j$, $P^j_0 = 0$, and $P^j - \int \kappa^j dt$ are local martingales for some locally bounded predictable processes $\kappa^j > 0$. Assume further that $\mathbb{E}\exp(\sum_{j=1}^k \int_0^T (\frac{\lambda^j_t}{\kappa^j_t} - 1)^2\kappa^j_t dt) < \infty$.

Due to the above growth condition, the positive local martingale

$$M := \mathcal{E}(\sum_{j=1}^{k}\int(\frac{\lambda^j}{\kappa^j} - 1)(dP^j - \kappa^j dt)) = e^{-\sum_{j=1}^k \int(\lambda^j - \kappa^j)dt}\prod_{s\le\cdot}(1 + \sum_{j=1}^{n}(\frac{\lambda^j_s}{\kappa^j_s} - 1)\Delta P^j_s)$$

is a martingale. Define the measure $\mathbb{Q}$ by $d\mathbb{Q} = M_T d\mathbb{P}$. As in Sec. 2.10, $\int \lambda^j dt$ are the $\mathbb{Q}$-compensator of P^j. This, (3.37), and boundedness of λ^j imply that X^i are $\mathbb{Q}$-square integrable martingales. *Thus A is arbitrage free.* As before, the SDE (3.37) integrates to

$$X^i_t = X^i_0 e^{\sum_{j=1}^k \beta_{ij}P^j_t - (e^{\beta_{ij}} - 1)\int_0^t \lambda^j_s ds}. \tag{3.38}$$

Now assume λ^j are constant. Then P^j are $\mathbb{Q}$-Poisson processes with intensities λ^j and are independent since $[P^j, P^l] = 0$, $j \ne l$. Sec. 3.9 applied under $\mathbb{Q}$ implies that δ given by (3.35) (with $\delta^m = F - \sum_{i=1}^n \delta^i X^i$) is a SFTS for $(X, 1)$ with price process $F = (f(t, X_t))$ satisfying $F_T = g(X_T)$, where $f(t, x)$ is defined explicitly by the long equation in (3.33), or equivalently, $f(t, x) = \mathbb{E}^{\mathbb{Q}} g(xX_T/X_t)$. Therefore, by numeraire invariance *δ is a SFTS for A with price process $C := A^m F$ satisfying $C_T = A^m g(X_T) = h(A_T)$* by homogeneity.

Assume finally that the payoff function $h(a)$ is such that the functions $\gamma_i(x)$ defined in (3.36) are bounded (e.g., $h(a) = \max(a^1, \cdots, a^m)$). By Sec. 3.9, *if $k = n$ then δ is the unique bounded SFTS for A with payoff $C_T = h(A_T)$.* In general, since A is arbitrage free, $\hat{C} = C$ for any other bounded SFTS $\hat{\delta}$ for A with payoff $\hat{C}_T = h(A_T)$, where $\hat{C} := \sum_i \hat{\delta}^i A^i$.

References

[1] Black, F., M. Scholes, M.: The Pricing of Options and Corporate Liabilities. *Journal of Political Economics*, 81, 637-59, (1973).
[2] Delbaen, F. Schachermayer, W.: *The Mathematics of Arbitrage*, Springer (2006).
[3] Duffie, D.: *Dynamic Asset Pricing Theory*, third edition, Princeton University Press (2001).
[4] El-Karoui, N., Geman, H., Rochet, J.C.: Change of numeraire, change of probability measure, and option pricing, *Journal of Applied Probability* 32, 443-458 (1995).
[5] Harrison, M.J., Kreps , D.M.: Martingales and arbitrage in multiperiod securities markets. *J. Econ. Theory* 20, 381-408 (1979).
[6] Harrison, M.J., Pliska, S.: Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes Appl*, 11, 215-260 (1981).
[7] Jamshidian, F: Options and Futures Evaluation with Deterministic Volatilities. *Mathematical Finance 3*(2), 149-159 (1993).
[8] Margrabe, W.: The Value of an Option to Exchange One Asset for Another. *Journal of Finance* 33, 177-86 (1978).
[9] Merton, R: Theory of Rational Option Pricing. *Bell Journal of Economics* 4(1), 141-183 (1973).
[10] Neuberger, A.: Pricing Swap Options Using the Forward Swap Market. IFA Preprint (1990).