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Abstract

We present an application of the Dynamic Programming (DP) and of the Maximum Principle (MP) to solve an optimization over time when the production function is linear in the stock of capital (Ak model). Two views of capital are considered. In one, which is embraced by the great majority of macroeconomic models, capital is homogenous and depreciates at a constant exogenous rate. In the other view each piece of capital has its own finite productive life cycle (vintage capital). The interpretation of the time patterns of macroaggregates is quite different betweeen the two cases. A technological shock generates an oscillatory movement in the time pattern of per capita output when capital has a vintage structure; conversely an instantaneous adjustment with no transitional dynamics occurs when capital is homogenous.

From a methodological point of view it emerges that the DP approach delivers sharper results than the MP approach (for instance it delivers a closed form solution for the optimal investment strategy) under slacker parameter restrictions.

Cross-time and cross-country data on investments, income, and consumption drawn from the Penn World Table version 6.2 are used to evaluate the vintage and standard Ak model.

Keywords: Vintage Capital, Penn World Table, Maximum Principle, Hilbert Space.

JEL Classifications: E22, E37, O47

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Introduction

Historic slowdowns or great leaps forward are almost invariably explained in first approximation with a deceleration or an acceleration in the accumulation of capital. However, considerable disagreement exists among empirical macroeconomists and growth theorists on whether capital accumulation has a short or a long run effect on the growth rate of output. One class of endogenous growth models, known as AK-models[1], posits that the productivity of capital does not diminish with accumulation. Jones (1995) has undertaken the task of testing the consistency of the main implications of the AK-model by comparing the time series behavior of investments and GDP for a subset of OECD countries. He concluded that in several countries, including the US, the growth rate of income is loosely related to the investment ratio – a finding that seems to contradict the prediction of the AK model. However, some authors have recently questioned Jones' finding on two grounds. First,the data used to measure the investment ratio do not account for tax or subsidies to investments are not adjusted to reflect movements in the relative price of capital. In places where this has declined – for instance in the United States – the actual investment ratio is higher than the one recorded (See McGrattan (1998)). Secondly, the standard Ak model is based on the premise that capital is homogeneous and the depreciation rate is constant over time. As a result, only variations in the gross investment ratio – the one usually recorded in aggregate statistics – can affect per capita output. From a theoretical point of view constant depreciation is a quite convenient simplification, but from an empirical point of view it seems quite restrictive, especially in periods in which a whole class of machines embodying an obsolete technologies are being scrapped. Boucekkine et al. (2005) argue that relaxing the assumption of constant depreciation may reduce the odds that the Ak model is rejected by the evidence because income growth rate with no trend (or stationary) is compatible with rising investment ratios for long stretches of time. Indeed they show that if capital has a vintage structure and depreciation is of a horse-shoe form – that is a machine is as good as new for a predetermined number of years after which becomes useless – capital, and therefore output, may exhibit an oscillatory behavior whenever investments deviate from a the value consistent with a balanced growth path. Recently a small theoretical literature[2] has emerged that tries to solve a dynamic optimization problem of the kind posed by an Ak model with vintage structure in a Hilbert space. This environment allows for solving the problem under more general conditions than those set out in the work of Boucekkine et al. (2005). The purpose of this paper is to provide a comprehensive account of these new developments and to evaluate the extent to which the new techniques are helpful in understanding the post-war growth

[1] See Barro and Sala-i-Martin (2004) for a review.

[2] See Boucekkine et al. (2005), Fabbri and Gozzi (2006), Feichtinger et al. (2006), Barucci and Gozzi (2001)

experience of a large number of economies. Since we want to introduce the methodology to as many scholars as possible, we first provide an overview of two well-known optimization approaches and then move to solve the problem in a Hilbert space. In particular, next section introduces a typical Ak model presents its solution following an intuitive maximum principle (henceforth MP) approach. In section (2) this is contrasted with the DP approach. Section (3) discusses the extent to which the evidence is in line with the implication of the model, as far as the historical patterns of consumption, investment, and output are concerned. Section (4) introduced the vintage version of the model and provides a solution for the optimal investment policy following the MP approach. The DP solution in a Hilbert space is content of Section (5) (the more technical aspects of the solution are collected in the appendix). In order to gain further insights on the sequence of macroeconomic adjustments that follow a shock affecting the scrapping time or the state of the technology, a few simulation exercises are developed in section (6). Section (7) reconsiders the empirical evidence through the lens of the vintage model. Section (8) concludes.

1 Optimization with Homogeneous Capital

In this section we state the standard optimization problem when capital is homogeneous and depreciates at a constant rate. The economy is populated by a group of identical infinitely lived individuals of measure one. Each individual runs a firm that produces final goods according to the technology

$$y = Ak, \tag{1}$$

where k is the amount of capital and $A > 0$ is a constant. The optimization problem of an individual whose preferences are given by $\frac{c^{1-\sigma}}{1-\sigma}$, where c denotes units of consumption goods and σ is the inverse of intertemporal elasticity of substitution, is

$$\max_{c(\cdot)} U(k_0, c(\cdot)) = \int_0^\infty e^{-\rho t} \left[\frac{c(t)^{1-\sigma}}{1-\sigma}\right] dt \tag{2}$$

subject to the constraints:

$$\begin{aligned} \dot{k}(t) &= Ak(t) - c(t) - \delta k(t), \qquad (3) \\ k(0) &= k_0, \\ \lim_{t\to+\infty} e^{-\bar{r}(t)t} k(t) &\geq 0, \end{aligned}$$

where the index $t \geq 0$ refers to time. The function $U(k_0, c(\cdot))$ is the value of the objective function for a consumption strategy $c(\cdot)$,when the initial stock of capital is k_0. $k(t)$ is value of capital at time t, δ its (instantaneous) depreciation rate, $\bar{r}(t) = \frac{1}{t}\int_0^t r(\tau)d\tau$is the average rate of interest between today and time t. A dot on a variable denotes the derivative of that variable with respect to time. The first constraint starting from the top, says that the part of output not consumed is accumulated in the form of physical capital. The second constraint defines the exogenous endowment of initial capital and the third one is the No-Ponzi game condition that puts a restriction on the consumption strategy: In the long run the firm cannot end up with negative capital.

1.1 Maximum Principle

The idea is to find a path of consumption such that the value of the integral contained in problem (2) is maximum. We follow the Maximum Principle (henceforth MP) approach. We construct a Lagrangian-type function for a similar problem in finite horizon starting from $k = k_0$ and perturbate it around an admissible consumption-capital path $(c(\cdot), k(\cdot))$:

$$U^{\tau}(k_0, c(\cdot)) = \int_0^{\tau} e^{-\rho t} \left[\frac{c(t)^{1-\sigma}}{1-\sigma}\right] dt + \int_0^{\tau} \lambda(t)[Ak(t) - c(t) - \delta k(t) - \dot{k}(t)]dt + \nu e^{-\bar{r}(\tau)\tau} k(\tau), \quad (4)$$

where $\lambda(\cdot)$ is a dynamic Lagrange multiplier associated with the budget constraint and ν is the multiplier associated with the end value of capital and τ is the end of time. These multipliers are to be interpreted as the shadow values of capital in term of units of utility. Specifically, $\lambda(t)$ measures the change of utility associated with a marginal variation of net investments. Integrating by parts the term $\lambda(t)\dot{k}(t)$ appearing in the right-hand side of the previous equation we obtain

$$U^{\tau}(k_0, c(\cdot)) = \int_0^{\tau} e^{-\rho t} \left[\frac{c(t)^{1-\sigma}}{1-\sigma}\right] dt + \lambda(t) \int_0^{\tau} [Ak(t) - c(t) - \delta k(t)]dt -$$
$$- \lambda(t)k(t)|_0^{\tau} + \int_0^{\tau} \dot{\lambda}(t)k(t)dt + \nu e^{-\bar{r}(\tau)\tau} k(\tau). \quad (5)$$

For an admissible modification $(\Delta c(\cdot), \Delta k(\cdot))$ the first variation of (5) is[3]:

$$\Delta U^{\tau} = \int_0^{\tau} \left(e^{-\rho t} c(t)^{-\sigma} - \lambda(t)\right) \Delta i(t)dt +$$
$$+ \int_0^{\tau} \left((A-\delta)\lambda(t) + \lambda'(t)\right) \Delta k(t)dt + \left(\nu e^{-\bar{r}(\tau)\tau} - \lambda(\tau)\right) \Delta k(\tau). \quad (6)$$

If the optimal-consumption path is optimal we have $\Delta U^{\tau} \leq 0$ for any admissible modification. Namely, for internal optimal paths $(c^*(t), k^*(t))$ we have

$$e^{-\rho t} c^*(t)^{-\sigma} - \lambda^*(t) = 0, \quad (7)$$

$$\lambda^*(t)(A - \delta) + \dot{\lambda}^*(t) = 0, \quad (8)$$

[3]Note that we use the notation ΔU^{τ} for the first variation instead of the more common δU^{τ} since we have already used δ for the depreciation rate.

and

$$\nu^* e^{-\bar{r}(\tau)\tau} - \lambda^*(\tau) = 0. \tag{9}$$

In addition we have the Khun-Tucker complementarity-slackness condition

$$\nu^* e^{-\bar{r}(\tau)\tau} k^*(\tau) = 0.$$

Taking logs and time derivative of (7) we get:

$$\frac{\dot{c}^*(t)}{c^*(t)} = \frac{1}{\sigma}[-\frac{\dot{\lambda}(t)}{\lambda(t)} - \rho], \tag{10}$$

that combined with (8) yields

$$\frac{\dot{c}^*(t)}{c^*(t)} = \frac{1}{\sigma}[A - \delta - \rho]. \tag{11}$$

Finally, combining $\nu^* e^{-\bar{r}(\tau)\tau} = \lambda^*(\tau)$ with the complementarity-slackness condition yields the so called transversality condition

$$\lambda^*(\tau) k^*(\tau) = 0. \tag{12}$$

Eqs. (11) and (12), along with the initial condition and the budget constraint are the necessary conditions for the optimum in finite horizons. The only difference with the infinite horizon is that the transversality condition needs to hold only for a time arbitrarily large:

$$\lim_{\tau \to \infty} \lambda^*(\tau) k^*(\tau) = 0. \tag{13}$$

Notice that the parameter restriction $\frac{1-\sigma}{\sigma}[A - \delta - \rho] < \rho$ is to be posed so as to be sure that the criterion (2) has an upper bound. The condition simplifies to

$$\rho > (1 - \sigma)(A - \delta). \tag{14}$$

Although we do not prove it, Eqs. (3), (11), and (13) are also sufficient conditions for the optimization problem because the criterion function (2) is strictly concave whereas the constraint (3) is convex. Therefore this is a case of concave programming in which necessary and sufficient conditions coincide.

Good sources where maximum principle technique is fully developed and well done are Fleming and Rishel (1975), and Bensoussan et al. (1974). The short-cuts allowed us to obtain results with which most readers are familiar with. In the next session the same problem will be solved following the Bellman's principle of dynamic programming. But before moving to that topic we want to elaborate more on an important feature of the solution.

1.1.1 Smooth Patterns

A situation in which capital, consumption and output all grow at a constant, possibly different, rates is referred to as balanced growth path (BGP). One important aspect of the solution obtained above is that the economy is *always* on the BGP. An inspection of Eq. (11) reveals the regularity of the consumption pattern. The behavior of capital can be learned from the budget constraint (3): If the consumption-capital ratio (and therefore the consumption output ratio) is constant so is the growth rate of capital. Next we show that this is the case. For simplicity the "*" is dropped from the relevant variables. It is understood that these are on their optimal path. The budget constraint on the optimal trajectory is

$$\dot{k}(t) = (A-\delta)k(t) - c(0)e^{\frac{1}{\sigma}(A-\delta-\rho)t},$$

where $c(0)$ is a constant to be determined. Multiplying both sides by $\exp(-(A-\delta)t)$ we get:

$$-e^{-(A-\delta)t}[(A-\delta)k^*(t) - \dot{k}^*(t)] = -c(0)e^{[\frac{1}{\sigma}(A-\delta-\rho)-(A-\delta)]t},$$

or

$$\partial[e^{-(A-\delta)t}k^*(t)]/\partial t = -c(0)e^{[\frac{1}{\sigma}(A-\delta-\rho)-(A-\delta)]t}.$$

By integrating both sides between 0 and $t \leq \tau$ and after some rearrangements we get:

$$\begin{aligned} k(t) &= -\frac{c(0)}{[\frac{1}{\sigma}(A-\delta-\rho)-(A-\delta)]}e^{[\frac{1}{\sigma}(A-\delta-\rho)]t} + \\ &\quad +[\frac{c(0)}{[\frac{1}{\sigma}(A-\delta-\rho)-(A-\delta)]} + k(0)]e^{(A-\delta)t}. \end{aligned} \tag{15}$$

The initial condition $c(0)$ needs still to be determined (recall that the initial condition $k(0)$ is given). The transversality condition states that $\lim_{t\to+\infty}\lambda(t)k(t) = 0$. Replacing $k(t)$ in the previous expression with the right hand-side of Eq. (15) and noticing that from (10) and (11) we have $\lambda(t) = \lambda(0)e^{-(A-\delta)t}$, we get

$$\begin{aligned} \lambda(t)k(t) &= -\frac{\lambda(0)c(0)}{\frac{1}{\sigma}(A-\delta-\rho)-(A-\delta)}e^{\frac{1}{\sigma}(A-\delta-\rho)t}e^{-(A-\delta)t} + \\ &\quad +[\frac{c(0)}{\frac{1}{\sigma}(A-\delta-\rho)-(A-\delta)} + k(0)]\lambda(0). \end{aligned} \tag{16}$$

Since $\frac{1}{\sigma}(A-\delta-\rho) < (A-\delta)$, the first term of the right hand side of the previous equation goes to zero as t approaches infinity. Hence, for the transversality condition to be satisfied the sum of the two terms contained in the square brackets must be equal to zero (as long as $\lambda(0) > 0$). This implies

$$c(0) = k(0)[(A-\delta) - \frac{1}{\sigma}(A-\delta-\rho)]. \tag{17}$$

It is easy to verify that by replacing $c(0)$ with the previous expression Eq. (15) simplifies to

$$k(t) = k(0)e^{[\frac{1}{\sigma}(A-\delta-\rho)]t}. \tag{18}$$

Hence capital expands at a constant pace. Incidentally notice that combining Eqs. (16) and (17) we pin down the initial value of the capital shadow price: $\lambda(0) = k(0)^{-\sigma}[(A-\delta) - \frac{1}{\sigma}(A-\delta-\rho)]^{-\sigma}$. Consequently,

$$\lambda(t) = \alpha^{-\sigma}k(0)^{-\sigma}e^{-(A-\delta)t}, \tag{19}$$

where $\alpha = \frac{1}{\sigma}[\rho - (A-\delta)(1-\sigma)]$.

In sum, regardless of the initial level of capital ($k(0)$), the optimal path is always characterized by a constant growth rate of capital, consumption, and output. Such a rate is equal to $\frac{1}{\sigma}(A-\delta-\rho)$. The only parameter restriction needed for this result is $\rho > (1-\sigma)(A-\delta)$.

2 A Dynamic Programming Approach (Without Vintage Capital)

Bellman (1956) proposed an alternative approach to solve dynamic choices as the one described at the outset of the previous section. It is known as dynamic programming (DP). This is usually preferred over the MP whenever uncertainty is an integral part of the problem or when the problem is described in discrete rather than in continuous time. Nevertheless the method is presented here in continuous time so as to lay out a solution technique which will be extended in Section 4 to include situations in which capital has a vintage structure.

Three main mathematical instruments form the building blocks of Bellman's method: The *Value Function*, the *Hamiltonians*, and the *Hamilton-Jacobi-Bellman (HJB)* equation. In what follows these tools are described and then applied to solve the individual's problem of utility optimization. Before proceeding we wish to emphasize that the problem studied in this section is slightly different than the one described in (2), because a non negativity constraint is imposed both on consumption and on the stock of capital. The solution via the Bellman method could be carried out under the milder transversality condition. However, in Sections 2 and 3 a non-negative condition on the stock of capital becomes an essential simplification of the problem.

2.1 New Statement of the Problem

Given and initial level of capital $k > 0$ and a function

$$c\colon \mathbb{R}^+ \to \mathbb{R}^+,$$

where $\mathbb{R}$ is the set of real number, and $\mathbb{R}^+$ the set of positive real number, a *trajectory* for capital is given by the solution of the differential equation

$$\begin{cases} \dot{k}(t) = (A-\delta)k(t) - c(t) \\ k(0) = k_0 > 0 \end{cases} \tag{20}$$

where $k(t)$ is the value of capital at time t, which depends on the control variable consumption, $c(\cdot)$. We focus our attention only on consumption *strategies* that satisfy the two following conditions:

$$\begin{cases} c(t) \geq 0 & \text{for all } t \geq 0 \\ k(t) \geq 0 & \text{for all } t \geq 0. \end{cases} \tag{21}$$

In particular, we will search for a consumption strategy in the set of "admissible consumption strategies"

$$\mathcal{U}_{k_0} \stackrel{def}{=} \{c\colon [0,+\infty) \to \mathbb{R}^+ \ : \ k(t) \geq 0 \ for \ all \ t \geq 0\}, \tag{22}$$

where c is locally integrable, namely it is integrable on a closed and bounded set. The index k_0 emphasizes the dependence of the set from the initial condition. We will drop the index k_0 from $\mathcal{U}_{k_0}$ whenever ambiguities are unlikely to arise. The objective is to find an admissible control (that is a consumption strategy) that maximizes the functional $U(k_0, c(\cdot)) = \int_0^\infty e^{-\rho t}\left[\frac{c(t)^{1-\sigma}}{1-\sigma}\right] dt$. For this purpose we introduce three mathematical tools.

Value function Let the *value function* associated with an optimization problem of an individual with preferences given by $\frac{c(\cdot)^{1-\sigma}}{1-\sigma}$ and non negativity constraints on capital and consumption be

$$\begin{cases} V\colon \mathbb{R}^+ \to \mathbb{R} \\ V(k_0) \stackrel{def}{=} \sup_{c(\cdot)\in\mathcal{U}_{k_0}} \left[\int_0^{+\infty} e^{-\rho s}\frac{c(s)^{1-\sigma}}{1-\sigma} \mathrm{d}s\right], \end{cases}$$

where the set $\mathcal{U}_{k_0}$ is the one defined in (22). This says that the value function is the *supremum* value of discounted flow of utility which is possible to achieve for a given initial condition k_0. If an optimal trajectory exists (and we will prove that it does) the value function is exactly the value of the functional $\int_0^{+\infty} e^{-\rho s}\frac{c(s)^{1-\sigma}}{1-\sigma}\mathrm{d}s$ along the optimal trajectory.

Hamiltonians Let the *current value Hamiltonian* be the following mapping

$$\begin{cases} H_{CV}\colon \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \\ H_{CV}(k,\lambda,c) \stackrel{def}{=} \left[((A-\delta)k - c)\lambda + \frac{c^{1-\sigma}}{1-\sigma}\right], \end{cases} \tag{23}$$

where λ is the shadow price of capital, as in Section (1.1). The *current value Hamiltonian* is the current value of utility given by the flow of consumption and of net investments converted utility units by the shadow value λ. The expression is similar to the Lagrangian contained in equation (5), except that here the terms associated to the no-Ponzi game condition do not appear because we imposed the constraint $k(\cdot) \geq 0$.

The *maximum value Hamiltonian* is given by

$$\begin{cases} H : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \\ H(k, \lambda) \overset{def}{=} \sup_{c \geq 0} \left[H_{CV}(k, \lambda, c) \right]. \end{cases} \tag{24}$$

Notice that the supremum is taken on the set of positive real numbers ($c \geq 0$), whereas in the definition of the value function, the supremum was picked on a set of functions ($\mathcal{U}$).

The HJB equation If $(c^*(\cdot), k^*(\cdot))$ is an optimal strategy, from the definition of the value function V it follows that

$$V(k_0) = \int_0^\varepsilon e^{-\rho t} \frac{c^*(t)^{1-\sigma}}{1-\sigma} \mathrm{d}t + e^{-\rho\varepsilon} V(k^*(\varepsilon)). \tag{25}$$

Rearranging this equation and dividing all terms by ε we get

$$\frac{V(k_0) - e^{-\rho\varepsilon} V(k^*(\varepsilon)}{\varepsilon} - \frac{\int_0^\varepsilon e^{-\rho t} \frac{c^*(t)^{1-\sigma}}{1-\sigma} \mathrm{d}t}{\varepsilon} = 0.$$

For $\varepsilon \to 0$ this expression leads to an ordinary differential equation called Hamilton-Jacobi-Bellman equation:

$$\rho V(k_0) - H(k_0, V'(k_0)) = 0, \tag{26}$$

where $V'(k_0)$ denotes $\frac{\mathrm{d}}{\mathrm{d}k_0} V(k_0)$. This can be viewed as Eq. (25) in differential form. Below we solve Eq. (26) in order to obtain the explicit expression for the value function and then such expression will be used to solve the problem in feedback form.

2.1.1 Towards a Solution of the Optimization Problem: The Steps of the DP Method

We solve the optimization problem in three steps

Step (1) find a solution to the HJB contained in Eq. (26);

Step (2) find an optimal feedback rule that indicates, at each point in time, the optimal choice of consumption for a given stock of capital;

Step (3) determine an explicit form of the optimal consumption and capital on the basis of the optimal feedback rule elaborated in the previous step.

To ensure that the discounted value of the utility does not grow too quickly we keep imposing the same parameter restriction contained in Eq. (14).

Step (1) It has been shown elsewhere (see Fleming and Rishel (1975), Yong and Zhou (1999), and Zabczyk (1992)) that the value function is the only solution of the HJB in a wide range of cases[4]. Here we can give an explicit expression for a solution of the (26) and we will verify in Proposition 2.2 that it is an optimal one. The solution on the set $\mathbb{R}^+$ of the HJB in Eq. (26) is given by

$$\begin{cases} v\colon \mathbb{R}^+ \longrightarrow \mathbb{R} \\ v(k) \stackrel{def}{=} \frac{1}{1-\sigma}\left(\frac{\rho-(A-\delta)(1-\sigma)}{\sigma}\right)^{-\sigma} k^{1-\sigma} \end{cases} . \tag{27}$$

One can show, upon differentiation, that the function $v(.)$ is a solution of the HJB.

The following remark, which will be used in step 2, links the optimal consumption to the shadow price λ.

Remark 2.1 *On the set* $\mathbb{R}^+ \times \mathbb{R}^+$

$$argmax_{c\geq 0} H_{CV}(k, \lambda, c) = \lambda^{-1/\sigma}$$

and $H(k, \lambda)$ *assumes an explicit form given by*

$$H(k, \lambda) = \Big((A-\delta)k - \lambda^{-1/\sigma}\Big)\lambda + \frac{\lambda^{1-\frac{1}{\sigma}}}{1-\sigma}.$$

To prove this fact it is enough to use simple concavity arguments on $\mathbb{R}$.

Step (2) We define the feedback function as

$$\begin{cases} \phi\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \\ \phi(k) \stackrel{def}{=} argmax_{c\in\mathbb{R}^+} H_{CV}(k, v'(k), c) = (v'(k))^{-1/\sigma} \\ \qquad = \alpha k \end{cases} ,$$

where the equality of the middle row follows from Remark 2.1 and where $\alpha = \left(\frac{\rho-(A-\delta)(1-\sigma)}{\sigma}\right)$ (this result is derived from Eq. (27)). Notice that the condition in Eq. (14) guarantees that $\alpha > 0$. We will prove that such function is an "optimal feedback" function of the state (capital k), which does not depend on time, and that gives the optimal consumption: At any point in time, for a given level of capital $\bar{k}$ the optimal strategy is to consume $\phi(\bar{k}) = \alpha\bar{k}$. Therefore, along the *optimal trajectory*, namely the path of capital – the state variable – when consumption is chosen optimally, the quantity $\frac{c(t)}{k(t)}$ is constant and equal to α.

[4] Also in the non-regular case it is true: the value function is, under quite general assumptions, the only (viscosity) solution of the HJB, see Yong and Zhou (1999) or Fleming and Rishel (1975), a regular solution is always a viscosity solution, so, when it is regular, the value function is the only regular solution of the HJB).

Step (3) Finally, we prove that the feedback strategy is admissible – it satisfies the constraints $k \geq 0$ and $c \geq 0$ – and optimal. To show the admissibility we replace c with $\phi(k) = \alpha k$ in equation (20),

$$\begin{cases} \dot{k}(t) = (A-\delta)k(t) - \phi(k(t)) = \left(\frac{(A-\delta)-\rho}{\sigma}\right) k(t) \\ k(0) = k_0 \end{cases} . \tag{28}$$

By using the feedback rule $c = \alpha k$ the (candidate-optimal) trajectory of $k(\cdot)$ then is

$$k^*(t) = k_0 e^{(A-\delta-\rho)t} = k_0 e^{\frac{(A-\delta-\rho)}{\sigma}t}, \tag{29}$$

which is always positive and therefore an admissible trajectory.

Along such (candidate-optimal) trajectory, the (candidate-optimal) consumption is given by $c^*(t) = \alpha k^*(t)$ for all t, which is the same expression obtained with the maximum principle (see equations (11), (17), (18)). Next, we find the optimal consumption path.

Proposition 2.2 *The admissible feedback $\phi(\cdot)$ is optimal. Hence $c^*(\cdot)$ is the optimal control and $k^*(\cdot)$ is the optimal trajectory. Moreover v, defined in (27), is the value function of the problem. So $v = V$.*

Proof. Define $v_E(t,k) = e^{-\rho t}v(k)$ where v defined in (27) and solves the HJB equation. Let k_ς be the trajectory of the capital when the consumption is $\varsigma(\cdot)$, which denotes a generic admissible control $\varsigma(t)$. Then

$$v(k) - e^{-\rho T}v(k_\varsigma(T)) = v_E(0,k) - v_E(T, k_\varsigma(T)) =$$
$$= \int_0^T -\frac{\mathrm{d}}{\mathrm{d}t}v(t, k_\varsigma(t))\mathrm{d}t = \int_0^T \rho e^{-\rho t}v(k_\varsigma(t)) - e^{-\rho t}v'(k(t))\dot{k}_\varsigma(t)\mathrm{d}t. \tag{30}$$

For $T \to +\infty$, Hypothesis (14) implies that the term $e^{-\rho T}v(k_\varsigma(T))$ goes to zero. Therefore

$$v(k) = \int_0^{+\infty} \rho e^{-\rho t}v(k_\varsigma(t)) - e^{-\rho t}v'(k_\varsigma(t))\dot{k}_\varsigma^*(t)\mathrm{d}t \tag{31}$$

Now we are going to prove that the control given in feedback form $c = \phi(k) = \alpha k$ is optimal, that is that the value of the utility function $U(k, \varsigma(\cdot))$ cannot be greater than the value function $v(k)$. More formally

$$v(k) - U(k, \varsigma(\cdot)) = v(k) - \int_0^\infty e^{-\rho t}\frac{(\varsigma(t))^{1-\sigma}}{1-\sigma}\mathrm{d}t, \tag{32}$$

and substituting $v(k)$ with the right side of Eq. (31) we get

$$v(k) - U(k, \varsigma(\cdot)) = \int_0^{+\infty} e^{-\rho t}\left(\rho v(k_\varsigma(t)) - v'(k_\varsigma(t))\dot{k}_\varsigma(t) - \frac{(\varsigma(t))^{1-\sigma}}{1-\sigma}\right)\mathrm{d}t, \tag{33}$$

which, by using the v that solves Eq. (26) and H_{CV} in Eq. (23)), can be expressed as

$$= \int_0^{+\infty} e^{-\rho t} \left(H(k_\varsigma(t), v'(k_\varsigma(t))) - H_{CV}(k_\varsigma(t), v'(k_\varsigma(t)), \varsigma(t)) \right) \mathrm{d}t.$$

The original maximization problem is equivalent to the problem of finding a control $c^*(\cdot)$ that minimizes

$$v(k) - U(k, c^*(\cdot))$$

Since $H(k_\varsigma(t), v'(k_\varsigma(t))) - H_{CV}(k_\varsigma(t), v'(k_\varsigma(t)), \varsigma(t)) \geq 0$ in view of definition of H (equation (24)) then for all admissible control $\varsigma(\cdot)$:

$$v(k) - U(k, \varsigma(\cdot)) \geq 0.$$

Because $c^*(t)$ maximizes at every point in time the current value Hamiltonian, the feedback strategy $c(t) = \phi(k(t)) = \alpha k(t)$ satisfies

$$H(k^*(t), v'(k^*(t))) - H_{CV}(k(t), v'(k(t)), \varsigma(t)) = 0.$$

Hence, $v(k) - U(k, c(\cdot)) = 0$. Therefore, $c^*(\cdot)$ is an optimal control, $k^*(\cdot)$ is an optimal trajectory and v is the value function. ■

Remark 2.3 *In the dynamic programming knowing $\lambda^*(t)$ is not as crucial for the sake of characterizing the solution as it is in the MP approach. But for comparative purposes we work out its explicit solution following the general results in Fleming and Rishel (1975). Since $\lambda^*(t) = \frac{d}{dk} v_E(t, k^*(t))$, and this is equal to $e^{-\rho t} \frac{d}{dk} v(k^*(t))$,so we have that*

$$\lambda^*(t) = e^{-\rho t} \left(\frac{\rho - (A - \delta)(1 - \sigma)}{\sigma} \right)^{-\sigma} (k^*(t))^{-\sigma},$$

which combined with Eq. (29) gives $\alpha^{-\sigma}(k_0)^{-\sigma} e^{-(A-\delta)t}$, namely the same expression contained in Eq. (19).

2.2 Graphical Illustration

For a given level of initial stock of capital k there is an admissible (i.e. non negative) path of capital and consumption that expands at a constant rate. Such optimal trajectories depend only on the initial capital and on the parameters of the problem. We found that the optimal growth rate of consumption is equal to the optimal growth rate of output and capital, that such rate is equal to $\frac{(A-\delta-\rho)}{\sigma}$ that the ratio between $c(t)/k(t) = \frac{\rho-(A-\delta)(1-\sigma)}{\sigma}$. Thus for a given k_0 it is very easy to characterize the state of the economy on the optimal trajectory (sometimes also called 'saddle path'). To understand the feature of such optimal trajectory it is useful to consider situations in which the economy is not there. This can happen only if the consumer 'makes mistakes'.

Imagine that the consumer guesses correctly the growth rate of consumption $\frac{(A-\delta-\rho)}{\sigma}$ but picks an initial level of consumption, $\tilde{c}_0$, above the optimal one, $\frac{\rho-(A-\delta)(1-\sigma)}{\sigma}k_0$. Effectively the consumer is savings less that he should. From the budget constraint we know that $\dot{k}(t)/k(t) = A - c(t)/k(t) - \delta$. Hence $c(0)/k(0) > \frac{\rho-(A-\delta)(1-\sigma)}{\sigma}$, namely the ratio $c(t)/k(t)$ rises over time whereas $\dot{k}(t)/k(t)$ declines up to the point of becoming negative (in a way the consumer starts 'eating' the stock of capital). Because the Ponzi scheme is ruled out at a certain point consumption must go to zero and remain at that level forever.

Conversely, when the initial choice is too parsimonious, $\tilde{c}_0 < \frac{\rho-(A-\delta)(1-\sigma)}{\sigma}k_0$, $\dot{k}(t)/k(t)$ grows faster than consumption. If this behavior is protracted forever, in the long run the consumer ends up with too much capital, that is $\lambda(\tau)k(\tau) > 0$ for $\tau \longrightarrow +\infty$, a violation of the transversality condition in the MP approach and of the optimal feedback rule in the DP setting.

Fig. (1) illustrates the reasoning in a phase diagram, with k and c, running on the horizontal and vertical axis, respectively. The solid straight line represents the balanced growth path, along which $c(t) = \alpha k(t)$. Along the other two lines the growth rate of consumption is constant and equal to $\frac{(A-\delta-\rho)}{\sigma}$ (or to zero in the kinked-growth-path after that capital dries out), whereas the evolution of capital is driven by the budget constraints and the production function.

Consequences of a Shock. The absence of a transitional dynamics can be shown by varying any of the parameters of the model. We chose to reduce the depreciation rate from 5 to 4 percent. The left plots of Fig. (2) illustrate the effects of such a change on the phase diagram representing the balanced growth path. The path jumps up at the shock's arrival and becomes more sloped. The right plot of the figure shows that as a response to the shock, consumption jumps to a higher level instantaneously and grows at a faster pace, whereas investment declines as an immediate response to the shock, but grows more quickly than before. The immediate drop of investment is due to a 'wealth' effect: households anticipate that capital is going last longer and therefore can consume a larger share of production. Output and capital exhibit a similar qualitative response (not shown) to that of consumption. Fig. (3) compares the outcome of this experiment with one obtained through an alteration of the subjective discount rate, ρ, from 2% to 3%. The direction of the 'jump' is the same in both scenarios. If the consumer becomes more impatient he will want to consume more, therefore consumption will increase. But this means that fewer resources are being invested, and a lower growth rate of output and of consumption will be compatible with the new set of preferences.

3 An Intermezzo: A First Look at the Empirical Evidence

Are the implications of the Ak's model corroborated by the empirical evidence? This question has generated a small literature since the Jones (1995) undertook the task of testing several hypotheses implied by the model. The answer is not straightforward because of the quality of the data, the disagreement on how to measure investments and capital, and the choice of the most appropriate econometric approaches to test the behavior of time series of macro aggregates of several countries which potentially interact with each other. These issues are addressed in a work that one of the authors of this paper is developing with Rehim Kilic. Nevertheless we think that it is instructive to apply descriptive statistics and graphical analysis to cross-time and cross-country data in order to gain insight on the plausibility of the model's predictions.

Parallelism between Consumption and Output Patterns. During the period 1950-2004 per capita consumption and per capita gross domestic product in the United States have increased at an annual rate of 2.28% and 2.21%, respectively, according to data provided by the Penn World Table (PWT) 6.2. Similar estimates are obtained with data from the the Bureau of Economic Analysis (BEA). These numbers seem to agree with the prediction of the Ak model that consumption and income grow at the same rate. Moreover Fig. (4), which plots the logs of the two time series, shows that the two rates of growth have remained roughly constant over the post war period.

This prediction however is not specific to the Ak model. For instance the Solow model, which assumes decreasing return to capital, actually assumes that consumption and income stay in the same proportion. A similar assumption is found in Keynesian models.

Positive Association between Capital Productivity and Growth Rates. An interesting prediction specific to the model is that the difference in growth rates across countries is positively correlated with differences in the returns to capital. Given the value of δ, ρ and σ the productivity of capital, A, can be inferred through the growth rate of consumption (see Eq. (11)). Let the preference parameters $\rho = 0.02$ and $\sigma = 1.5$, and let the depreciation rate be $\delta = 0.05$. Then the (marginal) productivity of capital for the U.S., A, is predicted to be 0.1051 and that the net return, $A - \delta$ is 0.055. Table (1) reports similar calculations for a group of 20 countries which in 1975 had a per capita level of income above 50% of that of the US. Unfortunately there isn't an easy way to infer the absolute value of the technological index A. But at least it can be obtained as a relative term. If we know how countries stand with each other in terms of output and capital we can infer the relative productivity of capital. This kind of calculation was performed relating each country's output and capital per worker to that of the U.S. The outcome is shown in Fig. (5) that plots the percentage average growth rate of per capita income against the average relative productivity of capital (the US is normalized to 100), over the 1950-2004 period. It suggests that a positive relationship exists between the two variables. The correlation coefficient is 0.55. The highest return is recorded in Japan where it is about twice as large as in the U.S., whereas the lowest one is about half of that of the US and is found in UK.

Initial Consumption and Intertemporal Elasticity of Substitution. The model conjectures that individuals choose their level of initial consumption on the basis of the expected future expansion of income, which in turn depends on the returns on capital. In particular, Eq. (17) implies that the initial consumption-income ratio is equal to $[(1-\sigma)(\delta-A)+\rho]/\sigma A$. It follows that this ratio is decreasing in A as long as the elasticity is smaller than $(1+\rho/\delta)$, otherwise it rises with A. Intuitively, a larger return generates a wealth and a substitution effect. The former tends to raise initial consumption, whereas the latter acts in the opposite direction. The smaller σ,which represents the inverse of the intertemporal elasticity of substitution, the stronger is the substitution effect. In order to verify which one of the two effects is more likely to dominate, we computed the association between the consumption-per-capita-income ratio, averaged over the decade 1950-60, and the growth rate of consumption between 1950-2004. We selected countries that had at least five observations in the initial decade, and at least 40 on the overall period. The resulting sample of 67 countries yielded a correlation of -0.185, indicating that the substitution effect slightly dominates and that the elasticity of substitution in this model is close to one.

Investments and Growth. An intensely debated issue is the assumption of constant marginal productivity of capital. In most pre-1990 growth models this is assumed to decline and consequently accumulation alone cannot lead to sustained growth. To test the hypothesis of constant returns it is useful to differentiate the production function (1) with respect to time. This operation yields $\dot{y}/y = A\frac{\dot{k}}{y}$, which can be rearranged as

$$\frac{\dot{y}}{y} = A\frac{i}{y} - \delta. \tag{34}$$

where i is the per capita gross investment. If the depreciation rate is time invariant, the previous relationship predicts that the rate of growth of average income and the investment rate should move in lockstep. We verify this statement in three simple ways. First, we look at the association between the post-war average investment ratio and average post-war income growth rates of countries with at least 40 years of data (99 countries satisfy the criterion). Fig. (6) shows that for a large subset of countries the correlation is positive, although there are a few outliers with a relatively low investment rate and impressive growth record (since the outliers are developing countries one might question the quality of the data). A second method is to consider the cross-time dimension of the data and search for consistency between the time trend of income growth rates and that of investment ratios. Fig. (7) plots the two time trends – the plot is restricted to countries whose estimates are within the [-0.5, 0.5] interval. Evidence in favor of the Ak model is represented by observations that fall in the first and third quadrant. Overall the data convey a mixed message. A large group of countries falls in the predicted quadrants, but there is a good number of observations in the second and fourth quadrants as well. However, if we restrict our attention only to countries whose trends are significantly different than zero for both variables, the number of observations declines dramatically, and the outcome, shown in Fig. (8), is remarkably in line with the prediction of the Ak model: the slopes of the two variables are roughly aligned around a 45-degree line. A third method is to test whether there is any structural break in the time series, that is if a permanent change in the investment ratios is paralleled by a similar occurrence in the growth rate of income. We tested the hypothesis for both variables that the mean has not changed over the post-war period against the alternative, that in the post-1980 period it was lower than in the pre-1980 period. Out of the 99 countries with 40 data points, in 13 cases the null hypothesis was rejected. (We also considered an upward shift of the means but could not detect any such a case). For 21 countries the hypothesis of difference in means was not rejected for either variable. If the break point is 1970 instead, the rejection cases in favor of a (joint) downward shift are 20 instead of 13 and the rejection failures go down from 21 to 14. In sum, regardless whether the break point is 1970 or 1980, in about one third of the cases the outcome is in line with the prediction of the AK model and in two thirds of the cases it is not. The part of this study that follows the coming section explores a more sophisticated version of the Ak model, with the purpose of verifying whether its ability to account for the growth experiences is enhanced. In particular, we will show how to endogenize the depreciation rate, currently represented by the constant parameter δ.

4 AK model with vintage capital

We now expand the analysis by taking the view that capital is the ensemble of machines that are taken out of production after T years of service. They are productive as new while they are in place; in other words capital is not subject to wear and tear but to a horse-shoe type of depreciation. Here we do not consider obsolescence – the scrapping time is exogenously given – and there is no quality difference across vintages. Formally, the stock of capital is defined as

$$k_t = \int_{t-T}^{t} i(s)ds, \tag{35}$$

where $i(s)$ is the investment at time s. The instantaneous net investment, $\dot{k}(t)$, is now the difference between the gross investment and the value of equipment that was put in place T

periods ago, that is

$$\dot{k}(t) = i(t) - i(t - T). \tag{36}$$

This law of motion is different than the standard one because the depreciation of capital at time t is not proportional to the overall stock of capital but depends only on the amount of investment carried out T periods ago. An investment boom is going to be followed at a depreciation spike sometime in the future. Unless an extra amount of resources are saved, the stock of capital, and with it labor productivity, is going to decline after the machines introduced in the boom years are scrapped. If the depreciation were to follow these mechanics, the proportionality between growth rates and investment ratio implied by the simple version of the Ak model no longer holds. To see this more formally, if both sides of Eq. (36) are divided by $k(t)$ one gets $\dot{k}(t)/k(t) = i(t)/k(t) - i(t-T)/k(t)$. Since A is constant

$$\frac{\dot{y}(t)}{y(t)} = A\frac{i(t)}{y(t)} - \delta(t, T), \tag{37}$$

where $\delta(t, T) \equiv \frac{i(t-T)}{k(t)}$. This expression is similar to that of Eq. (34) except that here the depreciation rate is not constant.

We are interested in determining the optimal consumption and investment path for an individual with the same utility function as in section (1). The below analysis draws mostly on Boucekkine et al. (2005). It is easier to focus on the choice variable $i(t)$, rather than consumption. Thus the optimization problem to be studied is

$$\max_{i(\cdot)} U(.) = \int_0^\infty e^{-\rho t} \left(\frac{(Ak(t) - i(t))^{1-\sigma}}{1 - \sigma} \right) dt \tag{38}$$

subject to the state equations

$$\begin{aligned} \dot{k}(t) &= i(t) - i(t - T), \\ i(s) &= i_0(s) \; for \; s \in [-T, 0), \\ k(0) &= \int_{-T}^{0} i_0(s) ds, \end{aligned} \tag{39}$$

where $i_0(s)$ for $s \in [-T, 0)$ is a given distribution of preexisting investments at time $t = 0$,. Furthermore we impose the non-negativity constraints on $i(s)$ for all $s > 0$: it is not possible to consume more than current production. Note that such a condition guarantees that $k(\cdot)$ remain always positive and then it includes the no-Ponzi condition we have imposed in the one-dimensional case.

There are two departures from the problem posed in section (1). First of all, the initial stock of capital k_0 is replaced by the sum of the past flow of investments. Secondly, the accumulation of capital depends on the amount of equipment that was installed T years ago, which is about to be scrapped, as well as on the addition of new capital. These modifications of the original Ak model where capital was homogeneous will yield oscillatory trajectories for investment and output.

4.1 The Maximum Principle Approach

4.1.1 Interior Solution

We deal with an interior solution. We consider a investment-capital path $(i(\cdot), k(\cdot)$ with $i(\cdot) > 0$ and $c(t) = Ak(t) - i(t) > 0$ for every t. (5) changes to

$$U^\tau(i_0, i(\cdot)) = \int_0^\tau e^{-\rho t}[\frac{[Ak(t) - i(t)]^{1-\sigma} - 1}{1-\sigma}]dt + \int_0^\tau \lambda(t)[i(t) - i(t-T)]dt - \\ \lambda(t)k(t)|_0^\tau + \int_0^\tau \dot{\lambda}(t)k(t)dt. \quad (40)$$

We compute the first variation for an admissible modification $(\Delta i(\cdot), \Delta k(\cdot))$ obtaining

$$\Delta U^\tau(i_0, i(\cdot)) = \int_0^\tau \left(e^{-\rho t}A[Ak(t) - i(t)]^{-\sigma} + \dot{\lambda}(t)\right)\Delta k(t)dt + \\ + \int_0^\tau \left(-e^{-\rho t}(Ak(t) - i(t))^{-\sigma} + \lambda(t)\right)\Delta i(t)dt - \\ - \int_0^\tau \lambda(t)\Delta i(t-T)dt - \lambda(\tau)\Delta k(\tau). \quad (41)$$

Following Boucekkine et al. (2005) (page 52) we can use a change of variable (and use the fact that $i(t)$ is given for $t < 0$ and then $\Delta i(t) = 0$ for $t < 0$) to obtain

$$\int_0^\tau \lambda(t)\Delta i(t-T)dt = \int_0^{\tau - T} \lambda(t+T)\Delta i(t)dt.$$

Using such an expression in (41) and imposing that, along an optimal path $(i^*(\cdot), k^*(\cdot))$, $\Delta U^\tau(i_0, i^*(\cdot)) \leq 0$ for every admissible variation, we obtain

$$e^{-\rho t}[Ak^*(t) - i^*(t)]^{-\sigma} = [\lambda^*(t) - \lambda^*(t+T)], \quad (42)$$

$$A[\lambda^*(t) - \lambda^*(t+T)] - \dot{\lambda}^*(t) = 0, . \quad (43)$$

which is the equivalent of Eq. (8). The novelty here is that we are dealing with an advanced differential equation (ADE). Moreover we have the transversality condition

$$\lambda(\tau)k(\tau) = 0$$

that gives, letting $\tau \to +\infty$

$$\lim_{\tau\to\infty} \lambda(\tau)k(\tau) = 0. \quad (44)$$

As in the non-delay setting it can be proved that, since the functional is strictly concave, the conditions (42), (43) and (44) are sufficient for an internal path to be optimal (see Boucekkine et al. (2005) (Proposition 9) for details).

4.2 Balance growth paths

Let g_x be the growth rate of the variable x at time t. We focus for now the attention on a situation in which $g_x(t)$ are constant for every positive t. This implies that the evolution of all the variables is exponential and the initial datum i_0 is exponential as well. If g_λ is constant Eq. (43) implies that it is a root of the equation

$$A[1 - e^{-g_\lambda T}] = g_\lambda. \tag{45}$$

To ensure that such an equation admits exactly one positive root we impose

$$AT > 1$$

(see also Section 5). For the determination of the value of consumption and investments on the balanced growth path, notice that since $c(t) = Ak(t) - i(t)$ Eq. (42) can be written as:

$$e^{-\rho t} c^*(t)^{-\sigma} = \lambda^*(t) - \lambda^*(t+T), \tag{46}$$

which is the same as (7) except for the additional term $\lambda^*(t+T)$. Under the assumption that $g_\lambda(t)$ is constant the above equation becomes

$$e^{-\rho t} c^*(t)^{-\sigma} = \lambda^*(t)[1 - e^{g_\lambda}]$$

Taking logs and differentiating with respect to time the above equation one obtains the equivalent of Eq.(10). On the balanced growth path the stock of capital is $k^*(t) = i^*(t)\frac{1}{g_i}[1 - e^{-g_i T}]$, where g_i is constant (that $i(t) = i(t-s)\exp(g_i)s$ for $s < T$). The budget constraint implies that on the BGP

$$c^*(t) = i^*(t)[A\frac{1}{g_i}(1 - e^{-g_i T}) - 1],$$

and hence $g_c = g_i$. Therefore, consumption, gross investments and capital grow at the same common rate g, which, according to Eq. (8) is

$$g_c = \frac{1}{\sigma}(-g_\lambda - \rho), \tag{47}$$

which is the same as Eq. (10). In conclusion, on the balanced growth path the following equalities hold

$$g_i = g_c = \frac{1}{\sigma}(-g_\lambda - \rho),$$

where g_λ satisfies (45). As for the initial conditions, from the budget constraint we get that $c(0) = Ak(0) - i(0)$, where $k(0) = \int_{-T}^{0} i_0(s)ds$ is given, whereas $i(0)$ is to be determined. If the initial investments are spread across vintages in the same way as they are on the balanced growth path (except for a scalar) then $k(0) = i(0)\frac{1}{g_i}(1 - e^{-g_i T})$, where $g_i = \frac{1}{\sigma}(-g_\lambda - \rho)$ and

g_λ satisfies (45). Therefore, the budget constraint and the condition just obtained on the composition of capital imply that the initial level of consumption is:

$$c(0) = k(0)[A - \frac{\frac{1}{\sigma}(-g_\lambda - \rho)}{(1 - e^{-\frac{1}{\sigma}(-g_\lambda - \rho)T})}].$$

The initial value of the shadow value can be recovered by eq. (42) when $t = 0$:

$$\lambda(0) = [c(0)]^{-\sigma}[1 - e^{-g_\lambda T}]^{-1}.$$

4.3 The dynamics out of the BGPs

In sum, the MP allows for the characterization of the solution of the Ak model in a straightforward way even when capital is not homogeneous, as long as investments are distributed across vintage in a very special way. What happens if in one period, for whatever exogenous factor, investments are above or below the one compatible with the balanced growth path? Interestingly, the model generates some oscillatory trajectories. Boucekkine et al. (2005) analyze with great care the features of these trajectories.

In the general case (with a non-exponential initial datum) it can be proved that along an hypothetical internal optimal path (characterized by equations (42) (43) and (44)) g_λ and g_c remain always constant (see Boucekkine at al. (2005) Proposition 11 for a detailed proof). To study the dynamics of $i^*(\cdot)$ and $k^*(\cdot)$ they use a numerical method. See also Boucekkine et al. (1997) and Boucekkine et al. (2001).

However we find that Dynamic Programming illuminates more sharply the patterns of adjustment of macrovariables as a response to a shock that generates an investment book or slum. Therefore, we turn our attention to the presentation of this approach.

5 Dynamic Programming in an Hilbert Space

The objective of this section is to solve a problem of the type contained in (38) using a dynamic programming approach in Hilbert space (henceforth DPHS) instead of the maximum principle. We draw from (Fabbri and Gozzi (2006)) where a more formal account of this methodology can be found. From this section it will emerge that the DPHS approach has some advantages over the more popular maximum principle. First it could be the case that the interior solutions of the problem do not exist because the constraints of the problem are not satisfied (this issue was mentioned by the authors of Boucekkine and al. (2005) p. 60). Secondly, the DPHS analysis is carried out under milder parameter restrictions. Thirdly, the DPHS conveniently delivers AN explicit form of the optimal consumption path and a closed loop solution for the optimal trajectory of capital and investment, which will depend only on the initial conditions and the parameters' values. Similarly the value function and the optimal feedback function can be written in an explicit form. We will proceed as follows: introduction of the tools of dynamic programming in Hilbert setting; re-proposal of the consumer's optimization problem as in (38); re-derivation of the optimal path of consumption in feedback form, as well as the optimal feedback function.

5.0.1 Instruments of DP in vintage capital model

Three technical conditions are needed.

Hypothesis

$$AT > 1. \tag{H1}$$

This hypothesis guarantees the existence of a unique strictly positive root of the equation[5]

$$\xi = A(1 - e^{-\xi T}). \tag{48}$$

As it will be clarified below, ξ plays a key role in the characterization of the value function (see Proposition 5.2) and of the optimal feedback (Proposition 5.4). Notice that an identical restriction was imposed in the MP approach to guarantee the existence of a root to equation Eq. (45).

A second restriction similar to Hypothesis (14) imposed in the one dimensional case is needed to rule out trajectories that lead to unbounded utility, that is we want

$$\int_0^{+\infty} e^{-\rho t} \frac{(c(t))^{1-\sigma}}{1-\sigma} \mathrm{d}t < +\infty$$

for every control $c(.)$. As a way to determine such restriction, imagine that in each period the overall level of output is reinvested – an admissible, but hardly optimal strategy. Then capital expands at the fastest possible pace, given the state of technology. Let the accumulation of capital along such trajectory be described by the delayed differential equation (DDE)

$$\dot{k}_M(t) = Ak_M(t) - Ak_M(t-T). \tag{49}$$

Clearly the capital associated with the actual choice of investments, denoted with $k_{i(\cdot)}(t)$, does not exceed the one of the maximum- accumulation trajectory, namely $k_{i(\cdot)}(t) \leq k_M(t)$ for every choice of an admissible control $i(\cdot)$. Since the economy is closed $c(t) \leq Ak(t)$. Therefore the chain of relationships $c(t) \leq Ak(t) \leq Ak_M(t)$ implies that

$$\int_0^{+\infty} e^{-\rho t} \frac{(c(t))^{1-\sigma}}{1-\sigma} \mathrm{d}t \leq A^{1-\sigma} \int_0^{+\infty} e^{-\rho t} \frac{(k_M(t))^{1-\sigma}}{1-\sigma} \mathrm{d}t.$$

[5] This existence and uniqueness can be seen using simple concavity arguments.

An upper bound on the right hand side of the inequality would guarantee a finite utility's value associated with any admissible trajectory. The maximal root of the DDE in (49) is ξ (see Eq. (48)) implying that the $\xi-$ detrended value of $k_M(t)$ is always a finite number[6]

$$\left|\frac{k_M(t)}{e^{\xi t}}\right| \leq M \quad for\ all\ t \geq 0.$$

Therefore a sufficient condition for bounded utility is that

Hypothesis

$$\rho > \xi(1-\sigma). \tag{H2}$$

A third condition guarantees that consumption and investment in optimum are always positive, namely

Hypothesis

$$\frac{(\rho - \xi(1-\sigma))}{\sigma} \leq A. \tag{H3}$$

Notice that the conditions corresponding to (H2) and (H3) in the MP approach were more restrictive (that is $\rho > (1-\sigma)A$ and $\frac{\rho-\xi}{\sigma} < 0$), whereas the restriction $AT > 1$ is the same in both cases.[7] As we will see later (Eq. (53)) $g = \frac{\xi-\rho}{\sigma}$ is the growth rate of the consumption on the optimal trajectories, then the condition $\frac{\rho-\xi}{\sigma} < 0$ ensure positive growth while the condition (H3) allows to treat also non-positive growth cases.

5.1 Solution for the vintage model

Reformulating the problem in an infinite dimensional space has the major advantage of sidestepping the issue of dealing with a delay-state equation. Appendix (A), which is based on Fabbri and Gozzi (2006), presents the main aspect of its solution. The methodology closely follows the Bellman's DP principle, except that it is developed in a Hilbert space setting. To keep the presentation comparable with that in Section 2) we list three propositions dealing respectively with the value function, the optimal consumption trajectory, and the optimal investment strategy. The value function informs us of the maximum utility at a given time for a given history of investment in the period $[-T, 0)$.

The set of the admissible controls is

$$\mathcal{I}_{i_0} = \left\{ i(\cdot) \in L^2_{loc}([0, +\infty); \mathbb{R}^+) : \ i(t) \in [0, ak_{i_0,i}(t)], \ a.e. \right\},$$

where L^2 and L^2_{loc} are spaces discussed in the Appendix A. The value function of the problem is defined as

$$V(i_0) = \sup_{i(\cdot)\in\mathcal{I}_{\bar{\imath}}} \left\{ \int_0^\infty e^{-\rho s} \frac{(ak_{\bar{\imath},i}(t) - i(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}s \right\}. \tag{50}$$

[6]See Diekmann and al. (1995), page 27 for a proof.

[7]From Eq. (48) it follows that $\xi < A$ (it can be easily seen) and (H3) can be rearranged as $\frac{\rho-\xi}{\sigma} < A - \xi$.

In the non-delay case V depended on the one dimensional variable k_0; here it depends on how the distribution of initial capital is spread across vintages as indicated by the function $i_0 : [-T, 0) \to \mathbb{R}^+$. For this reason the problem is to be defined in a Hilbert space. We stress this point in the following observation.

Remark 5.1 *In the one dimensional case the stock of capital describes the state of the system: The set of admissible controls depends only on the stock of capital at each t, and the optimal trajectory as well as the optimal control in the set $[t, +\infty)$ depend only on the value $k(t)$. In particular, the value function depends only on $k_0 \in \mathbb{R}$. In the delay setting the state of the system is described by the history of investments in the interval $[t-T, t]$, which is a more detailed kind of information than $k(t)$. Also, the value function depends on i_0 as function $[-T, 0] \to \mathbb{R}$. For this reason we use as a state space the functional space the space L^2 (see Appendix A).*

Proposition 5.2 *The explicit expression of the value function V defined in (50) is*

$$V(i_0) = \nu \left(\int_{-T}^{0} (1 - e^{\xi s})\, i_0(-T-s) \mathrm{d}s \right)^{1-\sigma}$$

where the constant ν is given by

$$\nu = \left(\frac{\rho - \xi(1-\sigma)}{\sigma \xi / A} \right)^{-\sigma} \frac{1}{(1-\sigma)\xi / A}.$$

Proof. See the Appendix (A.2.1) **Step (1)**. ■

This expression is similar to that in Eq. (27) for the non-delay case, except that this one depends on a function of the history of i_0 in the interval $[-T, 0]$ whereas in the non-delay case the value function depended only on a real number (aggregate capital). The term $(1 - e^{\xi s})$ inside the integral can be interpreted as the value (in utils) of machines: older machines are worth less then younger ones. More precisely the value ranges between 0 and $1 - e^{-\xi T}$.

Consumption grows is an exponential way, even when the state variables (capital and investment) exhibit an irregular behavior. Intuitively, the concavity of the utility function and the ability to make intertemporal transfers through investments allow the individual to smooth out his consumption path.

Proposition 5.3 *The optimal growth rate of consumption is constant and equal to $g = \frac{\xi - \rho}{\sigma}$. Moreover given an initial distribution of investments $i_0(\cdot) \in L^2([-T, 0); \mathbb{R}^+)$ in equation (39), the optimal initial value of consumption is*

$$c_0 = \left(\frac{\rho - \xi(1-\sigma)}{\sigma \xi / A} \right) \int_{-T}^{0} (1 - e^{\xi s})\, i_0(-T-s) \mathrm{d}s. \tag{51}$$

Proof. See Appendix (A.2.1) in particular **Step (2)** and **Step (3)** for a sketch of the proof and for the references. ■

We wish to emphasize that propositions (5.2) and (5.3) did not assume that the economy is on the balanced growth path.

Contrary to consumption, both output and investment exhibit oscillatory movements when the economy is not on the balanced growth path. Interestingly we are able to describe these movements with explicit functional forms that link the optimal investment at time t as to the stock of capital at time t and to the sequence of investment in the period $[t-T, t)$.

Proposition 5.4 *The optimal investment strategy $i^*(\cdot)$ and the capital stock trajectory $k^*(\cdot)$ satisfy for all $t \geq 0$:*

$$i^*(t) = \frac{A}{\sigma}\left[1 - \frac{\rho}{\xi}\right] k^*(t) + \frac{\rho - \xi(1-\sigma)}{\sigma\xi/A} \int_{-T}^{0} e^{\xi s}\, i^*(-T+t-s)\mathrm{d}s.$$

Moreover along the optimal trajectories we have

$$i^*(t) = Ax^{*0} - \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi/A}\right)\left(\int_{-T}^{0} e^{\xi s} x^{*1}(s)\mathrm{d}s + x^{*0}\right). \tag{52}$$

Proof. Appendix (A) shows a sketch of the proof. In particular **Step (2)** and **Step (3)** give indications on how to go from an explicit form of the value function to the optimal feedback . Therein are also contained references of previous works where these results have been derived. ■

Finally, we observe that from the budget constraint it follows that $i^*(\cdot)$ is connected with the state trajectory $k^*(\cdot)$ by the following optimal feedback strategy for all $t \geq 0$:

$$i^*(t) = Ak^*(t) - c_0 e^{gt} \tag{53}$$

where c_0 is given by Eq.(51) and $g = \frac{\xi - \rho}{\sigma}$. Alternatively, Eq. (53) can be also be expressed as

$$i^*(t) = A\int_{-T+t}^{t} i^*(s)ds - c_0 e^{gt},$$

where instead of capital the history of investments appears. In differential form the above equation becomes

$$\begin{cases} \frac{di(t)}{dt} = Ai(t) - Ai(t-T) - gc_0 e^{gt} \\ i(0) = A\int_{-T}^{0} i_0(s)ds - c_0 \\ i(s) = i_0(s) \quad \textit{for all } s \in [-T, 0). \end{cases} \tag{54}$$

(See Appendix A for more details). When capital was homogeneous $k^*(t)$ expanded at a constant rate g and so did investments. Here the optimal investment and the capital stock may have an irregular growth pattern. Only a very special initial vintage distribution would lead to smooth exponential growth.

6 Simulations

This section illustrates the adjustment process of the economy when a shock affects the productivity of capital and its retirement time. The mathematical details of the dynamics are collected in appendix (B).

6.1 Technological Shock

In the Ak model the parameter A is usually kept constant, because the growth process is exclusively accounted for by the accumulation of physical and human capital. Nevertheless, as a way to illustrate the working of the model in the transitional dynamics, consider a scenario in which a one-time technological shock hits the economy. This might be the consequence of a major institutional change, such as the introduction of property rights, or of a modification of the form of government; somehow *all* capital in place becomes more productive. How does a drastic innovation of this kind affect the behavior of macroaggregates?

Let $t_0 = 0$ be the time when the shock occurs, and denote with A_0 and A the technological parameter before and after the shock, respectively. The other parameters, T, ρ, and σ remain constant. Let g_0 (g) be the asymptotic growth rate associated to the parameters A_0 $(A), T, \sigma$, and ρ. And imagine that the system is along a balance growth path before the shock's arrival, so that the initial datum is $i_0(s) = Ie^{g_0 s}$ for $0 \leq s \leq t_0$. Fig. (9) illustrates the $g-$discounted patterns of consumption, output, investment and the behavior of the investment ratio when the parameters are set according to Table (2) (g_0 and g are 4.86% and 5.57%, respectively the after shock ξ is 0.7998).

Before the shock the ratio is constant, because the system is along a balanced grow path. Following Eq. (52) the investment ratio for $t < 0$ is

$$\frac{i(t)}{y(t)} \equiv \frac{1}{\frac{A_0}{g_0}\left(1 - e^{-g_0 T}\right)} \sim 18.04\% \qquad \text{for all } t < 0.$$

As a consequence of the technological shock, Proposition (5.3) tells us that the investment ratio decreases. To disentangle the forces behind this outcome, it is useful to see separately the reaction of investments and consumption at time 0^+ (an instant after the shock). The initial optimal level of consumption determined through Eq. (52):

$$c_{0^+} = \left(\frac{\rho - \xi(1-\sigma)}{\sigma}\frac{A}{\xi}\right) e^{-g_0 T} \left(\frac{-1}{g_0}\left(1 - e^{g_0 T}\right) - \frac{1}{\xi - g_0}\left(1 - e^{-(\xi - g_0)T}\right)\right)$$

(thereafter expands at a constant rate g). Therefore investments and the investment ratio are respectively

$$A\frac{1}{g_0}\left(1 - e^{-g_0 T}\right) - c(0^+),$$

and

$$1-\frac{c_{0^+}}{A\frac{1}{g_0}\left(1-e^{-g_0 T}\right)}\sim 16.54\%.$$

Although consumption expands at a constant rate g thereafter, output, and consequently investments and the investment ratio, follow an irregular pattern for a while. For $t\to\infty$ the system tends to a new BGP with growth of the investment g. So, asymptotically, we have

$$\frac{i(t)}{y(t)}\xrightarrow{t\to\infty}\frac{1}{\frac{A}{g}\left(1-e^{-gT}\right)}\sim 16.30\%.$$

The discounted investment and production have a non-differentiability point at $t=T$ $(=10)$. This fact can be understood seeing to (54): the derivative of the function in the point T depends on the value of the function at 0, but in 0 the function as a discontinuity and then a discontinuity of the derivative appears in T. In the same way we will have a discontinuity in the second derivative in $t=2T$ etc...

Why do growth-discounted investments and production show a cyclical-like behavior? Since discounted consumption is always constant during the transition, from Eq. (52) it follows that the same holds for the following quantity:

$$e^{-gt}\left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi/A}\right)\left(\int_{-T}^{0}e^{\xi s}x^{*1}(s)\mathrm{d}s+x^{*0}\right),$$

which can be written as

$$H(i_g)(t):=\int_{-T}^{0}e^{gs}\left(1-e^{-\xi T}e^{-\xi s}\right)i_g(s+t)ds$$

where $i_g(\cdot)$ is the discounted investment $i_g(t):=e^{-gt}i(t)$. Note that the function $s\mapsto e^{gs}\left(1-e^{-\xi T}e^{-\xi s}\right)$ is continuous and strictly positive in $(-T,0)$ and it weights the history of the discounted investment. This means that the evolution of the discounted investment cannot be strictly increasing or decreasing. In particular, since $H(i_g)(t)$ is constant, we have only two possibilities: either i_g is constant – a balance growth situation – or it oscillates forever, with smaller and smaller oscillations – as in Fig. (9.B). Fig. (9.C) reports the qualitatively similar behavior of the growth rate of the income. Note that two discontinuities appear in its dynamics: the first on 0 and the second in T. This fact can be easily understood if we write the expression of the growth rate of the income: it is $\frac{y'(t)}{y(t)}$. So, there is a discontinuity of $y(t)$ in 0, arguing as above, we have a discontinuity of $y'(t)$ in T and then we have a discontinuity of $\frac{y'(t)}{y(t)}$ in T. Later we have a discontinuity of the derivative of the growth rate of the income in $t=2T$. The initial value of the growth rate of the income is constant with value g_0, since the system is along a BGP with growth g_0, the asymptotic value is g.

Intuitively, part of the excess production is consumed (wealth effect) and part of it is invested, leading to an amount of investments which is larger than would have been otherwise. The investment ratio may decline however, because the change in investments may not be as large as the increased productivity of past investments (Fig. (9.D)).

After the shock, investments must be chosen in a way to keep consumption growing at a constant rate. This requires a quite sophisticated choice. Imagine first that the rule is to expand investments according to the new balanced growth path rate. Then production would increase not only faster than before the shock, but also faster than the rate compatible with the new balanced growth path. This is hardly optimal, because it means that actually consumption can increase less than what the long run path requires. Indeed any other strategy of smooth investment will deliver a bumpy consumption profile. It then must be that the growth rate of investment fluctuates. In the case simulated, the smoothness of consumption is obtained by keeping investment below the long run trend in the periods that follow the shock. By doing so, the consumer keeps eating an increasing fraction of output, and the post-shock investment profile looks flatter than the new balanced growth path. Although the flattening of the investment profile serves the purpose of consumption smoothing, at a certain point it poses a constraint on the growth of output. Indeed, if the individual stubbornly kept the same investment strategy he would end up having output growing at the same lower rate, which is not compatible with the desired higher consumption rate. It must be that at a certain point in time $\tau < T$, it is actually optimal to reverse the strategy by steepening the investment profile. The investment pattern thereon is chosen in a way to counter balance the flat profile observed in the aftermath of the shock. When all the pre-shock investments have been scrapped out, a new smooth distribution of investments over age is observed. However it does not look like the long run one: The left (right) side is still flatter (steeper) than that of the long run distribution. All the subsequent $T-$ cycle look similar, except that the deviation of investments with respect to the long run pattern required to keep consumption on a smooth profile are smaller and smaller. Next we consider an exogenous shock on the equipment retirement's age.

6.2 Shock on Replacement Time

The composition of investment during the post-war period has changed in favor of equipment and software, as it is illustrated by Fig. (10), which plots the ratio between equipment and structure from 1950 to 2004. The ratio has almost doubled in the second part of the century, and especially in the 1990s. Indeed the upward trend of the ratio shown in the diagram is quite conservative, because the price of equipment relative to that of structure has declined in the same period, implying an even steeper rise of investment over structures in 'real' terms. Since equipment are kept in service for a shorter period than structures, it is reasonable to assume that the average scrapping time of capital has gradually declined in recent decades. The vintage AK model allow us to see how the optimal investment ratio responds to a variation in the replacement time.

We propose a simulation consisting in a reduction of the replacement time at a rate of 0.05 for fifty periods, which amounts to a gradual decline from 10 to 7.5 in the retirement age. The other parameters are the same collected in Table (1) before the technological shock. The system is along a balance growth path for $t < 0$. Then the dynamics from $t = 0$ and $t = 1$ can be obtained as in the previous simulation. For the period between 0 and 1 the initial datum is investment history that unfolded in the time interval $(-9, 0)$. Between between 1 and 2 the initial condition is the (non-BGP) history of investments between $t = -9$ and $t = 1$. Through this recursive method are built the dynamics for the following periods $(2, 3)$, $(3, 4)$, etc..

The patterns associated with output growth and the investment ratio are shown in Fig. (11). The two dynamics go in opposite directions. The growth rate of the capital slightly decreases, going from 4.53% to 4.49%. On the contrary, the investment ratio shows a strong tendency to increase (from the 17.8% of time $t = 0$ to over the 22% of time $t = 50$.)

The qualitative patterns just shown are somehow consistent with those observed in the US in the postwar period, during which the growth rate of GDP has remained constant or slightly declined whereas the investment share has increased at a regular pace. This is shown in Fig (12) (each variable is deflated by its own deflator). A more rigorous test confirms the visual impression of a rise in the investment ratio. A linear least square regression of the investment ratio against a constant and time suggests a positive and statistically significant time trend, whereas the growth rates exhibit a negative and statistically insignificant trend.

7 Another Look at the Cross-Country Evidence

In section (3) we found that the tight relationship between investment ratios and income growth predicted by the standard Ak model is only partially supported by the evidence. How does a variable depreciation rate change that conclusion?

Eq. (37) calls for the use of additional information on the investment series relative to Eq. (34). The former one can be written as $\frac{\dot{y}(t)}{y(t)} = A[\frac{i(t)}{y(t)} - \frac{i(t-T)}{y(t)}]$,which can be rearranged as

$$\frac{\dot{y}(t)}{y(t)} = A[\frac{i(t)}{y(t)} - \frac{i(t-T)}{y(t-T)}\frac{y(t-T)}{y(t)}]. \tag{55}$$

The ratio preceded by a negative sign times A corresponds to what used to be the depreciation rate in Eq. (34). Here today's depreciation rate is determined by what events occurred in the economy T periods ago. We replicate the two tests proposed in section (3). Figs. (13) and (14) present the scatter plot of the time-trends of the two sides of the Eq. (55) In one, only countries whose coefficients are statistically different than zero for both variables appear, whereas the other one includes all the countries for which the required data were available, regardless of the statistical significance – we eliminate a few observations that yield extremely large estimates (in absolute terms)of the right hand side of Eq. (55), so as to locate more precisely on the diagram the remaining countries. A visual inspection of the two diagrams indicates that the Ak model with vintage structure is more easily rejected by this kind of cross-country evidence, for in both cases most of the countries fall in the 'wrong' quadrants. However, the vintage Ak model is more likely to pass the consistency test than the standard Ak model. This result can be grasped by comparing the two plots in Fig. (15). The association between income growth rates and investment ratios is greater in the graph were these appear in differences, as in Eq. (55), rather than in levels. The partial correlation is about zero and 0.68, respectively.

8 Conclusion

This paper presented and evaluated the use of mathematical optimization tools applied to the Ak growth model. We have neither developed new methodologies, nor supplied new data. Yet, the journey was able to provide suggestive conclusions, one of which is presently revisited.

The mathematical representation of the stock of capital has important consequences for the interpretation of the time patterns of macrovariables. When capital has a vintage structure, more information from a given set of data can be extracted. A spike of investment today has positive *and* negative consequences over time. The positive ones, which are usually picked up by models with no vintage structure, is given by the augmented productivity of labor which is endowed with more capital. The negative ones appear when the machines of a given age are being scrapped.

Despite the fact that the Ak vintage model contains a more detailed – although not necessarily more realistic – representation of the stock of capital, its predictive power does not seem to be any better than the original Ak model. Quite the contrary a cross-country analysis showed that the association between the behavior of investments and per capita growth rates is more likely to come out negative when differences in investment ratios are being used (as suggested by the vintage model) than the current investment ratios (as implied by the model with homogenous capital). But this outcome does not diminish the potential returns of using optimization techniques in which the state variable is not a real number but rather a distribution of an attribute. One potential area of application is the extensive literature on inequality and growth.

From the methodological point of view, it emerged that the use of Dynamic Programming to solve the vintage Ak model is more appealing that the MP for it delivers a closed form solution to the investment strategy which can be easily compared with the actual investment time series of an economy. Unfortunately, if the production function is not linear in the stock of capital, neither of the two methodologies surveyed here can be readily applied. This means that for the time being the dynamics of vintage growth models with increasing or diminishing returns cannot be fully described – unless, the production function can somehow be transformed to look linear in the stock of capital. Finally, an issue that hopefully will be addressed in the future is the choice of the scrapping time. The decision of when to adopt a new piece of equipment is as important as that about the amount of investments to be undertaken.[8]

A DP for vintage capital model

The content of this appendix is based on Fabbri and Gozzi (2006). The purpose is to describe how to use dynamic programming to solve an optimization problem when capital has a vintage structure.

A.1 Statement of the Problem

First we need to define a set that contains function mapping from the vintage span of investment $[-T, 0)$ into the set of real numbers. Let

$$L^2([-T,0);\mathbb{R}^+) = \left\{ f\colon [-T,0) \to \mathbb{R}^+ \ : \int_{-T}^{0} |f(s)|^2 \mathrm{d}s < +\infty \right\},$$

For example every continuous (and every bounded) function on $[-T, 0]$ is in $L^2(-T, 0)$. Similarly,

$$L^2_{loc}([0,+\infty);\mathbb{R}^+) = \left\{ f\colon [0,+\infty) \to \mathbb{R}^+ \ : \ \int_{0}^{b} |f(s)|^2 \mathrm{d}s < +\infty \ \textit{for all}\ b > 0 \right\}$$

Secondly, we define the Hilbert space in which we will set the problem. This is $M^2 \overset{def}{=} \mathbb{R} \times L^2([-T,0);\mathbb{R})$. Thirdly, we refer to $x(t) \in M^2$ as an Hilbert state. Therefore, if we call $\gamma(t)[\cdot] \in L^2([-T,0);\mathbb{R})$ the function:

$$\begin{cases} \gamma(t)[\cdot]\colon [-T,0) \to \mathbb{R} \\ \gamma(t)[s] \overset{def}{=} -i(t-s-T), \end{cases}$$

the Hilbert state is

$$x(t) = (k(t), \gamma(t)[\cdot]). \tag{56}$$

[8] A recent discussion of this point can be found in Iacopetta (2007).

Below we will denote the two components of $x(t)$ as $x^0(t) \in \mathbb{R}$ and $x^1(t) \in L^2([-T, 0); \mathbb{R})$.

The set of the admissible controls is

$$\mathcal{I}_{i_0} = \left\{ i(\cdot) \in L^2_{loc}([0, +\infty); \mathbb{R}^+) : \ i(t) \in [0, ak_{i_0,i}(t)], \ a.e. \right\}$$

and we can rewrite the functional we want to maximize in terms of the Hilbert state:

$$J(i_0, i(\cdot)) = \int_0^\infty e^{-\rho s} \frac{(Ax^0(t) - i(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}s.$$

A.2 The tools of the dynamic programming and the comparison with the unidimensional case

The state equation The evolution of the system in the one-dimensional case is described by the differential equation (20). Here the evolution of the system is described by a differential equation in the Hilbert space M^2. Namely it can be proved, following Delfour (1986) and Bensoussan and al. (1992), that the Hilbert state defined in (56) satisfies the following differential equation in M^2 that the state equation of our system:

$$\begin{cases} \frac{d}{\mathrm{d}t} x(t) = G^* x(t) + i(t)(0, \delta_0 - \delta_{-T}), \quad t > 0 \\ x(0) = x_0, \end{cases} \tag{57}$$

where δ_0 and δ_{-T} is the Dirac measure in 0 and $-T$, and G^* a suitable generator of a C_0-semigroup. See Pazy (1983) for a definition of C_0 semigroup and for a definition of the solution of the evolution equation in Hilbert space. The complete expression of G^* is the following:

$$\begin{cases} D(G^*) \stackrel{def}{=} \{(\psi^0, \psi^1) \in M^2 \ : \ \psi^1 \in W^{1,2}(-T, 0; \mathbb{R}), \ \psi^0 = \psi^1(0)\} \\ G^* \colon D(G^*) \to M^2 \\ G^*(\psi^0, \psi^1) \stackrel{def}{=} (0, \frac{\mathrm{d}}{\mathrm{d}s} \psi^1) \end{cases}$$

where $W^{1,2}(-T, 0; \mathbb{R})$ is the Sobolev space of indexes $1, 2$ (see Ziemer (1990)).

The value function . Once we have described the set of the admissible controls as

$$\mathcal{I}_{i_0} = \left\{ i(\cdot) \in L^2_{loc}([0, +\infty); \mathbb{R}^+) : \ i(t) \in [0, ak_{i_0,i}(t)], \ a.e. \right\}$$

the value function of the problem is defined similarly to the one dimensional case (Eq. (27) as

$$V(i_0) = \sup_{i(\cdot) \in \ \mathcal{I}_{\bar{\imath}}} \left\{ \int_0^\infty e^{-\rho s} \frac{(ak_{\bar{\imath},i}(t) - i(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}s \right\}$$

The Hamiltonians and the HJB equation To write properly the Hamiltonians it is useful to recall the definition of the scalar product in M^2: Given two elements (x^0, x^1) and (y^0, y^1) of M^2 we define $\langle (x^0, x^1), (y^0, y^1) \rangle_{M^2}$ as

$$\left\langle (x^0, x^1), (y^0, y^1) \right\rangle_{M^2} := x^0 y^0 + \int_{-T}^{0} x^1(s) y^1(s) ds.$$

The *current value Hamiltonian* is defined as

$$H_{CV}(x, p, i) = \langle (\ x^0,\ x^1), Gp \rangle_{M^2} + \langle i, \delta_0(p)^1 - \delta_{-T}(p)^1 \rangle_{\mathbb{R}} + \frac{(A\ x^0 - i)^{1-\sigma}}{(1-\sigma)} \tag{58}$$

while the *maximum value Hamiltonian* is given by

$$H(x, p) \stackrel{def}{=} \sup_{i \in [0, Ax^0]} \left[H_{CV}(x, p, i) \right] \tag{59}$$

Note the formal analogy with the one-dimensional case: $H_{CV}(x, p, i)$ is obtained pairing the state equation with a co-state variable (that here we call p) and adding the current utility. The only difference is that in the one-dimensional case the pairing is obtained through a real multiplication, here we need the scalar product in M^2. Note also that the costate variable in this case belongs to the space M^2 (instead of $\mathbb{R}$). Other difficulties arise from the non-regularity of the term $\delta_0 - \delta_{-T}$ that does not belong to a functional space but is a distribution (see for example Yosida (1995)), this means that the term $\langle i, \delta_0(p)^1 - \delta_{-T}(p)^1 \rangle_{\mathbb{R}}$ makes sense only if p^1 is regular enough.

$H_{CV}(x, p, i)$ and $H(x, p)$ are both written in quite an informal way (for example we have not specified the domains of such functions) indeed here we want only to give a scheme of the approach, avoiding details (that can be found in Fabbri and Gozzi (2006)).

The HJB equation The HJB is formally not very different from the HJB we introduced in the one-dimensional case (26)

$$\rho v(x^0, x^1) - H((x^0, x^1), Dv(x^0, x^1)) = 0. \tag{60}$$

Nevertheless note that in the above HJB the differential $Dv(x^0, x^1)$ is intended as a differential in a Hilbert space (see for example Yosida (1995)) and the Hamiltonian H has the more complex formulation we described believe.

A.3 The steps of the DP method

Similar to the one-dimensional case (section 2) the application of the dynamic programming to the solution of the vintage model is described in three steps.

Step (1) - Solution of HJB We first look for a solution of the HJB. The solution, as in the one-dimensional case, is found though a "guess". A candidate solution is given and, through direct computations, such function can be proved to be a solution of the HJB. A solution of the HJB is given by

$$v(\ x^0,\ x^1) \stackrel{def}{=} \nu\left(\int_{-T}^{0} e^{\xi s}\ x^1(s)\mathrm{d}s +\ x^0\right)^{1-\sigma} \tag{61}$$

with

$$\nu = \left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi/A}\right)^{-\sigma}\frac{1}{(1-\sigma)\xi/A}.$$

To check that such function is a solution of the HJB (60) we have to differentiate it in M^2,to check that its differential is in the domain of G^* and prove by direct computations that the (60) is satisfied.

Step (2) - The feedback function The optimal feedback is given as in the one-dimensional case by the argmax of the current value Hamiltonian (58). Namely it is defined as

$$\phi(x) \stackrel{def}{=} argmax_{i\geq 0}H_{CV}(x,\nabla v(x),i) = Ax^0 - \left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi/A}\right)\left(\int_{-T}^{0} e^{\xi s}x^1(s)\mathrm{d}s + x^0\right). \tag{62}$$

This feedback function has to proved to be admissible and optimal, namely in order to prove that $\phi(x)$ is an optimal feedback we should prove first that the trajectory driven by $\phi(x(t))$ satisfies the positivity constraints for all the times and then, using the fact that v solves the HJB (the idea is similar to the one used in Proposition 2.2 but some technicalities appear) prove that such trajectory is optimal. This means that $\phi(\cdot)$ is an optimal feedback and that the function v is in fact the value function of the problem, thus $v = V$.

Step (3) - The explicit optimal trajectory To find the optimal trajectory we use, as in the one-dimensional case (see (29)), the optimal feedback $\phi(x)$ and we express the optimal control as

$$i^*(t) = \phi(x^*(t))$$

that can be explicitly written as

$$i^*(t) = Ax^{*0} - \left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi/A}\right)\left(\int_{-T}^{0} e^{\xi s}x^{*1}(s)\mathrm{d}s + x^{*0}\right). \tag{63}$$

Now, recalling the definition of the Hilbert state x that was given in (56) $\phi(x^*(t))$ can be expressed in terms of the history of the investment and find a delay differential equation (DDE) solved by the optimal investment. It is

$$i^*(t) = \frac{A}{\sigma}\left[1-\frac{\rho}{\xi}\right]k^*(t) + \frac{\rho-\xi(1-\sigma)}{\sigma\xi/A}\int_{-T}^{0} e^{\xi s}\ i^*(-T+t-s)\mathrm{d}s.$$

Differently from the one-dimensional case such DDE cannot be solved explicitly and numerical techniques will be used to obtain the graphics.

Nevertheless it is possible, considering the element of $\psi = (1, s \mapsto e^{\xi s}) \in M^2$, to define the function

$$\begin{cases} f \colon \mathbb{R}^+ \to \mathbb{R} \\ f \colon t \mapsto \langle \psi, x^*(t) \rangle_{M^2} \end{cases}$$

and calculate its derivative. The computations are quite technical because they involve the differential in M^2 but it is possible to prove that $c^*(t) = cf(t)$ for some constant c and that $\frac{df}{dt}(t) = gf(t)$ for $g = \frac{\xi - \rho}{\sigma}$. So that $c^*(t)$ is an exponential and its initial value can be computed using (63). Eventually we have that

$$c^*(t) = Ak^*(t) - i^*(t) = A(x^*)^0(t) - i(t) = c_0 e^{gt}$$

where

$$c_0 = \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi/A} \int_{-T}^{0} \left(1 - e^{\xi s}\right) i_0(-T - s) \mathrm{d}s \right)$$

and $g = \frac{\xi - \rho}{\sigma}$.

A.4 The limit for $T \to \infty$

It can be seen that the model for $T \to \infty$ tends to the one-dimensional AK model. Note that in the vintage case the term A included also the discounting factor δ that appears in the classical case (or, that is formally the same, $\delta = 0$). We have

$$\begin{array}{ll} \xi \xrightarrow{T \to +\infty} A & \nu \xrightarrow{T \to +\infty} \frac{\alpha^{-\sigma}}{1-\sigma} \\ g \xrightarrow{T \to +\infty} A - \alpha = \frac{A - \rho}{\sigma} & c_0 \xrightarrow{T \to +\infty} \alpha k \end{array}$$

Moreover the value function tends to the one-dimensional one and the optimal trajectories tend to the one-dimensional ones, indeed if $i_0 \in L^2(-\infty, 0)$ the term

$$\left(\int_{-T}^{0} \left(1 - e^{\xi s}\right) \bar{\imath}(T - s) \mathrm{d}s \right) \xrightarrow{T \to +\infty} k$$

and so the value function:

$$V(\bar{\imath}) = \nu \left(\int_{-T}^{0} \left(1 - e^{\xi s}\right) \bar{\imath}(T - s) \mathrm{d}s \right) \xrightarrow{T \to +\infty} \frac{\alpha^{-\sigma}}{1 - \sigma} k^{1-\sigma}$$

that is the value function in the non-delay case.

B The tools used in the simulations

As observed in Proposition 5.4 the problem of finding an optimal investment trajectory i^* is reduced to the one of solve the DDE (53). This can be rearranged as

$$i^*(t) = A \int_{-T+t}^{t} i^*(s)ds - c_0 e^{gt},$$

where instead of capital the history of investments appear. In differential form the above equation becomes

$$\begin{cases} \frac{di(t)}{dt} = Ai(t) - Ai(t-T) - gc_0 e^{gt} \\ i(0) = A\int_{-T}^{0} i_0(s)ds - c_0 \\ i(s) = i_0(s) \;\; for\ all\ s \in [-T, 0). \end{cases} \tag{64}$$

In order to solve it numerically we first need to find ξ as root of the equation (48) using a simple bisection algorithm. Secondly we use Matalb 6.5 DDE solver called 'dde23' developed by L.F. Shampine and S. Thompson. The routine allows to solve DDEs of the form

$$\dot{y}(t) = f(t, y(t), y(t-\tau_1), ...y(t-\tau_n))$$

and use an explicit Runge-Kutta triple. See Shampine and Thompson (2000) for a description of the algorithm and to learn more on convergence results.

In both simulations presented in the text it was assumed that the system was on a balance growth path before the shock. So the initial datum i_0 is equal to $i_0(s) = Ie^{g_0 s}$ where I is a constant. The constant I does not alter the qualitative dynamics of the system, for it appears in equations as a scale factor. In the simulation involving a shock in the technological parameter the growth rate of the consumption and the asymptotic growth rate of capital, output and investment go to a higher level, but in the short run oscillatory movements are detectable for output and investments.

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Country	Growth Rate of per Capita Consumption	Marginal Product. of Capital (A)	Net Return of Capital	Average Investment Ratios: 1950-2000
Argentina	1.00	8.51	3.51	18.88
Australia	2.10	10.16	5.16	26.88
Austria	4.11	13.17	8.17	27.45
Belgium	2.92	11.39	6.39	26.27
Canada	2.24	10.36	5.36	23.85
Switzerland	1.65	9.48	4.48	30.85
Denmark	2.17	10.25	5.25	25.17
Spain	3.25	11.87	6.87	25.84
Finland	3.32	11.99	6.99	28.73
France	3.89	12.84	7.84	26.02
United Kingdom	0.08	7.12	2.12	19.32
Iceland	3.37	12.06	7.06	29.59
Israel	2.18	10.27	5.27	32.28
Italy	4.21	13.32	8.32	28.03
Japan	5.85	15.78	10.78	30.68
Luxembourg	3.19	11.79	6.79	26.95
Netherlands	3.09	11.64	6.64	26.23
Norway	2.89	11.33	6.33	32.48
New Zealand	1.24	8.86	3.86	23.64
USA	2.35	10.52	5.52	19.67

Source: Authors' elaboration based on PWT 6.2
Note: The rates are in percentage. A country's growth rate of consumption is the estimated time trend coefficient of its log of per capita consumption time series. The value of A is obtained from Eq. (11), using the following parameters' values: $\rho = 0.02, \delta = 0.05$, and $\sigma = 1.5$.

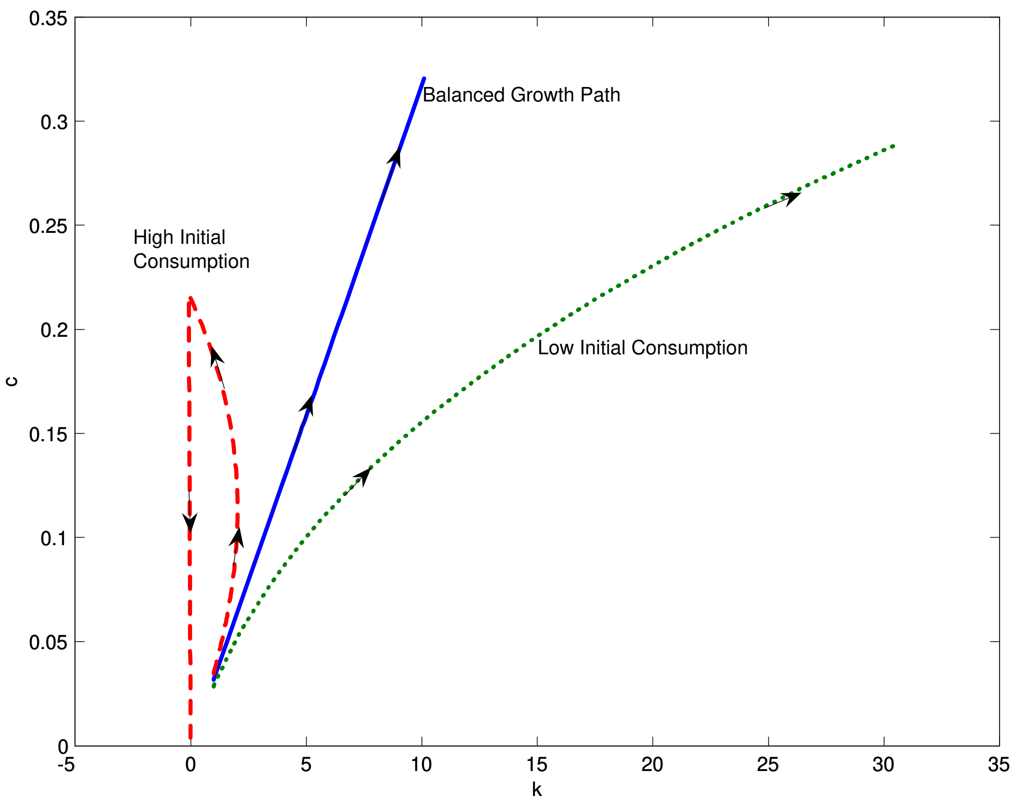
Table 1: Growth Rates of Consumption and Capital Returns

Parameter	Value
T	10
A_0	0.70
A	0.80
ρ	0.02
σ	14

Note: Parameters used for Simulation described in Section (6).

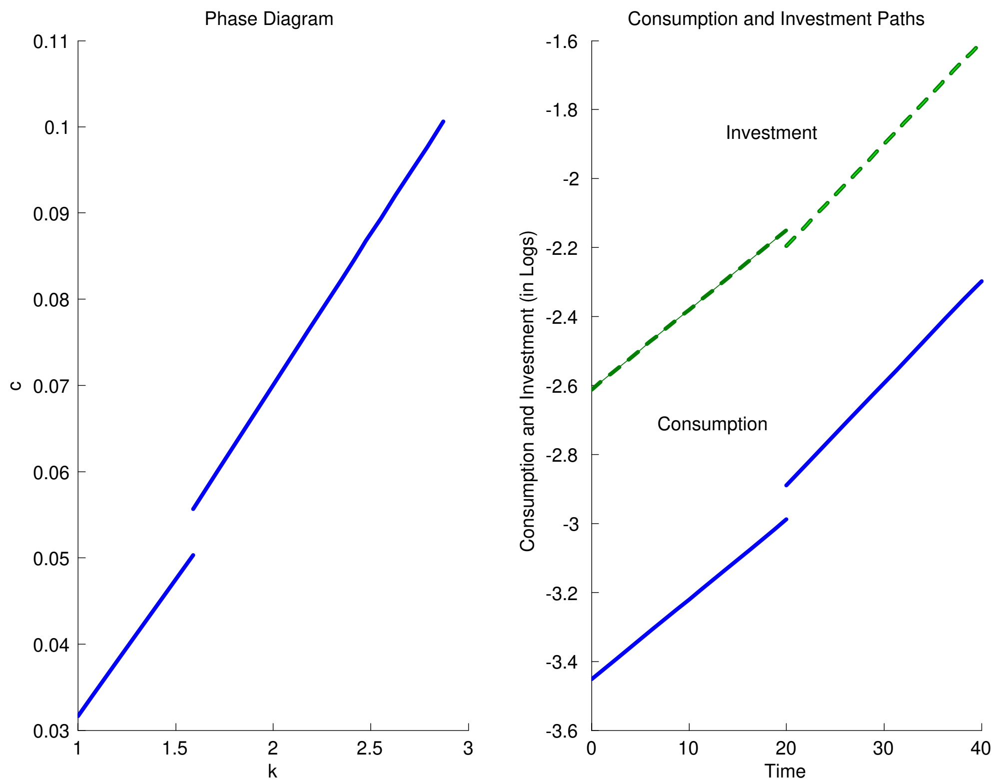
Table 2: Technological Shock

Figure 1: Phase diagram for the simple Ak model



Source: Authors' elaboration. Initial condition: $k_0 = 1$.
Parameters' values: $\rho = 0.02, \delta = 0.05$, $\sigma = 1.5$, and $A = 0.1051$.
The error in choosing the initial consumption, $\tilde{c}_0$, is 10% above or below the optimal one.

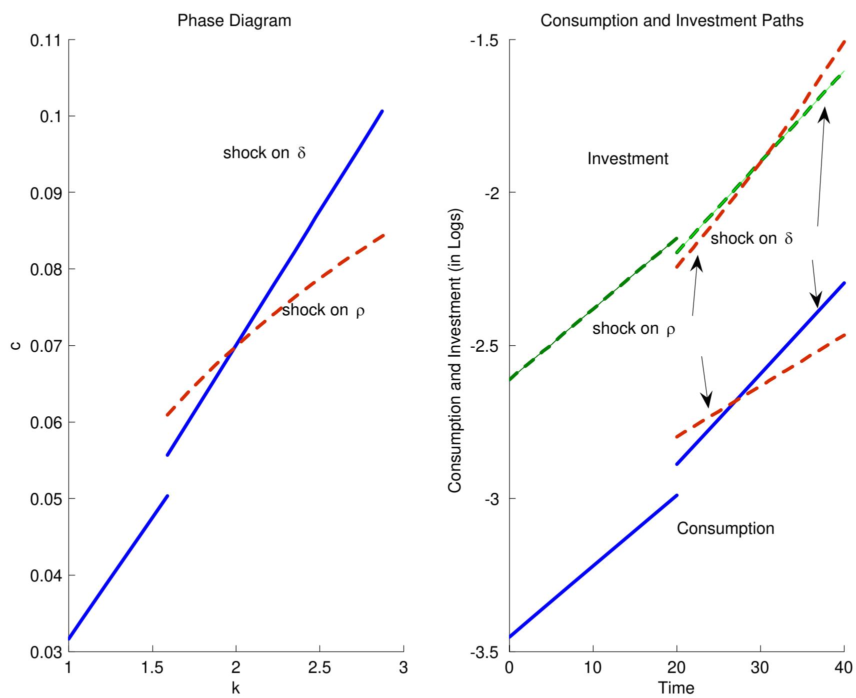
Figure 2: No Transitional Dynamics in the Simple Ak model



Source: Authors' elaboration.

Note: The shock consists in a one-time reduction of the depreciation rate from 5 to 4 percent. For other parameters's values see note in Fig. (1).

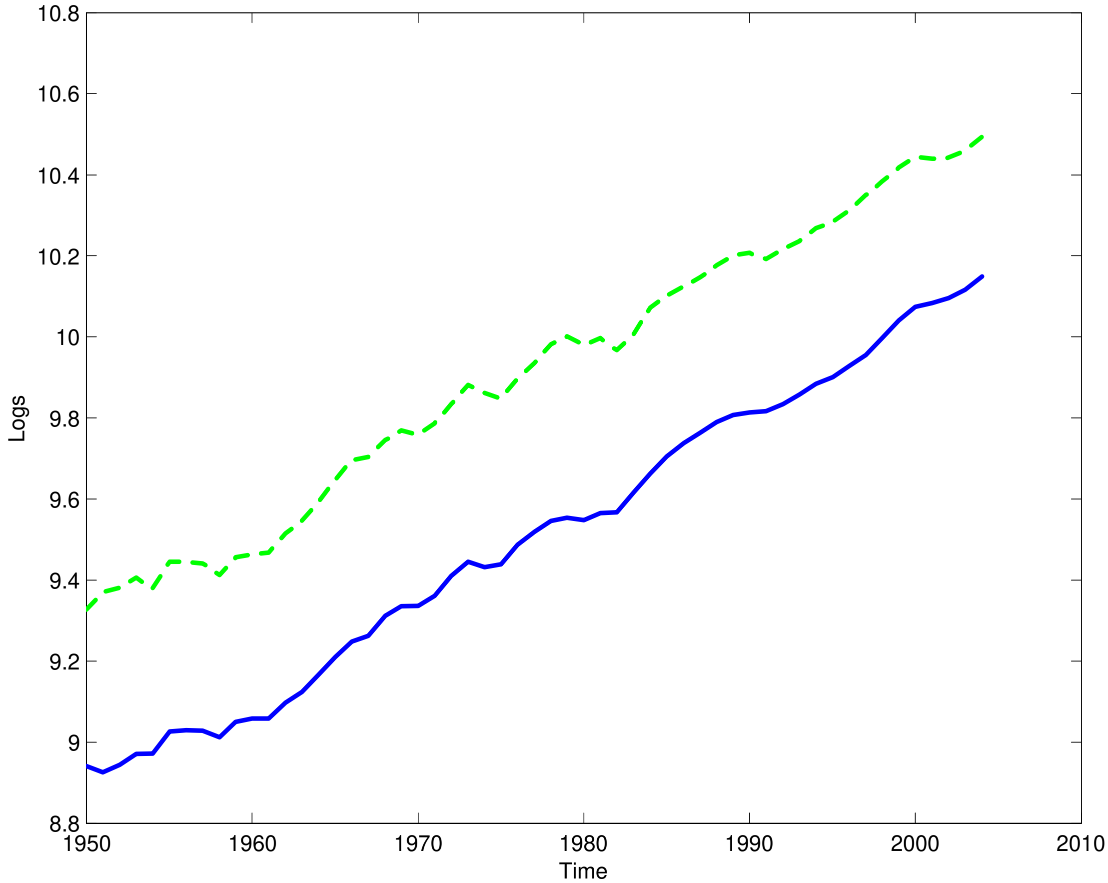
Figure 3: Comparing the Effects of a shock on rho and on delta



Source: Authors' elaboration.

Note: The two plots compare the effect of a decline of δ from 5 to 4 percent. with an increase of ρ from 2 to 3 percent.

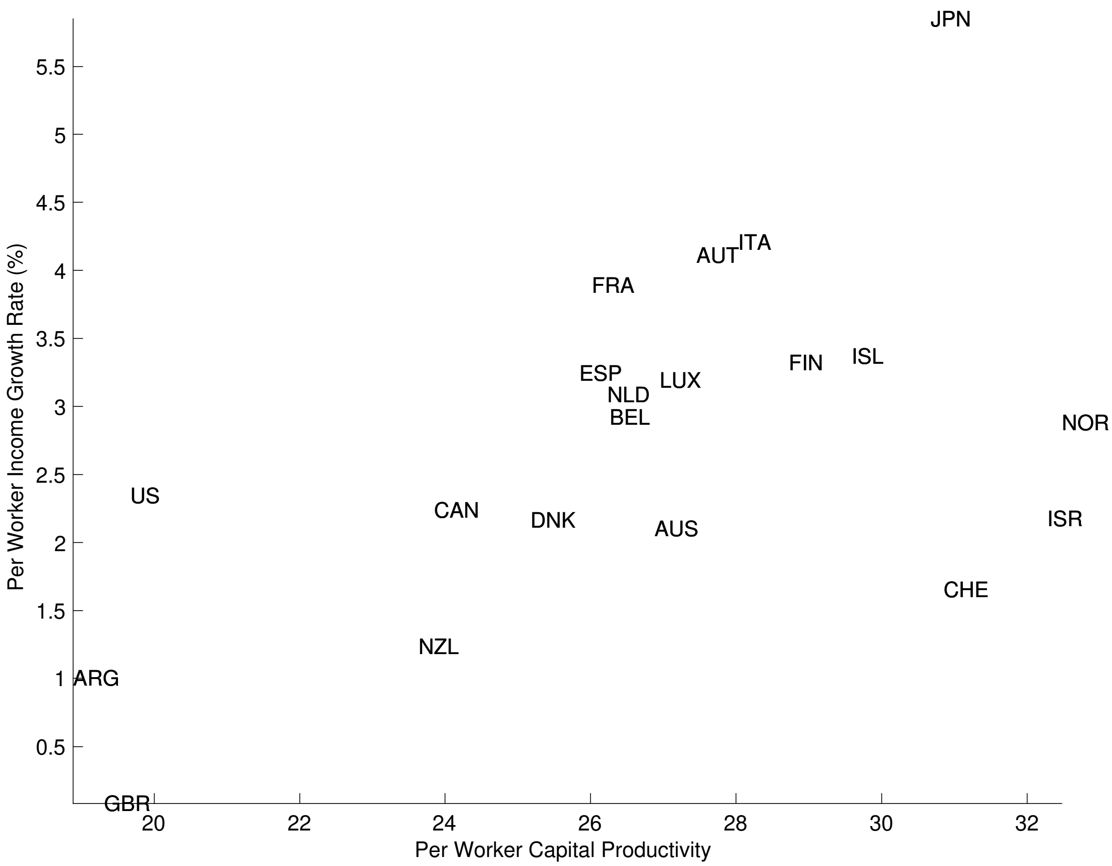
Figure 4: Per Capita Consumption and GDP in Constant US dollars



Source: Authors' Elaboration based on BEA data.

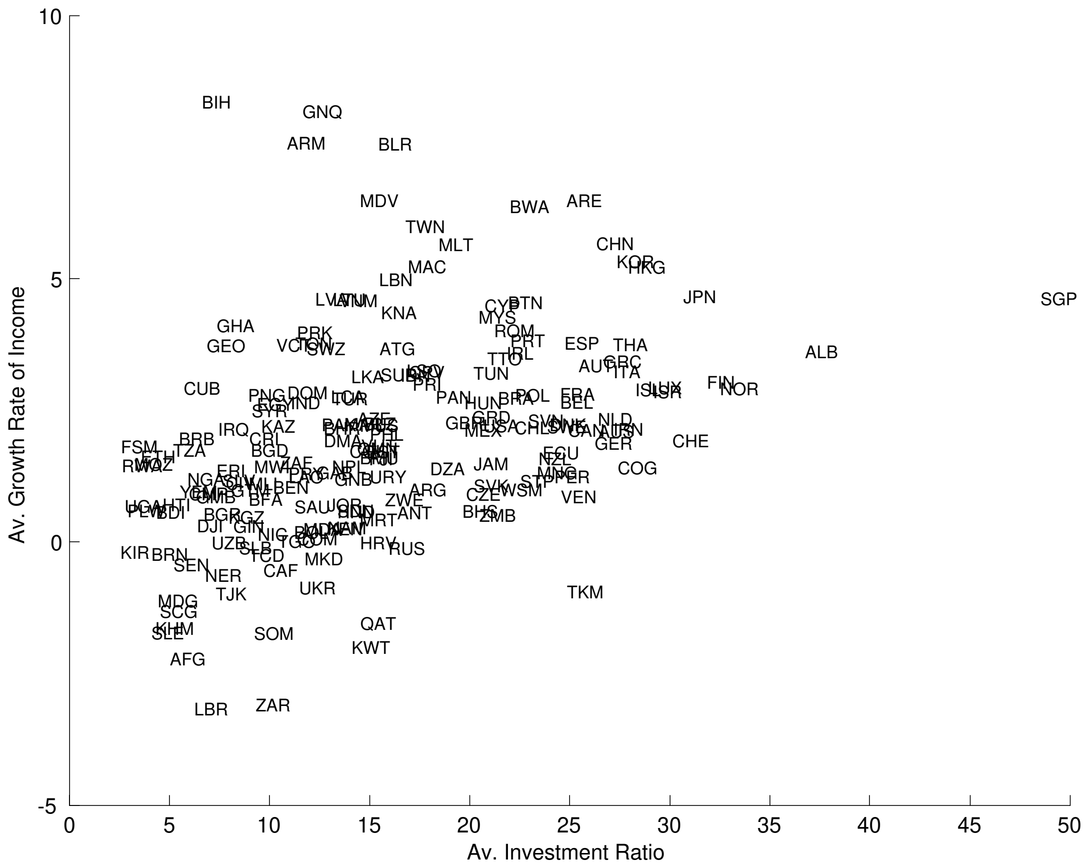
Note: The top (bottom) line is the log of per capita income (consumption).

Figure 5: Growth Rates and Capital Productivity



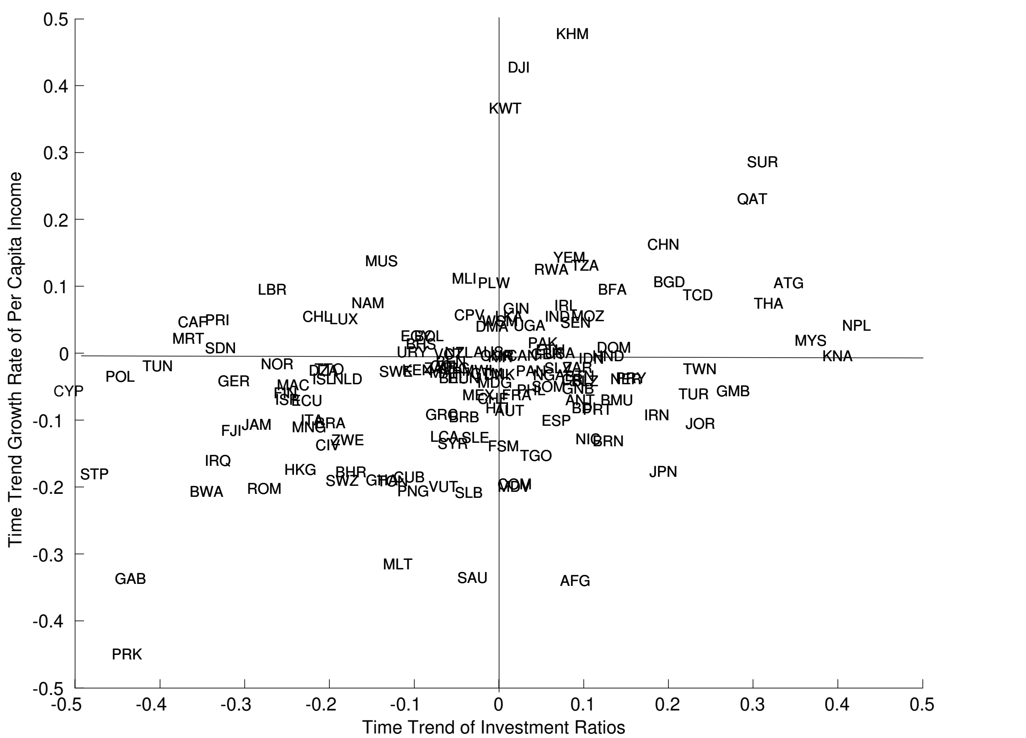
Source: Authors' elaboration based on PWT 6.2 data.

Figure 6: Income Growth and Investment Ratio: 1950-2004



Source: Authors' elaboration based on data drawn from PWT 6.2.
Note: The averages are taken over the 1950-2000 period.

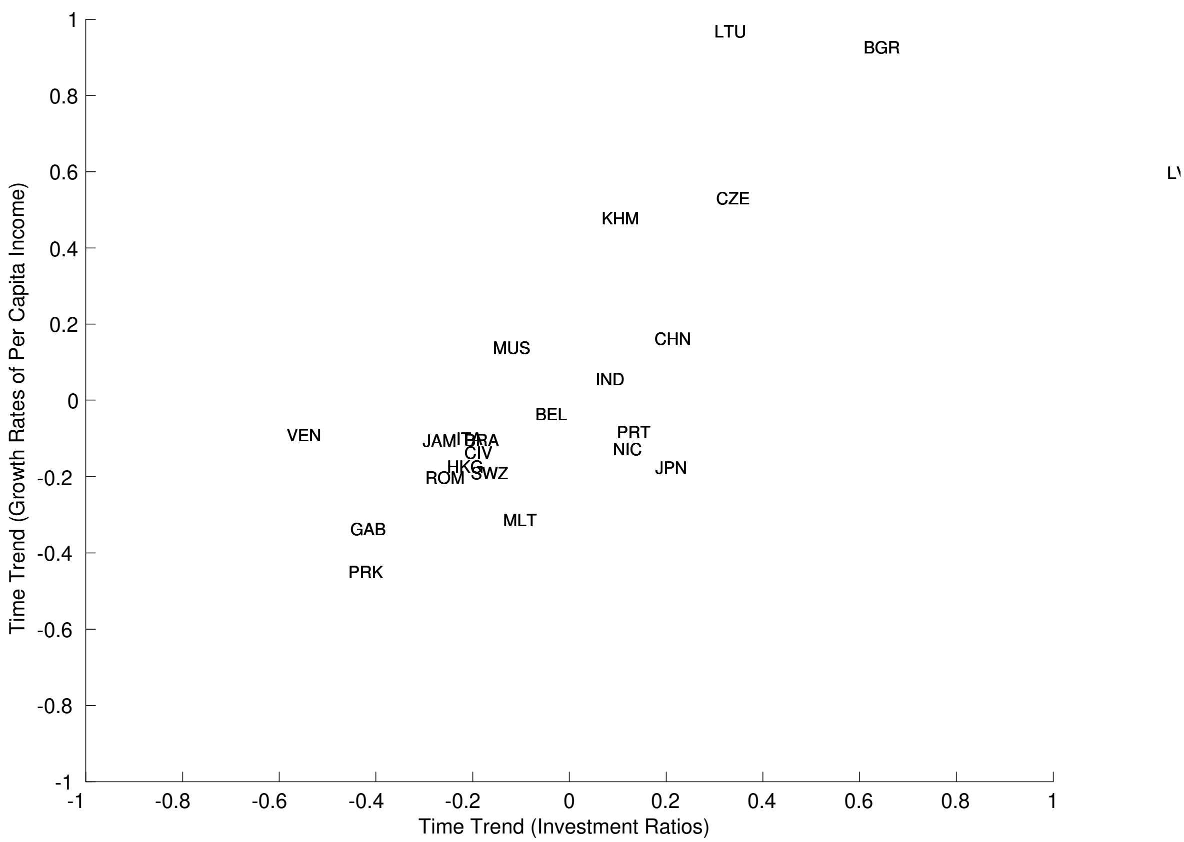
Figure 7: Consistency of Time Trends



Source: See Fig. (6).

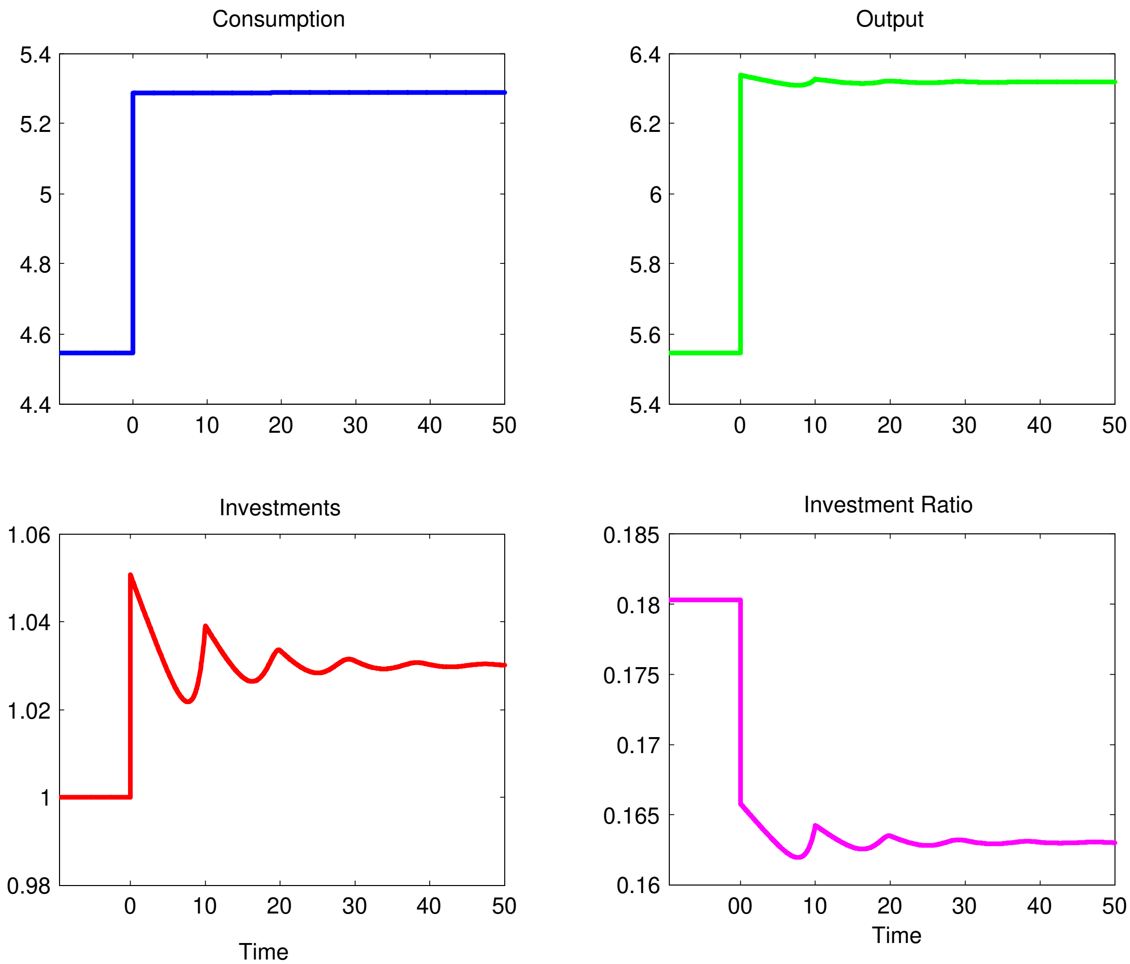
Note: the time trend is estimated over the period 1950-2000.

Figure 8: Consistency of "Significant" Time Trends



For source and notes see Fig. (7). This plot selected only countries that showed a statistically significant time trend coefficient for both variables.

Figure 9: Positive Technological Shock



Note: The simulations are based on parameters reported in Table (2).

Figure 10: Share of Equipment over Structures in the US: 1950-2000

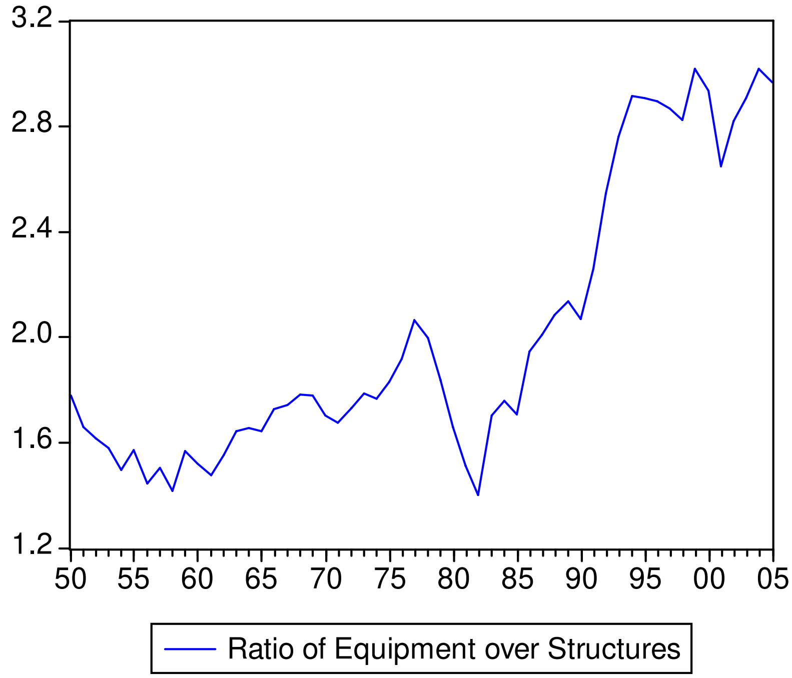
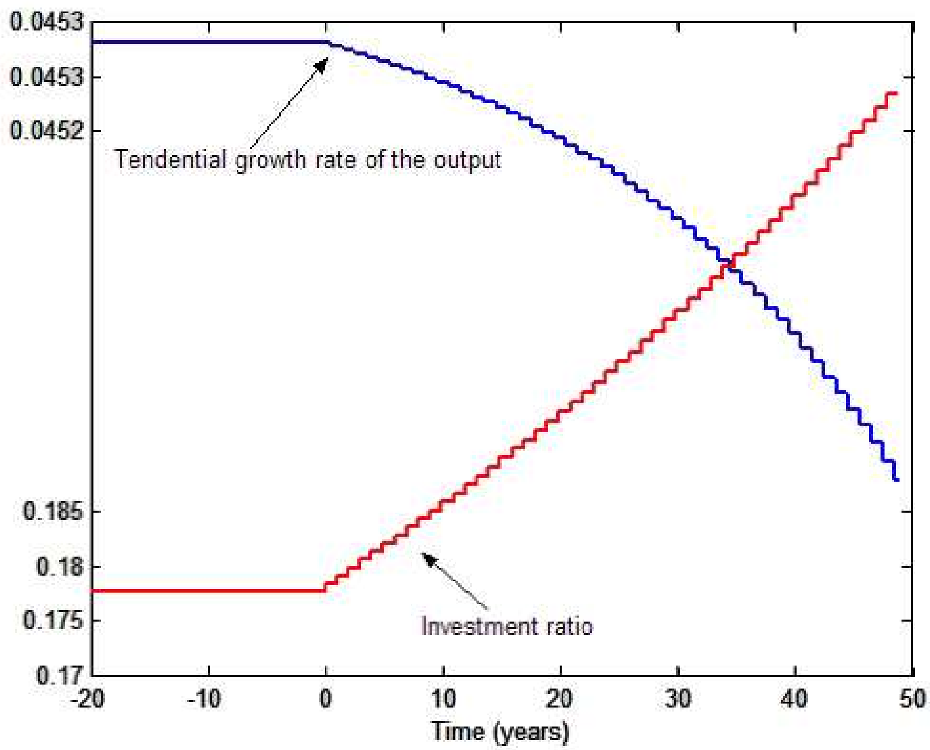
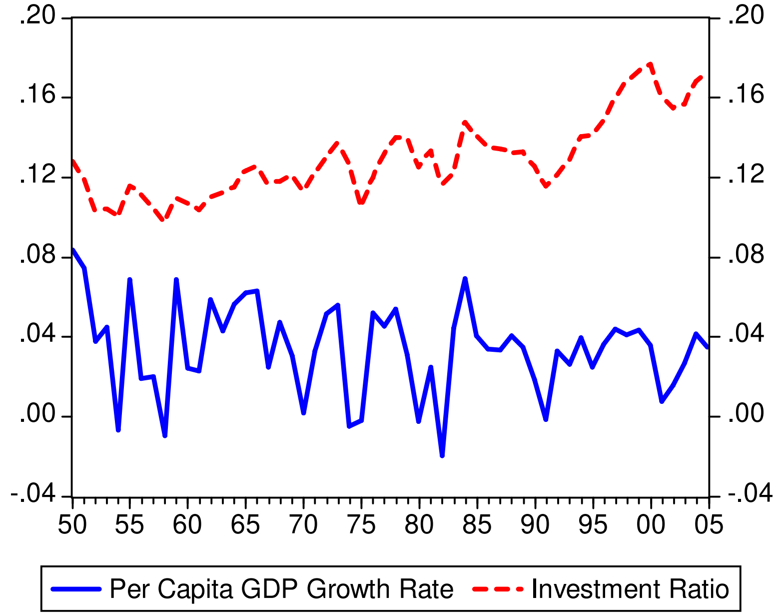


Figure 11: Gradual Reduction of Scrapping Time



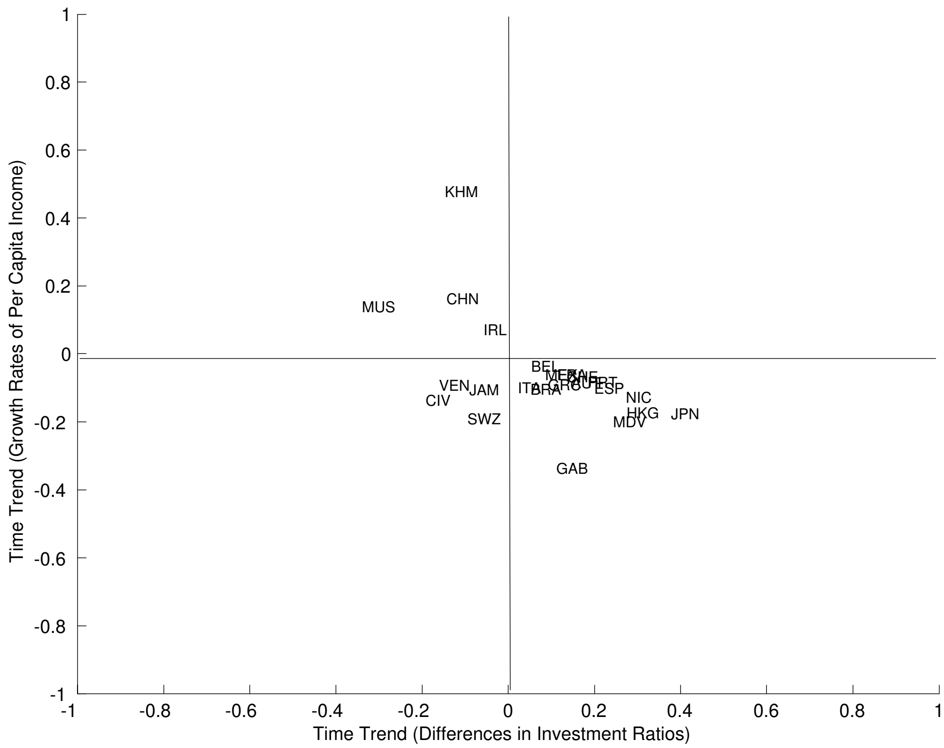
Note: Reduction of the replacement time at a rate of 0.05 for fifty periods, from 10 to 7.5 years (see text).

Figure 12: Growth Rate of per Capita Real GDP and Investment Ratio in the US: 1950-2000



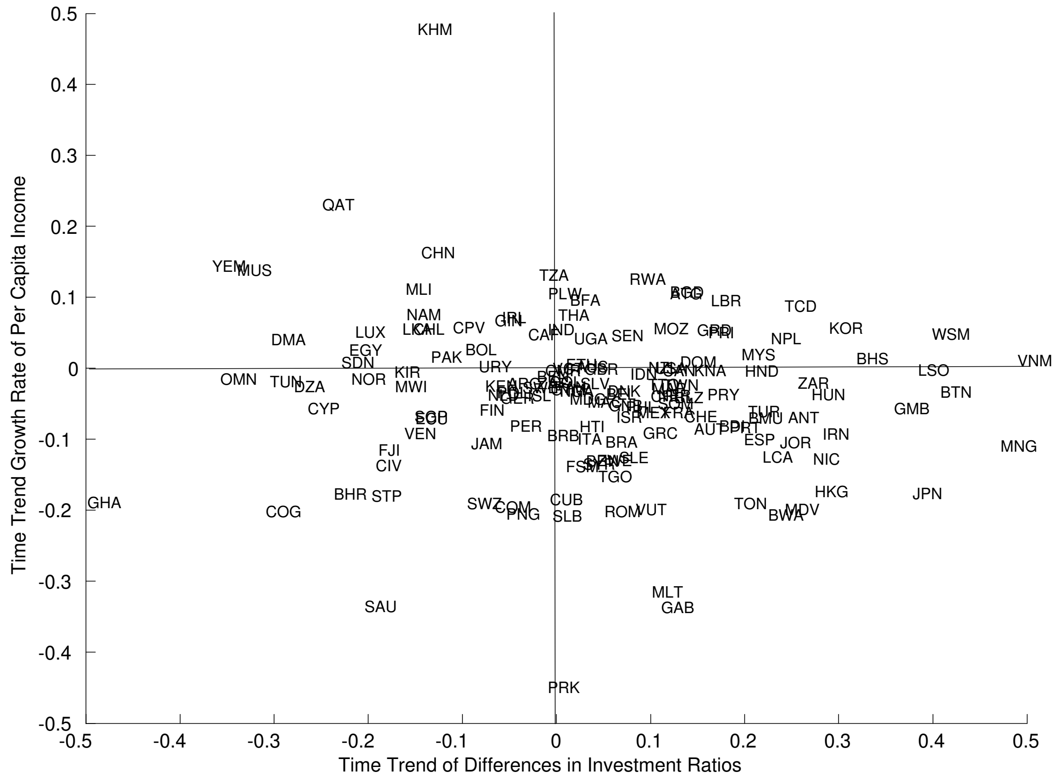
Source: Authors' elaboration based on BEA data.

Figure 13: Consistency of "Significant" Time Trends (Vintage-Version)



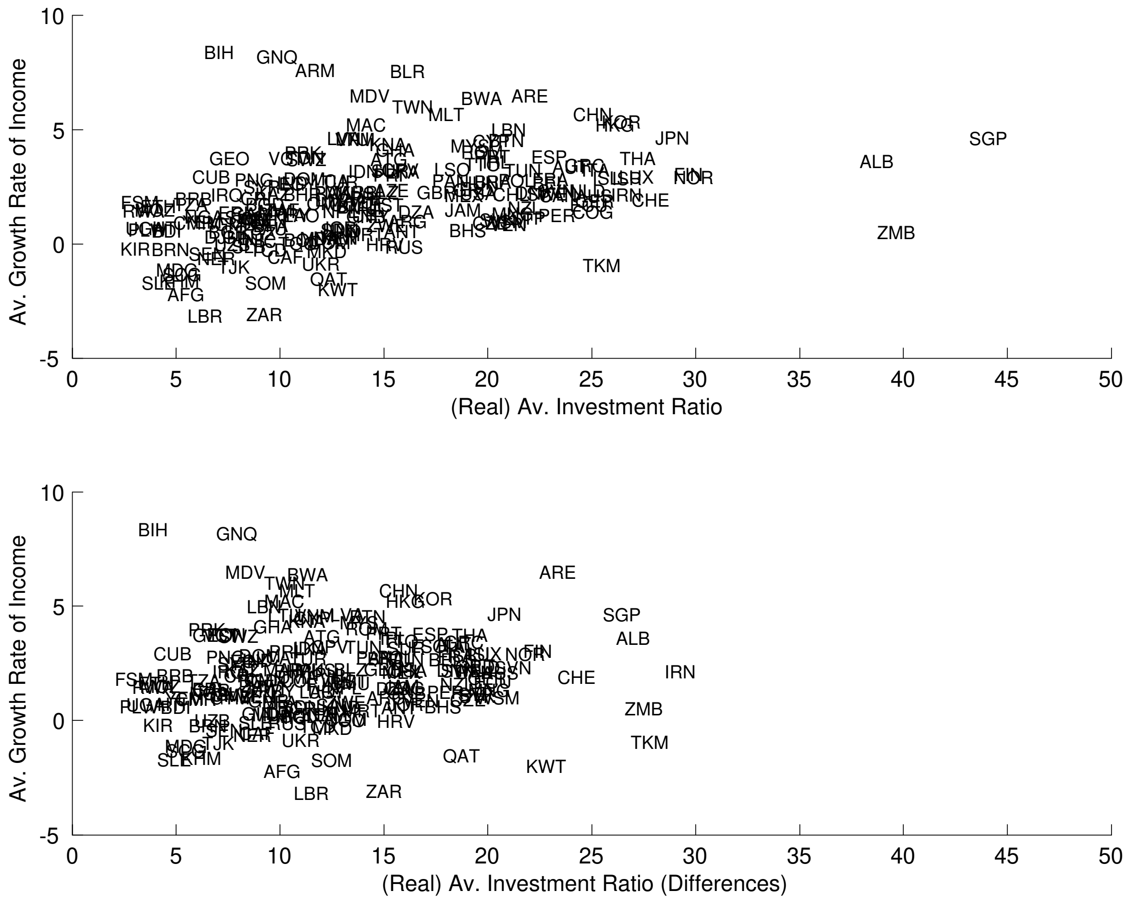
Source: See Fig. (7).

Figure 14: Consistency of Time Trends (Vintage-Version)



Source: See Fig. (7).

Figure 15: Vintage vs. non Vintage with Average Growth Rates



Source: See Fig. (7)

Note: The horizontal axis of the bottom plot considers changes in investment ratio over a 10-year window.