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Abstract

This paper proposes conditions for the existence and uniqueness of solutions to systems of linear differential or algebraic equations with delays or advances, in which some variables may be non-predetermined. These conditions represent the counterpart to the Blanchard and Kahn conditions for the functional equations under consideration. To illustrate the mathematical results, applications to an overlapping generations model and a time-to-build model are developed.

1 Introduction

Two common characteristics of many dynamic models in economics are that
the initial values of some variables are unknown and that certain asymptotic
properties–notably convergence toward a steady state–must be accounted for.
Mathematically, these are boundary value problems. The analytical resolution
method consists of projecting the trajectory onto the stable eigenspace of the dy-
namic system. By comparing the dimensions of the space of non-predetermined
variables with those of the unstable eigenspace, one can deduce the properties of
the existence and determinacy of a solution to the system under consideration
(Blanchard and Khan, 1980; Buiter, 1984). The equilibrium is said to be in-
determinate when there is more than one solution, potentially causing sunspot
fluctuations to appear (Azariadis, 1981; Benhabib and Farmer, 1999). However,
the mathematical theorems that characterize these properties were only estab-
lished for systems of finite dimensions comprising ordinary differential equations
(ODEs) or difference equations. In this paper, we generalize these theorems
to include some systems of delay or advanced differential equations (DDEs or
ADEs).

   As Burger (1956) pointed out, many dynamic systems in economics can be
written as DDEs. Since his work, DDEs have been used in the demographic
economics, vintage capital, time-to-build, and monetary policy literatures (see
Boucekkine et al., 2004 for an excellent survey of the use of DDEs in economics).
However, for want of a theorem, researchers have had to either confine their
work to very specific cases where the stability properties of the dynamics can be
proven[1] or use numerical methods or other mathematical tools (most notably
optimal control with the Hamilton-Jacobi-Bellman equation).[2]

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[1] See, among others, Gray and Turnovsky (1979); Boucekkine et al. (2005); Bambi (2008);
Augeraud-Véron and Bambi (2011); and d'Albis et al. (2012).
[2] See Fabbri and Gozzi (2008); Freni et al. (2008); Boucekkine et al. (2010); Federico et
al. (2010); and Bambi et al. (2012).

DDE systems, which are characterized by a stable manifold of infinite dimensions, have generated an abundance of mathematical literature (see the textbooks by Bellman and Cooke, 1963; and Diekmann et al. 1995). However, the existing theorems are only valid for systems where all the variables are predetermined and defined as continuous functions. Our first objective is to extend these theorems to cases where some variables are non-predetermined (i.e., their past values are given but their value when the system is initiated is unknown) and to cases where some predetermined variables are discontinuous. To do so, we use the mathematical results of d'Albis et al. (2012). In that paper, we defined an operator that acts on a multivalued space and studied its properties. In the present paper, we use the properties of this operator to rewrite a spectral projection formula according to the initial conditions and compute the jump made by non-predetermined variables. We set the projection on the unstable manifold to zero and deduce the magnitude of the jump that nullifies the projection on the unstable manifold. The spectral projection formula then enables us to establish the conditions for the existence and uniqueness of a solution. Most notably, we prove that it is possible to come to a conclusion by comparing the dimensions of the space of the unknown initial conditions to those of the unstable eigenspace. Our results also apply to systems of algebraic equations with delays if their nth derivative is a DDE. In this case, the constraints imposed by such equations must be accounted for by the conditions for existence and uniqueness.

Our second objective is to extend these theorems to differential equations with advances. Systems of ADEs are more similar to ODE systems as they have a stable eigenspace of finite dimensions. We demonstrate that the solution is generated by a finite number of eigenvalues simply by projecting the trajectory onto the stable eigenspace. Conditions for existence and determinacy are ob-

tained by comparing the number of roots with negative real parts to the number of initial conditions. We further study the case of systems that include algebraic equations and define the additional constraints that must be considered.

In Section 2, we present the kind of equations we are interested in and relate them to the literature in economics. In Section 3, our main theorems are presented. The conditions for the existence and uniqueness of solutions to systems of DDEs are in Section 3.1, whereas those of systems of ADEs are in Section 3.2. In Section 4, we solve two economic models in order to illustrate our results and show how to apply our theorems. An overlapping generations model whose dynamics are given by an algebraic equation with delay is studied in Section 4.1 and the decentralized economy of a time-to-build model that can be written with a system of DDEs is studied in Section 4.2. Section 5 concludes.

2   Presentation of the problem

To fix matters, we consider a DDE. Letting $t \in \mathbb{R}_+$ denote time, the dynamic problem can be written as:

$$\begin{cases} x'(t) = \int_{t-1}^{t} d\mu \left(u - t \right) x \left(u \right), \\ x \left(\theta \right) = \bar{x} \left(\theta \right) \text{ given for } \theta \in [-1, 0], \end{cases} \tag{1}$$

where x is a variable with initial value given by a continuous function over the interval $[-1, 0]$, x' denotes its derivative with respect to time, and μ is a measure on $[-1, 0]$. Equation (1) features dynamics that depend on past variables (i.e., delays) on the interval $[t - 1, t]$.[3] In economics, the Johansen (1959) and Solow (1960) vintage capital models are well know examples of dynamic problems described by (1). Classical results for such dynamics are presented in Diekmann et al. (1995).

In economic models, we may have other types of systems. Herein, we will

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[3] Note that the largest delay is normalized to one even though it could be any positive real number. However, we do not consider systems with infinite delays as their characteristic roots may not be isolated.

consider three dynamics that differ from (1). First, we study algebraic equations with delay that reduce to DDEs upon (a finite number of) differentiations with respect to time. This problem can be written as:

$$\begin{cases} x(t) = \int_{t-1}^{t} d\mu\,(u - t)\,x\,(u)\,, \\ x\,(\theta) = \bar{x}\,(\theta) \ \text{given for } \theta \in [-1, 0]\,. \end{cases} \qquad (2)$$

The main difference with the DDE presented above comes from a discontinuity that is allowed at time $t = 0$: $x\,(0^+)$ is given but may be different from $x\,(0^-)$. Indeed, $x\,(0^+)$ is given through the algebraic equation:

$$x(0^+) = \int_{-1}^{0} d\mu\,(u)\,x\,(u)\,. \qquad (3)$$

To summarize, the initial value is provided by $x\,(0^+)$ and a continuous function over the interval $[-1, 0)$ where $x\,(0^-)$ exists. In both problems (1) and (2), the variable is predetermined and is usually backward-looking. Examples of such dynamics are given in Benhabib (2004) and d'Albis et al. (2014) for interest rate policy models and by de la Croix and Licandro (1999) and Boucekkine et al. (2002) for vintage human capital issues. We will study the latter as an illustrative example in Section 4.1.

The second kind of dynamics we consider allows for non-predetermined variables (i.e., forward-looking variables) that do not have a given initial value at time $t = 0$. For a DDE, this dynamic problem can be written as:

$$\begin{cases} x'(t) = \int_{t-1}^{t} d\mu\,(u - t)\,x\,(u)\,, \\ x\,(\theta) = \bar{x}\,(\theta) \ \text{given for } \theta \in [-1, 0). \end{cases} \qquad (4)$$

The initial value is now given by a function that is continuous on $[-1, 0)$ and bounded in 0. Growth theory examples of such dynamics can be found in d'Albis et al. (2012) and Bambi et al. (2012). An example for vintage capital theory can be found in Jovanovic and Yatsenko (2012).

Finally, the third type of dynamics considers equations with advances rather

than delays. For instance, an ADE can be written as:

$$x'(t) = \int_t^{t+1} d\mu(u-t) x(u). \tag{5}$$

ADEs appear as the Euler equation of some vintage capital models studied using optimal control (Boucekkine et al., 2005) or dynamic programming (Boucekkine et al., 2010). Depending on whether or not $x(0)$ is given, the dynamics characterize a backward-looking or a forward-looking variable. Finally, algebraic equations with advances can also be considered in monetary theory models, as in d'Albis et al. (2014).

3 Main theorems

In this section, we study functional differential-algebraic systems with delays and then we study those with advances.

3.1 Functional systems with delays

Let us consider the following linear system:

$$\begin{cases} \mathbf{x}_0'(t) = \int_{t-1}^t d\bar{\mu}_1(u-t) W(u), \\[2mm] \mathbf{x}_1(t) = \int_{t-1}^t d\bar{\mu}_2(u-t) W(u), \\[2mm] \mathbf{y}'(t) = \int_{t-1}^t d\bar{\mu}_3(u-t) W(u), \\[2mm] \mathbf{x}_i(\theta) = \bar{\mathbf{x}}_i(\theta) \text{ given for } \theta \in [-1,0] \text{ and } i = \{0,1\}, \\[2mm] \mathbf{y}(\theta) = \bar{\mathbf{y}}(\theta) \text{ given for } \theta \in [-1,0). \end{cases} \tag{6}$$

Here, $\mathbf{x}_0 \in \mathbb{R}^{n^b}$ is a vector of n^b backward variables whose dynamics are characterized by DDEs and $\mathbf{x}_0'$ denotes its gradient. $\mathbf{x}_1 \in \mathbb{R}^{n_1^b}$ is a vector of n_1^b backward variables characterized by an algebraic equation with delays. $\mathbf{y} \in \mathbb{R}^{n^f}$ is a vector of n^f forward variables characterized by a DDE and $\mathbf{y}'$ denotes its gradient. The $\bar{\mathbf{x}}_i$ are continuous on $[-1,0]$ and $\bar{\mathbf{y}}(\theta)$ is continuous on $[-1,0)$ and bounded in 0. Moreover, $W = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{y})$ is a vectorial function.

We assume there exists a steady-state normalized to zero. We define a solution to system (6) as a function $W(t)$ whose restriction for positive time belongs to $C([0, +\infty))$, satisfies (6), and has $\lim_{t \to +\infty} W(t) = 0$.

Note that the results presented below can be easily extended to study solutions that converge to a Balanced Growth Path (BGP) where all variables grow asymptotically at a given growth rate by considering the detrended variables.

Let n^+ denote the number of eigenvalues with positive real parts of the characteristic function of system (6). Further, let s be the number of independent adjoint eigenvectors of the characteristic function generated by the n^+ eigenvalues. By definition, $s \leq n^b + n_1^b + n^f$.

Assumption H1. There are no eigenvalues with real parts equal to zero and all eigenvalues are simple.

These restrictions are often assumed for ordinary differential equations; the absence of pure imaginary roots excludes a central manifold while simple roots imply a one-dimensional Jordan block. System (6) displays a configuration with a stable manifold of infinite dimension and an unstable manifold of dimension s. Hence, provided that $s \geq 1$, the configuration has a saddle point but multiple solutions may emerge. By multiple solutions, we implicitly mean an infinity of solutions since it features a continuum of initial values for forward variables that initiate a trajectory satisfying system (6) and converging to the steady-state.

Assumption H2. The stable manifold is not transverse to the (x_0, x_1) coordinates.

This second assumption implies that the projection of initial conditions on the unstable manifold encounters the stable manifold. Using it, we conclude that $s \leq \min\{n^+, n^f\}$. Then, we obtain the following result.

Theorem 1. *Let H1 and H2 prevail. There exists a solution to system (6) if $n^+ = s$ and there may be no solution if $n^+ > s$. Upon existence, a solution is unique if and only if $n^f = s$.*

<u>Proof.</u> Given the assumption that algebraic equations reduce to DDEs when differentiated a finite number of times, system (6) can be rewritten as:

$$
\begin{cases}
\mathbf{x}'(t) = \int_{t-1}^{t} d\mu_1 (t - u) V(u), \\[2mm]
\mathbf{y}'(t) = \int_{t-1}^{t} d\mu_2 (t - u) V(u), \\[2mm]
\mathbf{x}(\theta) = \bar{\mathbf{x}}(\theta) \text{ given for } \theta \in [-1,0], \\[2mm]
\mathbf{y}(\theta) = \bar{\mathbf{y}}(\theta) \text{ given for } \theta \in [-1,0),
\end{cases}
\tag{7}
$$

where $\mathbf{x} \in \mathbb{R}^{n-n^f}$ is a vector of backward variables (with $n \equiv n^b + n_1^b + n^f$), $\mathbf{y} \in \mathbb{R}^{n^f}$ is a vector of forward variables, and $V = (\mathbf{x}, \mathbf{y})$. Let us first rewrite system (7) in a compact way using the linear operator L_-, acting on $C([-1,0], \mathbb{R}^n)$ and defined as follows:

$$
L_- (V(t)) = \int_0^1 d\mu(u) V(t - u).
$$

To study system (7), which incorporates forward variables, d'Albis et al. (2012) suggest extending the set of initial conditions to $C([-1,0], \mathbb{R}^n) \times \mathbb{R}^n$. A solution to (7) is defined as a function $V(t) \in T$ where:

$$
T = C([-1,0], \mathbb{R}^n) \times \{V \in C([0,\infty), \mathbb{R}^n) : \|V\|_\infty < \infty\},
$$

with initial conditions $(\bar{\mathbf{x}}(\theta), \bar{\mathbf{y}}(\theta))$ defined on $C([-1,0], \mathbb{R}^n)$ by $\bar{\mathbf{y}}(0) = \bar{\mathbf{y}}(0^-)$ and where $V(t)$ satisfies (7). Note that the solution may be multivalued at $t = 0$ as $\mathbf{y}(0^+)$ may be different from $\bar{\mathbf{y}}(0^-)$. D'Albis et al. (2012) further allows one to consider a problem where an initial jump is possible; to compute a possible jump at $t = 0$, we modify the definition of L_- to make it act on $C([-1,0], \mathbb{R}^n) \times \mathbb{R}^n$:

$$
L_- (V(t), v) = \int_0^1 d\mu(u) V(t - u) + \left(\mu(0^+) - \mu(0^-)\right) v.
$$

Moreover, the initial conditions $X = (\bar{\mathbf{x}}(\theta), \bar{\mathbf{y}}(\theta), (\mathbf{x}(0^+), \mathbf{y}(0^+)))$ now belong to $C([-1,0], \mathbb{R}^n) \times \mathbb{R}^n$.

Information concerning the local existence and multiplicity of solutions is contained in the characteristic function. Let $\Delta_{L_-}(\lambda) = \lambda I - \int_{-1}^{0} d\mu(u) e^{\lambda u}$ denote the characteristic function of (7). It can be computed as follows:

$$\Delta_{L_-}(\lambda) = \prod_{i=1}^{N_1^b} (\lambda - \alpha_i) \delta_{L_-}(\lambda),$$

where $\delta_{L_-}(\lambda)$ is the characteristic function of system (6) and where the $(\alpha_i)_{1 \le i \le N_1^b}$ denote the N_1^b roots that appear as a consequence of the differentiation of the algebraic equations of system (6). If an algebraic equation reduces to a differential equation when differentiated once with respect to time, $N_1^b = n_1^b$. If this reduction needs more than one differentiation, $N_1^b > n_1^b$ but N_1^b conditions are now provided at $t = 0$.

Let $Q_{\alpha_i}(X)$ be the spectral projection on the vector space spanned by $e^{\alpha_i t}$. Then $Q_{\alpha_i}(X) = e^{\alpha_i t} H_{\alpha_i} R_{\alpha_i}(X)$, where:

$$R_{\alpha_i}(X) = (\mathbf{x}(0^+), \mathbf{y}(0^+)) + \sum_{j=1}^{3} \int_{-1}^{0} d\bar{\mu}_j(u) e^{\alpha_i u} \int_{u}^{0} e^{-\alpha_i s} d\bar{\mu}_j(s)(\bar{\mathbf{x}}(s), \bar{\mathbf{y}}(s))$$

and where H_{α_i} is a matrix such that $\Delta_{L_-}(\alpha_i) H_{\alpha_i} = H_{\alpha_i} \Delta_{L_-}(\alpha_i) = 0$. The computation of $H_{\alpha_i} R_{\alpha_i}(X)$ (see Theorem 3.16 in d'Albis et al., 2012) shows that it is proportional to:

$$\mathbf{x}_1(0) - \int_{-1}^{0} d\bar{\mu}_2(u) W(u).$$

This implies that $H_{\alpha_i} R_{\alpha_i}(X) = 0$.

Let us assume in what follows that $\Delta_{L_-}(\lambda) = 0$ has no roots with real part equal to 0 and let n^+ denote the number of roots with positive real parts that are distinct to any α_i.

If $n^+ = 0$, there is no unstable manifold, implying that the set of initial conditions leading to a solution is $C([-1,0], \mathbb{R}^n) \times \mathbb{R}^n$. For any initial condition

$(\mathbf{x}(\theta), \mathbf{y}(\theta)) \in C([-1, 0], \mathbb{R}^n)$, with $\mathbf{y}(0) = \mathbf{y}(0^-)$, and any $(\mathbf{x}(0^+), \mathbf{y}(0^+))$, a continuous and bounded solution can be found.

If $n^+ > 0$, there exists an unstable manifold and one must use the spectral projection formula to describe the solutions to system (7). Let $(\lambda_j)_{1 \leq j \leq n^+}$ be the characteristic roots with positive real part of $\delta_{L_-}(\lambda) = 0$. The spectral projection $Q_{\lambda_j}(X)$ on the vector space spanned by $e^{\lambda_j t}$ is $Q_{\lambda_j}(X) = e^{\lambda_j t} H_{\lambda_j} R_{\lambda_j}(X)$ where:

$$R_{\lambda_j}(X) = (\mathbf{x}(0^+), \mathbf{y}(0^+)) + \sum_{j=1}^{2} \int_{-1}^{0} d\mu_j(u) e^{\lambda_j u} \int_u^0 e^{-\lambda_j s} d\mu_j(s) (\bar{\mathbf{x}}(s), \bar{\mathbf{y}}(s))$$

and where H_{λ_j} satisfies $\Delta_{L_-}(\lambda_j) H_{\lambda_j} = H_{\lambda_j} \Delta_{L_-}(\lambda_j) = 0$. As the dynamics belong to the stable manifold, the projection on the unstable manifold should be null:

$$Q_{\lambda_j}(X) = 0. \tag{8}$$

We thus obtain a system of n^+ equations with n^f unknowns, which are given by $\mathbf{y}(0^+)$. Since eigenvectors may be linearly dependent, system (8) can be decomposed into two parts: a system of s equations with n^f unknowns, and $(n^+ - s)$ conditions on the initial known conditions $(\bar{\mathbf{x}}(.), \bar{\mathbf{y}}(.))$, where $\bar{\mathbf{x}}(0^-), \bar{\mathbf{x}}(0^+)$ and $\bar{\mathbf{y}}(0^-)$ are given. As the adjoint eigenvectors (denoted $(W_i^*)_{1 \leq i \leq s}$) are linearly independent, we can write this formally as:

$$W_i^*\left(0, \mathbf{y}(0^+) - \bar{\mathbf{y}}(0^-)\right) = M_i(\bar{\mathbf{x}}(.), \bar{\mathbf{y}}(.)) \text{ for } 1 \leq i \leq s,$$

and

$$0 = M_i(\bar{\mathbf{x}}(.), \bar{\mathbf{y}}(.)) \text{ for } s+1 \leq i \leq n^+,$$

where $M_i(\bar{\mathbf{x}}(.), \bar{\mathbf{y}}(.))$ is an operator acting on the initial conditions, which is defined using the fact that the spectral projection on the unstable manifold has to be null. The first equation implies that W_i^* should not be colinear to the x-axis if we want to avoid degeneracies. As the W_i^* are orthogonal to the stable

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manifold, the stable manifold should not be orthogonal to the x-axis. If $s < n^f$, there are multiple solutions as some components of $y(0^+)$ can be freely chosen to have a solution. If $n^+ > s$, there is no solution generically; whatever $y(0^+)$, the system of n^+ equations with n^f unknowns cannot be solved unless the initial condition happens to satisfy the conditions, which is not guaranteed. If $s = n^f$, the system for $\mathbf{y}\left(0^+\right) - \bar{\mathbf{y}}\left(0^-\right)$ has the same number of equations as unknowns. Thus, as the W_i^* are linearly independent, if a solution exists, it is unique. $\square$

Corollary 1. *Provided that adjoint eigenvectors are linearly independent, the following holds. If $n^f < n^+$, system (6) may have no solution. If $n^f = n^+$, the system always has a unique solution. If $n^f > n^+$, it always has multiple solutions.*

To establish a rule for existence and uniqueness, the proof of Theorem 1 finds initial conditions for forward variables (i.e., $\mathbf{y}\left(0^+\right)$) such that the projection of the dynamics on the unstable manifold is the null vector. In our case, the number of unknowns has the same dimension as $\mathbf{y}$. The number of forward variables is hence compared to the number of conditions obtained by setting the considered projection to zero; these conditions are linked to the number of eigenvalues with positive real parts. Conversely, as the dimensions of the stable manifold and the set of initial conditions are infinite, the information on the number of backward variables is not involved in the argument. As in finite dimensional system, multiple solutions implies indeterminacy.

3.2 Functional systems with advances

Let us now study a linear system written as:

$$
\begin{cases}
\mathbf{x}'(t) = \int_t^{t+1} d\bar{\mu}_1\,(u-t)\,W\,(u)\,, \\[2mm]
\mathbf{y}_0'(t) = \int_t^{t+1} d\bar{\mu}_2\,(u-t)\,W\,(u)\,, \\[2mm]
\mathbf{y}_1(t) = \int_t^{t+1} d\bar{\mu}_3\,(u-t)\,W\,(u)\,, \\[2mm]
\mathbf{x}\,(0) = \bar{\mathbf{x}}\,(0)\ \text{ given},
\end{cases}
\tag{9}
$$

where $\mathbf{x} \in \mathbb{R}^{n^b}$ is a vector of n^b backward variables whose dynamics are characterized by ADEs and where $\mathbf{x}'$ denotes its gradient. Here $\mathbf{y}_0 \in \mathbb{R}^{n^f}$ and $\mathbf{y}_1 \in \mathbb{R}^{n_1^f}$ are vectors of n^f and n_1^f forward variables characterized, respectively, by differential and algebraic equations with advances. Moreover, $W = (\mathbf{x}_0, \mathbf{y}_0, \mathbf{y}_1)$ is a vectorial function. A solution is defined as in the previous section.

Let n^- denote the number of eigenvalues with negative real parts of the characteristic function of system (9) and let s be the number of independent eigenvectors of the characteristic function generated by the n^- eigenvalues. Assuming H1 and provided that $s \geq 1$, system (9) displays a saddle point configuration with an unstable manifold of infinite dimension and a stable manifold of dimension s.

Assumption H3. The unstable manifold is not transverse to the (y_0, y_1) coordinates.

We obtain the following result.

Theorem 2. *Let H1 and H3 prevail. There exists a solution to system (9) if $n^b = s$ and there may be no solution if $n^b > s$. A solution is unique if and only if $n^- = s$.*

<u>Proof.</u> Since algebraic equations reduce to ADEs when differentiated a finite

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number of times, system (9) can be rewritten as:

$$
\begin{cases}
\mathbf{x}'(t) = \int_t^{t+1} d\mu_1 \left(t - u\right) V\left(u\right), \\[2mm]
\mathbf{y}'(t) = \int_t^{t+1} d\mu_2 \left(t - u\right) V\left(u\right), \\[2mm]
x\left(0\right) = \bar{x}\left(0\right) \ \text{given},
\end{cases}
$$

where $\mathbf{x} \in \mathbb{R}^{n^b}$ is a vector of backward variables, $\mathbf{y} \in \mathbb{R}^{n-n^b}$ is a vector of forward variables (with $n \equiv n^b + n_1^b + n^f$), and $V = (\mathbf{x}, \mathbf{y})$. Let n^- be the number of eigenvalues with negative real parts, and let s be the number of linearly independent eigenvectors. Any element of the stable space can be written as:

$$
V\left(t\right) = \sum_{j=0}^{n^-} \alpha_j v_j e^{\lambda_j t}, \tag{10}
$$

where the $(\lambda_j)_{1 \leq j \leq n^-}$ are eigenvalues with negative real parts, the $(v_j)_{1 \leq j \leq n^-}$ are eigenvectors, and the $(\alpha_j)_{1 \leq j \leq n^-}$ are residues.

Evaluating system (10) implies solving a system with n^- unknowns and n^b equations. Since the $(v_j)_{1 \leq j \leq n^-}$ may be linearly dependent, the system can be split into two parts. Let $(w_j)_{1 \leq j \leq s}$ be the family of linearly independent eigenvectors. The first subsystem we obtain can be rewritten as:

$$
\sum_{j=0}^{n^-} \beta_j w_j = \bar{\mathbf{x}}\left(0\right),
$$

yielding a system of s unknown $(\beta_j)_{1 \leq j \leq s}$ and n^b constraints. And, when the $(\beta_j)_{1 \leq j \leq s}$ are defined, we obtain a second system that can be rewritten as:

$$
\sum_{j=0}^{n^-} \alpha_j v_j = \sum_{j=0}^{s} \beta_j w_j,
$$

yielding a system of s equations and n^- unknowns, namely the $(\alpha_j)_{1 \leq j \leq n^-}$. $\square$

Corollary 2. *Given linearly independent eigenvectors, the following hold. If $n^- < n^b$, system (9) may have no solution. If $n^- = n^b$, the system always has a unique solution. And if $n^- > n^b$, it always has multiple solutions.*

Here, the rule that establishes the existence and uniqueness of solutions is different from that presented in Theorem 1. With advances, as the dimension of the unstable manifold is infinite, the idea is to find initial conditions for forward variables that permit one to write the dynamics on the stable manifold. This is why we use the number of eigenvalues with negative real parts to determine whether the solution exists and is unique. Since we rewrite the system as a finite dimensional system, the proof of Theorem 2 is similar to what can be found for ordinary differential equations.

4   Economic examples

In this section, we solve two economic models that give rise to the kind of equations we are interested in. The first, due to Boucekkine et al. (2002), is an overlapping generations model in which the dynamics of human capital is characterized by an algebraic equation with delay. The second considers the decentralized economy of a time-to-build model. The dynamics are characterized by a two-dimensional system with both a backward and a forward variable.

4.1   A scalar algebraic equation with a backward variable

Boucekkine et al. (2002) consider an overlapping generations model where agents make schooling and retirement decisions. By summing individual human capital accumulation over cohorts, the authors find that the aggregate human capital, denoted $H(t)$, is given by:

$$H(t) = A \int_{t-P}^{t-T} m(t-z) H(z) \, dz, \tag{11}$$

where $A > 0$ denotes the product of a scaling parameter of the production function and the birth rate, $T > 0$ is the age at which the agent leaves school, and $P > T$ is the age at retirement.[4] $m(t-z)$ is a survival function, which

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[4]See equation (25) in Boucekkine et al. (2002).

is parameterized in Boucekkine et al. (2002), but that we keep general in the following. We simply assume $m(x) > 0$ and $m'(x) < 0$ for all $x \in [T, P]$.

Equation (11) is an algebraic equation with delay. Moreover, $H(t)$ is a backward variable and its initial condition can be written as $H(\theta) = \bar{H}(\theta)$ where $\theta \in [-P, 0)$ and $\bar{H}(.) \in C^b([-P, 0))$. Note that $\bar{H}(0^-)$ can be different from $H(0)$, which is given by:

$$H(0) = A \int_{-P}^{-T} m(-z) \bar{H}(z) \, dz. \tag{12}$$

Boucekkine et al. (2002) propose an analytical study of the BGP and a numerical stability assessment. Below, we prove the existence and uniqueness of a trajectory that converges to the BGP.

Lemma 1. *There exists a unique solution to equation (11).*

<u>Proof.</u> To study the dynamic properties of $H(t)$ for $t \geq 0$, we consider the characteristic equation $\delta(\lambda) = 0$, where $\delta(\lambda)$ is defined by:

$$\delta(\lambda) = 1 - A \int_T^P m(z) e^{-\lambda z} dz.$$

Studying the real roots is straightforward; since $\delta' > 0$, $\lim_{\lambda \to -\infty} \delta(\lambda) = -\infty$ and $\lim_{\lambda \to \infty} \delta(\lambda) = 1$, there exists a unique real root γ that is positive if and only if $1 < A \int_T^P m(z) \, dz$. This root represents the growth rate of the BGP. The existence and uniqueness of trajectories that converge to the BGP depends on the number of complex roots with real parts greater than γ. Let us now prove that no complex root, denoted $\lambda = p + iq$, exists that satisfies $p > \gamma$. Suppose the contrary–that such a root exists. As $\mathrm{Re}(\delta(p + iq)) = 0$, we would have:

$$1 = A \left| \int_T^P m(z) e^{-pz} \cos(qz) \, dz \right| < A \int_T^P m(z) e^{-pz} dz,$$

which contradicts the fact that $\delta > 0$ for $\gamma < p$. Consequently, $n^+ = 0$ and we use Theorem 1 to conclude. $\square$

Lemma 1 and its proof demonstrate that our theorems can be easily applied to a given economic problem. It shows that the solution of the problem considered by Boucekkine et al. (2002) always exists, even when a general survival function is considered. Further, note that the dynamics of $H(t)$ will be characterized by short term fluctuations generated by the complex roots of the characteristic equation. These fluctuations are a natural output of models featuring a realistic age structure of the population and can be found in the pioneering works on stable population theory of Lotka (1939) and Feller (1941).

4.2 A two-dimensional system with a DDE and a forward variable

We now consider a neoclassical model with a time-to-build assumption, meaning that new investments take time to become productive capital. Most of the earlier contributions in continuous time (El Hodiri et al., 1972; Asea and Zak, 1997; and Bambi, 2008) focused on optimal paths. Recently Bambi et al. (2012) also considered the decentralized economy of an endogenous growth model. Below we study the decentralized economy of a neoclassical growth model.

The representative agent is infinitely lived and solves the following problem:

$$\max \int_0^{+\infty} e^{-\rho t} u\left(c\left(t\right)\right) dt,$$

$$s.t. \left| \begin{array}{l} a'\left(t\right) = r\left(t\right) a\left(t\right) + w\left(t\right) - c\left(t\right), \\ a\left(0\right) \text{ given, } \lim_{t \to +\infty} e^{-\int_0^t r(s)ds} a\left(t\right) \geq 0, \end{array} \right. \tag{13}$$

where $c(t)$ is consumption at time t, $a(t)$ is financial wealth (and $a'(t)$ its derivative with respect to time), $r(t)$ is the interest rate, $w(t)$ is the wage, and $\rho > 0$ is the discount rate. The utility function is given by:

$$u\left(c\right) = \frac{c^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}}, \tag{14}$$

where $\sigma > 0$ is the elasticity of intertemporal substitution. The optimal path of consumption is given by the traditional Euler equation:

$$c'\left(t\right) = \sigma\left[r\left(t\right) - \rho\right] c\left(t\right). \tag{15}$$

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Markets are perfectly competitive and, in particular, firms have access to a competitive rental market for capital goods. $k(t)$ denotes the stock of capital and $f(k(t))$ is output, with $f' > 0$ and $f'' < 0$. The optimization problem of a representative firm is static and the optimal behavior is to equalize the marginal productivity of each factor to its cost. One obtains:

$$f'(k(t)) = r(t) + \delta \text{ and } f(k(t)) - k(t) f'(k(t)) = w(t), \qquad (16)$$

where $\delta \geq 0$ is the rate of depreciation.

The time-to-build assumption means that the stock of capital at time t is the aggregation of all investments made before time $t - \tau$:

$$k(t) = k(0) + \int_0^{t-\tau} e^{-\delta(t-\tau-s)} i(s) \, ds. \qquad (17)$$

This implies that the law of motion for capital is written as:

$$k'(k) = i(t - \tau) - \delta k(t), \qquad (18)$$

where $i(t - \tau)$ is the investment made at time $t - \tau$ and where $\tau \geq 0$ is the time lag. Obviously, for $\tau = 0$ one has the standard neoclassical model.

The market clearing condition on the goods market is given by:

$$f(k(t)) = c(t) + i(t). \qquad (19)$$

The market clearing condition on the asset market is:

$$a(t) = k(t) + \int_{t-\tau}^{t} i(s) \, ds. \qquad (20)$$

Investment is valued on the asset market, as it has already been transformed into capital or is not yet productive. By differentiating (20) with respect to time and substituting (18) and (16), one obtains the differential equation on assets given by (13).

Substituting (16) into (15) and (19) into (18), the problem can be written as the following two-dimensional system:

$$\begin{cases} c'(t) = \sigma \left[f'(k(t)) - \delta - \rho \right] c(t), \\ k'(k) = f(k(t-\tau)) - c(t-\tau) - \delta k(t). \end{cases} \tag{21}$$

Since $k(\theta) = \bar{k}(\theta)$ for $\theta \in [-\tau, 0]$ is given, there is one forward variable, $c(t)$, and one backward variable, $k(t)$. We obtain the following result.

Lemma 2. *For τ sufficiently small, there exists a unique solution to system (21).*

<u>Proof.</u> Let the pair (c^*, k^*) denote the steady state of system (21). The characteristic function, denoted $\Delta(\lambda)$, is the determinant of the Jacobian matrix of the system linearized in the neighborhood of the steady state. Simple algebra gives:

$$\Delta(\lambda) = \lambda^2 + \delta\lambda + e^{-\lambda\tau} \left[A - \lambda(\rho + \delta) \right], \tag{22}$$

with $A := \sigma c^* f''(k^*)$. According to Theorem 1, there exists a unique solution to the system if $\Delta(\lambda) = 0$ has one root with positive real part. It is simple to prove that $\Delta(\lambda) = 0$ has, at least, one positive real root. We indeed have $\Delta(0) < 0$ and $\lim_{\lambda \to +\infty} \Delta(\lambda) = +\infty$. We now want to establish conditions under which the latter root is the only one with a positive real part. We proceed in three steps.

Step 1. We show that for $\tau = 0$, there is only one positive root. As $\Delta(\lambda)|_{\tau=0} = \lambda^2 - \rho\lambda + A$, the proof is immediate.

Step 2. We give conditions for the existence of pure imaginary roots, denoted iq_0. Using (22), such roots solve $\Delta(iq_0) = 0$, where:

$$\Delta(iq_0) = -q_0^2 + \delta i q_0 + \left[\cos(q_0\tau) - i\sin(q_0\tau) \right] \left[A - iq_0(\rho + \delta) \right].$$

By separating the real and the imaginary parts, one finds that q_0 must solve:

$$\begin{cases} q_0^2 = \cos(q_0\tau) A - \sin(q_0\tau) q_0 (\rho + \delta), \\ q_0\delta = \cos(q_0\tau) q_0 (\rho + \delta) + \sin(q_0\tau) A, \end{cases}$$

or, equivalently:

$$q_0^4 + q_0^2 \left[\delta^2 - (\rho + \delta)^2 \right] - A^2 = 0. \tag{23}$$

We show that there exists a positive q_0 that solves (23) by setting $q_0^2 = x$ and studying the equation $\Phi(x) = 0$, where $\Phi(x) = x^2 + x \left[\delta^2 - (\rho + \delta)^2 \right] - A^2$. We let τ_0 denote the lowest positive τ associated with q_0.

Step 3. We show that $d\lambda/d\tau > 0$ for $\lambda = iq_0$. Applying the implicit function theorem to $\Delta(\lambda) = 0$ gives:

$$\frac{d\lambda}{d\tau} = \left(\frac{\delta + e^{-\lambda\tau}(\rho + \delta)}{\lambda^2(\lambda + \delta)} - \frac{\tau}{\lambda} - \frac{2}{\lambda^2} \right)^{-1}.$$

This expression is then evaluated for $\lambda = iq_0$. Using (23), one obtains:

$$\left. \frac{d\lambda}{d\tau} \right|_{\lambda = iq_0} = \frac{\delta^2 + q_0^2}{1 + \frac{A^2}{q_0^4}},$$

which is positive.

We conclude that for all $\tau \in [0, \tau_0)$, $\Delta(\lambda) = 0$ has a unique root with positive real part. $\square$

According to Lemma 2, the length of time between when the investment expenditure is made and when the invested goods are transformed into productive capital is crucial for the existence of an intertemporal equilibrium. If it takes too much time to produce capital, the equilibrium may not exist. Moreover, multiple solutions cannot arise in such an economy, as there always exists at least one positive eigenvalue. But, as in the previous example, the dynamics are characterized by fluctuations whose magnitude decreases with time. These fluctuations are due to the time-to-build assumption, and can be explained by an over-investment made at date 0 to adjust to a capital stock that is too low. Oscillations in investment thus have the effect of generating cycles, which in turn trigger an "echo effect" (Boucekkine et al., 1997) or "wave-like" business fluctuations (Bambi et al., 2012).

5 Conclusion

This paper proposes theorems for the existence and uniqueness of solutions to systems of differential or algebraic equations with delays or advances. These theorems propose conditions that link the space of unknown initial conditions to the sign of the roots of the characteristic equation, just like the well-known Blanchard-Kahn conditions. They could therefore encourage the use of DDEs and ADEs in economics, which would enable the analytical study of many phenomena. However, certain economic dynamics are characterized by differential equations that have both delays and advances. In such cases, both the stable and unstable manifolds are of infinite dimensions and hence the theorems developed in this paper do not apply. We leave this problem for future research.

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