Higher-order volatility

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HIGHER-ORDER VOLATILITY

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An often overlooked, but nonetheless important purpose of derivatives modelling is to provide practitioners with actionable measures of risk, the "thinkable quantities" that Emanuel Derman has referred to.[1] Dollar prices generally convey little information in the world of derivatives, and option pricing models are used — and abused — to convert them to and from a view on the market.

The historical risk measure, the Black and Scholes (1973) volatility, remains a favourite on trading floors in spite of well-known model-inconsistent biases, embodied in an implied volatility skew or smile. One popular approach to addressing these biases has been to make volatility a function of time and the underlying asset price, as in the local volatility models of Dupire (1994), Derman and Kani (1994) and Rubinstein (1994). This offers a model-consistent fit to market prices, without introducing fundamentally new or overly esoteric quantities into the risk interface, as can happen with other models.

In this paper, we present an alternative extension of volatility. Working first with a general stochastic process, we define a sequence of statistical parameters with an

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[1] Emanuel Derman, "A guide for the perplexed quant", Quantitative Finance **1**(5) (September 2001), page 477.

intuitive gambling interpretation. We then derive moment formulae for the case when they are deterministic. Applied to a generic market quantity, the resulting risk interface features the familiar Black-Scholes handle on the variance of the underlying, along with "higher-order" analogues which capture departures from lognormality while retaining the look and feel of the original quantity. We provide snapshot implied values for the S&P 500 index options market.

1. j-TH ORDER VOLATILITY

We begin by considering a general adapted process (X_t) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$. Let $\delta t > 0$ denote a finite period of time, and define $\delta X_t = X_{t+\delta t} - X_t$, so that the relative change of the process over the interval t to $t + \delta t$ reads $\delta X_t / X_t$. We let $E_t \cdot$ denote expectation conditional on $\mathcal{F}_t$, and j is a generic positive integer.

Consider an agreement by which two parties undertake to exchange the amount $\left(\delta X_t / X_t \right)^j$, as yet unknown but to be revealed imminently, and a predetermined amount which we tentatively write in the form $\left(\Sigma_{j,t} \right)^j \times \delta t$. We suppose that both parties are in possession of the information $\mathcal{F}_t$, and that given this information, the agreement is a fair gamble under the probability measure Q, by which we mean:

$$E_t \left[\left(\delta X_t / X_t \right)^j - \Sigma_{j,t}^j \delta t \right] = 0 \, .$$

This leads us to formally define the quantity $\Sigma_{j,t}$ via the identity:

$$\Sigma_{j,t}^j \equiv E_t \left(\delta X_t / X_t \right)^j \Big/ \delta t \, ,$$

with the convention that $\Sigma_{j,t}$ equals the nonnegative root when j is even.

For an intuitive interpretation of this quantity, consider the following extrapolation. Assume for convenience that $m \times \delta t = 1$ for some integer m, and consider a second agreement by which the original arrangement is extended to m successive periods, but with the predetermined side remaining a fixed $\Sigma_{j,t}^{j} \delta t$ per period. In other words, the parties agree to exchange an unknown $\left(\delta X_u / X_u \right)^j$ and a predetermined $\Sigma_{j,t}^{j} \delta t$ for each $u = t, t + \delta t, ..., t + 1 - \delta t$, with the unknown leg to be revealed imminently.[2] Observing that the predetermined leg sums to $m \times \Sigma_{j,t}^{j} \delta t = \Sigma_{j,t}^{j}$, suppose next that the process variable has changed exactly once between times t and $t + 1$. Then if the relative change equals $\Sigma_{j,t}$, the unknown leg equals the predetermined leg and the parties are even. Thus, for such an agreement the quantity $\Sigma_{j,t}$ can be viewed as a break-even relative change. We shall refer to $\Sigma_{j,t}$ as j-*th order finite-period volatility*.

Of particular interest will be the limit of vanishingly small δt, for which we define the quantity $\sigma_{j,t}$ via:

$$\sigma_{j,t}^{j} \equiv \lim_{\delta t \searrow 0} \Sigma_{j,t}^{j},$$

again with $\sigma_{j,t}$ nonnegative when j is even. Equivalently, $\sigma_{j,t} = \lim_{\delta t \searrow 0} \Sigma_{j,t}$. We shall refer to $\sigma_{j,t}$ as j-*th order instantaneous volatility*.

2. MOMENTS

Now in the general case the volatilities we have defined are stochastic. In the appendix we show that when the finite-period volatilities of orders one to n exist and are

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[2] This variation on the first agreement need not itself be a fair gamble.

deterministic, the n-th conditional moment of the process can be obtained as:

$$E_t X_{t+\tau}^n = X_t^n \prod_u \left[1 + \sum_{j=1}^{n} \binom{n}{j} \Sigma_{j,u}^j \delta t \right], \tag{1}$$

for any positive multiple τ of δt, where the product is over $u = t, t+\delta t, ..., t+\tau-\delta t$. If we next suppose that there exists a $\Delta > 0$ such that this assumption holds for every $\delta t \leq \Delta$, then clearly the instantaneous volatilities, when they exist, are also deterministic. Further, fixing $\tau > 0$ and taking the limit $\delta t \searrow 0$ in (1) yields:

$$E_t X_{t+\tau}^n = X_t^n \exp\left[\tau \sum_{j=1}^{n} \binom{n}{j} \sigma_j^j \right], \tag{2}$$

where the σ_j, which are assumed to exist, are defined via:

$$\sigma_j^j \equiv \frac{1}{\tau} \int_t^{t+\tau} \sigma_{j,u}^j du,$$

again with the convention that σ_j is nonnegative when j is even. We shall refer to σ_j as *j-th order average volatility*.[3] We emphasise here that we have not established that deterministic instantaneous volatilities imply (2) in and of themselves. However, since the upper bound Δ in the above argument can be chosen arbitrarily small, this would seem to be more of a technical than a conceptual issue.

It is clear from (2) that the n-th moment of the process at time t is fully determined by the average volatilities of orders one to n. Thus, first-order volatility σ_1 determines the first moment, and we may take the view that given σ_1, second-order volatility σ_2 governs the second moment, and so on. It is also apparent that if $\sigma_j = 0$ for every $j > 2$, then $\ln\left(X_{t+\tau}/X_t \right)$ is normally distributed (with mean $\left(\sigma_1 - \frac{1}{2}\sigma_2^2 \right)\tau$ and variance $\sigma_2^2\tau$).[4] Thus, nonzero values for higher-order average volatilities indicate

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[3] Admittedly a misnomer since it is the j-th power that is averaged.

[4] We caution that an ad-hoc, higher-order truncation of the sequence of volatilities may not be compatible with any probability measure.

and quantify deviations from the lognormal distribution. In particular, σ_3 and σ_4 govern auxiliary skewness and kurtosis.

3. AN IMPLEMENTATION

We now take Q to be an equivalent martingale measure, $(\mathcal{F}_t)$ to be the market information structure, and (X_t) to be an adapted market process. For expediency we limit ourselves to the basic equity setting, with X_t as the price at time t of a stock or index, with the money-market account as numeraire, and with the instantaneous dividend yield and riskless rate of interest both deterministic. In this context, it is easily verified that first-order volatility $\sigma_{1,t}$ equals the instantaneous cost of carry, which is itself deterministic.

In what follows, we shall treat the volatilities exclusively as attributes of the market measure, to be implied from a set of option prices, rather than as inputs to a pricing scheme. We begin with the assumption that the volatilities of orders two through n are deterministic as per the previous section, and thus that the first n moments have the form (2) (no assumption is necessary regarding higher orders). Note here that the first two model moments are now identical to those of the Black-Scholes model, with second-order volatility in the role of Black-Scholes volatility. We then propose to substitute a known distribution in place of the unknown one, fit it to the option prices, and compute the volatilities from the fitted statistics. For this it will prove convenient to use the canonical Merton (1976) jump-diffusion, which offers both a ready-to-use pricing capability and a reasonably good fit to market prices.

Under the cost-of-carry parameterisation of Bates (1991), the asset price experiences the usual standard diffusive innovations with coefficient σX_t, along with

Poisson-driven jumps of size κX_t at the rate λ. Here σ and λ are nonnegative constants, and $\ln(1+\kappa)$ is normally distributed with mean $\gamma - \frac{1}{2}\delta^2$ and variance δ^2. Letting m_n and m_n' denote the n-th moments (about zero) of κ and $1+\kappa$ respectively, we have $m_1 = e^\gamma - 1$ and $m_n' = \exp\left[n\gamma + \frac{1}{2}n(n-1)\delta^2\right]$, and the n-th conditional moment of the process reads:

$$E_t X_{t+\tau}^n = X_t^n \exp\left[n(b - \lambda m_1)\tau + \tfrac{1}{2}n(n-1)\sigma^2\tau + \lambda\tau(m_n' - 1)\right], \qquad (3)$$

where b is the instantaneous cost of carry. For convenience we recall the Merton (1973) version of the Black-Scholes formula for the price at time t of a plain vanilla option:

$$BSM = \varepsilon e^{-r\tau}\left[e^{b\tau}X_t N(\varepsilon d_1) - KN(\varepsilon d_2)\right],$$

with:

$$d_1 = \frac{\ln\left(X_t/K\right) + \left(b + \frac{1}{2}\nu^2\right)\tau}{\nu\sqrt{\tau}}, \qquad d_2 = d_1 - \nu\sqrt{\tau},$$

where K is the exercise price, r is the rate of interest, τ is the time to expiration, ν is the volatility, $N\cdot$ is the standard normal distribution function, and ε equals one for a call option, negative one for a put. The price of the same option under the jump-diffusion can then be written:

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} BSM_n,$$

where BSM_n is the Black-Scholes-Merton formula, but with cost of carry $b_n = b - \lambda m_1 + n\gamma/\tau$ and volatility $\nu_n = \sqrt{\sigma^2 + n\delta^2/\tau}$.[5] As for the relationship between

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[5] This formula, found in Bates (1991), differs superficially from the one in Merton (1976), which involves the ordinary Black-Scholes formula, and applies only when the cost of carry equals the rate of interest. In the latter case, the two formulae are of course strictly equivalent.

the jump-diffusion parameters and the volatilities, it is easily verified that (2) agrees with (3) when:

$$\sigma_j^j = b\mathbf{1}_{j=1} + \sigma^2\mathbf{1}_{j=2} + \lambda m_j \mathbf{1}_{j\geq2}\,,$$

where $\mathbf{1}_.$ is the indicator function, and m_j can be computed from the m_k' via:

$$m_j = \sum_{k=0}^{j}\binom{j}{k}(-1)^{j-k}\,m_k'\,.$$

4. AN EXAMPLE

For an example we turn to the S&P 500 (European-exercise) index options from the Chicago Board Options Exchange. The option price data is sourced from the exchange's web-based quote service on May 13, 2005, time-stamped 14:43 ET (fifteen minutes delayed), and consists of a snapshot of bid-ask quotes for all call and put contracts with under one year to expiration. The index level is 1,152.66. We discard all strikes for which the bid of either option is less than one-half a point, upon which for each contract the bid and ask are averaged to produce a price estimate. Rates of interest are linearly interpolated from the most recent Libor curve. For each expiration, the estimation of the jump-diffusion parameters (cost of carry included) is carried out by minimizing the sum of squared errors between market and model prices. Table 1 reports the estimated jump-diffusion parameters, and table 2 the corresponding average volatilities up to order four. Figure 1 shows the market price data in the form of implied Black-Scholes volatilities (computed with the cost of carry from table 1), and figure 2 plots the implied average volatilities as per table 2.

Four aspects of these results stand out. The first is that the volatility estimates of orders one and two are broadly as expected, given that they correspond respectively to

the cost of carry and to the Black-Scholes volatility.[6] A second feature is that the two higher-order volatilities show themselves to be of the same order of magnitude as second-order volatility, making good on their initial promise. In absolute value, all three are within a two-percent range at the 18-week mark. A third feature is that third-order volatility is markedly negative, which is to be expected since equity indices tend to drop more sharply than they rise. Fourth and finally, the third and fourth-order volatility estimates show a marked expiration dependence, with sharply lower absolute values at the short end compared with the long one. On this feature we limit ourselves to two comments. First, our assumption that the volatilities are deterministic is clearly counterfactual, and should be expected to result in parameter bias. This is the unhappy lot of most financial models. However, as our second comment we ask whether it is reasonable to expect perfect rationality from derivatives markets. For example, the writing of short-dated, out-of-the-money vanilla options is an ordinarily profitable operation, which could lead some participants to take chances, wittingly or not, for comparatively less remuneration than is demanded at longer expirations. This kind of misjudgement is all the more plausible in the absence of suitable risk metrics beyond those measuring and pricing ordinary market variability.

5. CONCLUSION

We have introduced a set of risk measures which translate and convey the information in option market prices in a new way. Whether skew and smile exposures can be managed effectively via these quantities remains to be seen. The primary function that is envisaged for them is as an alternative to implied volatility surfaces for the

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[6] To be precise, under our estimation methodology the second-order volatility estimate is unchanged whether we assume zero average volatilities for $j > 2$ (tantamount to the Black-Scholes specification) or not.

monitoring of market conditions. This risk interface is suitable for other products besides equity derivatives. It applies in a straightforward way to foreign exchange options, as well as interest-rate caplets and swaptions, subject to the standard parameterisations and assumptions.

In the implementation of this interface, we have treated the volatilities as attributes of the market measure, and presented snapshot estimates for the first four. A potential next step is to investigate the volatilities as inputs into a pricing scheme. While those of order higher than four can be expected to have an impact on valuations, it could be that for some purposes, they need not be known with precision. A number of models offer enhanced versions of the Black-Scholes formula based on higher-order moments, for example the Edgeworth expansion of Jarrow and Rudd (1982).

APPENDIX

To establish (1), we note that:

$$\left(X_{t+\tau}/X_t\right)^n = \prod_u \left(1 + \delta X_u/X_u\right)^n = \prod_u \left[1 + \sum_{j=1}^n \binom{n}{j}\left(\delta X_u/X_u\right)^j\right], \qquad \text{(A1)}$$

using the binomial theorem. Now $E_t \cdot = E_t E_{t+\delta t}...E_{t+\tau-\delta t}\,\cdot\,$, and assuming that the $\Sigma_{j,u}^j$ exist and are deterministic, taking expectations in (A1) and simplifying yields (1). To derive (2) we note that:

$$1 + \sum_{j=1}^n \binom{n}{j}\Sigma_{j,u}^j \delta t = \exp\left[o(\delta t) + \sum_{j=1}^n \binom{n}{j}\Sigma_{j,u}^j \delta t\right],$$

where $o(\delta t)$ represents terms which vanish with δt faster than δt (that is, $o(\delta t)/\delta t \to 0$ as $\delta t \searrow 0$). Replacing in (1), we obtain:

$$E_t\left(X_{t+\tau}/X_t\right)^n = \exp \sum_u \left[o(\delta t) + \sum_{j=1}^n \binom{n}{j}\Sigma_{j,u}^j \delta t\right]. \qquad \text{(A2)}$$

Now by definition $\Sigma_{j,u}^j \to \sigma_{j,u}^j$ as $\delta t \searrow 0$, hence $\sum_u \Sigma_{j,u}^j \delta t \to \sigma_j^j \tau$, and since $\sum_u o(\delta t) = o(\delta t)\tau/\delta t \to 0$, taking limits in (A2) yields (2).

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Table 1

Jump-diffusion parameters

Weeks to expiration	b (%)	σ (%)	λ	γ (%)	δ (%)
1	0.73	11.38	9.09	-3.06	2.23
5	1.25	10.13	2.47	-6.46	5.18
9	1.46	10.16	1.12	-10.33	7.38
18	1.53	9.79	0.68	-13.71	10.04
31	1.65	9.64	0.46	-18.22	12.65
44	1.71	9.64	0.33	-22.13	15.51

Table 2

Average volatilities (%)

Weeks to expiration	σ_1	σ_2	σ_3	σ_4
1	0.73	15.95	-8.56	7.74
5	1.25	16.05	-11.83	11.97
9	1.46	16.15	-13.48	14.40
18	1.53	16.09	-14.87	16.56
31	1.65	16.45	-16.45	18.81
44	1.71	16.58	-17.50	20.51

Figure 1
S&P 500 implied Black-Scholes volatilities by expiration

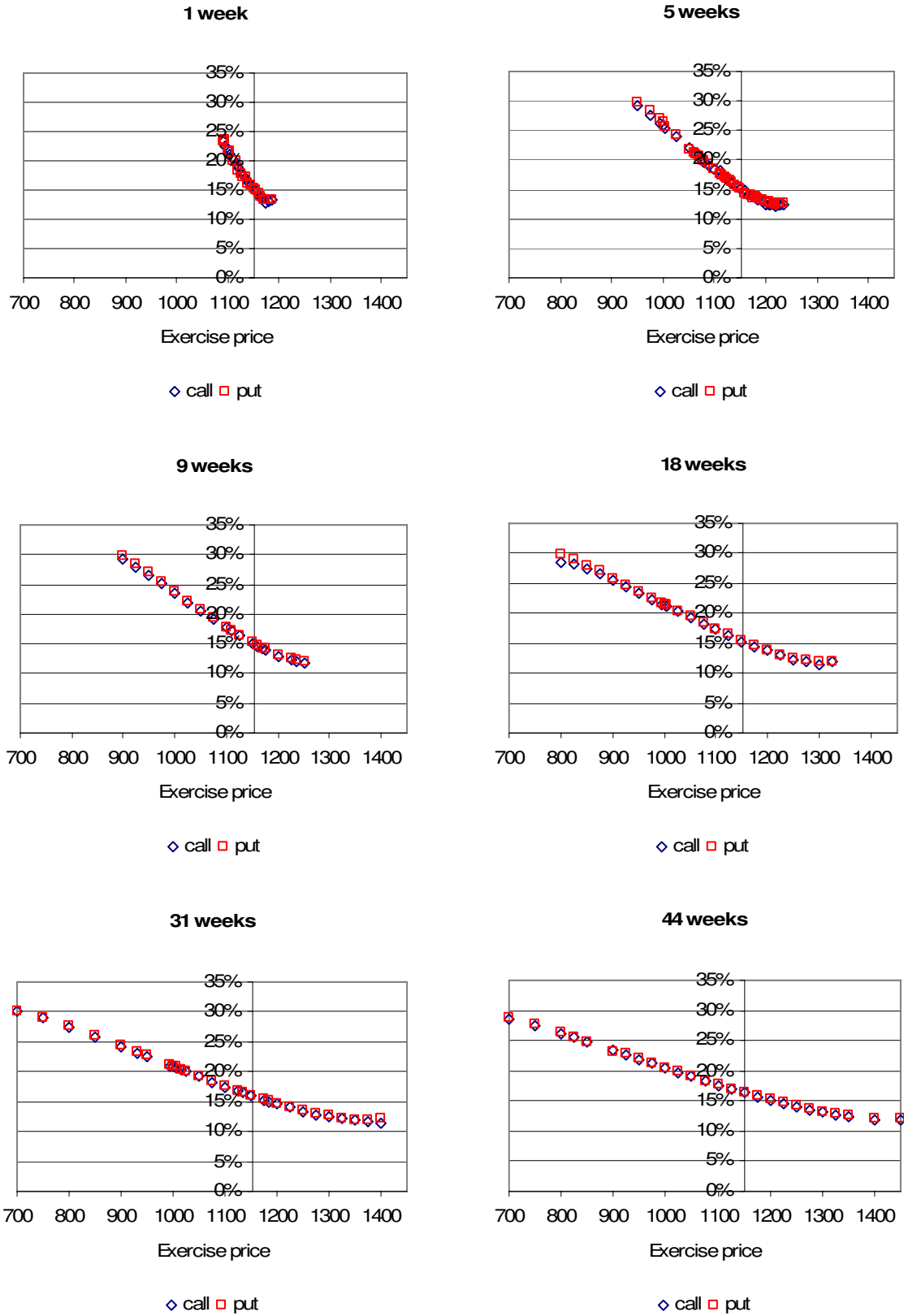**1 week**

5 weeks

9 weeks

18 weeks

31 weeks

44 weeks

Figure 2

Implied average volatilities