# A STRUCTURAL MODEL FOR CREDIT-EQUITY DERIVATIVES AND BESPOKE CDOs

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# A STRUCTURAL MODEL FOR CREDIT-EQUITY DERIVATIVES AND BESPOKE CDOS

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ABSTRACT. We present a new structural model for single name equity and credit derivatives which we also correlate across reference names to produce a model for bespoke synthetic CDOs. The model captures volatility and outlook risk along with correlation risk for small and for large moves separately. We show that the model calibrates well to both equity structured products and credit derivatives. As examples, we discuss a number of single name derivatives on IBM spanning the credit-equity spectrum and ranging from volatility swaps, to cliquets, CDS options and CDSs on leveraged loans with pre-payment risk. We also use the model to price tranches on the investment grade DJ.CDX.IG index along with tranches on the high yield index DJ.CDX.HY. We show that the model gives consistent and high precision pricing across all these derivative asset classes. We show that this can be achieved consistently, with the very same parameter choices across these diverse derivative assets and making use of only minor explicit time dependencies.

## 1. INTRODUCTION

Devising a structural model for bespoke CDOs, equity structured products and the new generation of volatility sensitive credit derivatives is an exercise at reconciling opposites: equity versus CDS spread volatilities, correlations between equity returns versus systematic industry sector risk, single name long-dated path-dependent derivatives versus tranche spreads, investment grade versus high-yield synthetic credit portfolios, equity versus mezzanine and senior tranches. In this paper we present a 5-factor structural model which accomplishes these multiple objectives. The stochastic factors are firm equity value, name outlook, equity volatility, recovery rates and number of shares outstanding. As a single name example we take an IBM dataset and calibrate to high precision to American options, variance swaps, CDS spread curves and historical CDS spread volatilities. Within this framework we price single name derivatives probing vega convexity such as volatility swaps, capped/floored cliquets and EDSs along with volatility sensitive credit derivatives such as CDS options, CDSs on leverage loans and soft risky annuities.

We then extrapolate the IBM model to other reference names by resetting the initial condition. This way we model names in the DJ.CDX.IG standard investment grade portfolio and the DJ.CDX.HY standard high yield portfolio achieving a perfect fit to their CDS curve. We introduce two separate correlation factors for each industry sector, one for equity returns and one for systematic credit risk and calibrate to high precision to the 10-year term structure of tranche spreads across the various segments of the capital structure for both portfolios with the very same parameters.

Model parameters are time-homogeneous with only few exceptions: an exogenous interest rate, a term structure of recovery rates to fit CDS spread curves and a slowly varying term structure of the systematic credit correlation factor to fit senior tranche spreads. We make use of operator methods, the theory of Abelian path dependent options and dynamic conditioning. We avoid Monte Carlo simulations altogether for both path-dependents and basket derivatives and achieve a noiseless and efficient numerical framework which yields smooth price sensitivities and is ideally suited for the emerging multi-core architectures.

The quest for a satisfactory modeling framework for bespoke CDOs is attracting considerable interest and we can hardly make justice of the literature being generated. A sample of papers which we found stimulating are (Andersen and Sidenius 2004),(Giesecke and Goldberg 2005), (Hull and White 2003), (Joshi and Stacey 2005), (Lucas *et al.* 2001), (O'Kane and Livesey 2004), (Li 2000), (Schonbucher 2006), (di Graziano and Rogers n.d.). On our account, we first developed models combining jumps and local volatility for single name credit derivatives using analytically solvable schemes (Albanese *et al.* 2003), (Albanese and Chen 2005*a*), (Albanese and Chen 2005*b*), (Albanese and Chen 2004). Then we shifted towards rating models specified non-parameterically (Albanese *et al.* 2005-2006). Finally, we concluded it was necessary to incorporate detailed volatility information in our model and developed an interest in equity derivatives and structured equity products which has led to the present paper. We came to conclude that to meet the many objectives of this complex problem, nothing short of a unified credit-equity structural model can meet all the requirements and set out to build one.

The model we present here is robust and parameterized in a quite intuitive and controllable way. Our main focus in this article is to explain the modeling assumptions and user interface of the model which are indeed quite simple. The internal numerical workings we devised to support this modeling framework are innovative and it would be difficult to make justice of their complexities in a short paper as this one is intended to be. Instead we review some key technical concepts and give snippets of derivations which we found of key importance.

## 2. The single name model

The single-name modeling framework we adopt is a 3-factor defaultable equity model with stochastic volatility and stochastic outlook dynamics. The factors are subsequently augmented to 5 by incorporating stochastic recovery rates and a stochastic number of shares outstanding. We find that postulating a slowly varying stochastic outlook dynamics alongside a faster mean-reverting volatility dynamics is essential in order to reconcile within a time-homogeneous framework implied volatilities of equity options and variance swap rates along with the term structure of historical CDS spread volatilities. The two differ by a substantial factor of order 2 to 4. The gap in volatilities can be reconciled within a 3-factor model like ours because the information affecting equity volatilities is shorter lived than the information affecting spread volatilities.

We calibrate the stochastic outlook term affecting long time scales by fitting the CDS curve and use this information for long-dated extrapolations of the implied volatility surface for equity options and for pricing equity structure products affected by vega convexity, from capped and floored cliquets to volatility swaps.

Along with equity structure product we also obtain a model fitting CDS spread volatilities to high precision which is thus suitable to price CDS options, soft risky annuities and CDSs on leveraged, amortizing loans affected by pre-payment risk. We stress time-homogeneity as a key property and requirement for our single name model. Apart from an exogenously specified term structure of interest rates, we achieve high precision calibration to all of our objectives within the bounds of a time-homogenous model. We take the view that whenever precision needs to be absolute and perfectly accurate fits are required, then one should still insist on high precision calibration of a time-homogeneous model and allow for only small time-inhomogeneous adjustments having a minor impact on forward volatilities and forward correlations.

We use operator methods and express our model in terms of a Markov generator given by a time dependent matrix of the form

$$(1) \quad \mathcal{L}^\alpha(x,a,b;x',a',b';t) = \big(\mathcal{D}_{\text{out}}(x;x'|a,b)\delta_{aa'} - \mathcal{J}_{\text{out}}(x,a;x',a'|b)\big)\delta_{bb'}$$

$$(2) \quad + \mathcal{J}_{\text{vg}}(x;x'|b)\delta_{aa'}\delta_{bb'} + \mathcal{V}_{\text{sv}}(b;b'|x,a)\delta_{xx'}\delta_{aa'} + \mathbf{J}^\alpha_{\text{jtd}}(t)(\delta_{x'0} - \delta_{xx'})\delta_{aa'}\delta_{bb'}.$$

A state variable is given by a triplet $y = (x,a,b)$ whereby $x$ is related to the stock price by a function $S(x)$, $a$ is an outlook indicator and $b$ is associated to volatility regimes. In our example, $x = 0,...69$, the variable $a$ takes two values and distinguishes between a stable and a negative outlook and the variable $b$ takes three values corresponding to three levels for volatility: low, medium and high. The index $\alpha$ denotes the reference name. As we explain in more detail below, we assume that only the last jump-to-default term is name specific.

The Markov generator satisfies the following three conditions corresponding to probability positivity and conservation and to arbitrage freedom, respectively:

$$(3) \quad \mathcal{L}^\alpha(x,a,b;x',a',b';t) \geq 0 \quad \text{if} \quad (x,a,b) \neq (x',a',b')$$

$$(4) \quad \sum_{x',a',b'} \mathcal{L}^\alpha(x,a,b;x',a',b';t) = 0$$

$$(5) \quad \sum_{x',a',b'} \mathcal{L}^\alpha(x,a,b;x',a',b';t)(S(x') - S(x)) = r(t)S(x)$$

for all triples $(x',a',b')$. Here $r(t)$ is an exogenously specified interest rate that we assume to be piecewise constant. The matrix $\mathcal{D}_{out}(x;x'|a,b)$ is tridiagonal and is chosen in such a way that the arbitrage free condition above is satisfied.

The operator $\mathcal{J}_{\text{out}}(x,a;x',a'|b))$ models jumps downwards which, in our specific example, are chosen to be of size 12% if the outlook is negative and small symmetric jumps of size 2% if the outlook is stable. Both jumps are finite activity and occur with yearly frequency. The operator $\mathcal{J}_{\text{vg}}(x;x'|b)\delta_{aa'}\delta_{bb'}$ describes a variance-gamma process of log-normal volatility dependent on the state variable $b$ and of the form $S^{\beta(S)-1}$ where $\beta(S)$ is the function in Fig. 1. The variance rate is chosen to be 4% and is of the order of the rate estimated econometrically, not sufficient to sustain an implied volatility skew over medium or long time horizons. The first two terms admits the state $x = 0$ as an absorbing point. The operator $\mathcal{V}_{\text{sv}}(b;b'|x,a)\delta_{xx'}\delta_{aa'}$ describes a mean reverting process on the volatility regimes.

The operator $\mathbf{J}_{\mathtt{jtd}}^{\alpha}(t)$ is represented as the sum of three terms describing jump to default, i.e.:

$$(6) \qquad \mathbf{J}_{\mathtt{jtd}}^{\alpha}(t) = \mathbf{J}_{\mathtt{jtd}}^{\mathtt{is}\alpha}(t) + \mathbf{J}_{\mathtt{jtd}}^{\mathtt{sec}}(t) + \mathbf{J}_{\mathtt{jtd}}^{\mathtt{jur}}(t)$$

We assume that these matrix elements are non-negative for all times $t \geq 0$. The function $\mathbf{J}_{\mathtt{jtd}}^{\mathtt{is}\alpha}(t)$ describes idiosyncratic jump to default, is name dependent and uncorrelated across different names. For times $t$ up to two years from current time, this first term is chosen in a minimalistic way so to fit the term structure of digital CDS spreads. For such short maturities, this term is essential to achieve a perfect fit as the other terms in the generator describing a gradual process to default are unable to generate sufficiently large default probabilities. For longer time horizons, $\mathbf{J}_{\mathtt{jtd}}^{\mathtt{is}\alpha}(t)$ is extrapolated as a constant function. This ensures that for maturities exceeding two years the shape of the CDS curve is controlled by the function $\beta(S)$ and by the definition of the outlook dynamics, namely the process to default prevails on the jump-to-default mechanism as the leading contributor to the total probability of default. The term $\mathbf{J}_{\mathtt{jtd}}^{\mathtt{sec}}(t)$ is shared by all names in the same sector, equals zero during the first two years period and then grows slowly. Similarly for the term $\mathbf{J}_{\mathtt{jtd}}^{\mathtt{jur}}(t)$ which is shared by all names in the portfolio. These latter jump-to-default terms model sector-wide and jurisdiction-wide default event, these are very rare events whose implied probability is chosen to fit the senior tranche spreads.

To find option prices along with CDO tranche spreads under this model we need to accomplish three basic tasks. The first is to evaluate numerically the single name propagator satisfying the equation

$$(7) \qquad \frac{d}{dt_1} U^{\alpha}(t_1, t_2) + \mathcal{L}^{\alpha}(t_1) U^{\alpha}(t_1, t_2) = 0,$$

with final time condition $U^{\alpha}(t_2, t_2) = \mathbb{I}$, the identity operator, for all time ordered pairs $t_1 \leq t_2$. The second task is to evaluate more general propagators carrying path information for path-dependent options. The third is to correlate across a large number of names, possibly in the hundreds, while keeping the numerical complexity within reasonable bounds.

The single-name propagator satisfying equation (7) is given by the path-ordered exponential

$$(8) \qquad U^{\alpha}(t_1, t_2) = \mathcal{P} \exp\left( \int_{t_1}^{t_2} \mathcal{L}^{\alpha}(s) ds \right).$$

There are two useful methods to express a path-ordered exponential. One is the Feynman path-integral expansion

(9)
$$\mathcal{P} \exp\left( \int_{t_1}^{t_2} \mathcal{L}^{\alpha}(s) ds \right) = \lim_{N \to \infty} \left( \mathbb{I} + \delta t_N \mathcal{L}^{\alpha}(t_1) \right) \left( \mathbb{I} + \delta t_N \mathcal{L}^{\alpha}(t_1 + \delta t_2) \right) ... \left( \mathbb{I} + \delta t_N \mathcal{L}^{\alpha}(t_N) \right)$$

where $\delta t_N = \frac{t_2 - t_1}{N}$. The second is Dyson's formula

$$(10) \qquad \mathcal{P} \exp\left( \int_{t_1}^{t_2} \mathcal{L}^{\alpha}(s) ds \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{P} \left( \int_{t_1}^{t_2} \mathcal{L}^{\alpha}(s) ds \right)^n.$$

where

$$(11) \quad \mathcal{P}\left(\int_{t_1}^{t_2} \mathcal{L}^\alpha(s)ds\right)^n = n! \int_{t_1}^{t_2} ds_1 \int_{s_1}^{t_2} ds_2 .... \int_{s_{n-1}}^{t_2} ds_n \mathcal{L}^\alpha(s_1)\mathcal{L}^\alpha(s_2)...\mathcal{L}^\alpha(s_n).$$

Dyson's formula is useful mostly from a theoretical viewpoint. In the next section we provide examples of its use by deriving a moment method to price a number of Abelian path-dependent options, ranging from risky annuities to volatility swaps and cliquets.

A remark that is quite central in both our calibration method for CDS curves and in our model for industry sector systematic risk is that the operator $\mathcal{L}^\alpha(t)$ can be split as follows as the sum of two mutually commuting operators

$$(12) \quad \mathcal{L}^\alpha(t) = \mathcal{L}_0(t) + \mathcal{J}_{\mathtt{jtd}}^\alpha(t).$$

Here, $\mathcal{L}_0(t)$ consists of the sum of all the terms in (2) except the last, the time dependency being inherited exclusively by the exogenous interest rate. Notice that in our model we assume that these terms are not name specific. Moreover, $\mathcal{J}_{\mathtt{jtd}}^\alpha(t)$ is the name specific one-parameter family of operators of matrix elements

$$(13) \quad \mathcal{J}_{\mathtt{jtd}}^\alpha(x,a,b;x',a',b';t) = \mathbf{J}_{\mathtt{jtd}}^\alpha(t)(\delta_{x'0} - \delta_{xx'})\delta_{aa'}\delta_{bb'}.$$

Since the commutator $[\mathcal{L}_0(t), \mathcal{J}_{\mathtt{jtd}}(t)] = 0$ vanishes, single name propagators factorize into the product of propagators corresponding to these two terms. More precisely

$$U^\alpha(t_1, t_2) = \mathcal{P}\exp\left(\int_{t_1}^{t_2}(\mathcal{L}_0(s) + \mathcal{J}_{\mathtt{jtd}}^\alpha(s))ds\right)$$

$$(14) \qquad = \mathcal{P}\exp\left(\int_{t_1}^{t_2}\mathcal{L}_0(s)ds\right)\mathcal{P}\exp\left(\int_{t_1}^{t_2}\mathcal{J}_{\mathtt{jtd}}^\alpha(s)ds\right) = U_0(t_1,t_2)U_{\mathtt{jtd}}^\alpha(t_1,t_2)$$

The Feynman path integral representation is interesting in this context as this formula can be implemented numerically very efficiently by the method of fast exponentiation. The method works as follows. Assume that the name independent dynamic generators $\mathcal{L}_0(t)$ are piecewise constant as a function of time. Suppose $\mathcal{L}_0(t) = \mathcal{L}_{0i}$ in the time interval $[t_i, t_i + (\Delta t)_i]$. Assume $\delta t$ be chosen so small that the following two conditions hold:

$$(\text{FE1}) \quad \min_{y\in\Lambda}(1 + \delta t \mathcal{L}_{0i}(y,y)) \geq 1/2$$

$$(\text{FE2}) \quad \log_2 \frac{(\Delta t)_i}{\delta t} = n \in \mathbb{N}.$$

This condition leads to intervals $\delta t$ of the order of one hour of calendar time and this is indeed the choice we make. To compute $e^{(\Delta t)_i \mathcal{L}_{0i}}(x,y)$, we first define the elementary propagator

$$(15) \quad u_{\delta t}(x,y) = \delta_{xy} + \delta t \mathcal{L}_i(y,y)$$

and then evaluate in sequence $u_{2\delta t} = u_{\delta t} \cdot u_{\delta t}$, $u_{4\delta t} = u_{2\delta t} \cdot u_{2\delta t}$, ... $u_{2^n\delta t} = u_{2^{n-1}\delta t} \cdot u_{2^{n-1}\delta t}$. Matrix multiplication is accomplished numerically by invoking the routine $\mathtt{dgemm}$ in Level-3 BLAS.

This factorization property is crucial as it allows us to correlate among names the jump to default term in a very simple and straightforward way. The propagators $U_0(t)$ describing the remaining part of the process to default are instead more

difficult to correlate and for those we use the method of dynamic conditioning below.

In Fig. 2 we show our fit of price quotes for American equity options of maturities up to two years. We are within the bid-ask spreads with the exception of a couple of out-of-the-money call options of maturity 2 years. In Fig. 3 we graph the implied volatility surface for European options extrapolated to 12 years. The persistent skew is a result of the outlook dynamics. Notice that this fit and extrapolation are obtained with absolutely no explicit time dependencies with the exception of an exogenous, deterministic interest rate.

## 3. Abelian Path Dependencies

In (Albanese 2006), we introduced the notion of Abelian path dependent options. This theory identifies a class of path-dependent payoffs characterized by a commutativity property for a certain operator algebra associated to the path dependent option. More precisely, let $I_t$ be the process for the path dependent structured leg and let us approximate it as follows: $I_t = (\Delta I)m_t$ where $m_t$ is an integer value process. If $y_t = (x_t, a_t, b_t)$, the lifted Markov generator for the joint process $(y_t, m_t)$ has the following form:

$$(16) \qquad \tilde{\mathcal{L}}^\alpha(y, m; y', m'; t) = \mathcal{L}^\alpha(y, m; y', m'; t) + \mathcal{A}^\alpha(m; m'|y)\delta_{yy'}.$$

If the operators $\mathcal{A}^\alpha(y)$ of matrix elements $\mathcal{A}^\alpha(y)_{mm'} = \mathcal{A}^\alpha(m; m'|y)$ are mutually commuting for all $y$, i.e. they generate an Abelian operator algebra, then the path-dependent is called Abelian. The representation theory for Abelian $C^*$ algebras applies to this case, see (Bratteli and Robinson 2002) making the analysis straightforward. In fact, an Abelian $C^*$ algebra can be represented as an algebra of continuous functions on the set of maximal ideals endowed with a compact, Hausdorff topology. In our case this topological space is just finite. From a numerical viewpoint, what matters is that if the Abelian property is satisfied, then generically there exists a non-singular linear transformation $S(m; i)$ which diagonalizes all the operators $\mathcal{A}^\alpha(m; m'|y)$ simultaneously. Hence, the transformed lifted generator $S^{-1}\tilde{\mathcal{L}}^\alpha S$ is a complex, block-diagonal matrix which can be fast-exponentiated by means of the BLAS routine `zgemm` which implements a complex matrix-matrix multiplication. This exponential gives the joint distribution of the pairs $(y_t, m_t)$ for all time horizons $t$. Risky annuities, volatility swaps, cliquets are all examples of Abelian path dependent options along with lookback options, range accruals, TARNs, faders, etc.. (Notice here one of the differences between probability theory expressed via stochastic calculus as opposed to our formalism based on operator methods: in the former one focuses on measure changes which are given by one parameter families of positive linear transformations, while in our case we can make full use of general complex valued linear tansformations as the applicability of operator methods does not hinge on the existence of a probabilistic interpretation for the dynamic generators).

In addition to applying block-diagonalization techniques, for most Abelian payoffs one can also use a moment method based on Dyson's expansion. For instance, to price a risky annuity, a building block in the valuation of CDS contracts, one can use the following formula:

$$(17) \qquad \frac{d}{d\epsilon}\bigg|_{\epsilon=0} e^{(\Delta t)\mathcal{L}^\alpha_\epsilon}(y_1, y_2) = E_0\left[\left(\int_0^{\Delta t} 1(y_s^\alpha \neq 0)ds\right)\delta(y_{\Delta t}^\alpha = y_2)\big|y_0^\alpha = y_1\right]$$

where $y_t^\alpha$ is the lattice process followed by reference name $\alpha$ and

$$(18) \qquad \mathcal{L}_\epsilon^\alpha(y; y') = \mathcal{L}^\alpha(y; y') + \epsilon 1(y \neq 0)\delta_{yy'}.$$

Similarly one can proceed to evaluate risky annuities associated to EDS contracts. To value volatility swaps one can use the more general formula:

$$(19) \qquad \frac{d^n}{d\epsilon^n}\bigg|_{\epsilon=0} e^{(\Delta t)\mathcal{L}_\epsilon^\alpha}(y_1, y_2) = E_0\left[\left(\int_0^{\Delta t} v^\alpha(y_s^\alpha)ds\right)^n \delta(y_{\Delta t}^\alpha = y_2)\big|y_0^\alpha = y_1\right]$$

where

$$(20) \qquad \mathcal{L}_\epsilon^\alpha(y; y') = \mathcal{L}^\alpha(y; y') + \epsilon v^\alpha(y)\delta_{yy'}$$

where $v(y)$ is the state dependent variance given by

$$(21) \qquad v^\alpha(y) = \sum_{y'} \mathcal{L}^\alpha(y, y')\left(\log \frac{S(y')}{S(y)}\right)^2.$$

Notice that if only two moments are desired to reconstruct the joint distribution of the underlying and realized variance, then equation (19) needs to be evaluated only for $n = 1, 2$.

The credit counterparts to long-dated equity structured products are given by emerging volatility sensitive single-name credit derivatives such as CDSs on leveraged loans subject to pre-payment risk, CDS options and soft risky annuities paying a continuous cash-flow stream up to either maturity or up until a given constant maturity CDS spread hits a pre-assigned barrier, whichever comes first. Exposures of this nature are contained in bespoke CLOs and in managed synthetic CDOs and thus have to be captured and modeled in a controllable fashion and with high precision by a comprehensive model for bespoke baskets.

In Fig. 4 we show the term structure of CDS spreads for IBM assuming a constant 40% recovery rate. A perfect fit falls in place when superimposing a slightly time-varying term structure of recovery rates. In Figs. 10, 11 and 12 we show the aggregate fit of the 3 year, 5 year and 10 year CDS digital spreads for the DJ.CDX.IG components where market quotes are extrapolated assuming a flat 40% recovery rate. These graphs are obtained by using the IBM model as a proxy for other names, recycling the same choice of model parameters across the pool of names in the DJ.CDX.IG index while allowing ourselves only two modifications: we reset the initial condition and we recalibrate the jump-to-default term in such a way to obtain a perfect fit to digital CDS spread curve over the first two years period. For the purpose of this fit, digital CDS spreads are extrapolated from the vanilla ones by assuming 40% recovery rates. Since the jump-to-default operator commutes with the rest of the generator as noticed above, this allows us to recycle all the numerical evaluations across all names by using a factorization formula for the single name propagators of the form in equation (14).

In Fig. 5 we plot the term structure of EDS spreads. This curve may not be calibrated against very liquidly traded assets but nevertheless we find that it is an important object as the EDS/CDS ratio is tightly related to the degree of vega convexity.

As an example of an instrument sensitive to vega convexity, in Fig. 6 we plot the volatility versus variance swap curves as a function of the outlook and volatility regimes. The spread between these two rates is a function of the forward volatility of realized volatility. As a further example, in Fig. 7 we show the term structures of

cliquet price per-annum for both plain and capped and floored cliquets. Comparing the two one sees that although the cap and the floor may balance out for short maturities, as the maturity lengthens beyond 6-7 years the cap tends to appreciate more than the floor. This is precisely due to increased equity volatility in scenarios where the stock price falls.

Moving to the credit domain, in Fig. 8 we show the model implied term structures of at-the-money CDS option volatility with tenors 5, 7 and 10 years. Due to the presence of two extra factors, stochastic volatility and stochastic outlooks, it is possible to calibrate the model so that these at-the-money volatilities are at the historical realized level which is quite high, between 30% and 50%, while the equity return volatility is as low as 15%. The reason why these two different volatility levels are compatible within our model is that CDS spread volatilities are mostly affected by outlook dynamics while short term equity volatilities are mostly affected by the stochastic volatility dynamics.

Having gained good quantitative control on CDS spread volatilities, one is well positioned to price volatility sensitive single name credit derivatives with confidence. In Fig. 9 we see the model implied spreads for a CDS on a portfolio of three leveraged loans hypothetically entered into by the reference name IBM we chose for this series of examples. The loans are amortizing, floating rate and, most importantly, are subject to pre-payment risk. We model pre-payment by assuming that when the constant maturity 5 year CDS spread hits the 80% of the spot level, the pre-payment option is exercised by the borrower. Notice that the spreads on a CDS on leveraged loans are substantially higher than those on a vanilla CDS. A quantitative assessment of the stopping time for pre-payment is essential and for this one needs a model which reflects well the CDS spread volatility and its dynamics.

## 4. Dynamic Conditioning

The correlation model for CDOs is based on the notion that intra-sector correlations tend to be higher then inter-sector correlations. We thus build a sector based, multi-factor model. In the example in this paper, we consider the two standard portfolios DJ.CDX.IG and DJ.CDX.HY and group all reference names contained in either one of them into 9 investment grade sectors, namely: utilities, sovereigns, media, industrial, financial, energy, consumer, material and technology. In addition, we find it necessary to add a special high yield sector containing names whose 5 year spread is above 500 bp and a special cross-over sector with names whose 5 year spread is between 250 and 500 bp. These choices are arguably questionable and could greatly be refined. However we won't go into much further depth to analyze alternatives as our objective here is to demonstrate that, even with fairly simplistic economic assumptions, parameters can be selected to achieve a high precision fit with observed market tranche spreads. This calibration can be achieved across investment grade and high yield portfolios even with rather simple minded choices. If a different and more refined view about sector classification was used, we are confident the model could still be calibrated, although differences would emerge when pricing bespoke tranches and assessing sensitivities.

We distinguish between correlations related to small deviations and correlations related to large deviations or systematic sector risk. This reflects and is linked to a split of the form in equation (12) of the generator into a sum of two mutually

commuting operators, a jump to default generator plus a richer term describing a gradual process to default. Leveraging on the factorization formula between path-ordered exponentials, we can effectively correlate the two terms separately, as we explain below.

Let $\alpha$ be a reference name in sector sec and jurisdiction jur and let us split the generator for name $\alpha$ as follows:

$$(22) \qquad \mathcal{L}^\alpha = \left( \mathcal{L}_0 + \mathcal{J}_{\mathtt{jtd}}^{\mathtt{sec}}(t) \right) \ + \ \mathcal{J}_{\mathtt{jtd}}^{\mathtt{jur}}(t).$$

The propagator factorizes as follows:
(23)
$$U^\alpha(t_1, t_2) = \mathcal{P} \exp \left( \int_{t_1}^{t_2} \mathcal{L}_0(s) \right) \mathcal{P} \exp \left( \int_{t_1}^{t_2} \mathcal{J}_{\mathtt{jtd}}^{\mathtt{sec}}(s) ds \right) \mathcal{P} \exp \left( \int_{t_1}^{t_2} \mathcal{J}_{\mathtt{jtd}}^{\mathtt{jur}}(s) ds \right)$$

The second factor is then assumed to be shared across all names in each given sector while the third is shared across all names in a given jurisdiction. Letting the jump-to-default terms depend on time explicitly is a departure from our self-imposed rule of keeping all parameters time homogeneous but we find that this is needed in order to calibrate the systematic correlation terms to the term structure of the senior tranche spreads. A time-homogenous choice would lead to senior tranche spreads which are either to large for short maturities or too small for long maturities.

Correlations among small deviations as they are captured by the first factor in equation (23) is modeled using dynamic conditioning. This term is very relevant as this has a great impact on tranche values on the lower segments of the capital structure, such as equity, junior mezzanine and thin tranches. A version of dynamic conditioning was introduced in our previous paper on CDO modeling (Albanese *et al.* 2005-2006). However here we use a powerful multi-factor refinement of this methodology, also reviewed in (Albanese 2006).

The new version of the method is illustrated in Fig. 13. To model correlations we proceed in stages. At the first stage, we introduce a binomial processes for each name and condition the continuous time lattice process for each name to this one. To prevent a possible misunderstanding, let us emphasize that the binomial trees used for conditioning should not be confused with Cox-Ross trees. We are not approximating a very rich three-factor model with a binomial tree. Each vertex or bond in one of our conditioning trees does not correspond to a particular stock price. Instead, each vertex of the tree corresponds to a propagator from the root of the tree to the given vertex, summarizing the transition probabilities across all paths joining the root to the vertex itself. Each bond is instead associated to a propagator conditional to a transition between the two vertices identifying the bond itself. The conditioning is performed using rules which ensure that, upon resumming over all the tree branches, one reobtains the unconditional propagators to high numerical precision for each reference name. To greatly speed up the calculation of single name sensitivities, we go one step further and evaluate the Fourier transforms of the single name forward loss distribution on each bond. Fourier transform representations are useful in this context as they allow one to deconvolve by simply taking ratios between Fourier transforms and thus recycle conditional price information across calculations of sensitivities corresponding to different reference names.

To describe the technique of dynamic conditioning in more detail, consider the name-indepedent portion $U(y_1, t_1; y_2, t_2)$ of a single-name Markov propagator. Consider also a name specific discrete binomial process $h_t^\alpha \in \mathbb{Z}$ on the conditioning tree associated to name $\alpha$, which is constant over the time intervals $[T_i, T_{i+1})$ where $T_i = T + i\Delta T$. Here $i = 0, 1, 2, ...N$ and $\Delta T = (T' - T)/N$.

The elementary propagator for the process $h_t^\alpha$ across neighboring time points $T_i$ is

$$(24) \qquad V(h_i^\alpha, T_i; h_{i+1}^\alpha, T_{i+1}) = q_+ \delta_{h_i^\alpha+1, h_{i+1}^\alpha} + q_- \delta_{h_i^\alpha-1, h_{i+1}^\alpha}$$

where $q_+, q_- > 0$ and $q_+ + q_- = 1$. On a general time interval $[T_i, T_j]$, we have that

$$(25) \qquad V(h_i^\alpha, T_i; h_j^\alpha, T_j) = \sum_{h_t^\alpha : h_{T_i}^\alpha = h_i^\alpha, h_{T_j}^\alpha = h_j^\alpha} \prod_{k=i}^{j-1} V(h_{T_k}^\alpha, T_k; h_{T_{k+1}}^\alpha, T_{k+1}).$$

In our example, we find most effective to use symmetric trees with node splitting probabilities $q_\pm = 1/2$. We also use time intervals of one quarter of a calendar year.

Next we define a lifted conditional propagator $\bar{U}(y_i, h_i^\alpha, T_i; y_j, h_j^\alpha, T_j)$ in such a way to preserve unconditional transition probabilities from the starting time at $T_i$, i.e. so that

$$(26) \qquad \sum_{h_j^\alpha} \bar{U}(y_i, h_i^\alpha, T_i; y_j, h_j^\alpha, T_j) = U(y_i, T_i; y_j, T_j).$$

To do so, one strategy is to define two conditional propagators from each starting node $(h_i^\alpha, T_i)$, namely $U_+(y_i, h_i^\alpha, T_i; y_{i+1}, h_i^\alpha + 1, T_{i+1})$ and $U_-(y_i, h_i^\alpha, T_i; y_{i+1}, h_i^\alpha - 1, T_{i+1})$ so that

$$U(y_i, T_i; y_{i+1}, T_{i+1}) = q_+ U_+(y_i, h_i^\alpha, T_i; y_{i+1}, h_i^\alpha + 1, T_{i+1}) +$$
$$q_- U_-(y_i, h_i^\alpha, T_i; y_{i+1}, h_i^\alpha - 1, T_{i+1}).$$

The operator $\bar{U}(y_i, h_i^\alpha, T_i; y_j, h_j^\alpha, T_j)$ satisfying (26) is defined as the following sum over paths on the conditional binomial tree:

$$\bar{U}(y_i, h_i^\alpha, T_i; y_j, h_j^\alpha, T_j) = \sum_{h_t^\alpha : h_{T_1}^\alpha = h_i^\alpha, h_{T_j}^\alpha = h_j^\alpha} \sum_{y_1, ..y_{j-1}} \prod_{k=1..j} q_{h_{T_k}^\alpha - h_{T_{k-1}}^\alpha}(h_{T_{k-1}}^\alpha) \times$$
$$U_{h_{T_k}^\alpha - h_{T_{k-1}}^\alpha}(y_{k-1}, h_{T_{k-1}}^\alpha, T_{k-1}; y_k, h_{T_k}^\alpha, T_k).$$

Here, $q_1 = q_+$, $q_{-1} = q_-$, $U_1 = U_+$, $U_{-1} = U_-$.

Since in general the operators $U_+$ and $U_-$ do not commute with each other, each path $h_t^\alpha$ in the summation on the right hand side of this equation represents a different operator. In many situations one can however avoid the numerical complexities of a non-recombining tree by finding modified versions of such operators so that the kernels
$$(27)$$
$$U_{h_\cdot^\alpha}(y_i, y_k) = \sum_{y_i, ..y_{j-1}} \prod_{k=1..j} q_{h_{T_k}^\alpha - h_{T_{k-1}}^\alpha}(h_{T_{k-1}}^\alpha) \bar{U}(y_{k-1}, h_{T_{k-1}}^\alpha, T_{k-1}; y_k, h_{T_k}^\alpha, T_k)$$

are all equal to each other, for all paths originating from the root vertex, i.e. with $h_t^\alpha : h_{T_0}^\alpha = h_0^\alpha, h_{T_j}^\alpha = h_j^\alpha$ and at least one single fixed starting point $y_0 = \bar{y}_0$.

The reasons why we focus on the initial point $y_0 = \bar{y}_0$ only are manifold. Firstly, in applications one needs to condition marginals only to spot values, as the price of options in the hypothetical case asset prices were different is not known. Secondly,

if we insisted on the same property being valid for all initial starting points we would end up with a seriously ill posed problem of difficult solution. Because of these reasons, we settle for the more modest objective of conditional recombination. Namely, on each node $(h_i^\alpha, T_i)$ with $i > 0$ we define

$$(28) \qquad \bar{U}(y_{i-1}, h_{i-1}^\alpha, T_{i-1}; y_i, h_i^\alpha, T_i) = U_{h_i^\alpha - h_{i-1}^\alpha}(y_{i-1}, h_{i-1}^\alpha, T_{i-1}; y_i, h_i^\alpha, T_i)$$

in case $h_i^\alpha = \pm i$. Otherwise, we determine this operator so that

$$\bar{U}(\bar{y}_0, h_0^\alpha, T_0; y_i, h_i^\alpha, T_i) =$$
$$\sum_{y_{i-1} \in \Lambda} \bar{U}(\bar{y}_0, h_0^\alpha, T_0; y_{i-1}, h_{i-1}^\alpha, T_{i-1}) \bar{U}(y_{i-1}, h_{i-1}^\alpha, T_{i-1}; y_i, h_i^\alpha, T_i)$$

for all $y_i \in \Lambda$ and a fixed $\bar{y}_0$. This can be achieved in more than one way. Notice however that, by construction, the arbitrariness of this choice does not reflect in the transition probabilities from the root vertex at the initial condition.

It turns out that if we are interested only in synthetic CDOs on index portfolios as opposed to forward starting portfolios or tranche options, then it is not even necessary to define a full conditional dynamics. All that is needed is to define marginals originating from the root vertex. In this case, it is not even necessary to re-specify the dynamics on each particular bond, all that matters is to keep track consistently of the combinatorial coefficients.

As a next step in the construction, consider sector specific processes $c_t^\sigma$ which are piecewise constant across neighboring time points $T_i$. The dynamics of $c_t^\sigma$ is by construction correlated to that of $h_i^\alpha$. More precisely, the propagator for the joint process is

$$W(h_i^\alpha, c_i^\sigma, T_i; h_{i+1}^\alpha, c_{i+1}^\sigma, T_{i+1}) =$$
$$q_{++}(h_i^\alpha, c_i^\sigma, i)\delta_{h_i^\alpha + 1, h_{i+1}^\alpha}\delta_{c_i^\sigma + 1, c_{i+1}^\sigma} + q_{+-}(h_i^\alpha, c_i^\sigma, i)\delta_{h_i^\alpha + 1, h_{i+1}^\alpha}\delta_{c_i^\sigma - 1, c_{i+1}^\sigma}$$
$$+ q_{-+}(h_i^\alpha, c_i^\sigma, i)\delta_{h_i^\alpha - 1, h_{i+1}^\alpha}\delta_{c_i^\sigma + 1, c_{i+1}^\sigma} + q_{--}(h_i^\alpha, c_i^\sigma, i)\delta_{h_i^\alpha - 1, h_{i+1}^\alpha}\delta_{c_i^\sigma - 1, c_{i+1}^\sigma}$$

where $q_{\pm\pm}(h_i^\alpha, c_i^\sigma, i) \geq 0$ and

$$(29) \qquad q_{++}(h_i^\alpha, c_i^\sigma, i) + q_{+-}(h_i^\alpha, c_i^\sigma, i) + q_{-+}(h_i^\alpha, c_i^\sigma, i) + q_{--}(h_i^\alpha, c_i^\sigma, i) = 1.$$

Further equations to determine the coefficients $q_{\pm\pm}(h_i^\alpha, c_i^\sigma, i)$ can be obtained by postulating an intra-sector correlation factor $\rho_{\alpha\sigma} \in [0, 1]$. Given this correlation parameter, we obtain a new equation by setting

$$(30) \qquad q_{++}(h_i^\alpha, c_i^\sigma, i) - q_{+-}(h_i^\alpha, c_i^\sigma, i) - q_{-+}(h_i^\alpha, c_i^\sigma, i) + q_{--}(h_i^\alpha, c_i^\sigma, i) = \rho_{\alpha\sigma}.$$

Two more equations to complete the linear system follow by the requirements that marginals be recovered by unconditioning, i.e.

$$(31) \qquad\qquad\qquad q_{++}(h_i^\alpha, c_i^\sigma, i) + q_{+-}(h_i^\alpha, c_i^\sigma, i) = q_+$$
$$(32) \qquad\qquad\qquad q_{-+}(h_i^\alpha, c_i^\sigma, i) + q_{--}(h_i^\alpha, c_i^\sigma, i) = q_-.$$

In our example, to keep things simple we assume that $\rho_{\sigma\alpha} = \rho_\sigma$ depends only on the sector and not on the name in that sector. The intra-sector correlations in our example are chosen according to the following table:

| Sector | uti | sov | med | ind | fin | ene | con | mat | tec | hy | xor |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Correlation | 40% | 40% | 40% | 40% | 40% | 40% | 40% | 40% | 40% | 60% | 60% |

TABLE 1. Sector correlations, where $\mathtt{uti}$ = utilities, $\mathtt{sov}$ = sovereigns, $\mathtt{med}$ = media, $\mathtt{ind}$ = industrial, $\mathtt{fin}$ = financials, $\mathtt{ene}$ = energy, $\mathtt{con}$ = consumer, $\mathtt{mat}$ = materials, $\mathtt{tec}$ = technology, $\mathtt{hy}$ = high yield and $\mathtt{xor}$ = cross-over.

Due to the conditional recombination property, if $y_0 = \bar{y}_0$, then the conditional propagators resum with a simple formula

$$\sum_{h_t^\alpha : h_T^\alpha = 0, h_{T_j}^\alpha = h_j^\alpha} \sum_{y_1, .. y_{j-1}} \prod_{k=1..j} W(h_i^\alpha, c_i^\sigma, T_i; h_{i+1}^\alpha, c_{i+1}^\sigma, T_{i+1}) \times$$

$$q_{h_{T_k}^\alpha - h_{T_{k-1}}^\alpha}(h_{T_{k-1}}^\alpha) \bar{U}_{h_{T_k}^\alpha - h_{T_{k-1}}^\alpha}(y_{k-1}, h_{T_{k-1}}^\alpha, T_{k-1}; y_k, h_{T_k}^\alpha, T_k)$$

$$= \sum_{h_j^\alpha} W(0, 0, T_0; h_j^\alpha, c_j^\sigma, T_j) \bar{U}(y_0, 0, T_0; y_j, h_j^\alpha, T_j)$$

$$\equiv \tilde{U}(y_0, 0, T_0; y_j, c_j^\sigma, T_j).$$

Since dynamic conditioning is applied only to reference name to the reference name independent factor of single name propagators, it is straightforward to apply the construction to all names in any given sector $\sigma$. Consider the lattice processes $y_t^{(\alpha)}$, for each $\alpha \in \sigma$. Then, conditioned to starting all processes at fixed lattice points $y_0^{(\alpha)} = \bar{y}_0^{(\alpha)}$ and conditioned to $c_{T_j}^\sigma = c_j^\sigma$, the multi-factor propagator factorizes into the product of conditional single factor propagators

$$(33) \qquad \prod_{\alpha \in \sigma} U(y_0^{(\alpha)}, 0, T_0; y_j^{(\alpha)}, c_j^\sigma, T_j)$$

This is the key formula which we use in the following to correlate processes on a sector basis while ensuring that numerical complexity increases only linearly with the number of names in each sector $N_\sigma$.

The construction above can then be iterated to correlate sectors at a jurisdiction level and proceed to the second level of the tree in Fig. 13. At this level, one needs to introduce inter-sector correlation coefficients $\rho_{\sigma j}$, where $j$ represents the jurisdiction. In our example, again for the sake of simplicity, we assume that the $\rho_{\sigma j}$ are sector independent and are all equal to 60%.

One further extension of the above construction is to add additional risk factors such as recovery rates and total number of share outstanding. These factors can be added on a name by name basis simply by defining a process for these risk factors on a separate tree and a corresponding correlation factor. We find that these factors are not essential to achieve a high quality calibration but they do affect prices. Hence, if one can estimate their volatility and correlation independently, it is useful to know that they can be incorporated at marginal numerical cost.

In Fig. 16 we show the 10 year term structure of upfront fee for the 0-3 equity tranche of the DJ.CDX.IG against market quotes. The model fits market quotes

within the bid ask spread. In Fig. 17 we show how the 03 tranche splits into 0-1, 1-2 and 2-3 equity tranchelets assuming no upfront fees.

In Fig. 18 and 20 we show the 10 year term structure of spreads for the 3-7, 7-10, 10-15, 15-30 mezzanine and senior tranches of the DJ.CDX.IG against market quotes. In Fig. 19 we show how the 10 year term structure of the 3-7 junior mezzanine tranche splits into term structures of spreads for the 5-7 and 3-7 mezzanine tranchelets, based on our model.

Next we consider the High Yield index. In Fig. 21 we plot the 10 year term structure of upfront fee for the 0-10 and 10-15 equity tranches of the DJ.CDX.HY against the 5 year market quote. Both fits are excellent. In Fig. 22 we show the 10 year term structure of for the 15-25 and 25-35 mezzanine and senior tranches of the DJ.CDX.HY against the 5 year market quote. A small discrepancy is observed only in the latter.

## 5. NUMERICAL METHODS, COMPUTER ENGINEERING AND CONCLUSIONS

In our previous articles (Albanese *et al.* 2003), (Albanese and Chen 2005*a*), (Albanese and Chen 2005*b*), (Albanese and Chen 2004), we modeled credit derivatives by means of a new class of continuous time lattice models with numerical techniques based on spectral analysis and the ability to evaluate special functions numerically. As our research made progress, in a strive to gain flexibility we moved toward a non-parametric framework. We experimented with spectral methods based on numerical diagonalizations but found that Markov generators with stochastic drift are subject to pseudo-spectrum pathologies which render them nearly impossible to diagonalize within the confines of double precision arithmetics, see (Trefethen and Embree 2006). We succeeded to make progress past this stumbling block only by rediscovering the method of fast exponentiation we discussed above. This method had been known for decades but never quite made it into the numerical PDE literature. For historical reasons, PDE methods have focused on sparse BLAS level 2 techniques, mostly due to the preconception that large matrices cannot be stored in memory and manipulated as full matrices. Although this was true in the 60s and 70s when the field of numerical PDEs was created, it is certainly not true any longer nowadays.

In today's rapidly evolving technology landscape, enormous progress is being made. Because of applications to computer graphics, a central focus of current computer engineering is nowadays to improve on our ability of multiplying matrices, i.e. of optimizing `dgemm` and `zgemm` by building and mass-producing special purpose multi-core processors. Efficient, processor specific versions of `dgemm` devised for general purpose single core CPUs have been available for two decades at least (Goto and van de Geijn to appear) and have been improving. Emerging hardware such as the Clearspeed accelerator boards and the IBM CELL processor open a new era of rapid progress in this direction. As cluster computing hits a technology wall because of limited scalability and high power consumption rates, these special hardware architectures with remarkably low power consumption rates give staggering performance for matrix multiplication. It would seem that in Finance there is still plenty of room to ride Moore's law by refocusing on model building techniques based on operator methods.

On the wave of the new computer technology, it is our opinion that operator methods will come to play a central role in Mathematical Finance. We find that

it can be highly rewarding to recast the mathematical framework of Finance on the premise that we are able to evaluate matrix-matrix products very efficiently. What we achieve is a unified modeling framework for single name equity and credit derivatives based on a 3 factor model with the possibility to incorporate more factors and to correlate hundreds of names without resorting to any Monte Carlo simulation and with no numerical noise. In our understanding, the model here presented meets all the requirements to be a high quality framework to price bespoke CDOs, equity structured products and volatility sensitive credit derivatives, along with equity baskets and a broad array of hybrids.

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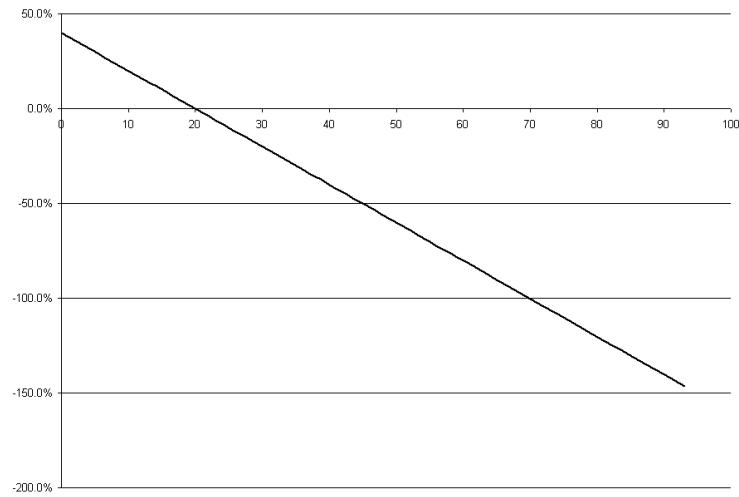
FIGURE 1. Plot of the function $\beta(S)$ defining the local volatility dependency.
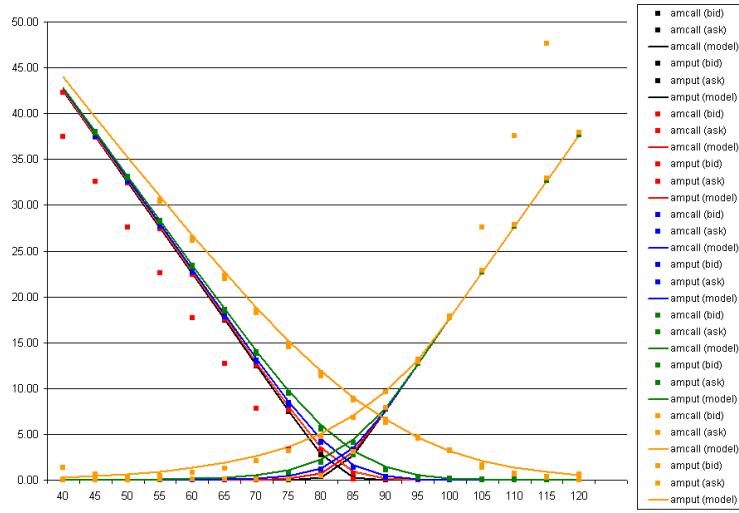


FIGURE 2. Fit of American equity option quotes for maturities up to two years.
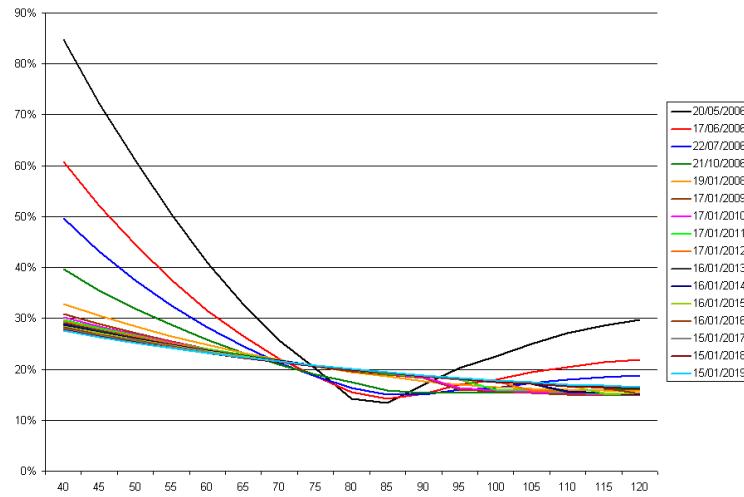
FIGURE 3. Implied volatility surface for European options extrapolated to 12 years.
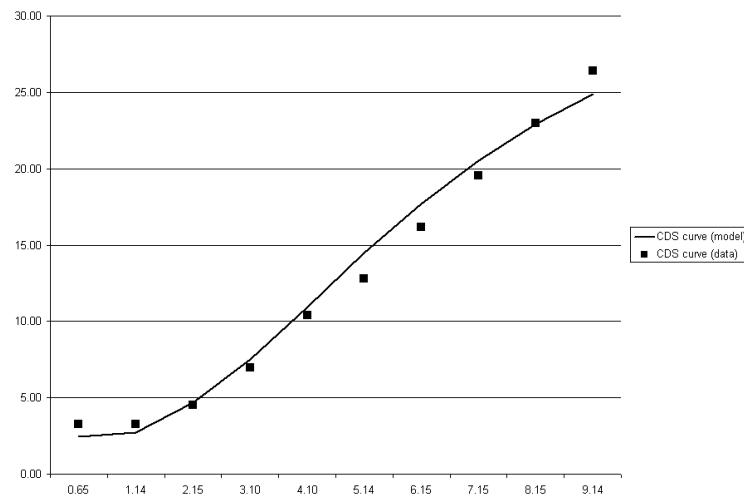


FIGURE 4. Term structure of CDS spreads for IBM assuming a constant 40% recovery rate. A perfect fit falls in place when superimposing a slightly time-varying term structure of recovery rates.
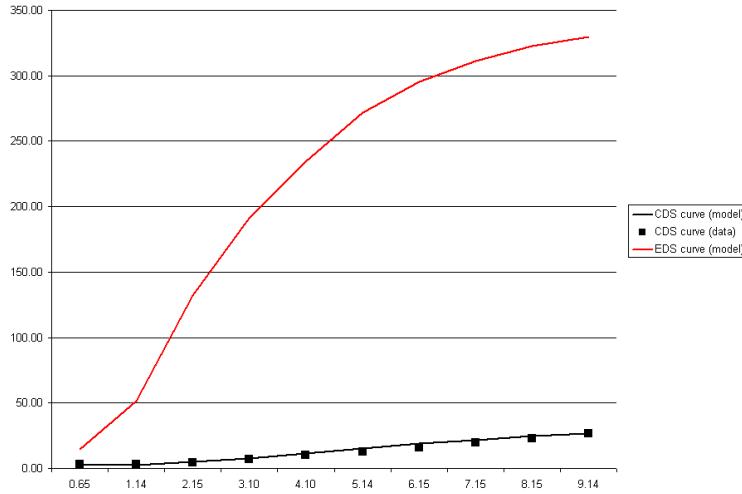
Figure 5. Term structure of EDS spreads. The calibration of this curve controls the degree of vega convexity.
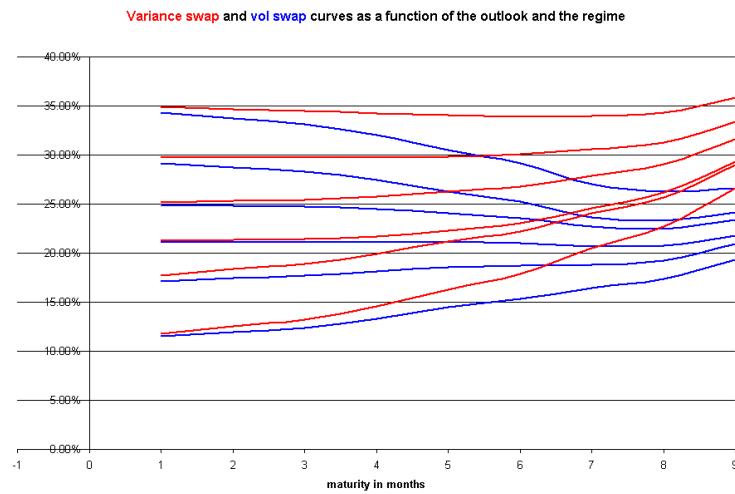


Figure 6. Volatility versus variance swap curves as a function of the outlook and volatility regimes.
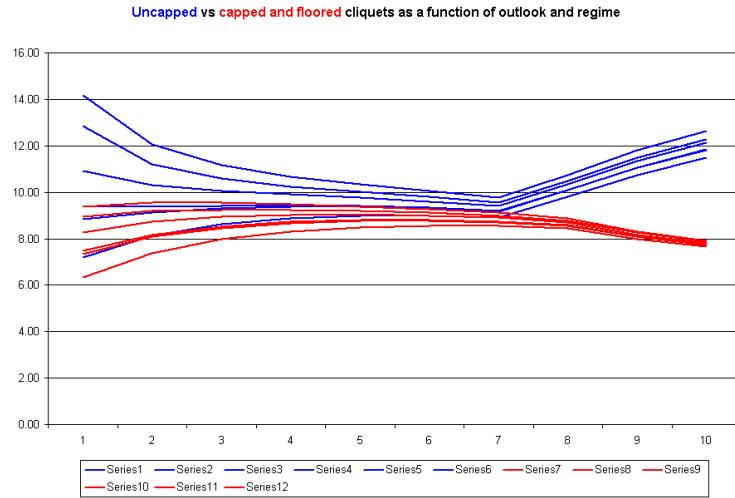
FIGURE 7. Term structures of cliquet price per-annum for both plain and capped and floored cliquets.
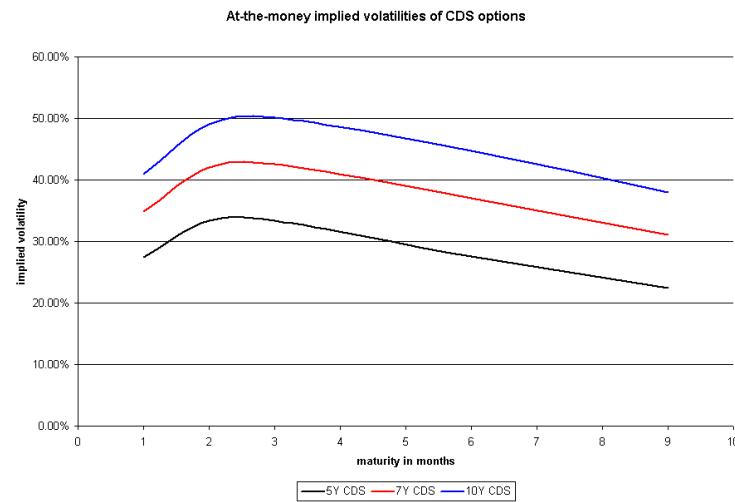


FIGURE 8. Model term structure of at-the-money CDS option volatility with tenors 5, 7 and 10 years.
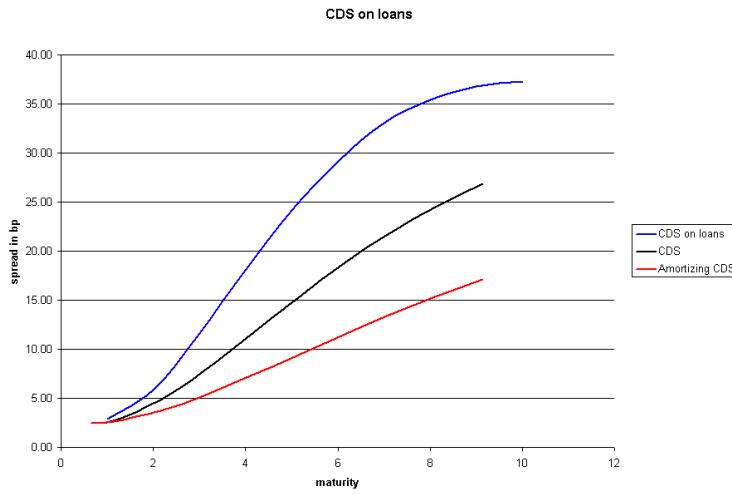
FIGURE 9. 10 year term structure of spreads for plain CDSs, amortizing CDSs and a CDS on a portfolio of 3 leveraged amortizing loans with pre-payment risk.
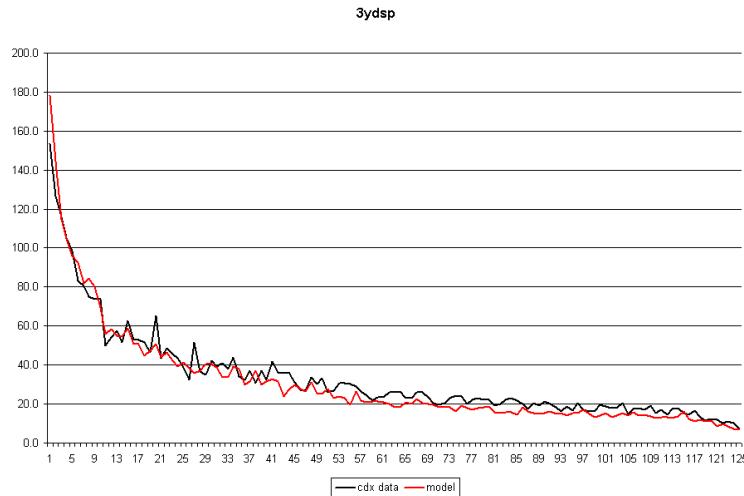


FIGURE 10. Aggregate fit of the 3 year CDS digital spreads for the DJ.CDX.IG components where market quotes are extrapolated assuming a flat 40% recovery rate.
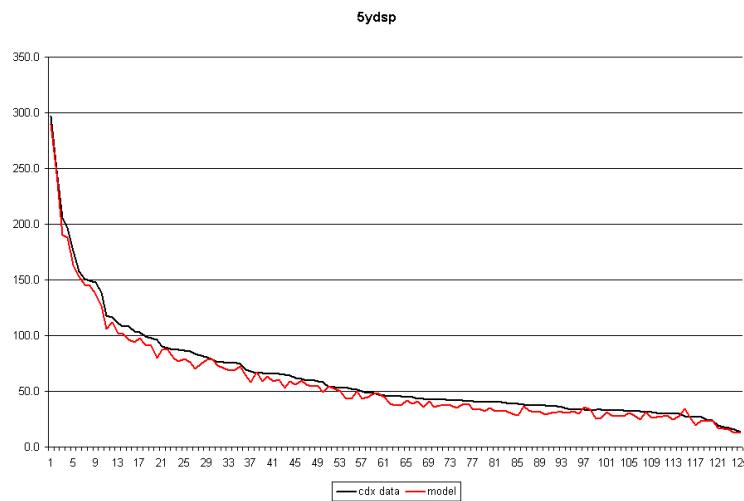
FIGURE 11. Aggregate fit of the 5 year CDS digital spreads for the DJ.CDX.IG components where market quotes are extrapolated assuming a flat 40% recovery rate.
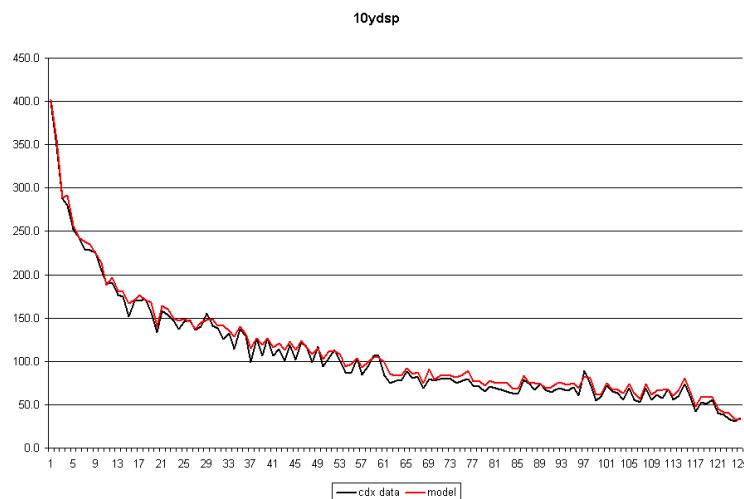


FIGURE 12. Aggregate fit of the 10 year CDS digital spreads for the DJ.CDX.IG components where market quotes are extrapolated assuming a flat 40% recovery rate.
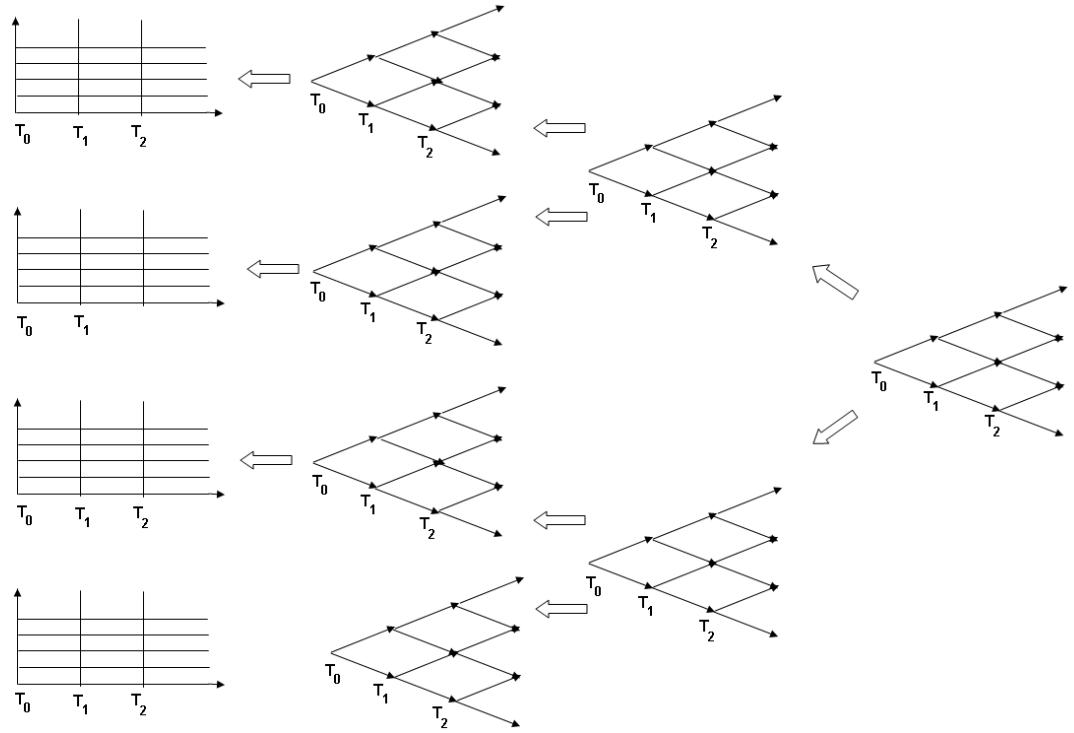
FIGURE 13. Dynamic conditioning scheme for multi-sector modeling.

**Term structure of loss distributions**
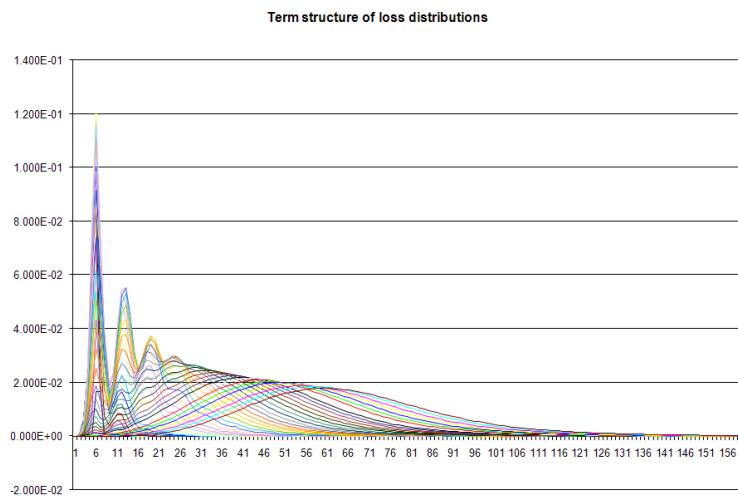


FIGURE 14. 10 year term structure of loss distribution for the DJ.CDX.IG portfolio, plotted quarter by quarter.

**Term structure of loss distributions**
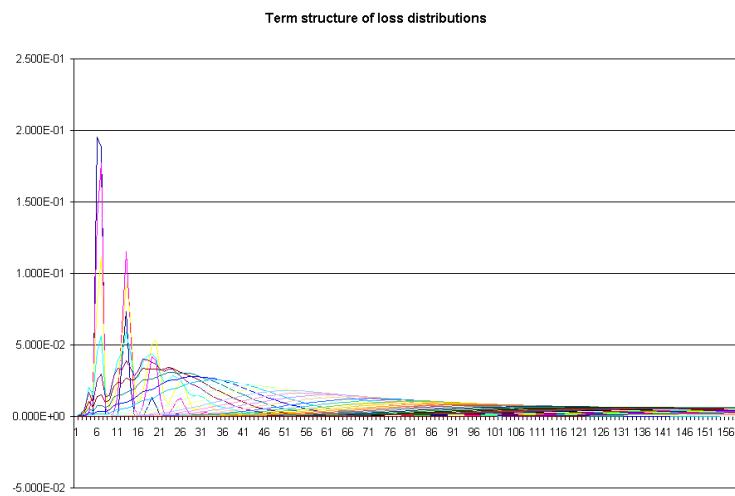


FIGURE 15. 10 year term structure of loss distribution for the DJ.CDX.HY portfolio, plotted quarter by quarter.
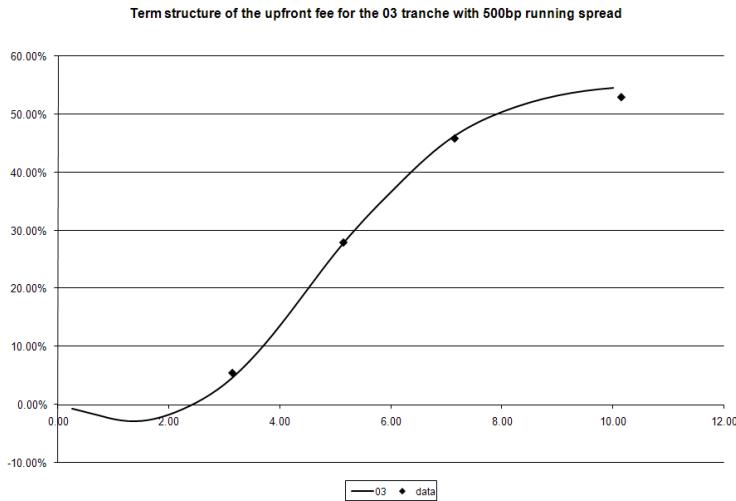
FIGURE 16. 10 year term structure of upfront fee for the 0-3 equity tranche of the DJ.CDX.IG against market quotes. The model fits market quotes within the bid ask spread.
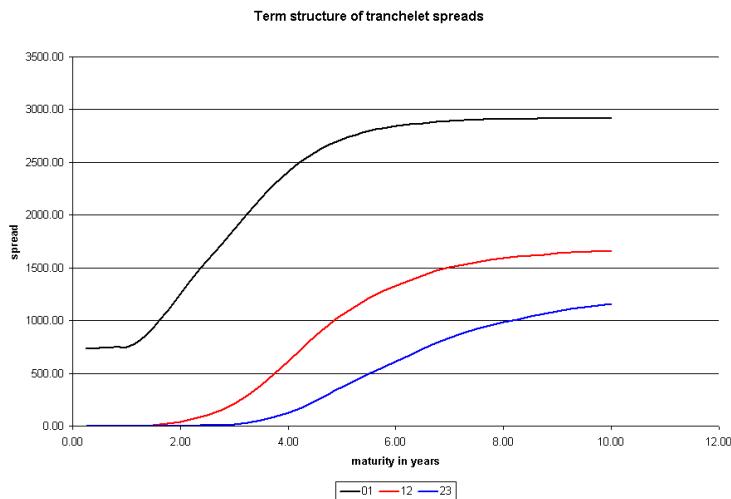


FIGURE 17. 10 year term structure of the 0-1, 1-2 and 2-3 equity tranchelets for the DJ.CDX.IG

Term structure of the 3-7, 7-10, 10-15, 15-30 tranche spreads
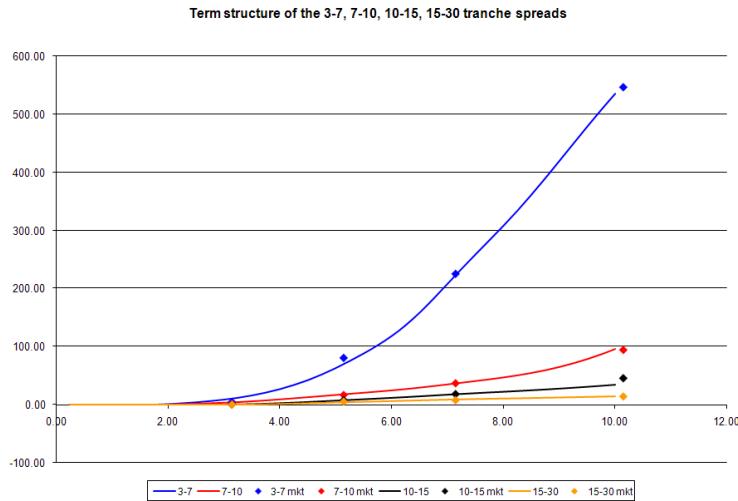


FIGURE 18. 10 year term structure of spreads for the 3-7, 7-10, 10-15, 15-30 mezzanine and senior tranches of the DJ.CDX.IG against market quotes.
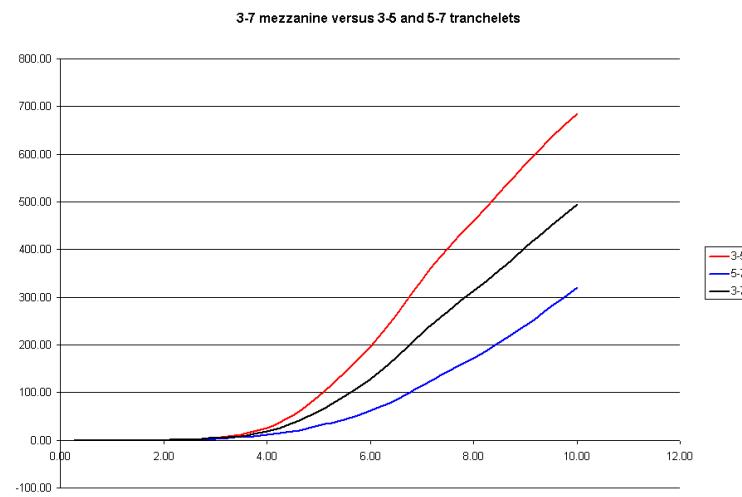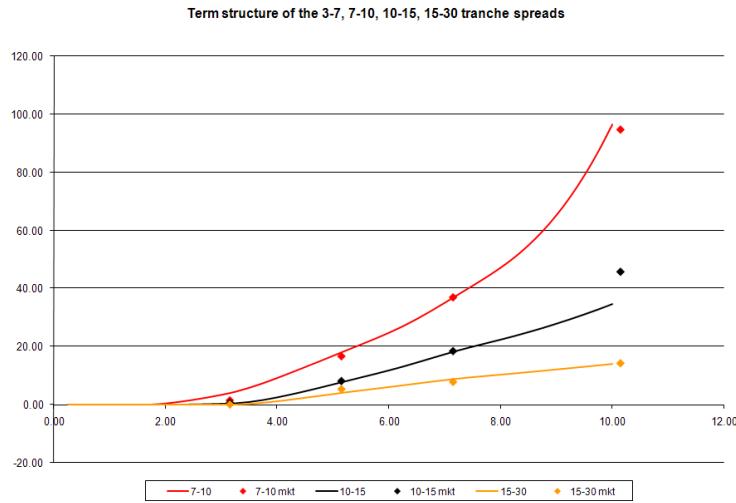
3-7 mezzanine versus 3-5 and 5-7 tranchelets



FIGURE 19. 10 year term structure of the 3-5, 5-7 and 3-7 mezzanine tranchelets for the DJ.CDX.IG

Term structure of the 3-7, 7-10, 10-15, 15-30 tranche spreads



FIGURE 20. 10 year term structure of spreads for the 7-10, 10-15, 15-30 mezzanine and senior tranches of the DJ.CDX.IG against market quotes.
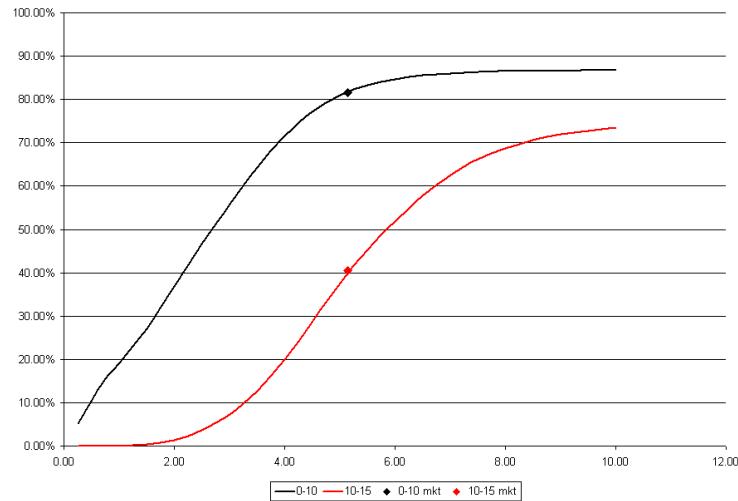


FIGURE 21. 10 year term structure of upfront fee for the 0-10 and 10-15 equity tranches of the DJ.CDX.HY against the 5 year market quote.
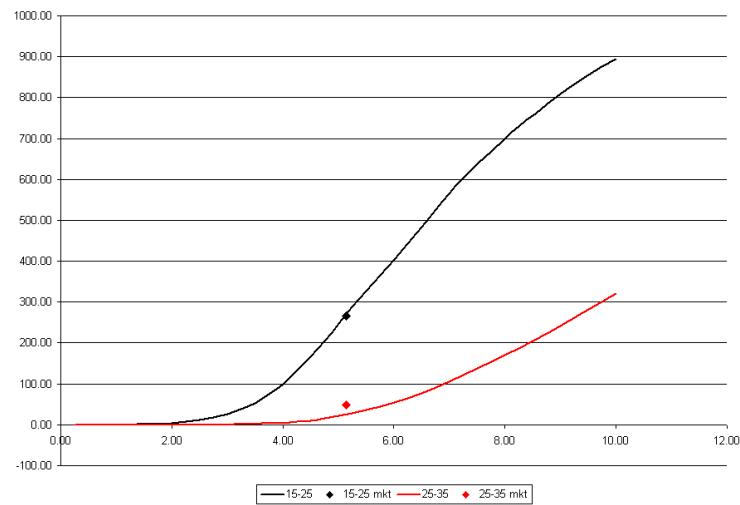
FIGURE 22. 10 year term structure of for the 15-25 and 25-35 mezzanine and senior tranches of the DJ.CDX.HY against the 5 year market quote.